Lecture 8: Value Function Methods

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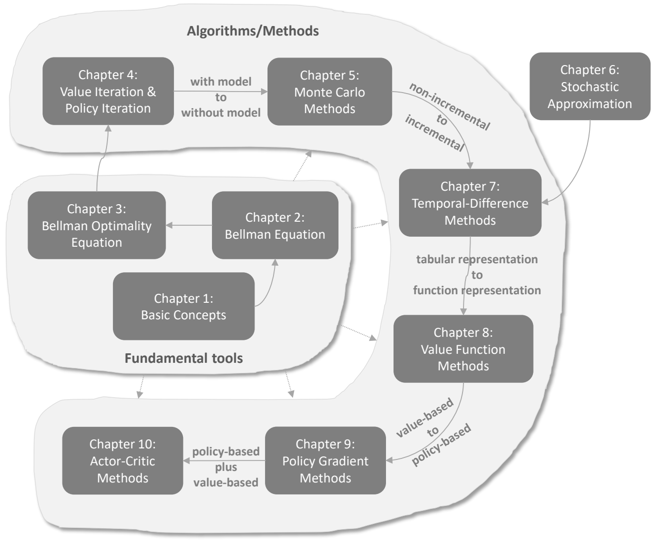
Outline



Algorithms/Methods
Chapter 4: Value Iteration & Policy Iteration
with model to without model
Chapter 5: Monte Carlo Methods
non-incremental to incremental
Chapter 6: Stochastic Approximation
Chapter 7: Temporal-Difference Methods
Chapter 3: Bellman Optimality Equation
Chapter 2: Bellman Equation
Chapter 1: Basic Concepts
Fundamental tools
tabular representation to function representation
Chapter 8: Value Function Methods
value-based to policy-based
Chapter 10: Actor-Critic Methods
policy-based plus value-based
Chapter 9: Policy Gradient Methods

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Motivating examples: from table to function

So far in this book, state and action values are represented by tables.

- For example, state value:

State	s_1	s_2	$\cdots$	s_n
Value	$v_\pi(s_1)$	$v_\pi(s_2)$	$\cdots$	$v_\pi(s_n)$

- For example, action value:

	a_1	a_2	a_3	a_4	a_5
s_1	$q_\pi(s_1,a_1)$	$q_\pi(s_1,a_2)$	$q_\pi(s_1,a_3)$	$q_\pi(s_1,a_4)$	$q_\pi(s_1,a_5)$
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s_9	$q_\pi(s_9,a_1)$	$q_\pi(s_9,a_2)$	$q_\pi(s_9,a_3)$	$q_\pi(s_9,a_4)$	$q_\pi(s_9,a_5)$

- Advantage: intuitive and easy to analyze
- Disadvantage: difficult to handle large or continuous state or action spaces. Two aspects: 1) storage; 2) generalization ability

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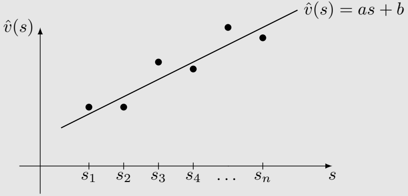
Motivating examples: from table to function

Consider an example:

- There are n states: $s_1, \dots, s_n$.
- The state values are $v_\pi(s_1), \dots, v_\pi(s_n)$, where π is a given policy.
- n is very large!
- We hope to use a simple curve to approximate these values.

Motivating examples: from table to function

For example, we can use a simple straight line to fit the dots.



Suppose the equation of the straight line is

$$\hat{v}(s,w) = as + b = \underbrace{[s,1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{w} = \phi^T(s)w$$

w is the parameter vector; $\phi(s)$ the feature vector of s; $\hat{v}(s,w)$ is linear in w.

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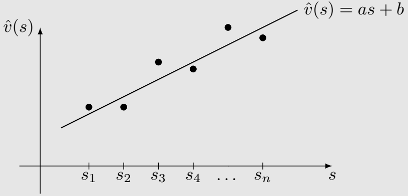

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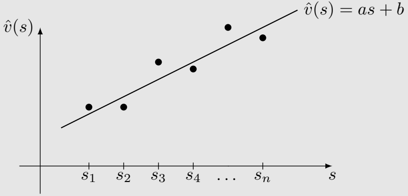

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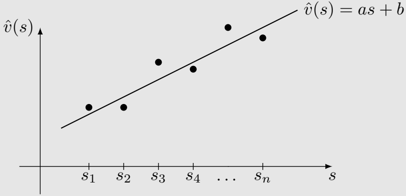

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Difference 1: How to retrieve the value of a state

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- When the values are represented by a function, we need to input the state index s into the function and calculate the function value.

For example, $s \to \phi(s) \to \phi^T(s)w = \hat{v}(s, w)$

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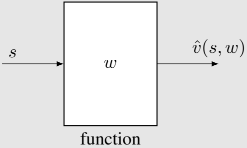

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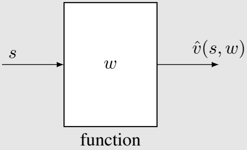

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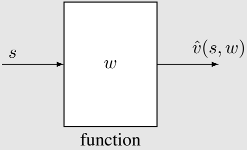

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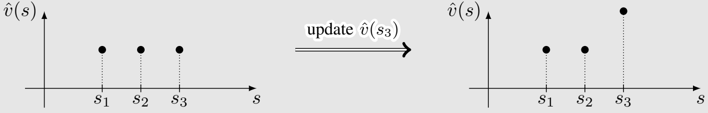

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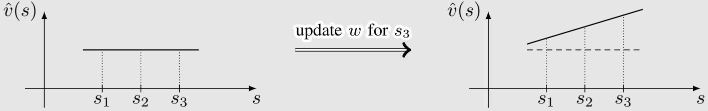

(b) Function method

Benefit: generalization ability. When we update $\hat{v}(s, w)$ by changing w, the values of the neighboring states are also changed.

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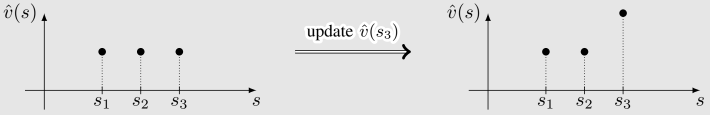

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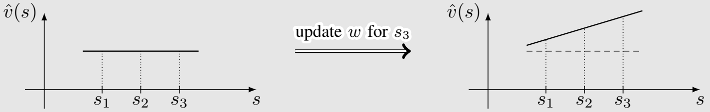

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The benefits are not free. It comes with a cost: the state values can not be represented accurately. This is why this method is called approximation.

We can fit the points more precisely using high-order curves:

$$\hat{v}(s,w) = as^2 + bs + c = \underbrace{[s^2, s, 1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{w} = \phi^T(s)w.$$

In this case,

- The dimensions of w and $\phi(s)$ increase; the values may be fitted more accurately.
- Although $\hat{v}(s,w)$ is nonlinear in s, it is linear in w. The nonlinearity is contained in $\phi(s)$.

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Quick summary:

- **Idea:** Approximate the state and action values using parameterized functions: $\hat{v}(s, w) \approx v_\pi(s)$ where $w \in \mathbb{R}^m$ is the parameter vector.
- **Key difference:** How to retrieve and change the value of $v(s)$
- **Advantages:**
  1) **Storage:** The dimension of w may be much smaller than $|\mathcal{S}|$.
  2) **Generalization:** When a state s is visited, the parameter w is updated so that the values of some other unvisited states can also be updated.

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Objective function

Introduce in a more formal way:

- Let $v_\pi(s)$ and $\hat{v}(s,w)$ be the true state value and the estimated state value, respectively.
- Our goal is to find an optimal w so that $\hat{v}(s,w)$ can best approximate $v_\pi(s)$ for every s.
- This is a policy evaluation problem. Later we will extend to policy improvement.

To find the optimal w, we need two steps.

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$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2].$$

- Our goal is to find the best w that can minimize $J(w)$.
- The expectation is with respect to the random variable $S \in \mathcal{S}$. What is the probability distribution of S?
  - This is new. We have not discussed the probability distribution of states so far.
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The first way is to use a uniform distribution.

- That is to treat all the states to be equally important by setting the probability of each state as $1/|\mathcal{S}|$.
- In this case, the objective function becomes

$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2] = \frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}(v_\pi(s) - \hat{v}(s, w))^2.$$

- **Drawback:**
  - The states may not be equally important. For example, some states may be rarely visited by a policy. Hence, this way does not consider the real dynamics of the Markov process under the given policy.

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The second way is to use the stationary distribution.

- Stationary distribution is an important concept that will be frequently used in this course. It describes the long-run behavior of a Markov process.
- Let $\{d_\pi(s)\}_{s\in\mathcal{S}}$ denote the stationary distribution of the Markov process under policy π. By definition, $d_\pi(s) \geq 0$ and $\sum_{s\in\mathcal{S}} d_\pi(s) = 1$.
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$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S,w))^2] = \sum_{s\in\mathcal{S}} d_\pi(s)(v_\pi(s) - \hat{v}(s,w))^2.$$

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- Since more frequently visited states have higher values of $d_\pi(s)$, their weights in the objective function are also higher than those rarely visited states.

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More explanation about stationary distribution:

- *Distribution*: Distribution of the state
- *Stationary*: Long-run behavior
- *Summary*: after the agent runs a long time following a policy, the probability that the agent is at any state can be described by this distribution.

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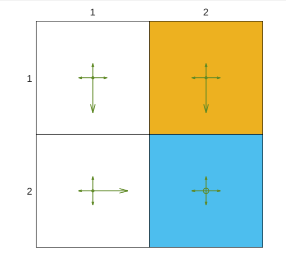
Illustrative example:

- Given a policy shown in the figure.
- Let $n_\pi(s)$ denote the number of times that s has been visited in a very long episode generated by π.
- Then, $d_\pi(s)$ can be approximated by

$$d_\pi(s) \approx \frac{n_\pi(s)}{\sum_{s' \in \mathcal{S}} n_\pi(s')}$$



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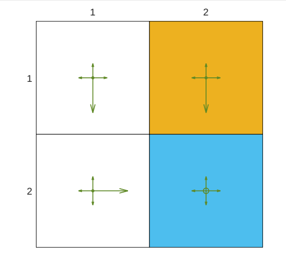

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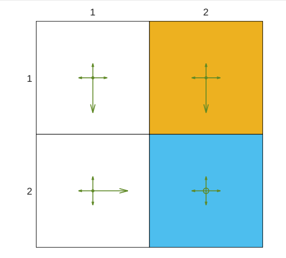

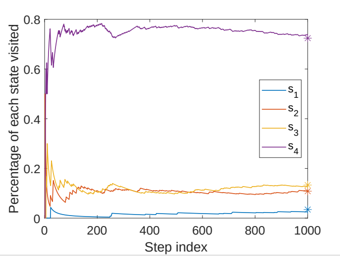

Figure: Long-run behavior of an ϵ-greedy policy with $\epsilon = 0.5$.

Objective function - Stationary distribution

The converged values can be predicted because they are the entries of d_π:

$$d_\pi^T = d_\pi^T P_\pi$$

For this example, we have P_π as

$$P_\pi = \begin{bmatrix} 0.3 & 0.1 & 0.6 & 0 \\ 0.1 & 0.3 & 0 & 0.6 \\ 0.1 & 0 & 0.3 & 0.6 \\ 0 & 0.1 & 0.1 & 0.8 \end{bmatrix}.$$

It can be calculated that the left eigenvector for the eigenvalue of one is

$$d_\pi = \left[0.0345, 0.1084, 0.1330, 0.7241\right]^T$$

A comprehensive introduction can be found in my book.

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Outline

Optimization algorithms

While we have the objective function, the next step is to optimize it.

- To minimize the objective function $J(w)$, we can use the gradient-descent algorithm:

$$w_{k+1} = w_k - \alpha_k \nabla_w J(w_k)$$

The true gradient is

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The true gradient above involves the calculation of an expectation.

We can use the stochastic gradient to replace the true gradient:

$$w_{k+1} = w_k + \alpha_k \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))\nabla_w \hat{v}(S, w)]$$

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$$w_{t+1} = w_t + \alpha_t (v_\pi(s_t) - \hat{v}(s_t, w_t))\nabla_w \hat{v}(s_t, w_t)$$

where s_t is a sample of S. Here, $2\alpha_t$ is merged to α_t.

- The samples are expected to satisfy the stationary distribution. In practice, they may not satisfy.
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Optimization algorithms

In particular,

- First, **Monte Carlo learning with function approximation**
  Let g_t be the discounted return starting from s_t in the episode. Then, g_t can be used to approximate $v_\pi(s_t)$. The algorithm becomes

$$w_{t+1} = w_t + \alpha_t (g_t - \hat{v}(s_t, w_t)) \nabla_w \hat{v}(s_t, w_t).$$

- Second, **TD learning with function approximation**
  By the spirit of TD learning, $r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t)$ can be viewed as an approximation of $v_\pi(s_t)$. Then, the algorithm becomes

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Optimization algorithms

Pseudocode: TD learning with function approximation

Initialization: A function $\hat{v}(s, w)$ that is a differentiable in w. Initial parameter w_0.
Aim: Approximate the true state values of a given policy π.

For each episode generated following the policy π, do
    For each step (s_t, r_{t+1}, s_{t+1}), do
        In the general case,
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    In the linear case,
    $w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \phi^T(s_{t+1}) w_t - \phi^T(s_t) w_t\right] \phi(s_t)$

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It can only estimate the state values of a given policy, but it is important to understand other algorithms introduced later.

Outline

Selection of function approximators

An important question that has not been answered: How to select the function $\hat{v}(s, w)$?

- The first approach, which was **widely used before**, is to use a linear function

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In the linear case where $\hat{v}(s, w) = \phi^T(s)w$, we have

$$\nabla_w \hat{v}(s, w) = \phi(s).$$

Substituting the gradient into the TD algorithm

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t) - \hat{v}(s_t, w_t)\right] \nabla_w \hat{v}(s_t, w_t)$$

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We next show that tabular representation is a special case of linear function representation. Hence, the tabular and function representations are unified!

- Consider a special feature vector for state s:

$$\phi(s) = e_s \in \mathbb{R}^{|\mathcal{S}|},$$

  where e_s is a vector with the sth entry as 1 and the others as 0.
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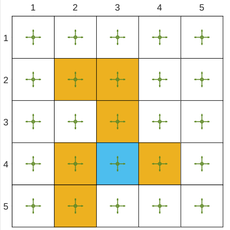
Outline

Illustrative examples

Consider a 5x5 grid-world example:



- Given a policy: $\pi(a|s) = 0.2$ for any s, a
- Our aim is to estimate the state values of this policy (policy evaluation problem).
- There are 25 state values in total. We next show that we can use less than 25 parameters to approximate 25 state values.
- Set $r_{\text{forbidden}} = r_{\text{boundary}} = -1$, $r_{\text{target}} = 1$, and $\gamma = 0.9$.

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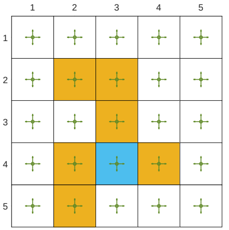

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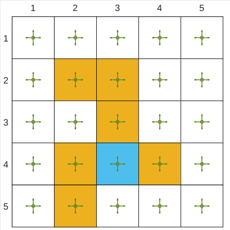

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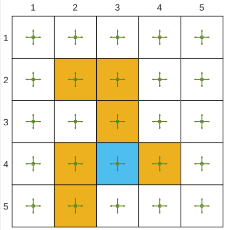

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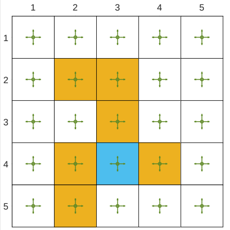

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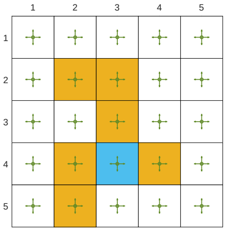
- The true state values and the 3D visualization



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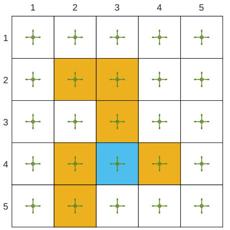

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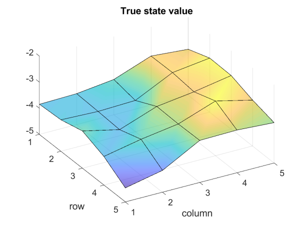

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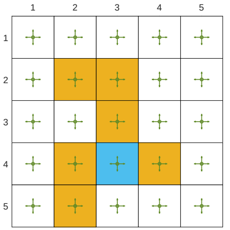

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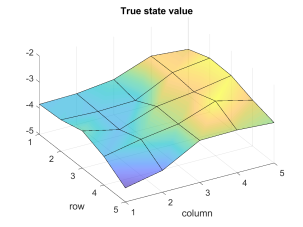

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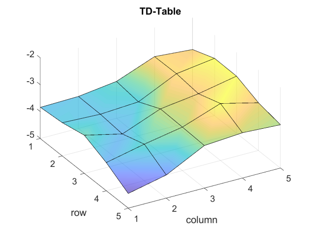
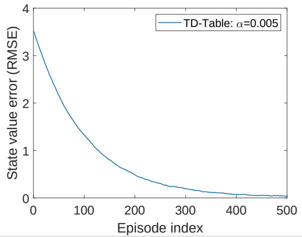
- For comparison, the results by the tabular TD algorithm (called TD-Table here):





Illustrative examples

TD-Linear:

- How to apply the TD-Linear algorithm?
  - Feature vector selection:

$$\phi(s)=\begin{bmatrix}1\\x\\y\end{bmatrix}\in\mathbb{R}^3.$$

  - In this case, the approximated state value is

$$\hat{v}(s,w)=\phi^T(s)w=[1,x,y]\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}=w_1+w_2x+w_3y.$$

Remark: $\phi(s)$ can also be defined as $\phi(s)=[x,y,1]^T$, where the order of the elements does not matter.

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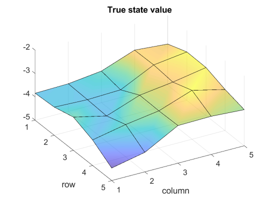

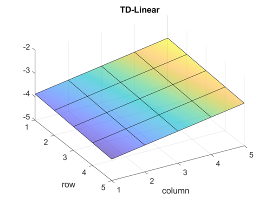

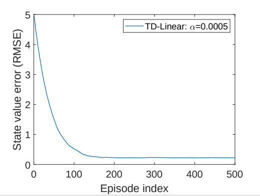

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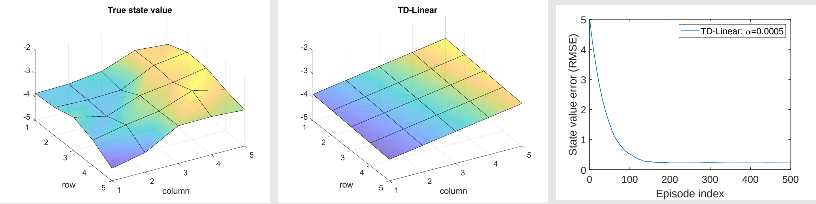

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Illustrative examples

To enhance the approximation ability, we can use high-order feature vectors and hence more parameters.

- For example, we can consider

$$\phi(s) = [1, x, y, x^2, y^2, xy]^T \in \mathbb{R}^6.$$

  In this case,

$$\hat{v}(s, w) = \phi^T(s)w = w_1 + w_2 x + w_3 y + w_4 x^2 + w_5 y^2 + w_6 xy$$

  which corresponds to a quadratic surface.

- We can further increase the dimension of the feature vector:

$$\phi(s) = [1, x, y, x^2, y^2, xy, x^3, y^3, x^2 y, xy^2]^T \in \mathbb{R}^{10}.$$

To enhance the approximation ability, we can use high-order feature vectors and hence more parameters.

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Illustrative examples

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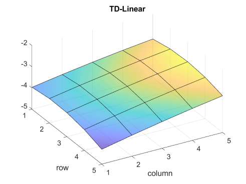

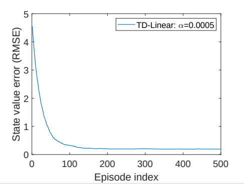

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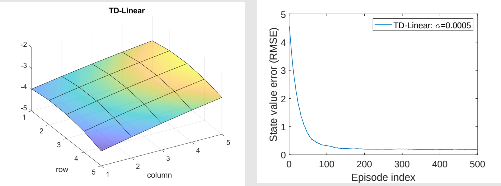

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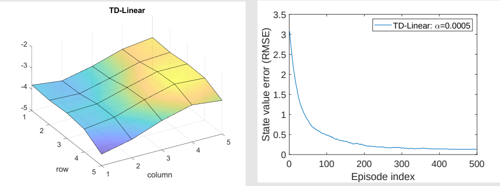

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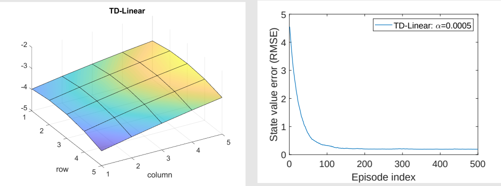

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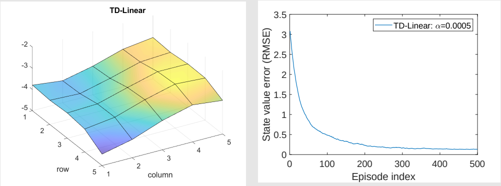

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Outline

Summary of the story

Up to now, we finished the story of TD learning with value function approximation.

1) This story started from the objective function:

$$J(w) = \mathbb{E}[(v_\pi(S) - \hat{v}(S, w))^2]$$

The objective function suggests that it is a policy evaluation problem.

2) The gradient-descent algorithm is

$$w_{t+1} = w_t + \alpha_t (v_\pi(s_t) - \hat{v}(s_t, w_t)) \nabla_w \hat{v}(s_t, w_t)$$

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Outline

Theoretical analysis (optional)

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Theoretical analysis (optional)

Different objective functions:

- **Objective function 1: True value error**

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- **Objective function 2: Bellman error**

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- **Objective function 3: Projected Bellman error**

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More details are omitted here. Interested readers can check my book.

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Sarsa with function approximation

So far, we merely considered the problem of state value estimation. That is we hope

$$\hat{v} \approx v_\pi$$

To search for optimal policies, we need to estimate action values.

The Sarsa algorithm with value function approximation is

$$w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t).$$

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Pseudocode: Sarsa with function approximation

Aim: Search a policy that can lead the agent to the target from an initial state-action pair (s_0, a_0).

For each episode, do

If the current s_t is not the target state, do

Take action a_t following $\pi_t(s_t)$, generate r_{t+1}, s_{t+1}, and then take action a_{t+1} following $\pi_t(s_{t+1})$

Value update (parameter update):

$$w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \hat{q}(s_{t+1}, a_{t+1}, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t)$$

Policy update:

$\pi_{t+1}(a|s_t) = 1 - \frac{\varepsilon}{|\mathcal{A}(s)|}(|\mathcal{A}(s)| - 1)$ if $a = \arg\max_{a \in \mathcal{A}(s_t)} \hat{q}(s_t, a, w_{t+1})$

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Sarsa with function approximation

Illustrative example:

- Sarsa with *linear function* approximation.
- $\gamma = 0.9$, $\epsilon = 0.1$, $r_{\text{boundary}} = r_{\text{forbidden}} = -10$, $r_{\text{target}} = 1$, $\alpha = 0.001$.

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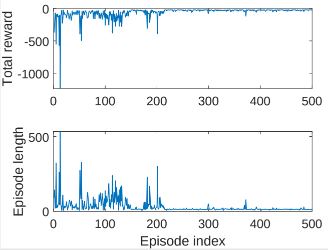

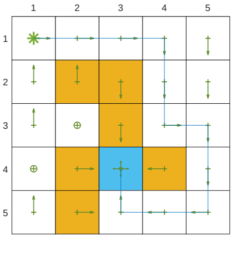

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Outline

Q-learning with function approximation

Similar to Sarsa, tabular Q-learning can also be extended to the case of value function approximation.

The q-value update rule is

$$w_{t+1} = w_t + \alpha_t \Big[r_{t+1} + \gamma \max_{a \in \mathcal{A}(s_{t+1})} \hat{q}(s_{t+1}, a, w_t) - \hat{q}(s_t, a_t, w_t) \Big] \nabla_w \hat{q}(s_t, a_t, w_t),$$

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Q-learning with function approximation

Pseudocode: Q-learning with function approximation (on-policy version)

Initialization: Initial parameter vector w_0. Initial policy π_0. Small $\varepsilon > 0$.
Aim: Search a good policy that can lead the agent to the target from an initial state-action pair (s_0, a_0).

For each episode, do

If the current s_t is not the target state, do

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Value update (parameter update):

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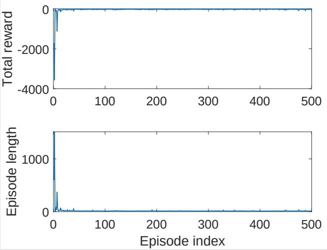

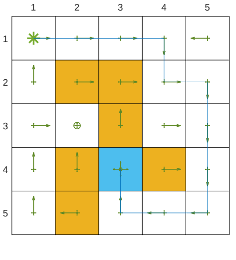

Outline

Deep Q-learning

Deep Q-learning or deep Q-network (DQN):

- One of the earliest and most successful algorithms that introduce deep neural networks into RL.
- The role of neural networks is to be a nonlinear function approximator.
- Different from the following algorithm:

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Deep Q-learning

Deep Q-learning aims to minimize the objective function/loss function:

$$J(w) = \mathbb{E}\left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w)\right)^2\right],$$

where (S, A, R, S') are random variables.

- This is actually the Bellman optimality error. That is because

$$q(s, a) = \mathbb{E}\left[R_{t+1} + \gamma \max_{a \in \mathcal{A}(S_{t+1})} q(S_{t+1}, a) \Big| S_t = s, A_t = a\right], \quad \forall s, a$$

  The value of $R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w)$ should be zero in the expectation sense

Deep Q-learning

Deep Q-learning aims to minimize the objective function/loss function:

$$J(w) = \mathbb{E}\left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w)\right)^2\right],$$

where (S, A, R, S') are random variables.

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How to minimize the objective function? Gradient-descent!

- How to calculate the gradient of the objective function? Tricky!
- That is because, in this objective function

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the parameter w not only appears in $\hat{q}(S, A, w)$ but also in

$$y \doteq R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w)$$

- Note that

$$\nabla_w y \neq \gamma \max_{a \in \mathcal{A}(S')} \nabla_w \hat{q}(S', a, w)$$

- To solve this problem, we can assume that w in y is fixed (at least for a while) when we calculate the gradient.

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To do that, we can introduce two networks.

- One is a main network representing $\hat{q}(s, a, w)$
- The other is a target network $\hat{q}(s, a, w_T)$.

The objective function in this case degenerates to

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- The basic idea of deep Q-learning is to use the gradient-descent algorithm to minimize the objective function.
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Deep Q-learning - Two networks

First technique:

- Two networks, a main network and a target network.

Why is it used?

- The mathematical reason has been explained when we calculate the gradient.

Implementation details:

- Let w and w_T denote the parameters of the main and target networks, respectively. They are set to be the same initially.
- In every iteration, we draw a mini-batch of samples $\{(s, a, r, s')\}$ from the replay buffer (will be explained later).
- The inputs of the networks include state s and action a. The target output is $y_T \doteq r + \gamma \max_{a \in \mathcal{A}(s')} \hat{q}(s', a, w_T)$. Then, we directly minimize the TD error or called loss function $(y_T - \hat{q}(s, a, w))^2$ over the mini-batch $\{(s, a, y_T)\}$.

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Another technique:

- Experience replay

Question: What is experience replay?

Answer:

- After we have collected some experience samples, we do NOT use these samples in the order they were collected.
- Instead, we store them in a set, called replay buffer $\mathcal{B} \doteq \{(s, a, r, s')\}$
- Every time we train the neural network, we can draw a mini-batch of random samples from the replay buffer.
- The draw of samples, or called experience replay, should follow a uniform distribution (why? no prior knowledge). Can we use stationary distribution here like before? No, since no policy is given.

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Question: Why is experience replay necessary in deep Q-learning? Why does the replay must follow a uniform distribution?

Answer: The answers lie in the objective function.

$$J = \mathbb{E}\left[\left(R + \gamma \max_{a \in \mathcal{A}(S')} \hat{q}(S', a, w) - \hat{q}(S, A, w)\right)^2\right]$$

- $R \sim p(R|S, A), S' \sim p(S'|S, A)$: R and S are determined by the system model.
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Answer (continued):

- However, the samples are not uniformly collected because they are generated consequently by certain policies.
- To break the correlation between consequent samples, we can use the experience replay technique by uniformly drawing samples from the replay buffer.
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Revisit the tabular case:

- Question: Why does not tabular Q-learning require experience replay?
  - Answer: Because it does not require any distribution of S or A.
- Question: Why does Deep Q-learning involve distributions?
  - Answer: Because we need to define a *scalar* objective function $J(w) = \mathbb{E}[*]$, where $\mathbb{E}$ is for all (S, A).
  - The tabular case aims to solve a set of equations for all (s, a) (Bellman optimality equation), whereas the deep case aims to optimize a scalar objective function.
- Question: Can we use experience replay in tabular Q-learning?
  - Answer: Yes, we can. And more sample efficient (why?)

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Pseudocode: Deep Q-learning (off-policy version)

Aim: Learn an optimal target network to approximate the optimal action values from the experience samples generated by a behavior policy π_b.

Store the experience samples generated by π_b in a replay buffer $\mathcal{B} = \{(s, a, r, s')\}$

For each iteration, do

Uniformly draw a mini-batch of samples from $\mathcal{B}$

For each sample (s, a, r, s'), calculate the target value as $y_T = r + \gamma \max_{a \in \mathcal{A}(s')} \hat{q}(s', a, w_T)$, where w_T is the parameter of the target network

Update the main network to minimize $(y_T - \hat{q}(s, a, w))^2$ using the mini-batch $\{(s, a, y_T)\}$

Set $w_T = w$ every C iterations

Remarks:

- Why no policy update?
- Why not using the policy update equation that we derived?
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Illustrative example:

- This example aims to learn optimal action values for every state-action pair.
- Once the optimal action values are obtained, the optimal greedy policy can be obtained immediately.

Deep Q-learning

Setup:

- One single episode is used to train the network.
- This episode is generated by an exploratory behavior policy shown in Figure (a).
- The episode only has 1,000 steps! The tabular Q-learning requires 100,000 steps.
- A shallow neural network with one single hidden layer is used as a nonlinear approximator of $\hat{q}(s, a, w)$. The hidden layer has 100 neurons.

See details in the book.

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- One single episode is used to train the network.
- This episode is generated by an exploratory behavior policy shown in Figure (a).
- The episode only has 1,000 steps! The tabular Q-learning requires 100,000 steps.
- A shallow neural network with one single hidden layer is used as a nonlinear approximator of $\hat{q}(s, a, w)$. The hidden layer has 100 neurons.

See details in the book.

Deep Q-learning

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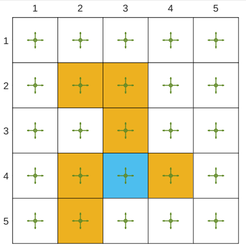
Deep Q-learning



The behavior policy.

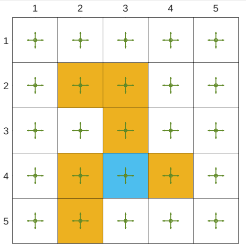
An episode of 1,000 steps.

The obtained policy.

The TD error converges to zero.

The state estimation error converges to zero.

Deep Q-learning



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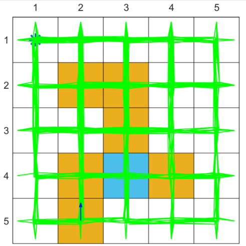

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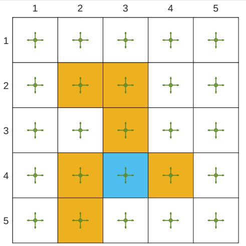

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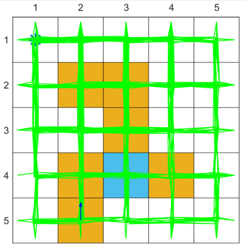

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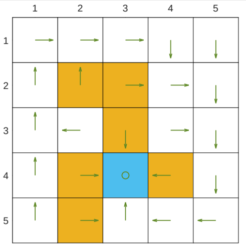

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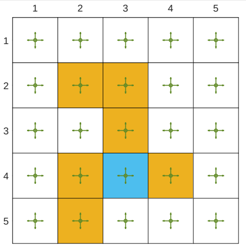

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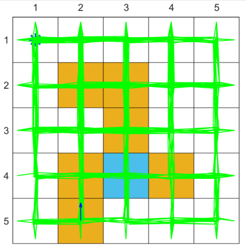

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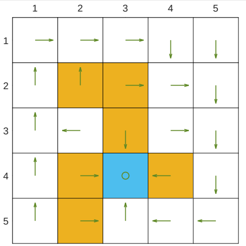

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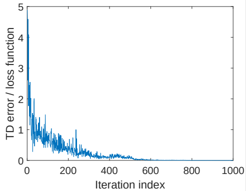

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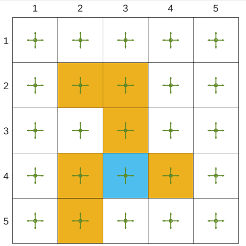

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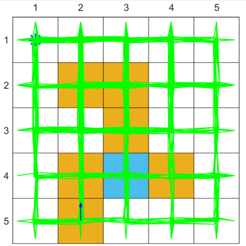

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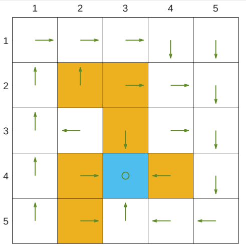

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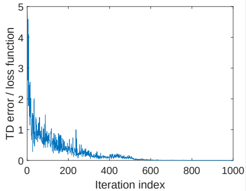

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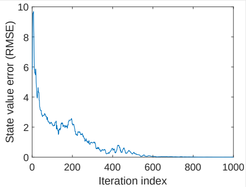
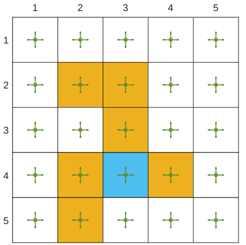


The state estimation error converges to zero.

What if we only use a single episode of 100 steps? Insufficient data



The behavior policy.

An episode of 100 steps.

The final policy.

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The state error does not converge to zero.

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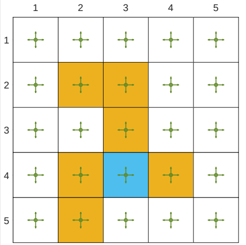

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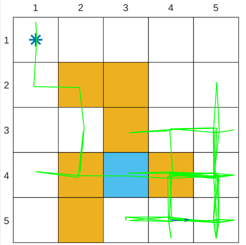

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Deep Q-learning

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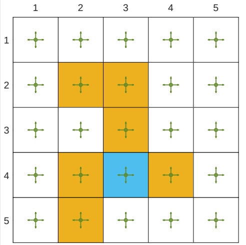

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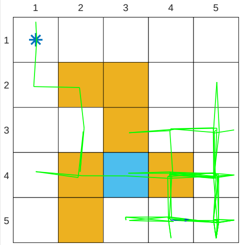

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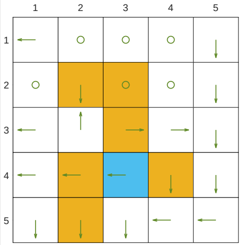

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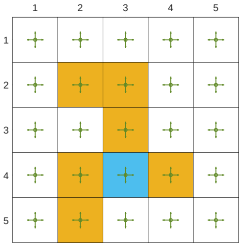

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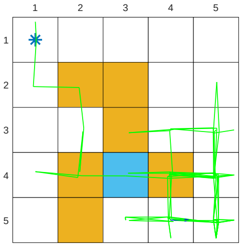

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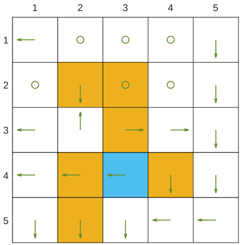

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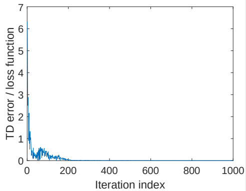

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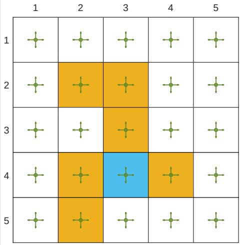

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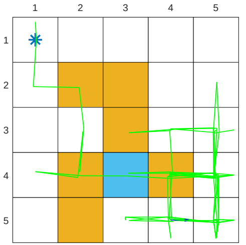

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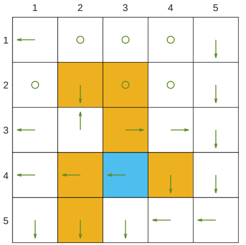

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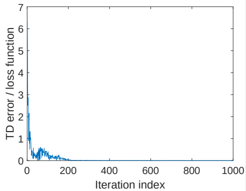

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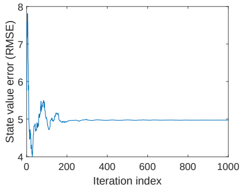

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Outline

Summary

This lecture introduces the method of value function approximation.

- First, understand the basic idea.
- Second, understand the basic algorithms.