Segunda diferencia finita

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Partiendo de la serie de Taylor

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + R_1 \Rightarrow R_1 = \frac{f''(\xi)h^2}{2!} \therefore O(h^2)$$

$$f(x_{i+2}) = f(x_i) + 2f'(x_i)h + \frac{(2h)^2 f''(x_i)}{2!} + R_2$$

$$\hookrightarrow R_2 = \frac{f'''(\xi)h^3}{3!} \therefore O(h^3)$$

$$f(x_{i+2}) = f(x_i) + 2hf'(x_i) + \frac{4h^2 f''(x_i)}{2!} + O(h^3)$$

$$\therefore \frac{\cancel{4}h^2 f''(x_i)}{\cancel{2}} = f(x_{i+2}) - f(x_i) - 2hf'(x_i) + O(h^3)$$

$$2h^2 f''(x_i) = f(x_{i+2}) - f(x_i) - 2hf'(x_i) + O(h^3)$$

Retomando

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$2h^2 f''(x_i) = f(x_{i+2}) - f(x_i) - \left(\frac{f(x_{i+1}) - f(x_i)}{\cancel{h}}(2\cancel{h})\right) + O(h^3)$$

$$h^2 f''(x_i) = f(x_{i+2}) - \cancel{f(x_i)} - 2(f(x_{i+1})) + \cancel{2}f(x_i) + O(h^3)$$

$$h^2 f''(x_i) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) + O(h^3)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + \left.\frac{O(h^3)}{h^2}\right\} O(h)$$

Hacia atrás $f(x_{i-1}) = f(x_i) - hf'(x_i) + R_1 \rightarrow O(h)$

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + \frac{2^2 h^2 f''(x_i)}{2!} + R_2 \rightarrow O(h^3)$$

$$f(x_{i-2}) = f(x_i) - 2hf'(x_i) + 2h^2 f''(x_i)$$

$$f(x_{i-2}) - f(x_i) = 2h^2 f''(x_i) - 2hf'(x_i)$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$\therefore f(x_{i-2}) - f(x_i) = 2h^2 f''(x_i) - 2f(x_i) + 2f(x_{i-1})$$

$$\Rightarrow h^2 f''(x_i) = f(x_{i-2}) \cancel{-f(x_i)} - 2f(x_{i-1}) + \cancel{2}f(x_i)$$

$$\Rightarrow \boxed{f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2}}$$

Centrada:

$$f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2 f''(x_i)}{2} + O(h^3)$$

$$f(x_{i-1}) = f(x_i) - hf'(x_i) + \frac{h^2 f''(x_i)}{2} + O(h^3)$$

$$\Downarrow$$

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + h^2 f''(x_i) + O(h^3)$$

$$\boxed{\frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)}{h^2} = f''(x_i)}$$

Tercera diferencia finita hacia adelante

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + O(h^2)$$

$$f(x_{i+2}) = f(x_i) + 2h f'(x_i) + \frac{(2h)^2 f''(x_i)}{2!} + O(h^3)$$

$$f(x_{i+3}) = f(x_i) + 3h f'(x_i) + \frac{(3h)^2 f''(x_i)}{2!} + \frac{(3h)^3 f'''(x_i)}{3!} + O(h^4)$$

$$\therefore \frac{f'''(x_i)\,27h^3}{6} = \left[f(x_{i+3}) - \left(f(x_i) + 3h\left(\frac{f(x_{i+1}) - f(x_i)}{h}\right) + \frac{9h^2}{2}\left(\frac{f(x_i) - 2f(x_{i+1}) + f(x_{i+2})}{h^2}\right)\right)\right]$$

$$\frac{f'''(x_i)\,27h^3}{6} = f(x_{i+3}) - f(x_i) - 3\left(f(x_{i+1}) - f(x_i)\right) - \frac{9}{2}\left(f(x_i) - 2f(x_{i+1}) + f(x_{i+2})\right)$$

$$\frac{f'''(x_i)\,27h^3}{6} = f(x_{i+3}) - f(x_i) - 3f(x_{i+1}) + 3f(x_i) - \frac{9}{2}f(x_i) + 9f(x_{i+1}) - \frac{9}{2}f(x_{i+2})$$

$$h^3 f'''(x_i) = f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)$$

finalmente

$$\boxed{f'''(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}}$$