Método de descomposición LU

La ecuación $Ax=b$ se puede escribir como

$$\left\{ (L\underbrace{U)x}_{Y} = b \right\} \Rightarrow \text{con } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ; \; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

y $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow$ Donde: $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$\Rightarrow$ tal que: $\left\{ LY = b \quad \text{ y } \quad Y = Ux \right\} \Rightarrow Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

U: es el resultado de la eliminación Gaussiana

LY: se construye con los factores multiplicativos de dicha eliminación

Ej: $\left\{ l_{21} = \dfrac{a_{21}}{a_{11}} \right\}$

Ejemplo

$$\left.\begin{aligned} 6x_1 + 18x_2 + 3x_3 &= 3 \\ 2x_1 + 12x_2 \qquad &= 19 \\ 4x_1 + 15x_2 + 3x_3 &= 0 \end{aligned}\right\} \quad A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 0 \\ 4 & 15 & 3 \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix} \; ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & -1 \\ 0 & 0 & 3/2 \end{bmatrix} \longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{bmatrix}$$

$$l_{21} = \frac{2}{6}$$

$$l_{31} = 4/6$$

$$l_{32} = 3/18$$

$$LU = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 0 \\ 4 & 15 & 3 \end{bmatrix}$$

Ahora con $Ly = b$

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/2 & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix} \quad \begin{cases} y_1 = 3 \\ y_2 = 18 & 19 - \frac{1}{3}(3) = 18 \\ y_3 = -11 & 0 - \frac{4}{6}(3) - \frac{3}{6}(10) \end{cases}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\underset{U}{\begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & -1 \\ 0 & 0 & 3/2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 18 \\ -11 \end{bmatrix}$$

$$x_3 = -11 \cdot \frac{2}{3} = -22/3$$

$$x_2 = 18 - 22/3 = 1.7$$

$$x_1 = \left[3 - 3\left(\frac{-22}{3}\right) = -0.93 - 18(1.7)\right] 6$$

Método de Gauss-Seidel

$$Ax = b \rightarrow x_i^{k+1} = M x_i^k + C$$

donde: $A = N - P$

$$M = N^{-1} P \quad y \quad C = N^{-1} b$$

N es tal que $n_{ij} = \begin{cases} a_{ij} & i \leq j \\ 0 & i > j \end{cases}$

$$P_{ij} \begin{cases} -a_{ij} & i > j \\ 0 & i \leq j \end{cases}$$

$$x_i^{k+1} = \frac{-\sum_{1 \leq j \leq i-1} a_{ij} x_j^{(k+1)} - \sum_{i+1 \leq j < n} a_{ij} x_j^{(k)} + b_i}{a_{ij}}$$

$i = 1, 2, \ldots, n$

$$\hookrightarrow x_0^{(k+1)} = \frac{-\sum_{1 \leq j \leq i-1} a_{0j} x_j^{(k+1)} - \sum_{i+1 \leq j < n} a_{0j} x_j^{(k)} + b_0}{a_{0j}}$$

$$\Rightarrow \underline{\underline{x_0}} = \frac{-\sum_{i+1 \leq j < n} a_{0j} x_j^{(k)} + b_0}{a_{0j}}$$