EXERCISE WEEK 6:
PRACTICAL COURSE
MODELING, SIMULATION, OPTIMIZATION

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Due date: Sunday June 19 2021, 24:00h (midnight)

Send your solutions to daniel.veldman@math.fau.de. Put all your files in one .zip-file and name it `Exercises_Week6_<your name>.zip`

The solutions to the exercises will be published on StudOn on Monday June 20. We cannot consider submissions after the solutions have been uploaded.

We consider heat conduction in the copper bar in Figure **??** with a length of $L = 0.3$ [m], a cross sectional area $A_{\text{cs}} = 10^{-4}$ [m^2], a thermal conductivity $k = 400$ [W/K/m], a mass density $\rho = 8960$, and a heat capacity $c = 385$ [J/kg/K]. The temperature increase in the bar (w.r.t. a constant temperature distribution at $t = 0$) is denoted by $T(x, t)$ [K]. In the part $[0, L/4]$, we can apply a uniform heat load $u(t)$ in [W]. We want to use the heat load $u(t)$ to keep the temperature at the right end $T(L, t)$ close to $T_d = 1$ [K].

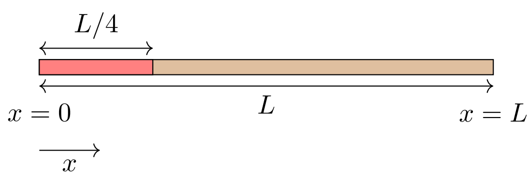

FIGURE 1. The considered copper bar

A finite element discretization of the bar with $M = 100$ elements and $N = 101$ nodes leads to a system of the form

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad \mathbf{x}(0) = \mathbf{x}_{\text{init}}.$$

We want to determine the applied heat load $u(t)$ that minimizes

$$J = \frac{1}{2}\int_0^T \left((\mathbf{x}(t) - \mathbf{1})^\top \mathbf{Q}(\mathbf{x}(t) - \mathbf{1}) + (u(t))^2 \right) \, \mathrm{d}t.$$

The matrices $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{x}_{\text{init}}$, and $\mathbf{Q}$ are given in the file `bar_model.mat`. We also fix $T = 8$ [minutes] $= 480$ [s] and $\mathbf{1}$ denotes a vector of ones of length N.

a. (2pt) Compute the state $\mathbf{x}(t)$ resulting from the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$ using the Crank-Nicolson scheme. Use a time grid with $N_T = 201$ points. Use the obtained solution to evaluate the cost functional J for the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$ discretized with the combination of the trapezoid and the midpoint rule explained in the lecture.
Hint: You can use the file `Week6_exerciseab` as a starting point.

b. (2pts) Compute the adjoint state $\boldsymbol{\varphi}(t)$ (for the input $u(t) = \sin(\frac{\pi}{2}\frac{t}{T})$) that leads to discretely consistent gradients. Follow the procedure given in the lecture. Use the obtained (discretized) adjoint state to compute the gradient ∇J.
Hint: You can again use the file `Week6_exerciseab`.

c. (2pts) Compute the coefficients G and H in the quadratic approximation of the cost functional $\beta \mapsto J(u_0 - \beta \nabla J)$. Compare the obtained quadratic approximation to the true values of the cost functional. Do you expect to see any difference?
Hint: you can again use the file `Week6_exercisec` as a starting point.

d. (3pts) Use the results from parts a,b, and c to develop and implement a gradient-based algorithm to minimize the functional J.
Hint: You can use the file `Week6_exercised` as a starting point.
Hint: Do you need a line search to assure that the cost functional decreases in every iteration?

e. (1pt) Run the algorithm you developed in d. with the matrix $\mathbf{Q}$ rescaled by a factor 0.01 and a factor 100. What do you observe in the obtained optimal controls? Try to explain your observations.