

# Stability

## Control Theory, Lecture 2

by Sergei Savin

Spring 2021

- Critical point (node)
- Stability
- Asymptotic stability
- Stability vs Asymptotic stability
- LTI and autonomous LTI
- Stability of autonomous LTI
  - ▶ Example: real eigenvalues
  - ▶ Example: complex eigenvalues
  - ▶ General case
  - ▶ Illustration
- Read more

# CRITICAL POINT (NODE)

Consider the following ODE:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

Let  $\mathbf{x}_0$  be such a state that:

$$\mathbf{f}(\mathbf{x}_0, t) = 0 \quad (2)$$

Then such state  $\mathbf{x}_0$  is called a *node* or a *critical point*.

Node  $\mathbf{x}_0$  is called *stable* iff for any constant  $\delta$  there exists constant  $\varepsilon$  such that:

$$\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta \longrightarrow \|\mathbf{x}(t) - \mathbf{x}_0\| < \varepsilon \quad (3)$$

Think of it as "for any initial point that lies at most  $\delta$  away from  $\mathbf{x}_0$ , the rest of the trajectory  $\mathbf{x}(t)$  will be at most  $\varepsilon$  away from  $\mathbf{x}_0$ ".

Or, more picturesque, think of it as "the solutions with different initial conditions do not diverge from the node"

Node  $\mathbf{x}_0$  is called *asymptotically stable* iff for any constant  $\delta$  it is true that:

$$||\mathbf{x}(0) - \mathbf{x}_0|| < \delta \longrightarrow \lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{x}_0 \quad (4)$$

Think of it as "for any initial point that lies at most  $\delta$  away from  $\mathbf{x}_0$ , the trajectory  $\mathbf{x}(t)$  will asymptotically approach the point  $\mathbf{x}_0$ ".

Or, more picturesque, think of it as "the solutions with different initial conditions converge to the node"

# STABILITY VS ASYMPTOTIC STABILITY

## Example

Consider dynamical system  $\dot{x} = 0$ , and solution  $x = 7$ . This solution is stable, but not asymptotically stable (other solutions do not diverge from  $x = 7$ , but do not converge to it either).

## Example

Consider dynamical system  $\dot{x} = -x$ , and solution  $x = 0$ . This solution is stable and asymptotically stable (other solutions converge to  $x = 0$ ).

## Example

Consider dynamical system  $\dot{x} = x$ , and solution  $x = 0$ . This solution is unstable (other solutions diverge from  $x = 0$ ).

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5)$$

This is called a *linear time-invariant system*, or *LTI*.

Consider the following linear ODE:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (6)$$

This is also an LTI, but it is also called an *autonomous system*, since its evolution depends only on the state of the system.

# STABILITY OF AUTONOMOUS LTI

## Example: real eigenvalues

Consider autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (7)$$

where  $\mathbf{A}$  can be decomposed via eigen-decomposition as  $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ , where  $\mathbf{D}$  is a diagonal matrix.

$$\dot{\mathbf{x}} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x} \quad (8)$$

Multiply it by  $\mathbf{V}^{-1} \longrightarrow \mathbf{V}^{-1}\dot{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\mathbf{x}$ .

Define  $\mathbf{z} = \mathbf{V}^{-1}\mathbf{x} \longrightarrow \dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$ .

Since elements of  $\mathbf{D}$  are real, we can clearly see, that iff they are *all negative* will the system be asymptotically stable. If they are non-positive, the system is stable. And those elements are eigenvalues of  $\mathbf{A}$ .



# STABILITY OF AUTONOMOUS LTI

## Example: complex eigenvalues, part 1

Assume that  $\mathbf{A}$  can be decomposed via eigen-decomposition as  $\mathbf{A} = \mathbf{U}\mathbf{C}\mathbf{U}^{-1}$ , where  $\mathbf{C}$  is a complex-valued diagonal matrix and  $\mathbf{U}$  is a complex-valued invertible matrix.

We can perform the same steps (multiply by  $\mathbf{U}^{-1}$ , then define  $\mathbf{z} = \mathbf{U}^{-1}\mathbf{x}$ ) to arrive at:

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \quad (9)$$

which falls into a set of independent equations, with complex coefficients  $c_i$ :

$$\dot{z}_i = c_i z_i \quad (10)$$

The solution is:

$$z_i = k_0 e^{c_i t} \quad (11)$$

# STABILITY OF AUTONOMOUS LTI

## Example: complex eigenvalues, part 2

The solution  $z_i = k_0 e^{c_i t}$ , where  $c_i = \alpha_i + i\beta_i$ , can be decomposed using Euler's identity:

$$z_i = k_0 e^{c_i t} = k_0 e^{(\alpha_i + i\beta_i)t} = k_0 e^{\alpha_i t} e^{i\beta_i t} = k_0 e^{\alpha_i t} (\cos(\beta_i t) + i \sin(\beta_i t))$$

As you can see, brackets  $(\cos(\beta_i t) + i \sin(\beta_i t))$  has a constant norm,  $\|(\cos(\beta_i t) + i \sin(\beta_i t))\| = 1$ . Therefore, norm of  $z_i$  depends entirely on the norm of  $e^{\alpha_i t}$ , which is:

- 1 constant if  $\alpha_i = 0$ , hence the system is stable.
- 2 decreasing if  $\alpha_i < 0$ , hence the system is asymptotically stable.
- 3 increasing if  $\alpha_i > 0$ , hence the system is unstable.

# STABILITY OF AUTONOMOUS LTI

## General case

Consider an autonomous LTI:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad (12)$$

### Definition

Eq. (12) is stable iff real parts of eigenvalues of  $\mathbf{A}$  are non-positive.

### Definition

Eq. (12) is asymptotically stable iff real parts of eigenvalues of  $\mathbf{A}$  are negative.

# STABILITY OF AUTONOMOUS LTI

## Illustration

Here is an illustration of *phase portraits* of two-dimensional LTIs with different types of stability:



Figure 1: phase portraits for different types of stability

Credit: [staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf](http://staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf)

- Control Systems Design, by Julio H. Braslavsky  
[staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf](http://staff.uz.zgora.pl/wpaszke/materialy/spc/Lec13.pdf)

# THANK YOU!

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Control-Theory-Slides-Spring-2021](https://github.com/SergeiSa/Control-Theory-Slides-Spring-2021)

Check Moodle for additional links, videos, textbook suggestions.