EXERCISE WEEK 5:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION

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Due date: Sunday May 29 2021, 24:00h (midnight)

Send your solutions to daniel.veldman@math.fau.de. Put all your files in one .zip-file and name it `Exercises_Week5_<your name>.zip`

The solutions to the exercises will be published on StudOn on Monday May 30. We cannot consider submissions after the solutions have been uploaded.

We consider the aluminum rod in Figure 1 with a length of $L = 0.3$ [m], a cross sectional area of $A_{\mathrm{cs}} = 0.01$ [m^2], and a thermal conductivity of $k = 237$ [W/m/K]. At five locations $x_{\mathrm{m},i} = (i-1)\frac{L}{4}$ $(i \in \{1, 2, 3, 4, 5\})$ along the rod, temperature sensors are installed. Our aim is to reconstruct the heat load applied to the rod based on the measured temperatures $\bar{T}_{\mathrm{m},i}$ at these five locations.

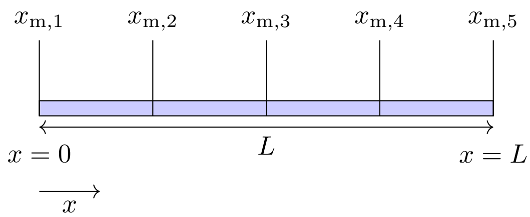

FIGURE 1. The considered aluminum rod

We reconstruct the applied heat load by minimizing the functional

$$\mathcal{J}(Q) = \frac{1}{2}\sum_{i=1}^{5}(T(x_{\mathrm{m},i}) - \bar{T}_{\mathrm{m},i})^2 + \frac{w}{2}\int_0^L (Q(x))^2 \, \mathrm{d}x,$$

where $w > 0$ is a weight and $T(x)$ is the steady-state temperature field resulting from the heat load $Q(x)$ which satisfies

$$kA_{\mathrm{cs}}\frac{\mathrm{d}^2T}{\mathrm{d}x^2}(x) + Q(x) = 0, \qquad kA_{\mathrm{cs}}\frac{\mathrm{d}T}{\mathrm{d}x}(0) = hT(0), \qquad -kA_{\mathrm{cs}}\frac{\mathrm{d}T}{\mathrm{d}x}(L) = hT(L).$$

A finite element discretization of this problem takes the form

$$J(\mathbf{u}) = \frac{1}{2}\mathbf{e}_{\mathrm{m}}^\top\mathbf{e}_{\mathrm{m}} + \frac{w}{2}\mathbf{u}^\top\mathbf{E}\mathbf{u},$$

$$\mathbf{e}_{\mathrm{m}} = \mathbf{E}_{\mathrm{m}}\mathbf{T} - \mathbf{T}_{\mathrm{m}}, \qquad \mathbf{A}\mathbf{T} + \mathbf{E}\mathbf{u} = \mathbf{0},$$

where $\mathbf{T}$ and $\mathbf{u}$ are vectors containing the nodal values of $T(x)$ and $Q(x)$, $\mathbf{T}_{\mathrm{m}} = [\bar{T}_{\mathrm{m},1}, \bar{T}_{\mathrm{m},2}, \bar{T}_{\mathrm{m},3}, \bar{T}_{\mathrm{m},4}, \bar{T}_{\mathrm{m},5}]^\top$ and the $5 \times N$-matrix $\mathbf{E}_{\mathrm{m}}$ (where N is the number of nodes) selects the nodal temperatures at the location of the temperature sensors. The matrices $\mathbf{E}$, $\mathbf{E}_{\mathrm{m}}$, and $\mathbf{A}$ for a FE model with $M = 100$ elements (and $N = 101$

nodes) are given in the file `rod_model.mat`. This file also contains the vector $\mathbf{T}_{\mathrm{m}}$ with the measured temperature values.

a. (2pts) Compute the gradient of the function J in the discretized problem w.r.t. the standard Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$ and the w.r.t. the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{E} \mathbf{y}$. Plot the obtained gradients in the point $\mathbf{u}_0 = \mathbf{0}$ against the spatial coordinate x for $w = 10^{-4}$.
Hint: there are two approaches to compute the Jacobian of J. One is to eliminate $\mathbf{e}_{\mathrm{m}}$ and $\mathbf{T}$ from the expression for the discretized cost function given above and differentiate the resulting expression. The other is to use the chain rule.
Hint: You can use the file `Week5_exerciseab` as a starting point.

b. (3pts) Determine the quadratic approximations of the functional $\beta \mapsto J(\mathbf{u}_0 - \beta \nabla J(\mathbf{u}_0))$ for the two gradients at $\mathbf{u}_0 = \mathbf{0}$ computed in part a. Compare the values of the quadratic approximation to the values of $\beta \mapsto J(\mathbf{u}_0 - \beta \nabla J(\mathbf{u}_0))$ for a range of step sizes β in a plot (again for $w = 10^{-4}$).
Hint: You can use the file `Week5_exerciseab` as a starting point.

c. (2pts) Develop and implement a gradient-based algorithm to minimize the functional $J(\mathbf{u})$. Use the gradient w.r.t. the weighted inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{E} \mathbf{y}$ computed in a. and use the quadratic approximation developed in b. to determine the step size β. Keep $w = 10^{-4}$ and terminate the algorithm when the relative change in $\mathbf{u}$ and $J(\mathbf{u})$ are below 10^{-5}.
Hint: You can use the file `Week5_exercisec` as a starting point.
Hint: Do you need a line search to assure that the cost functional decreases in every iteration?

d. (1pt) Vary the weight w in your code from c. to compute the optimal $\mathbf{u}$ for $w = 10^{-3}$ and $w = 10^{-6}$. Explain what you observe.

e. (2pts) Use the projected gradient method to assure that all entries of the vector $\mathbf{u}$ remain between 120 and 190. Take again $w = 10^{-4}$.
Hint: You can use the file `Week5_exercisee` as a starting point.
Hint: Do you now need a line search to assure that the cost functional decreases in every iteration?