Stability

Control Theory, Lecture 2

by Sergei Savin

Spring 2021

- Read more

Critical point (node)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Let $\mathbf{x}_0$ be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \tag{2}$$

Then such state $\mathbf{x}_0$ is called a *node* or a *critical point*.

Node $\mathbf{x}_0$ is called *stable* iff for any constant δ there exists constant ε such that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \;\longrightarrow\; ||\mathbf{x}(t) - \mathbf{x}_0|| < \varepsilon \tag{3}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the rest of the trajectory $\mathbf{x}(t)$ will be at most ε away from $\mathbf{x}_0$".

Or, more picturesque, think of it as "the solutions with different initial conditions do not diverge from the node"

Node $\mathbf{x}_0$ is called *asymptotically stable* iff for any constant δ it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \ \longrightarrow \ \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{x}_0 \tag{4}$$

Think of it as "for any initial point that lies at most δ away from $\mathbf{x}_0$, the trajectory $\mathbf{x}(t)$ will asymptotically approach the point $\mathbf{x}_0$".

Or, more picturesque, think of it as "the solutions with different initial conditions converge to the node"

Stability vs Asymptotic stability

Example

Consider dynamical system $\dot{x} = 0$, and solution $x = 7$. This solution is stable, but not asymptotically stable (other solutions do not diverge from $x = 7$, but do not converge to it either).

Example

Consider dynamical system $\dot{x} = -x$, and solution $x = 0$. This solution is stable and asymptotically stable (other solutions converge to $x = 0$).

Example

Consider dynamical system $\dot{x} = x$, and solution $x = 0$. This solution is unstable (other solutions diverge from $x = 0$).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5}$$

This is called a *linear time-invariant system*, or *LTI*.

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{6}$$

This is also an LTI, but it is also called an *autonomous system*, since its evolution depends only on the state of the system.

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{7}$$

where $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \tag{8}$$

Multiply it by $\mathbf{V}^{-1} \longrightarrow \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$.
Define $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \longrightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$.

Since elements of $\mathbf{D}$ are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of $\mathbf{A}$.

Assume that $\mathbf{A}$ can be decomposed via eigen-decomposition as $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$, where $\mathbf{C}$ is a complex-valued diagonal matrix and $\mathbf{U}$ is a complex-valued inevitable matrix.

We can perform the same steps (multiply by $\mathbf{U}^{-1}$, then define $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$) to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \tag{9}$$

which falls into a set of independent equations, with complex coefficients c_i:

$$\dot{z}_i = c_i z_i \tag{10}$$

The solution is:

$$z_i = k_0 e^{c_i t} \tag{11}$$

Example: complex eigenvalues, part 2

The solution $z_i = k_0 e^{c_i t}$, where $c_i = \alpha_i + i\beta_i$, can be decomposed using Euler's identity:

$$z_i = k_0 e^{c_i t} = k_0 e^{(\alpha_i + i\beta_i)t} = k_0 e^{\alpha_i t} e^{i\beta_i t} = k_0 e^{\alpha_i t}(\cos(\beta_i t) + i \sin(\beta_i t))$$

As you can see, brackets $(\cos(\beta_i t) + i \sin(\beta_i t))$ has a constant norm, $||(\cos(\beta_i t) + i \sin(\beta_i t))|| = 1$. Therefore, norm of z_i depends entirely on the norm of $e^{\alpha_i t}$, which is:

1. constant if $\alpha_i = 0$, hence the system is stable.
2. decreasing if $\alpha_i < 0$, hence the system is asymptotically stable.
3. increasing if $\alpha_i > 0$, hence the system is unstable.

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{12}$$

Definition

Eq. (12) is stable iff real parts of eigenvalues of $\mathbf{A}$ are non-positive.

Definition

Eq. (12) is asymptotically stable iff real parts of eigenvalues of $\mathbf{A}$ are negative.

Stability of autonomous LTI

Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:

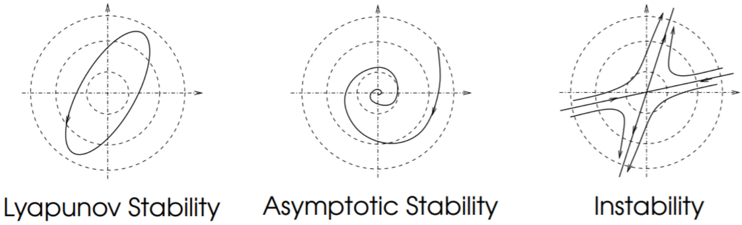

Figure 1: phase portraits for different types of stability

Credit: staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

Read more

- Control Systems Design, by Julio H. Braslavsky
  staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Control-Theory-Slides-Spring-2021

Check Moodle for additional links, videos, textbook suggestions.