Practical Course: Modeling, Simulation, Optimization

Week 3

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3.A Solutions Exercise Week 2

3.B The weak form in 1-D

The weak form

Suppose we want to discretize the following 1-D conservation law by finite elements

$$\frac{\partial u}{\partial t}(t,x) = -\frac{\partial}{\partial x}\left(\phi\left(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x)\right)\right) + f(t,x), \qquad x \in (0,L), \quad \text{(1a)}$$

$$u(t,0) = 0, \qquad \phi(t,L) = 0, \qquad u(0,x) = u_0(x). \quad \text{(1b)}$$

(e.g. $\phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x)) = v(t,x)u(t,x) - \kappa(t,x)\frac{\partial u}{\partial x}$).

The weak form

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(e.g. $\phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x)) = v(t,x)u(t,x) - \kappa(t,x)\frac{\partial u}{\partial x}$).
For simplicity, we write $\phi(t,x)$ for $\phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x))$.
Multiply by a test function $w(x)$ and integrate from $x=0$ to $x=L$:

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\int_0^L w(x)\frac{\partial \phi}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x.$$

The weak form

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$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\int_0^L w(x)\frac{\partial \phi}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x.$$

$$\int_0^L w(x)\frac{\partial \phi}{\partial x}(t,x)\,\mathrm{d}x = w(x)\phi(t,x)\Big|_{x=0}^{L} - \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\phi(t,x)\,\mathrm{d}x = -\int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\phi(t,x)\,\mathrm{d}x.$$

Weak solution of the problem (1)

A weak solution $u(t,x) \in L^2(0,T;V)$ of the problem (1) satisfies

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\phi(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x, \qquad u(0,x) = u_0(x),$$

for all $w(x) \in V = \{w \in H^1(0,L) \mid w(0) = 0\}$ and almost all time instances t.

An example: heat conduction

Consider the following 1-D diffusion problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = \kappa \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), && && x \in (0,L),\\
&u(t,0) = 0, && \frac{\partial u}{\partial x}(t,L) = 0, && u(0,x) = u_0(x).
\end{aligned}
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$$u(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,L) = 0, \qquad u(0,x) = u_0(x).$$

Multiply by a test function $w(x)$ and integrate from $x = 0$ to $x = L$:

$$\int_0^L w(x) \frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = \kappa \int_0^L w(x) \frac{\partial^2 u}{\partial x^2}(t,x)\,\mathrm{d}x + \int_0^L w(x) f(t,x)\,\mathrm{d}x.$$

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Use integration by parts and the BCs we may rewrite the first term on the RHS

$$\int_0^L w(x) \frac{\partial^2 u}{\partial x^2}(t,x)\,\mathrm{d}x = w(x) \frac{\partial u}{\partial x}(t,x)\Big|_{x=0}^{L} - \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x) \frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x.$$

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Consider the following 1-D diffusion problem

$$\frac{\partial u}{\partial t}(t,x) = \kappa \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \qquad x \in (0,L),$$
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Use integration by parts and the BCs we may rewrite the first term on the RHS

$$\int_0^L w(x)\frac{\partial^2 u}{\partial x^2}(t,x)\,\mathrm{d}x = w(x)\frac{\partial u}{\partial x}(t,x)\Big|_{x=0}^{L} - \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x.$$

So we arrive at following weak form

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\kappa \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x) f(t,x)\,\mathrm{d}x,$$
$$u(0,x) = u_0(x),$$

for all $w \in V := \{w \in H^1(0,L) \mid w(0) = 0\}$ and almost all time instances t.

Question 1

Now consider the following heat conduction problem with a Robin BC

$$\frac{\partial u}{\partial t}(t,x) = \kappa \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x), \qquad x \in (0,L),$$

$$u(t,0) = 0, \qquad -\kappa \frac{\partial u}{\partial x}(t,L) = au(t,L), \qquad u(0,x) = u_0(x).$$

Question: What is the weak formulation for this problem?

A)

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\kappa \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x,$$
$$u(0,x) = u_0(x),$$

for all $w \in V := \{w \in H^1(0,L) \mid w(0) = 0, \frac{\mathrm{d}w}{\mathrm{d}x}(L) = aw(L)\}$ and a.a. time instances t.

B)

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\kappa \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x - aw(L)u(L) + \int_0^L w(x)f(t,x)\,\mathrm{d}x,$$
$$u(0,x) = u_0(x),$$

for all $w \in V := \{w \in H^1(0,L) \mid w(0) = 0\}$ and almost all time instances t.

C) None of the above.

3.C Galerkin discretization in 1-D

Galerkin discretization

We thus arrive at the weak formulation of our problem, for example

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\kappa \int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x,$$
$$u(0,x) = u_0(x)$$

with $u \in L^2(0,T;V)$ and all $w \in V = \{w \in H^1(0,L) \mid w(0) = 0\}$.

Galerkin discretization

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The basic idea for a Galerkin discretization:
Replace the infinite dimensional space V by an N-dimensional subspace $V_N \subset V$.
Note: V_N must be a subspace of V.

This thus leads to a solution $u_N \in L^2(0,T;V_N)$ which satisfies

$$\int_0^L w_N(x)\frac{\partial u_N}{\partial t}(t,x)\,\mathrm{d}x = -\kappa\int_0^L \frac{\mathrm{d}w_N}{\mathrm{d}x}(x)\frac{\partial u_N}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w_N(x)f(t,x)\,\mathrm{d}x,$$
$$u_N(0,x) = u_{0,N}(x),$$

for all $w_N \in V_N$.

Galerkin discretization

We thus arrive at the weak formulation of our problem, for example

$$\int_0^L w(x)\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x = -\kappa\int_0^L \frac{\mathrm{d}w}{\mathrm{d}x}(x)\frac{\partial u}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w(x)f(t,x)\,\mathrm{d}x,$$
$$u(0,x) = u_0(x)$$

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The basic idea for a Galerkin discretization:
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$$\int_0^L w_N(x)\frac{\partial u_N}{\partial t}(t,x)\,\mathrm{d}x = -\kappa\int_0^L \frac{\mathrm{d}w_N}{\mathrm{d}x}(x)\frac{\partial u_N}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w_N(x)f(t,x)\,\mathrm{d}x,$$
$$u_N(0,x) = u_{0,N}(x),$$

for all $w_N \in V_N$.
Two remarks:

- The choice of the subspace V_N determines whether u_N is a good approximation of u.
- The original initial condition $u_0(x) \in L^2(0,L)$ was replaced by $u_{0,N}(x) \in V_N$.

Galerkin approximation: a basis for V_N

We want to find the function $u_N \in L^2(0,T;V_N)$ which satisfies

$$\int_0^L w_N(x)\frac{\partial u_N}{\partial t}(t,x)\,\mathrm{d}x = -\kappa \int_0^L \frac{\mathrm{d}w_N}{\mathrm{d}x}(x)\frac{\partial u_N}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w_N(x) f(t,x)\,\mathrm{d}x,$$
$$u_N(0,x) = u_{0,N}(x),$$

for all $w_N \in V_N$.

Choose a basis $\{\mathbf{N}_1(x), \mathbf{N}_2(x), \dots, \mathbf{N}_N(x)\}$ for $V_N \subset V$ and define the row-vector

$$\mathbf{N}(x) = \begin{bmatrix} \mathbf{N}_1(x) & \mathbf{N}_2(x) & \cdots & \mathbf{N}_N(x) \end{bmatrix}.$$

Because $u_N \in L^2(0,T;V_N)$ and $w_N \in V_N$, we can write

$$u_N(t,x) = \sum_{n=1}^N \mathbf{N}_n(x) u_n(t) = \mathbf{N}(x)\mathbf{u}(t), \qquad w_N(x) = \mathbf{N}(x)\mathbf{w} = \mathbf{w}^\top (\mathbf{N}(x))^\top,$$

where $\mathbf{u} \in L^2(0,T;\mathbb{R}^N)$ and $\mathbf{w} \in \mathbb{R}^N$ is a column vector.

Galerkin approximation: Mass and stiffness matrices

We want to find the function $\mathbf{u} \in L^2(0,T;V_N)$ which satisfies for all $\mathbf{w} \in \mathbb{R}^N$

$$\int_0^L w_N(x)\frac{\partial u_N}{\partial t}(t,x)\,\mathrm{d}x = -\kappa\int_0^L \frac{\mathrm{d}w_N}{\mathrm{d}x}(x)\frac{\partial u_N}{\partial x}(t,x)\,\mathrm{d}x + \int_0^L w_N(x) f(t,x)\,\mathrm{d}x,$$

$$u_N(0,x) = u_{0,N}(x),$$

$$u_N(t,x) = \mathbf{N}(x)\mathbf{u}(t), \qquad w_N(x) = \mathbf{w}^\top(\mathbf{N}(x))^\top,$$

Substitute the expressions for u_N and w_N into the above equations:

$$\begin{aligned}\int_0^L \mathbf{w}^\top(\mathbf{N}(x))^\top\mathbf{N}(x)\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t)\,\mathrm{d}x &= -\kappa\int_0^L \mathbf{w}^\top\frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x)\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\mathbf{u}(t)\,\mathrm{d}x \\ &\quad + \int_0^L \mathbf{w}^\top(\mathbf{N}(x))^\top f(t,x)\,\mathrm{d}x, \\ \mathbf{N}(x)\mathbf{u}(0) &= u_{0,N}(x).\end{aligned}$$

Which can be rewritten as

$$\mathbf{w}^\top\mathbf{E}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{w}^\top\mathbf{A}\mathbf{u}(t) + \mathbf{w}^\top\mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

where

$$\mathbf{E} = \int_0^L (\mathbf{N}(x))^\top\mathbf{N}(x)\,\mathrm{d}x,\ \mathbf{A} = -\kappa\int_0^L \frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x)\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\,\mathrm{d}x,\ \mathbf{f}(t) = \int_0^L (\mathbf{N}(x))^\top f(t,x)\,\mathrm{d}x$$

Galerkin approximation: result

We obtained the following equation for $\mathbf{u}(t)$ which should be satisfied for all $\mathbf{w} \in \mathbb{R}^N$

$$\mathbf{w}^\top \mathbf{E} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{w}^\top \mathbf{A}\mathbf{u}(t) + \mathbf{w}^\top \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

Taking $\mathbf{w} = \mathbf{e}_1, \mathbf{w} = \mathbf{e}_2, \ldots, \mathbf{w} = \mathbf{e}_N$,
we conclude that $\mathbf{u}(t)$ is the solution of the following system of ODEs

Result of Galerkin approximation

$$\mathbf{E} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{f}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0,$$

where

$$\mathbf{E} = \int_0^L (\mathbf{N}(x))^\top \mathbf{N}(x)\,\mathrm{d}x, \; \mathbf{A} = -\kappa \int_0^L \frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x) \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\,\mathrm{d}x, \; \mathbf{f}(t) = \int_0^L (\mathbf{N}(x))^\top f(t,x)\,\mathrm{d}x.$$

Observe: $\mathbf{E}$ is symmetric and positive definite, i.e. $\mathbf{E} = \mathbf{E}^\top$ and $\mathbf{u}^\top \mathbf{E}\mathbf{u} > 0$ for all $\mathbf{u} \neq 0$.
Because the Laplacian is self-adjoint, $\mathbf{A}$ is symmetric and negative semi-definite, i.e. $\mathbf{A} = \mathbf{A}^\top$ and $\mathbf{u}^\top \mathbf{A}\mathbf{u} \leq 0$ for all $\mathbf{u}$.

Question 2

We take $L = 1$ and consider two shape functions:

$$\mathbf{N}(x) = \begin{bmatrix} 1 & x \end{bmatrix}.$$

Compute

$$\mathbf{E} = \int_0^L (\mathbf{N}(x))^\top \mathbf{N}(x) \, \mathrm{d}x$$

A) $\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

B) $\mathbf{E} = \frac{4}{3}$

C) $\mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

D) $\mathbf{E} = 1$

E) None of the above

Question 3

We take $L = 1$ and consider two shape functions:

$$\mathbf{N}(x) = \begin{bmatrix} 1 & x \end{bmatrix}.$$

Compute

$$\mathbf{A} = -\int_0^L \frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x)\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\,\mathrm{d}x$$

A) $\mathbf{A} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B) $\mathbf{A} = -\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

C) $\mathbf{A} = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

D) $\mathbf{A} = -1$

E) None of the above

Question 4

We take $L = 1$ and consider two shape functions and the loading

$$\mathbf{N}(x) = \begin{bmatrix} 1 & x \end{bmatrix}, \qquad f(t,x) = 2 + t$$

Compute

$$\mathbf{f}(t) = \int_0^L (\mathbf{N}(x))^\top f(t,x)\,\mathrm{d}x$$

A) $\mathbf{f}(t) = \begin{bmatrix} 2+t & 1+\frac{1}{2}t \end{bmatrix}$

B) $\mathbf{f}(t) = (1+\frac{1}{2}t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

C) $\mathbf{f}(t) = (1+\frac{1}{2}t)\begin{bmatrix} 2 & 1 \end{bmatrix}$

D) $\mathbf{f}(t) = 2\frac{1}{2}$

E) None of the above

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3.D 1-D Finite elements

Finite Elements

Most finite element models are Galerkin discretizations.
but a specific choice of basis functions $\mathbf{N}(x)$.

We start by dividing the domain $(0, L)$ into M elements:

In 1-D, each element e corresponds to an interval $[x_{e-1}, x_e]$ of length L_e.

A function $v_N \in V_N$ is then of the form in element e

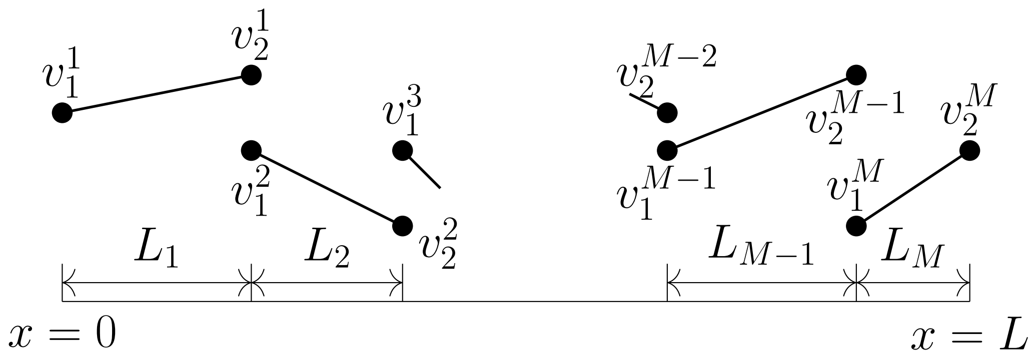
$$v_N(x) = \mathbf{N}^e\left(\frac{x - x_{e-1}}{x_e - x_{e-1}}\right)\mathbf{v}^e, \qquad x \in [x_{e-1}, x_e].$$

For example, we can choose

$$\mathbf{N}^e(\xi) = \begin{bmatrix} 1-\xi & \xi \end{bmatrix}, \qquad \mathbf{v}^e = \begin{bmatrix} v_1^e \\ v_2^e \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{N}^e(\xi) = (1-\xi)v_1^e + \xi v_2^e.$$

Finite elements: the function space (1/2)

We divide the domain $(0, L)$ into M elements and take an linear function in each interval



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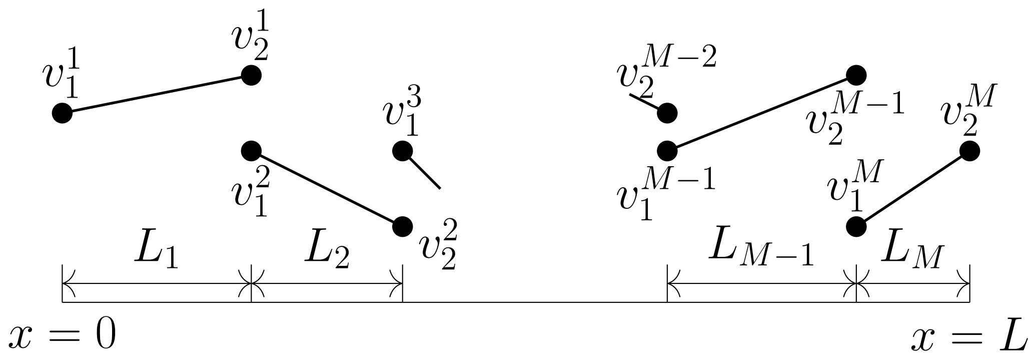
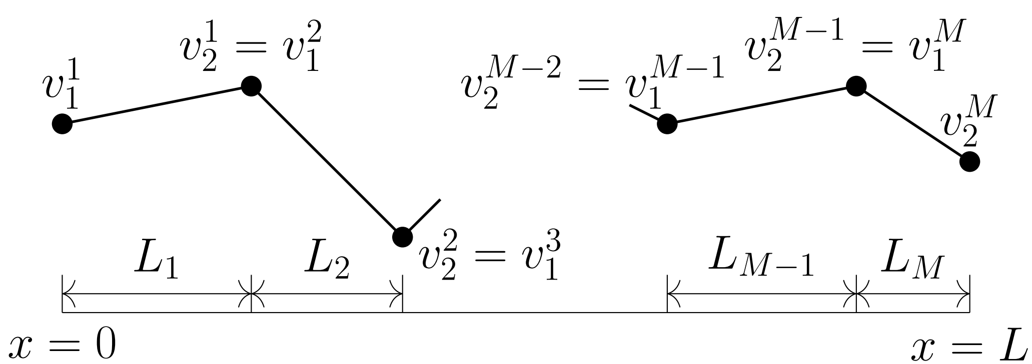


But we required in the Galerkin approximation that

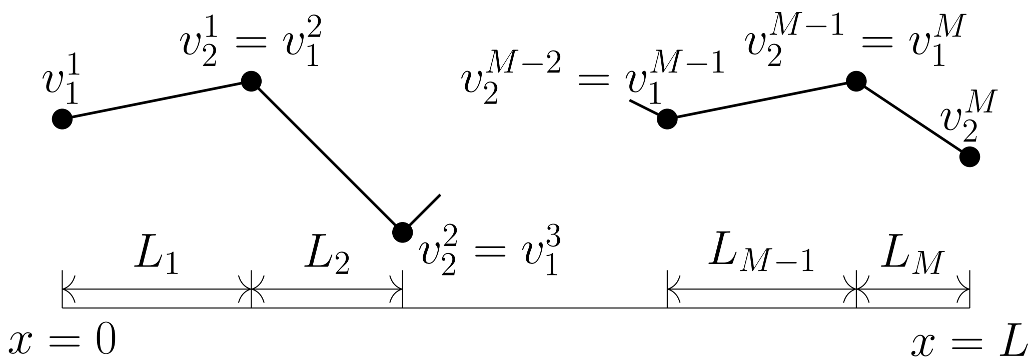
$$V_N \subset V \subset H^1(0, L).$$

Every function $v_N(x) \in V_N$ should thus be continuous: $v_2^{e-1} = v_1^e$.

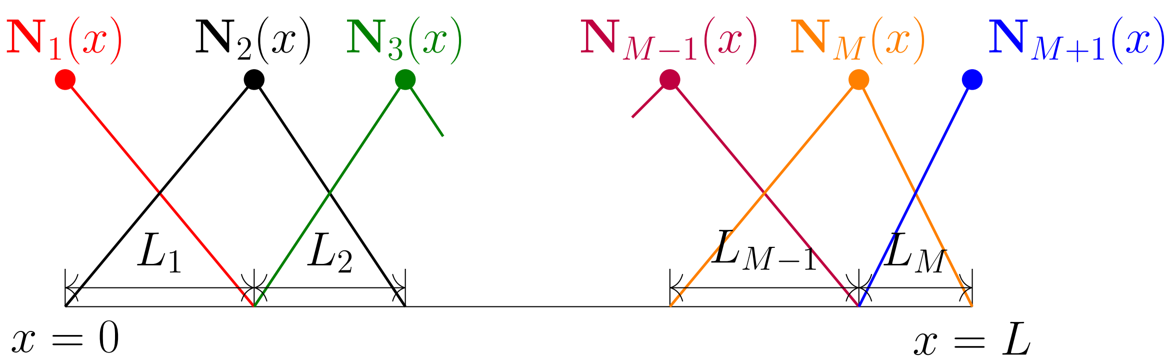


Note: $v_N(x)$ is defined by $M + 1$ values.

Finite elements: the function space (2/2)



The basis $\mathbf{N}(x) = [\mathbf{N}_1(x), \mathbf{N}_2(x), \ldots, \mathbf{N}_{M+1}(x)]$ for V_N is shown in the figure below.



The dimension N of the function space V_N is thus equal to $M + 1$ in this case.
This expression for $\mathbf{N}(x)$ can now be used in a Galerkin procedure to find the FE model.

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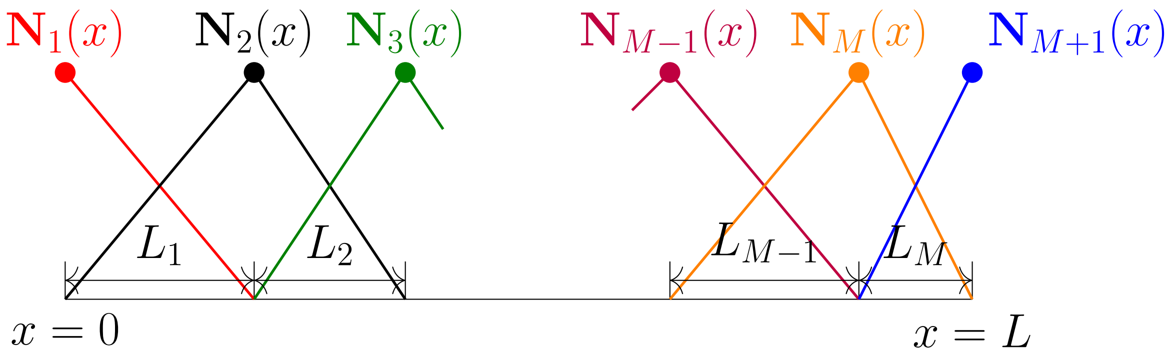
3.E Assembling the matrices for 1-D finite elements

Assembling the mass matrix $\mathbf{E}$

…but we can be more efficient

because the shape functions are similar in each element.



Observe that
$$\mathbf{E} = \int_0^L (\mathbf{N}(x))^\top \mathbf{N}(x)\,\mathrm{d}x = \sum_{e=1}^{M} \int_{x_{e-1}}^{x_e} (\mathbf{N}(x))^\top \mathbf{N}(x)\,\mathrm{d}x.$$

Inside (x_{e-1}, x_e), all shape functions are zero except for

$$[\mathbf{N}_e(x), \mathbf{N}_{e+1}(x)] = \left[\left(1 - \tfrac{x-x_{e-1}}{x_e-x_{e-1}}, \tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\right] = \mathbf{N}^e\left(\frac{x-x_{e-1}}{x_e-x_{e-1}}\right)$$

Using the change of variables $\xi = \frac{x-x_{e-1}}{x_e-x_{e-1}}$ (so $\mathrm{d}x = (x_e - x_{e-1})\,\mathrm{d}\xi = L_e \mathrm{d}\xi$)

$$\int_{x_{e-1}}^{x_e} \left(\mathbf{N}^e\left(\tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\right)^\top \mathbf{N}^e\left(\tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\,\mathrm{d}x = L_e \int_0^1 (\mathbf{N}^e(\xi))^\top \mathbf{N}^e(\xi)\,\mathrm{d}\xi.$$

Question 5

We have the following shape functions in the master/generic element

$$\mathbf{N}^e(\xi) = \begin{bmatrix} 1-\xi & \xi \end{bmatrix}.$$

Compute mass matrix for the master/generic element

$$\mathbf{E}^e = \int_0^1 (\mathbf{N}^e(\xi))^\top \mathbf{N}^e(\xi) \, \mathrm{d}\xi.$$

A) $\mathbf{E}^e = \frac{1}{3}$

B) $\mathbf{E}^e = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

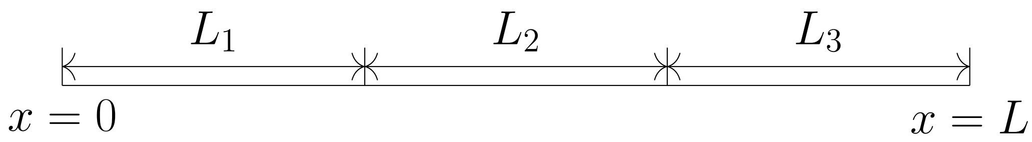
C) $\mathbf{E}^e = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$

D) $\mathbf{E}^e = \frac{2}{3}$

E) None of the above

Example: Assembling the mass matrix $\mathbf{E}$

We consider a domain divided into three elements of equal length.



Because we domain is 1-D and we use linear elements $N = M + 1 = 4$.

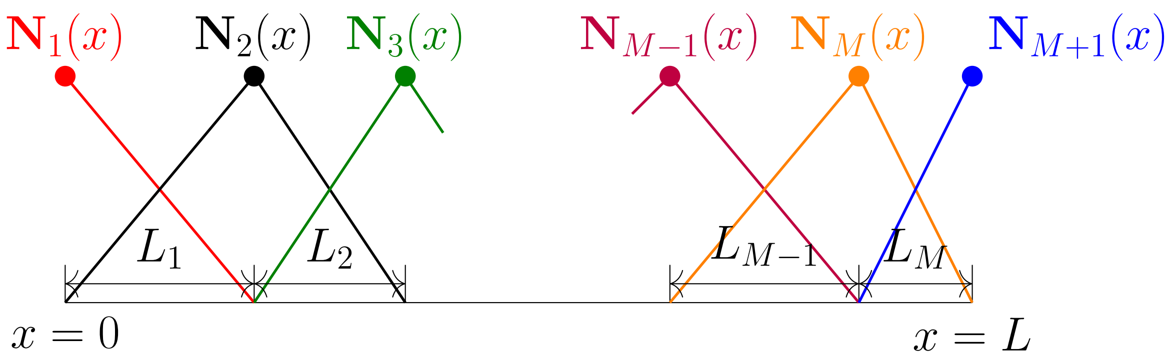
Assembly procedure:

$$\mathbf{E} = \sum_{m=1}^{3} \int_{x_{e-1}}^{x_e} (\mathbf{N}(x))^\top \mathbf{N}(x)\, \mathrm{d}x = \frac{L_1}{6} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{L_2}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{L_3}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \frac{L}{18} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Assembling the stiffness matrix $\mathbf{A}$

We can assemble the stiffness matrix in a similar way.



Observe that

$$\mathbf{A} = -\kappa \int_0^L \frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x) \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\, \mathrm{d}x = -\kappa \sum_{e=1}^{M} \int_{x_{e-1}}^{x_e} \frac{\mathrm{d}\mathbf{N}^\top}{\mathrm{d}x}(x) \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x)\, \mathrm{d}x.$$

Inside (x_{e-1}, x_e), all shape functions are zero except for

$$[\mathbf{N}_e(x), \mathbf{N}_{e+1}(x)] = \left[\left(1 - \tfrac{x-x_{e-1}}{x_e-x_{e-1}}, \tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\right] = \mathbf{N}^e\left(\frac{x - x_{e-1}}{x_e - x_{e-1}}\right)$$

Using the change of variables $\xi = \frac{x-x_{e-1}}{x_e-x_{e-1}}$ (so $\mathrm{d}x = (x_e - x_{e-1})\, \mathrm{d}\xi = L_e \mathrm{d}\xi$)

$$\int_{x_{e-1}}^{x_e} \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^e\left(\tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\right)^\top \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}^e\left(\tfrac{x-x_{e-1}}{x_e-x_{e-1}}\right)\, \mathrm{d}x = \dots$$

Question 6

Using the change of variables $\xi = \frac{x - x_{e-1}}{x_e - x_{e-1}}$ (so $\mathrm{d}x = (x_e - x_{e-1})\,\mathrm{d}\xi = L_e \mathrm{d}\xi$), we find that

$$\int_{x_{e-1}}^{x_e} \left(\frac{\mathrm{d}}{\mathrm{d}x} \mathbf{N}^e \left(\frac{x - x_{e-1}}{x_e - x_{e-1}} \right) \right)^\top \frac{\mathrm{d}}{\mathrm{d}x} \mathbf{N}^e \left(\frac{x - x_{e-1}}{x_e - x_{e-1}} \right) \,\mathrm{d}x = \ldots$$

A) $\ldots = L_e \int_0^1 \left(\frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \right)^\top \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \,\mathrm{d}\xi$

B) $\ldots = \int_0^1 \left(\frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \right)^\top \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \,\mathrm{d}\xi$

C) $\ldots = \frac{1}{L_e} \int_0^1 \left(\frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \right)^\top \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \,\mathrm{d}\xi$

D) $\ldots = \frac{1}{L_e^2} \int_0^1 \left(\frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \right)^\top \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi) \,\mathrm{d}\xi$

E) None of the above

Question 7

We have the following shape functions in the master/generic element

$$\mathbf{N}^e(\xi) = \begin{bmatrix} 1-\xi & \xi \end{bmatrix}.$$

Compute mass matrix for the master/generic element

$$\mathbf{A}^e = -\int_0^1 \left(\frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi)\right)^\top \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}\xi}(\xi)\,\mathrm{d}\xi.$$

A) $\mathbf{A}^e = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

B) $\mathbf{A}^e = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

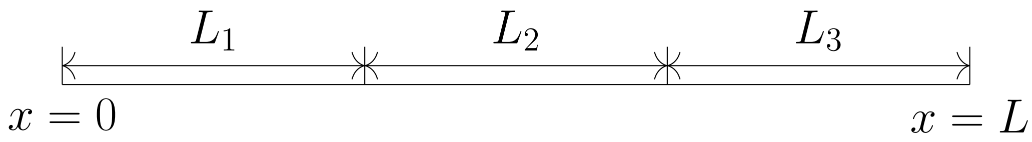
C) $\mathbf{A}^e = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

D) $\mathbf{A}^e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E) None of the above

Example: Assembling the stiffness matrix $\mathbf{A}$

We consider a domain divided into three elements of equal length.



Assembly procedure:

$$\mathbf{A} = -\kappa \sum_{m=1}^{3} \int_{x_{e-1}}^{x_e} \left(\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x) \right)^{\top} \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x}(x) \, \mathrm{d}x$$

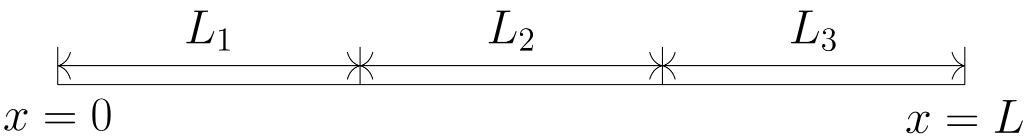
$$= \frac{\kappa}{L_1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\kappa}{L_2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\kappa}{L_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{3\kappa}{L} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Note: the structure of the $\mathbf{A}$ is similar to the structure of $\mathbf{A}$ in finite differences, but not exactly the same ...

Example: Assembling the load vector $\mathbf{f}$

We consider a domain divided into three elements of equal length.



The applied loading is $f(t, x) = 1$.

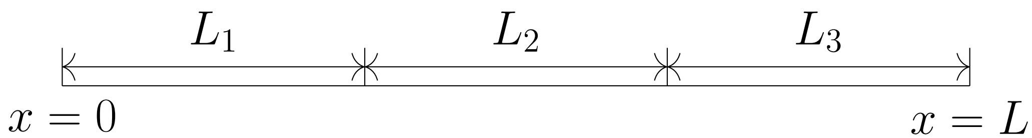
Assembly procedure:

$$\mathbf{f} = \sum_{m=1}^{3} \int_{x_{e-1}}^{x_e} (\mathbf{N}(x))^\top \,\mathrm{d}x = \frac{L_1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + \frac{L_2}{2}\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + \frac{L_3}{2}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \frac{L}{6}\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$$

Note: when $f(t, x) = 1$ we have that $\mathbf{f} = \mathbf{E}\mathbf{1}$. Do you see why?

Example: Applying Dirichlet boundary conditions

We consider a domain divided into three elements of equal length.



With the procedure of the previous slides, we have obtained the global mass, stiffness and forcing vector:

$$\mathbf{E} = \frac{L}{18}\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{A} = \frac{3\kappa}{L}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{f} = \frac{L}{6}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

Suppose that we have a zero Dirichlet boundary condition at the left end of the domain, i.e. $u(t, 0) = 0$.

Remove the first row and column to obtain the model with a zero Dirichlet BC:

$$\mathbf{E}_{\mathrm{ff}} = \frac{L}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad \mathbf{A}_{\mathrm{ff}} = \frac{3\kappa}{L}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{f}_{\mathrm{f}} = \frac{L}{6}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Note that this model only considers the free DOFs $\mathbf{u}_{\mathrm{f}}(t)$ which do not include $\mathbf{u}_1 = 0$.

$\mathbb{P}_1$-shape functions

The shape functions obtained by using the shape functions

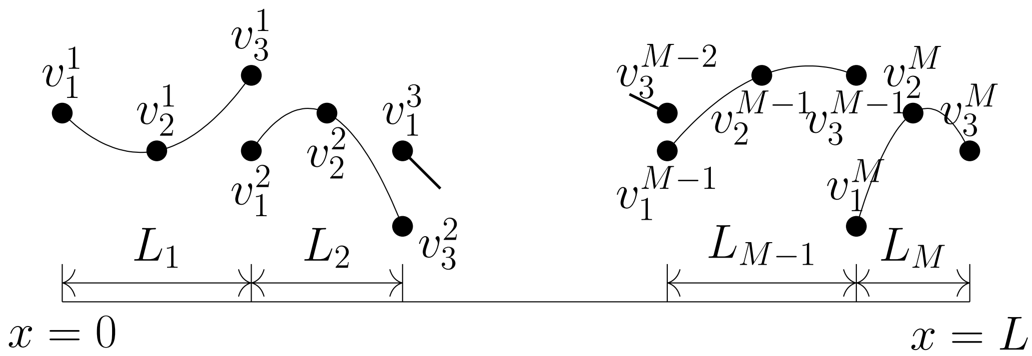
$$\mathbf{N}^e(\xi) = [1-\xi \qquad \xi], \qquad\qquad \xi \in (0,1),$$

in each element are also called $\mathbb{P}_1$-shape functions.
(The shape function in each element is a Polynomial of order 1)

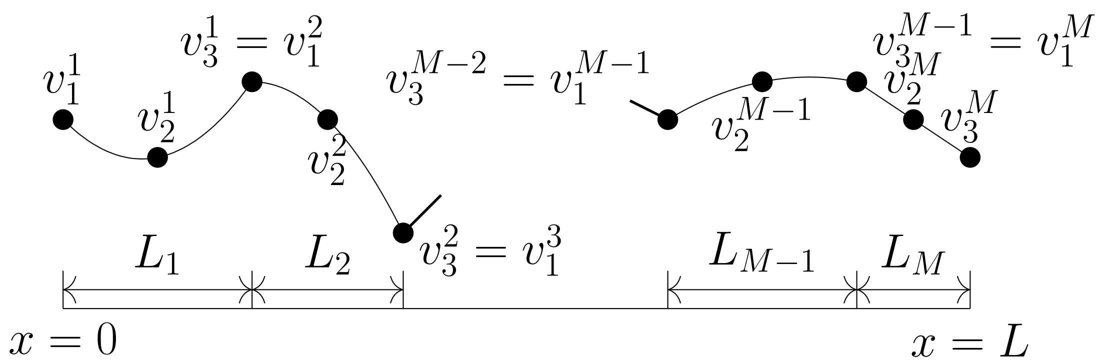
$\mathbb{P}_2$-shape functions

For $\mathbb{P}_2$ shape functions, the shape functions are polynomials of order 2, i.e.

$$\mathbf{N}^e(\xi) = [(1-\xi)(1-2\xi) \qquad 4\xi(1-\xi) \qquad \xi(2\xi-1)], \qquad \qquad \xi \in (0,1),$$



But we need that $V_N \subset V \subseteq H^1(0,L)$, so every $v_N(x)$ should be continuous.



A model with M elements now has $N = M+1+M = 2M+1$ nodes.
The element mass and stiffness matrices $\mathbf{E}^e$ and $\mathbf{A}^e$ now 3×3.

3.F Convergence analysis for finite elements

Stability: Cea's lemma

Original infinite dimensional problem:
find $u \in V$ such that

$$a(u, w) = b(w), \qquad \forall w \in V$$

Galerkin approximation:
find $u_N \in V_N \subset V$ such that

$$a(u_N, w_N) = b(w_N), \qquad \forall w_N \in V_N$$

Stability: Cea's lemma

Original infinite dimensional problem:
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Galerkin approximation:
find $u_N \in V_N \subset V$ such that

$$a(u_N, w_N) = b(w_N), \qquad \forall w_N \in V_N$$

Assume that there are $m, M > 0$ such that for all $u, w \in V$

$$a(u, u) \geq m|u|^2, \qquad |a(u, w)| \leq M|u||w|.$$

Lemma (Cea)

$$|u - u_N| \leq \frac{M}{m} \inf_{w_N \in V_N} |u - w_N|$$

Proof: Because $w_N \in V$,

$$a(u - u_N, w_N) = a(u, w_N) - a(u_N, w_N) = b(w_N) - b(w_N) = 0.$$

Using this result, we can then compute

$$m|u - u_N|^2 \leq a(u - u_N, u - u_N) = a(u - u_N, u - w_N + \underbrace{w_N - u_N}_{\in V_N})$$

$$= a(u - u_N, u - w_N) \leq M|u - u_N||u - w_N|.$$

Consistency: convergence rates

Using Cea's lemma, we just need to compute

$$\inf_{w_N \in V_N} |u - w_N|.$$

Idea: we can choose a specific mapping $r_N : V \to V_N$ find a bound $|u - r_N u| \leq Ch^p$.
The operator r_N is typically chosen as the interpolation operator.
Using Cea's lemma, we then find that

$$|u - u_N| \leq \frac{M}{m} \inf_{w_N \in V_N} |u - w_N| \leq \frac{M}{m} |u - r_N u| \leq \frac{M}{m} Ch^p.$$

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Using Cea's lemma, we then find that

$$|u - u_N| \leq \frac{M}{m} \inf_{w_N \in V_N} |u - w_N| \leq \frac{M}{m} |u - r_N u| \leq \frac{M}{m} Ch^p.$$

For **linear 1-D elements**, we have (see e.g. Allaire Lemma 6.2.10)

$$|u - u_N|_{L^2} \leq Ch^2 |u''|_{L^2}, \qquad |u' - u_N'|_{L^2} \leq Ch|u''|_{L^2}.$$

For **quadratic 1-D elements**, we have (see e.g. Allaire Theorem 6.2.14)

$$|u - u_N|_{H^1} \leq Ch^2 |u'''|_{L^2}.$$

More general, for $\mathbb{P}_k$ **rectangular elements**, we have (see e.g. Allaire Theorem 6.3.27)

$$|u - u_N|_{H^1} \leq Ch^k |u|_{H^{k+1}}.$$