Demostraciones

1 - Para la segunda derivada hacia adelante (dos puntos)

(1) $f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \dfrac{f''(x_i)(x_{i+1} - x_i)^2}{2!} + O(h)$

(2) $h = x_{i+1} - x_i = x_{i+2} - x_{i+1}$

(3) $f(x_{i+2}) = f(x_i) + f'(x_i)(x_{i+2} - x_i) + \dfrac{f''(x_i)(x_{i+2} - x_i)^2}{2!} + O(h)$

(4) $2h = x_{i+2} - x_i$

→ Sustituyendo (2) en (1) y (4) en (3)

(1) $f(x_{i+1}) = f(x_i) + f'(x_i)h + \dfrac{f''(x_i)h^2}{2} + O(h)$

(3) $f(x_{i+2}) = f(x_i) + f'(x_i)2h + \dfrac{f''(x_i)(2h)^2}{2} + O(h)$.

→ Multiplicando (1) por -2.

(1) $-2f(x_{i+1}) = -2f(x_i) - f'(x_i)2h - \dfrac{f''(x_i)\cancel{2}h^2}{\cancel{2}}$

→ Sumando (1) y (3)

$$
\begin{array}{rl}
 & -2f(x_{i+1}) = -2f(x_i) - f'(\cancel{x_i})2h - f''(x_i)h^2 + O(h^2) \\
+ & f(x_{i+2}) = f(x_i) + f'(\cancel{x_i})2h + 2f''(x_i)h^2 + O(h) \\
\hline
\end{array}
$$

(5) $f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)h^2 + O(h^3)$

→ Despejando $f''(x_i)$ en (5)

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + \underbrace{\frac{O(h^{\cancel{3}})}{\cancel{h^2}}}_{O(h)}$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

2 - Para la segunda derivada hacia atrás (dos puntos)

(1) $f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \dfrac{f''(x_i)(x_{i-1} - x_i)^2}{2!} + O(h)$

(2) $h = x_{i-1} - x_i = x_{i-2} - x_{i-1}$

(3) $f(x_{i-2}) = f(x_i) + f'(x_i)(x_{i-2} - x_i) + \dfrac{f''(x_i)(x_{i-2} - x_i)^2}{2!} + O(h)$

(4) $2h = x_{i-2} - x_i$

→ Sustituyendo ② en ① y ④ en ③

① $f(x_{i-1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + O(h)$

③ $f(x_{i-2}) = f(x_i) + f'(x_0)2h + \frac{f''(x_0)(2h)^2}{2} + O(h)$

→ Multiplicando ① por -2

① $-2f(x_{i-1}) = -2f(x_0) - f'(x_i)2h - \frac{f''(x_i)\cancel{2}h^2}{\cancel{2}} + O(h^2)$

→ Sumando ① y ③

$$
\begin{aligned}
-2f(x_{i-1}) &= -2f(x_i) - \cancel{f'(x_0)2h} + f''(x_0)h^2 + O(h^2) \\
+\ f(x_{i-2}) &= f(x_i) + \cancel{f'(x_i)2h} + f''(x_i)2h^2 + O(h)
\end{aligned}
$$

⑤ $f(x_{i-2}) - 2f(x_{i-1}) = -f(x_i) + f''(x_0)h^2 + O(h^3)$

→ Despejando $f''(x_i)$ en ⑤

$f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} + \frac{O(h^{\cancel{3}})}{\cancel{h^2}}$

$O(h)$

$\Rightarrow f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} + O(h)$

3 – Para la segunda derivada centrada (2 puntos)

① $f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)(x_{i+1} - x_0)^2}{2!} + O(h)$

② $h = x_{i+1} - x_i$

③ $f(x_{i-1}) = f(x_0) + f'(x_0)(x_{i-1} - x_i) + \frac{f''(x_0)(x_{i-1} - x_i)^2}{2!} + O(h)$

④ $-h = x_{i-1} - x_i$

→ Sustituyendo ② en ① y ④ en ③

① $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2} + O(h)$

③ $f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2} + O(h)$

→ Sumando ① y ③

$$
\begin{aligned}
f(x_{i+1}) &= f(x_i) + \cancel{f'(x_i)h} + \frac{f''(x_0)h^2}{2} + O(h) \\
+\ f(x_{i-1}) &= f(x_0) - \cancel{f'(x_i)h} + \frac{f''(x_i)h^2}{2} + O(h)
\end{aligned}
$$

⑤ $f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)h^2 + O(h)$

→ Sustituyendo $f''(x_i)$ de 5

$$f''(x_i) = \frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_0)}{h^2} + O(h)$$

4- Tercera derivada hacia adelante (tres puntos)

1 $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \frac{f^{(3)}(x_0)h^3}{3!} + O(h)$

2 $h = x_{i+1} - x_i$

3 $f(x_{i+2}) = f(x_i) + f'(x_0)2h + \frac{f''(x_i)(2h)^2}{2!} + \frac{f^{(3)}(x_i)(2h)^3}{3!} + O(h)$

4 $2h = x_{i+2} - x_i$

5 $f(x_{i+3}) = f(x_0) + f'(x_i)3h + \frac{f''(x_i)(3h)^2}{2!} + \frac{f^{(3)}(x_0)(3h)^3}{3!} + O(h)$

6 $3h = x_{i+3} - x_i$

→ Simplificando 1, 3 y 5

1 $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_0)h^2}{2} + \frac{f^{(3)}(x_i)h^3}{6} + O(h)$

3 $f(x_{i+2}) = f(x_i) + 2f'(x_i)h + f''(x_i)h^2 + \frac{4}{3}f^{(3)}(x_i)h^3 + O(h)$

5 $f(x_{i+3}) = f(x_i) + 3f'(x_i)h + \frac{9}{2}f''(x_i)h^2 + \frac{9}{2}f^{(3)}(x_i)h^3 + O(h)$

→ Multiplicando 3 por -3 y 1 por 3

1 $3f(x_{i+1}) = 3f(x_i) + f'(x_i)3h + \frac{3f''(x_i)h^2}{2} + \frac{3f^{(3)}(x_i)h^3}{6} + O(h)$

⇒ 1 $3f(x_{i+1}) = 3f(x_i) + 3f'(x_i)h + \frac{3}{2}f''(x_i)h^2 + \frac{1}{2}f^{3}(x_i)h^3 + O(h)$

3 $-3f(x_{i+2}) = -3f(x_i) - 6f'(x_i)h - 3f''(x_i)h^2 - 4f^{3}(x_i)h^3 + O(h)$

⇒ Sumando 1, 3 y 5

$3f(x_{i+1}) = 3f(x_i) + 3f'(x_i)h + \frac{3}{2}f''(x_i)h^2 + \frac{1}{2}f^{(3)}(x_i)h^3 + O(h)$

$+ \ -3f(x_{i+2}) = -3f(x_0) - 6f'(x_0)h - \frac{6}{2}f''(x_i)h^2 - \frac{8}{2}f^{(3)}(x_i)h^3 + O(h)$

$f(x_{i+3}) = f(x_0) + 3f'(x_i)h + \frac{9}{2}f''(x_i)h^2 + \frac{9}{2}f^{(3)}(x_i)h^3 + O(h)$

$3f(x_{i+1}) - 3f(x_{i+2}) + f(x_{i+3}) = f(x_i) + f^{(3)}(x_0)h^3 + O(h)$

→ Despejando $f^{(3)}(x_i)$

$$f^{(3)}(x_0) = \frac{3f(x_{i+1}) - 3f(x_{i+2}) + f(x_{i+3}) - f(x_i)}{h^3} + O(h)$$

$$f^3(x_i) = \frac{3f(x_{i+1}) - 3f(x_{i+2}) + f(x_{i+3}) - f(x_i)}{h^3} + O(h)$$ (4)