Theory: Low-Pass and High-Pass Filtering in the Frequency Domain

1. The Frequency Domain Representation of Images

In image processing, an image can be represented in two domains:

- **Spatial domain:** The image is represented by pixel intensities as a function of position

$$f(x,y)$$

- **Frequency domain:** The image is represented by its frequency components

$$F(u,v)$$

The frequency domain describes how rapidly the intensity values change across the image.

- **Low frequencies:** Represent slowly varying intensity regions such as smooth areas and backgrounds.
- **High frequencies:** Represent rapidly changing intensity regions such as edges, fine details, and noise.

Low frequencies correspond to the overall structure of the image, while high frequencies correspond to details and sharp transitions.

2. Fourier Transform in Image Processing

To convert an image from the spatial domain to the frequency domain, the **2D Discrete Fourier Transform (DFT)** is used:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Where:

- $f(x,y)$= input image in spatial domain
- $F(u,v)$= frequency domain representation
- M, N= image dimensions

- u, v= frequency coordinates
- j= imaginary unit

The inverse DFT reconstructs the original image:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

3. Frequency Domain Filtering

Filtering in the frequency domain is performed by multiplying the Fourier transform of the image by a filter function:

$$G(u,v) = H(u,v) \cdot F(u,v)$$

Where:

- $F(u,v)$= Fourier transform of input image
- $H(u,v)$= filter transfer function
- $G(u,v)$= filtered frequency image

The filtered image is then converted back using the inverse DFT.

4. Low-Pass Filtering (LPF)

A Low-Pass Filter allows low frequencies to pass and blocks high frequencies.

Mathematically:

$$G(u,v) = H_{LPF}(u,v) \cdot F(u,v)$$

Where:

$$H_{LPF}(u,v) = \begin{cases} 1, & D(u,v) \le D_0 \\ 0, & D(u,v) > D_0 \end{cases}$$

$D(u, v)$= distance from center
D_0= cutoff frequency

Effect of Low-Pass Filter

- Removes high-frequency components
- Smooths the image
- Reduces noise
- Causes blurring

Applications:

- Noise reduction
- Image smoothing
- Preprocessing for segmentation

5. High-Pass Filtering (HPF)

A High-Pass Filter allows high frequencies to pass and blocks low frequencies.

Mathematically:

$$G(u, v) = H_{HPF}(u, v) \cdot F(u, v)$$

Where:

$$H_{HPF}(u, v) = \begin{cases} 0, & D(u, v) \leq D_0 \\ 1, & D(u, v) > D_0 \end{cases}$$

Effect of High-Pass Filter

- Removes low frequencies
- Enhances edges
- Enhances fine details

- Sharpens the image

Applications:

- Edge detection
- Image sharpening
- Feature extraction

6. Distance Function in Frequency Domain

The distance from the center of the frequency domain is:

$$D(u, v) = \sqrt{(u - M/2)^2 + (v - N/2)^2}$$

This determines whether a frequency is low or high.

7. Summary of Effects

Filter Type	Filter Type	Blocks	Effect
Low-Pass	Low frequencies	High frequencies	Blurring, smoothing
High-Pass	High frequencies	Low frequencies	Edge enhancement, sharpening

8. Conceptual Interpretation

- Low-Pass Filter → removes detail → smooth image
- High-Pass Filter → removes smooth regions → highlights edges