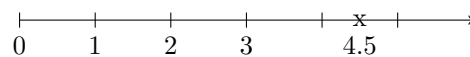


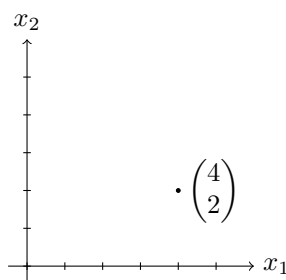
# Chapter 1

## Vectors

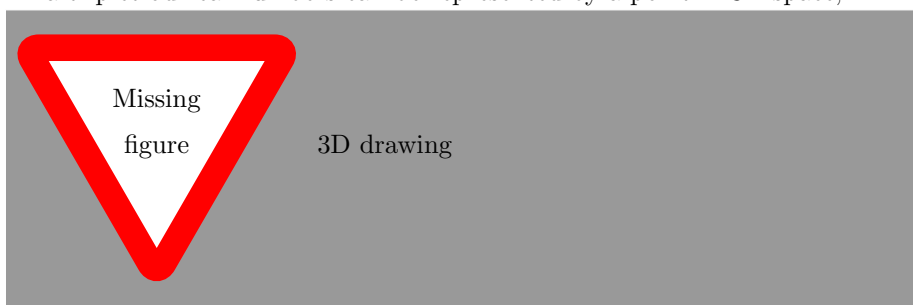
A real number can be represented by a point on a line, which is a 2-dimensional space,  $\mathbb{R}$



a pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space,  $\mathbb{R}^2$



a triplet of real numbers can be represented by a point in 3D space,  $\mathbb{R}^3$



### Definition

A vector is an ordered collection of  $n$  numbers

**Notation**

Usually vectors are given by letters, such as  $u, v, w$ . In textbooks vectors are written with bold font. In handwriting vectors are often written with a right arrow on top, such as  $\vec{u}$ . We will underline vectors, like so:  $\underline{u}$ .

□

**Definition**

Let us consider vector  $\underline{u} \in \mathbb{R}^n$ . The  $i$ -th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is  $u_i$

**Example**

$$\underline{u} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3 \Rightarrow u_1 = 3, u_2 = 7, u_3 = 11$$

**Definition**

Let us consider vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . Vector  $\underline{w} \in \mathbb{R}^n$  is a sum of  $\underline{u}$  and  $\underline{v}$ ,  $\underline{w} = \underline{u} + \underline{v}$ , if  $w_i = u_i + v_i$  for all  $i = 1, \dots, n$

**Example**

1.

$$\underline{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 + (-1) \\ 5 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

2.

$$\underline{u} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

$\underline{u} + \underline{v}$  is not defined! Both vectors should have the same number of components.

**Definition**

1. Vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$  are equal, if  $u_i = v_i$  for all  $i = 1, \dots, n$

2. A scalar is just another name for real number
3. Let us consider a scalar  $\alpha \in \mathbb{R}$  and vector  $\underline{u} \in \mathbb{R}^n$ . A product of  $\alpha$  and  $\underline{u}$  is defined as:

$$\alpha \underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot v_n \end{pmatrix}$$

**Example**

$$\alpha = 3, \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \Rightarrow \alpha \cdot \underline{u} = \begin{pmatrix} 3 \cdot -1 \\ 3 \cdot 2 \\ 3 \cdot 5 \\ 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

**Definition**

Let us consider scalars  $\alpha$  and  $\beta$ , and vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . A sum of  $\alpha \underline{u} + \beta \cdot \underline{v}$  is called a linear combination of vectors  $\underline{u}$  and  $\underline{v}$ .

**Example**

1.

$$2 \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

2.

$$\underline{u} - \underline{v} = 1 \cdot \underline{u} + (-1) \cdot \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_i - v_i \end{pmatrix}$$

3.

$$\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_i - u_i \end{pmatrix} = \underline{0}$$

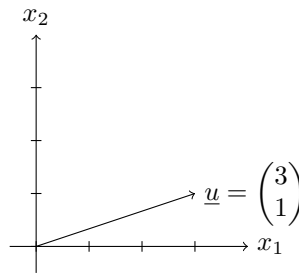
**Definition**

Vector  $\underline{u} \in \mathbb{R}^n$  is called a zero vector if all  $u_i = 0$ ,  $i = 1, \dots, n$ . The zero vector is often written as  $\underline{0} \in \mathbb{R}^n$

## 1.1 Graphic representation of vectors and vector operations

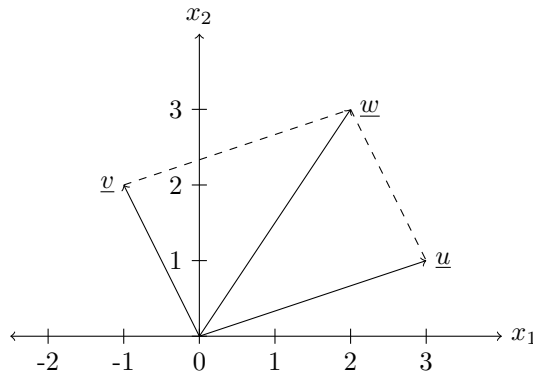
A vector can be represented in the following way:

1. An ordered collection of numbers,  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
2. As an arrow in space



3. A vector is a point in space, the endpoint of a vector from the origin.

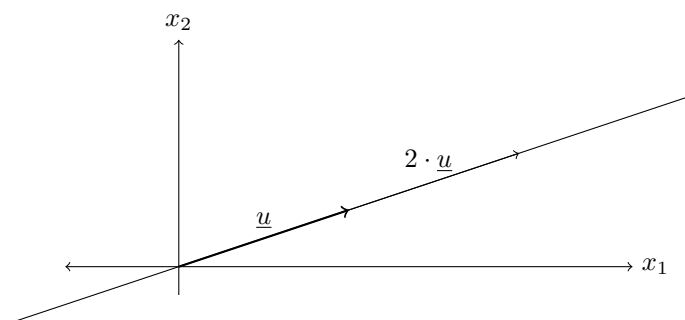
Let us consider vectors  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ,  $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and  $\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$



Let us consider vector  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ . What is  $2 \cdot \underline{u}$ ? We can calculate as follows:

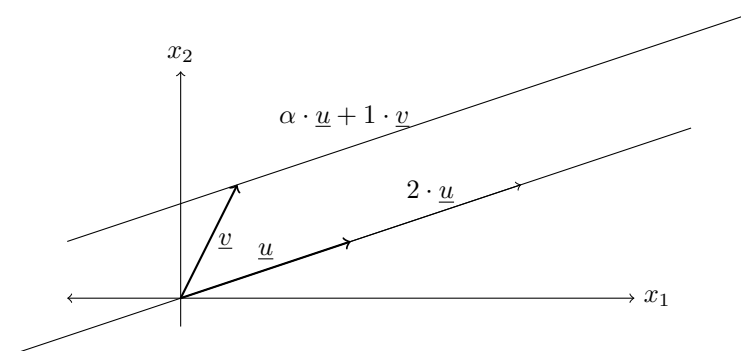
$$2 \cdot \underline{u} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We stretch vector  $\underline{u}$  2 times along the line defined by vector  $\underline{u}$ . What is  $-\underline{u}$ ? Simply reverse the direction. What will be the representation of  $\alpha \underline{u}$  for all possible values of  $\alpha$ ? An endless line



Let us consider two vectors  $\underline{u} \in \mathbb{R}^2$  and  $\underline{v} \in \mathbb{R}^2$ . What will be the representation of all linear combinations of  $\underline{u}$  and  $\underline{v}$ ,  $\alpha\underline{u} + \beta\underline{v}$

1. Plane:



2. Line:  $\underline{u}$  and  $\underline{v}$  are on the same line.

Note: Consider  $\underline{u}, \underline{v} \in \mathbb{R}^n$ .  $\underline{u}$  and  $\underline{v}$  are on the same line if there exists scalars  $\alpha$  and  $\beta$  such that  $\alpha\underline{u} + \beta\underline{v} = \underline{0}$ , when  $\alpha$  and  $\beta \neq 0$

3. Point: if  $\underline{u} = \underline{0}$  and  $\underline{v} = \underline{0} \Rightarrow \alpha\underline{u} + \beta\underline{v} = \underline{0}$

Consider  $\underline{v}, \underline{u}$ . They are on the same line if  $\alpha\underline{u} + \beta\underline{v} = \underline{0}$  and  $\alpha, \beta \neq 0$

## 1.2 Dot Product (Scalar product)

### Definition

Let us consider two vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . The dot (or scalar) product of vectors  $\underline{u}$  and  $\underline{v}$  is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^n u_iv_i$$

### Notation

We will use  $\langle \underline{u}, \underline{v} \rangle$  to denote the dot product, but sometimes  $\underline{u} \cdot \underline{v}$  is used

□

**Example**

1.

$$\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

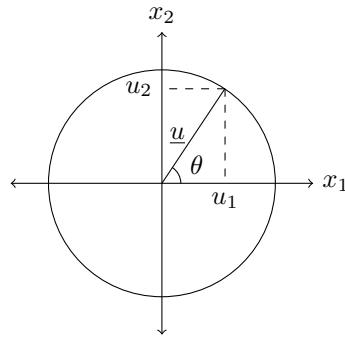
$$\langle \underline{u}, \underline{v} \rangle = 1 \cdot 0 + (-1) \cdot \frac{1}{2} + 3 \cdot (-1) = -3.5$$

2.

$$\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \langle \underline{u}, \underline{v} \rangle = 0$$

Let us consider  $\mathbb{R}^2$ . What is the set of all possible endpoints of unit vectors in  $\mathbb{R}^2$ , originating from the origin?

Fix positioning problem



$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\cos(\theta) = \frac{u_1}{\|\underline{u}\|} = u_1$$

$$\sin(\theta) = \frac{u_2}{\|\underline{u}\|} = u_2$$

$$\underline{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Now let us consider two unit vectors If  $\underline{u} \neq \underline{0}$  or  $\underline{v} \neq \underline{0}$  are not unit vectors

MISSING FIGURE PAGE 5, MIDDLE

$$\begin{aligned} \langle \underline{u}, \underline{v} \rangle &= \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \\ &= \cos(\theta - \varphi) = \cos(\psi) \\ &= \cos(\angle(\underline{u}, \underline{v})) \end{aligned}$$

we can find the angle between them as follows:

Add unit vectors underbrace Page 5

$$\begin{aligned}
\langle \underline{u}, \underline{v} \rangle &= \left\langle \|\underline{u}\| \cdot \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \|\underline{v}\| \cdot \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle \\
&= \|\underline{u}\| \|\underline{v}\| \left\langle \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle \\
&= \|\underline{u}\| \|\underline{v}\| \cos(\angle(\underline{u}, \underline{v}))
\end{aligned}$$

**Lemma**

If  $\underline{u} \neq \underline{0}, \underline{v} \neq \underline{0}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$ , then

$$\cos(\angle(\underline{u}, \underline{v})) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|}$$

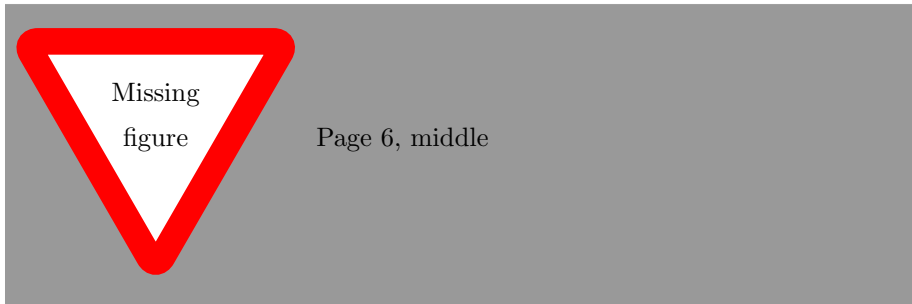
**1.3 Properties of dot product**

1.  $\langle \alpha \cdot \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$  for any  $\alpha \in \mathbb{R}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$ . Proof:

$$\begin{aligned}
\langle \alpha \cdot \underline{u}, \underline{v} \rangle &= (\alpha u_1) \cdot v_1 + \cdots + (\alpha u_n) \cdot v_n \\
&= \alpha \cdot (u_1 \cdot v_1 + \cdots + u_n \cdot v_n) \\
&= \alpha \cdot \langle \underline{u}, \underline{v} \rangle
\end{aligned}$$

2.  $\langle \underline{u}, \alpha \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$  for any  $\alpha \in \mathbb{R}, \underline{u}, \underline{v} \in \mathbb{R}^n$

3.  $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \cdot \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle, \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$

**Example**

Let us consider  $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \cdot 4 = 9 + 16 = 25 = 5^2$

**Definition**

The length of vector  $\underline{u} \in \mathbb{R}^n, \|\underline{u}\|$ , is defined as  $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$ . Sometimes it is also called the Euclidian norm of  $\underline{u}$ .

**Definition**

A vector with length equal to 1 is called a unit vector

If we take vector  $\underline{u} \neq \underline{0}$ , how to make it a unit vector? We should multiply vector  $\underline{u}$  by  $\frac{1}{\|\underline{u}\|}$ , we will get  $\frac{\underline{u}}{\|\underline{u}\|} = \text{unit vector}$ .

In our previous example:  $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . Unit vector is then

$$\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

We got  $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cdot \cos(\angle(\underline{u}, \underline{v}))$ . Let us take the absolute value of this

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \|\underline{v}\| \cdot |\cos(\angle(\underline{u}, \underline{v}))|$$

Notice that  $|\cos(\angle(\underline{u}, \underline{v}))| \leq 1$

**Lemma**

Cauchy Schwartz Inequality: for any  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It is easy to see that Cauchy - Schwartz inequality is correct also for zero vectors.



## Chapter 2

# Matrices

Let us consider a linear combination of vectors

$$x_1 \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrices times the vector.

### Example

1.

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

### Notation

Matrices are usually written with capital letters, i.e.  $A, B, C, \dots$

$A$  is an  $n$  by  $m$  matrix,  $A \in \mathbb{R}^{n,m}$  if it has  $n$  rows and  $m$  columns.

The element of matrix  $A$  located in row  $i$  and column  $j$  is written as  $a_{ij}$  or  $(A)_{ij}$ .

□

2.

$$\begin{aligned}
A &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
A \cdot \underline{x} &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (-1) \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}
\end{aligned}$$

For the product of matrix  $A$  with vector  $\underline{x}$  to exist, matrix  $A$  should have the same number of columns as vector  $\underline{x}$  components.

## 2.1 Matrix Operations

### Definition

Let us consider matrices  $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{n,m}$  where  $n$  = rows,  $m$  = columns. Matrix  $C \in \mathbb{R}^{n,m}$  is a sum of  $A$  and  $B$ ,  $C = A+B$  if  $C_{ij} = A_{ij} + B_{ij}$  for all  $i = 1, \dots, n$ ,  $j = 1, \dots, m$

### Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, C = A + B = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 4 & 5 \end{pmatrix}$$

### Definition

A product of a scalar  $\alpha$  and a matrix  $A \in \mathbb{R}^{n,m}$  is defined as  $(\alpha A)_{ij} = \alpha \cdot A_{ij}$ ,  $\forall i = 1, \dots, n; j = 1, \dots, m$ .

### Example

$$\alpha = 3, A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \alpha \cdot A = \begin{pmatrix} 0 & 0 & 3 \\ 6 & 9 & 15 \end{pmatrix}$$

### Properties

- $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{n,m}$ :  $A + B = B + A$

Proof:

$$\begin{cases} (A + B)_{ij} = A_{ij} + B_{ij} \\ (B + A)_{ij} = B_{ij} + A_{ij} \end{cases}$$

- $A, B, C \in \mathbb{R}^{n,m}$ :  $(A + B) + C = A + (B + C)$
- $\alpha \cdot (A + B) = \alpha A + \alpha B$  for  $\forall \alpha \in \mathbb{R}, A, B \in \mathbb{R}^{n,m}$

## **2.2 Matrix - Matrix multiplication**