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# Linear Algebra Class Notes

*Based on Professor Pivkin's Material*

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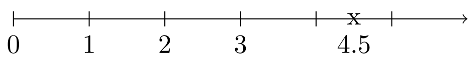
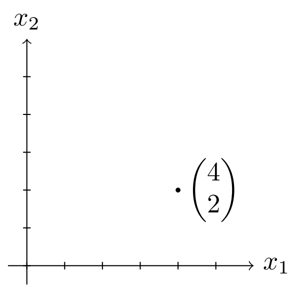
# Chapter 1

# Vectors

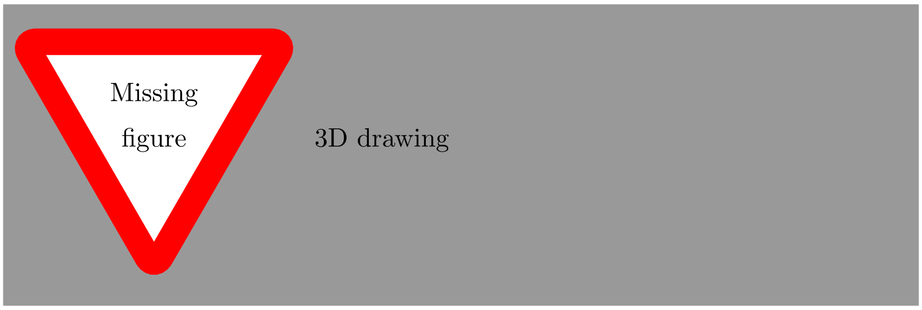
A real number can be represented by a point on a line, which is a 2-dimensional space, $\mathbb{R}$




a pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space, $\mathbb{R}^2$




a triplet od real numbers can be represented by a point in 3D space, $\mathbb{R}^3$




**Definition**

A vector is an ordered collection of $n$ numbers

### Notation

Usually vectors are given by letters, such as $u, v, w$. In textbooks vectors are written with bold font. In handwriting vectors are often written with a right arrow on top, such as $\overrightarrow{u}$. We will underline vectors, like so: $\underline{u}$.

□

**Definition**

Let us consider vector $\underline{u} \in \mathbb{R}^n$. The $i-$th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is $u_i$

### Example

$$\underline{u} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3 \Rightarrow u_1 = 3, u_2 = 7, u_3 = 11$$

**Definition**

Let us consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of $\underline{u}$ and $\underline{v}$, $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for all $i = 1, \ldots, n$

### Example

1.
$$\underline{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 + (-1) \\ 5 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

2.
$$\underline{u} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

$\underline{u} + \underline{v}$ is not defined! Both vectors should have the same number of components.

**Definition**

1. Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, if $u_i = v_i$ for all $i = 1, \ldots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$. A product of $\alpha$ and $\underline{u}$ is defined as:

$$\alpha\underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot v_n \end{pmatrix}$$

**Example**

$$\alpha = 3, \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \Rightarrow \alpha \cdot \underline{u} \begin{pmatrix} 3 \cdot -1 \\ 3 \cdot 2 \\ 3 \cdot 5 \\ 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

**Definition**

Let us consider scalars $\alpha$ and $\beta$, and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. A sum of $\alpha\underline{u} + \beta \cdot \underline{v}$ is called a linear combination of vectors $\underline{u}$ and $\underline{v}$.

**Example**

1. $$2 \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

2. $$\underline{u} - \underline{v} = 1 \cdot \underline{u} + (-1) \cdot \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_i - v_i \end{pmatrix}$$

3. $$\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_i - u_i \end{pmatrix} = \underline{0}$$

**Definition**

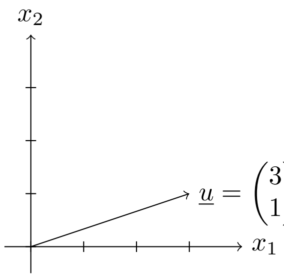
Vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1, \ldots, n$. The zero vector is often written as $\underline{0} \in \mathbb{R}^n$

## 1.1 Graphic representation of vectors and vector operations

A vector can be represented in the following way:

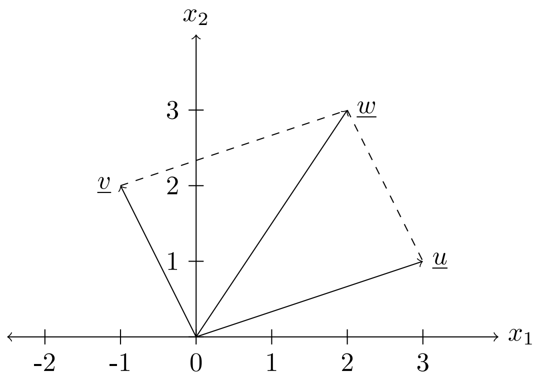
1. An ordered collection of numbers, $\underline{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2. As an arrow in space




3. A vector is a point in space, the endpoint of a vector from the origin.

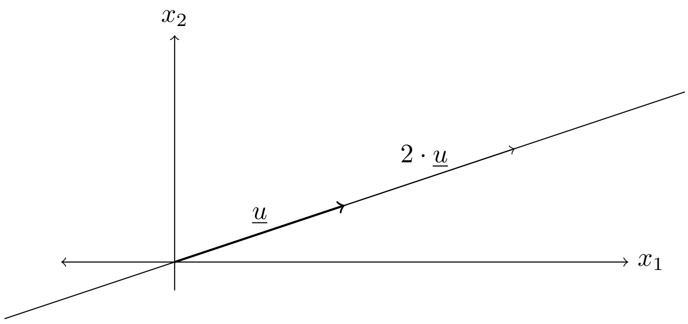
Let us consider vectors $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$




Let us consider vector $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What is $2 \cdot \underline{u}$? We can calculate as follows:

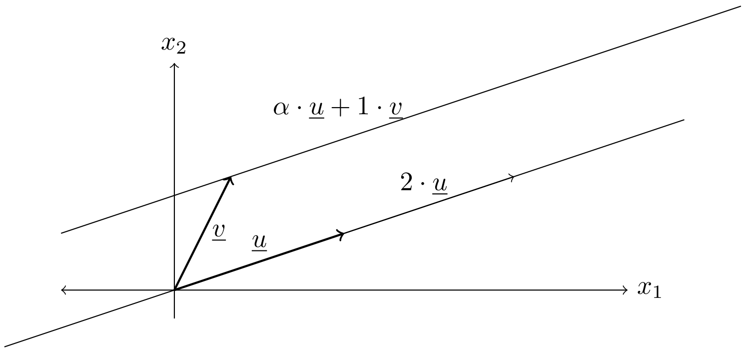
$$2 \cdot \underline{u} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We stretch vector $\underline{u}$ 2 times along the line defined by vector $\underline{u}$. What is $-\underline{u}$? Simply reverse the direction. What will be the representation of $\alpha\underline{u}$ for all possible values of $\alpha$? An endless line




Let us consider two vectors $\underline{u} \in \mathbb{R}^2$ and $\underline{v} \in \mathbb{R}^2$. What will be the representation of all linear combinations of $\underline{u}$ and $\underline{v}$, $\alpha\underline{u} + \beta\underline{v}$

1. Plane:




2. Line: $\underline{u}$ and $\underline{v}$ are on the same line.
   Note: Consider $\underline{u}, \underline{v} \in \mathbb{R}^n$. $\underline{u}$ and $\underline{v}$ are on the same line if there exists scalars $\alpha$ and $\beta$ such that $\alpha\underline{u} + \beta\underline{v} = \underline{0}$, when $\alpha$ and $\beta \neq 0$

3. Point: if $\underline{u} = \underline{0}$ and $\underline{v} = \underline{0} \Rightarrow \alpha\underline{u} + \beta\underline{v} = \underline{0}$

Consider $\underline{v}, \underline{u}$. They are on the same line if $\alpha\underline{u} + \beta\underline{v} = \underline{0}$ and $\alpha, \beta \neq 0$

## 1.2 Dot Product (Scalar product)

**Definition**

Let us consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. The dot (or scalar) product of vectors $\underline{u}$ and $\underline{v}$ is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + \underline{u}_n \underline{v}_n = \sum_{i=1}^{n} u_i v_i$$

### Notation

We will use $\langle \underline{u}, \underline{v} \rangle$ to denote the dot product, but sometimes $\underline{u} \cdot \underline{v}$ is used

$\square$

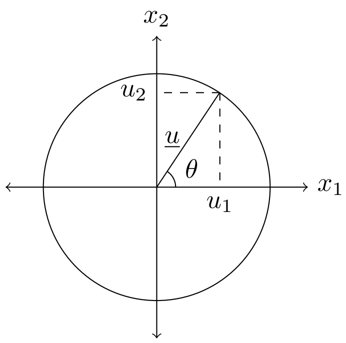
**Example**

1.
$$\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \underline{u} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$
$$\langle \underline{u}, \underline{v} \rangle = 1 \cdot 0 + (-1) \cdot \frac{1}{2} + 3 \cdot (-1) = -3.5$$

2.
$$\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{u} = \begin{pmatrix} 0 \\ 1 \\ , \end{pmatrix} \langle \underline{u}, \underline{v} \rangle = 0$$

Let us consider $\mathbb{R}^2$. What is the set of all possible endpoints of unit vectors in $\mathbb{R}^2$, originating from the origin?

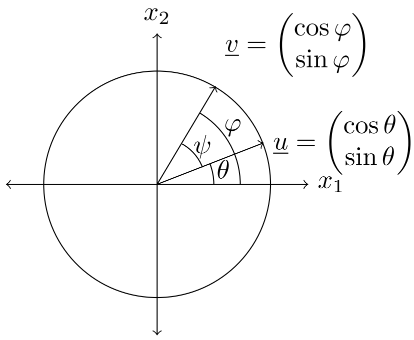
Fix positioning problem




$$\begin{aligned} \underline{u} &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \cos(\theta) &= \frac{u_1}{\|\underline{u}\|} = u_1 \\ \sin(\theta) &= \frac{u_2}{\|\underline{u}\|} = u_2 \\ \underline{u} &= \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \end{aligned}$$

Now let us consider two unit vectors If $\underline{u} \neq \underline{0}$ or $\underline{v} \neq \underline{0}$ are not unit vectors we can find the angle between them as follows:

$$\begin{aligned} \langle \underline{u}, \underline{v} \rangle &= \left\langle \|\underline{u}\| \cdot \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \|\underline{v}\| \cdot \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle \\ &= \|\underline{u}\|\|\underline{v}\| \left\langle \underbrace{\frac{1}{\|\underline{u}\|} \cdot \underline{u}, \frac{1}{\|\underline{v}\|} \cdot \underline{v}}_{\text{Unit Vectors}} \right\rangle \\ &= \|\underline{u}\|\|\underline{v}\| \cos\left(\angle(\underline{u}, \underline{v})\right) \end{aligned}$$




$$\begin{aligned} \langle \underline{u}, \underline{v} \rangle &= \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) \\ &= \cos(\theta - \varphi) = \cos(\psi) \\ &= \cos(\angle(\underline{u}, \underline{v})) \end{aligned}$$

**Lemma**

If $\underline{u} \neq \underline{0}, \underline{v} \neq \underline{0}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$, then

$$\cos\left(\angle(\underline{u}, \underline{v})\right) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\|\|\underline{v}\|}$$

## 1.3 Properties of dot product

1. $\langle \alpha \cdot \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$. Proof:

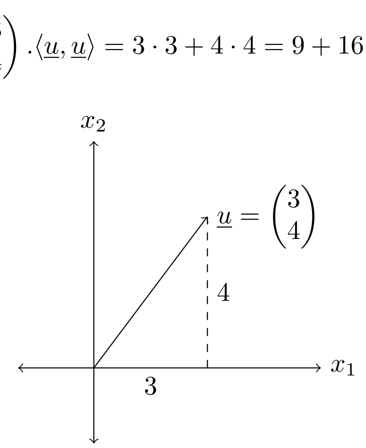
$$\begin{aligned} \langle \alpha \cdot \underline{u}, \underline{v} \rangle &= (\alpha u_1) \cdot v_1 + \cdots + (\alpha u_n) \cdot v_n \\ &= \alpha \cdot (u_1 \cdot v_1 + \cdots + u_n \cdot v_n) \\ &= \alpha \cdot \langle \underline{u}, \underline{v} \rangle \end{aligned}$$

2. $\langle \underline{u}, \alpha\underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u}, \underline{v} \in \mathbb{R}^n$

3. $\langle \alpha\underline{u} + \beta\underline{v}, \underline{w} \rangle = \alpha \cdot \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle, \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{vw} \in \mathbb{R}^n$

**Example**

Let us consider $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .$\langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \cdot 4 = 9 + 16 = 25 = 5^2$



**Definition**

The length of vector $\underline{u} \in \mathbb{R}^n$, $\|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidian norm of $\underline{u}$.

**Definition**

A vector with length equal to 1 is called a unit vector

If we take vector $\underline{u} \neq \underline{0}$, how to make it a unit vector? We should multiply vector $\underline{u}$ by $\frac{1}{\|\underline{u}\|}$, we will get $\frac{\underline{u}}{\|\underline{u}\|}$ = unit vector.

In our previous example: $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Unit vector is then

$$\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

We got $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cdot \cos\left(\angle(\underline{u}, \underline{v})\right)$ . Let us take the absolute value of this

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \|\underline{v}\| \cdot \left|\cos\left(\angle(\underline{u}, \underline{v})\right)\right|$$

Notice that $\left|\cos\left(\angle(\underline{u}, \underline{v})\right)\right| \leq 1$

**Lemma**

Cauchy Schwartz Inequality: for any $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It is easy to see that Cauchy - Schwartz inequality is correct also for zero vectors.

# Chapter 2

# Matrices

Let us consider a linear combination of vectors

$$x_1 \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrices times the vector.

**Example**

1.
$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

**Notation**

Matrices are usually written with capital letters, i.e. $A, B, c, \dots$

$A$ is an $n$ by $m$ matrix, $A \in \mathbb{R}^{n,m}$ if it has $n$ rows and $m$ columns.

The element of matrix $A$ located in row $i$ and column $j$ is written as $a_{ij}$ or $(A)_{ij}$.

$\square$

2.

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A \cdot \underline{x} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1) \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

For the product of matrix $A$ with vector $\underline{x}$ to exist, matrix $A$ should have the same number of columns as vector $\underline{x}$ components.

## 2.1 Matrix Operations

**Definition**

Let us consider matrices $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$ where $n =$ rows, $m =$ columns. Matrix $C \in \mathbb{R}^{n,m}$ is a sum of $A$ and $B, C = A + B$ if $C_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \dots, n$, $j = 1, \dots, m$

**Example**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, C = A + B = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 4 & 6 \end{pmatrix}$$

**Definition**

A product of a scalar $\alpha$ and a matrix $A \in \mathbb{R}^{n,m}$ is defined as $(\alpha A)_{ij} = \alpha \cdot A_{ij}$, $\forall i = 1, \dots, n; j = 1, \dots, m$.

**Example**

$$\alpha = 3, A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \alpha \cdot A = \begin{pmatrix} 0 & 0 & 3 \\ 6 & 9 & 15 \end{pmatrix}$$

**Properties**

- $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$: $A + B = B + A$
  Proof:
  $$\begin{cases} (A+B)_{ij} = A_{ij} + B_{ij} \\ (B+A)_{ij} = B_{ij} + A_{ij} \end{cases}$$
- $A, B, C \in \mathbb{R}^{n,m}$: $(A + B) + C = A + (B + C)$
- $\alpha \cdot (A + B) = \alpha A + \alpha B$ for $\forall \alpha \in \mathbb{R}, A, B \in \mathbb{R}^{n,m}$

## 2.2 Matrix - Matrix multiplication

**Definition**

Let us consider matrix $A \in \mathbb{R}^{n,m}$ and $A \in \mathbb{R}^{m,l}$. Then $C = A \cdot B$ is an $n$ by $l$ matrix, $C \in \mathbb{R}^{n,l}$ such that

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

**Example**

1.

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \in \mathbb{R}^{3,2}, B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{1,4} \\
C &= A \cdot B \in \mathbb{R}^{3,4} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 10 & 4 & 3 \end{pmatrix}
\end{aligned}$$

2.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}; AB = \text{ Not defined}$$

**Properties**

1. $AB$ is not always equal to $BA$. (most often, is the case).

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

2. $C(A+B) = CA + CB$

3. $(A+B)C = AC + BC$

4. $\alpha(AB) = A(\alpha B)$, $A \in \mathbb{R}^{n,m}$, $B \in \mathbb{R}^{m,l}$. Proof:

$$(\alpha(AB))_{ij} = \alpha \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{k=1}^{m} a_{ik} (\alpha b_{kj}) = A(\alpha B)$$

5. $(AB)C = A(BC)$

**Theorem**

Let us consider matrices $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,n}$, such that $A^{-1}$ and $B^{-1}$ exist. Then,

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

**Proof**

$$\left.\begin{aligned}(AB)(B^{-1}A^{-1}) = I \\ (B^{-1}A^{-1})(AB) = I\end{aligned}\right\} \text{ Prove this}$$

$$(AB)(B^{-1}A^{-1}) = A\underbrace{BB^{-1}}_{I}A^{-1} = A \cdot I \cdot A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{A^{-1}A}_{I}B = \cdot B^{-1} \cdot I \cdot B = I$$

$\Rightarrow$ According to the definition $B^{-1}A^{-1}$ is the inverse of $AB$

$\square$

**Lemma**

$A, B, C \in \mathbb{R}^{n,n}, \exists A^{-1}, \exists B^{-1}, \exists C^{-1}$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

**Theorem**

Let us consider $A \in \mathbb{R}^{n,n}$. Let us consider that $B \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,n}$ are both inverses of $A$.Then $B = C$. (The inverse is unique)

**Proof**

$$AB = BA = I \qquad\qquad AC = CA = I$$

$$\underbrace{BA \times C}\, = I \times C \qquad\qquad \underbrace{B \times AC}\, = B \times I$$

$$\underline{\underline{C = B}}$$

$\square$

## 2.3 Linear system of equations

Let us consider the following system of equations

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_2 + 2x_3 = 3 \\ 4x_3 = -1 \end{cases}$$

Find $x_1, x_2, x_3$. We can write this system in matrix form.

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \in \mathbb{R}^{3,3}, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \Rightarrow A\underline{x} = \underline{b}$$

$A$ is an upper triangular matrix. We can use backward substitution to find the solution:

1. $x_3 = -\frac{1}{4} = \frac{b_3}{a_{33}}$
2. $x_2 = \frac{3-2x_3}{1} = \frac{3-2\cdot\left(-\frac{1}{4}\right)}{1} = 3.5 = \frac{b_2 - a_{33}\cdot x_3}{a_{22}}$
3. $x_1 = \frac{2-4x_3-2x_2}{2} = -2 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$

In general, if $A \in \mathbb{R}^{n,n}$ is an upper triangular with $a_{ii} \neq 0, i = 1, \ldots, n$ then the backward substitution works as:

1. $x_n = \frac{b_n}{a_{nn}}$
2. $x_{n-1} = \frac{b_{n-1} - a_{n-1\ n}x_n}{a_{n-1\ n-1}}$ $\quad\ldots\quad$ $x_i =$

Cannot understand last calculation, page 12, LinearAlgebraNotes_1

## 2.4 Inverse Matrix

**Definition**

Let us consider a matrix $A \in \mathbb{R}^{n,n}$ (square matrix). Matrix $B \in \mathbb{R}^{n,n}$ is called an inverse of $A$, if

$$A \cdot B = I \text{ AND } B \cdot A = I$$

(Both conditions are vital)

### Notation

Usually, the inverse of $A$ is written as $A^{-1}$

### Note

Not all matrices have an inverse! In most cases, it is quite difficult to find an inverse matrix. But in some cases, the inverse is easy to find.

**Example**

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}, a_{ii} \neq 0, \forall i = 1, \dots, n$$

Then

$$\begin{aligned} A &= \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ A \cdot A^{-1} &= \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \\ A \cdot A^{-1} &= I \end{aligned}$$

## 2.5 Special Matrices

- Let us consider $A \in \mathbb{R}^{n,m}$ matrix. $A$ is called the zero matrix if all $a_{ij} = 0$, $\forall i = 1, \dots, n; j = 1, \dots, n$
- $D \in \mathbb{R}^{n,n}$ - square matrix is called diagonal matrix, if $d_{ij} = 0$ and if $i \neq j$
- Identity matrix:

$$I \in \mathbb{R}^{n,n}, I = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

- $L \in \mathbb{R}^{n,n}$ - lower triangular matrix, if

$$l_{ij} = 0, \forall i < j, L = \begin{pmatrix} * & \dots & 0 \\ \vdots & \ddots & \vdots \\ * & \dots & * \end{pmatrix}$$

- $U \in \mathbb{R}^{n,n}$ - upper triangular matrix, if

$$u_{ij} = 0, \forall i > j, L = \begin{pmatrix} * & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & * \end{pmatrix}$$

**Remark:**

If $A, B \in \mathbb{R}^{n,n}$ are both upper (lower) triangular matrices, then $C = A \cdot B$ is an upper triangular (lower).

If $A$ is lower triangular, $A \in \mathbb{R}^{n,n}, a_{ii} \neq 0, i = 1, \dots, n$ then we can use forward substitution, i.e.:

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \\ &\vdots \\ x_i &= \frac{b_i - a_{i1}x_1 - \dots - a_{ii-1}x_{i-1}}{a_{ii}} \qquad \forall i = 2, \dots, n \end{aligned}$$

## 2.6 Elementary Transition Matrices

Let us consider matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix}$$

and matrix

$$I_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$I_{21} \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

also

$$A \cdot I_{21} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

**Definition**

We can define the elementary transition matrix $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix $A \in \mathbb{R}^{n,n}$ then when calculating $I_{pq}$ we take row $q$ of $A$, put it into row $p$, replace everything else with 0.

We can also define:

$$E_{pq}(l) = I + l \cdot I_{pq}, l \in \mathbb{R} \text{ - scalar}$$
$$E_{pq}(l) \cdot A = (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A$$

We take row $q$ of $A$, multiply it by $l$, add it to row $p$ of $A$

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

If we have vector $\underline{b} \in \mathbb{R}^n$, then $I_{pq}\underline{b}$ - we take component $q$ of $\underline{b}$, put it into component $p$, replace everything else with zeros.

$$E_{pq}(l)\underline{b} \text{ - same as for matrices}$$

# Chapter 3

# Gaussian Elimination

**Example**

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} . \underline{b} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}, A\underline{x} = b$$

We can write this as a system of equations:

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 7x_3 = 10 \end{cases}$$

We can multiply equation 1 by $-\frac{a_{21}}{a_{11}} = -\frac{4}{2} = -2$, and add to equation 2. This is equivalent to multiplying $A\underline{x} = \underline{b}$ by $E_{21}\left(-\frac{a_{21}}{a_{11}}\right)$ on the left.

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 7x_3 = 10 \end{cases} \Leftrightarrow E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{21}\left(-\frac{a_{21}}{a_{11}}\right)\underline{b}$$

$$E_{21}\left(-\frac{a_{21}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ x_2 + 5x_3 = 12 \end{cases}$$

$$\Leftrightarrow E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right)\underline{b}$$

$$E_{31}\left(-\frac{a_{31}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are done with the first column. Let us denote the resulting matrix by $A^{(1)}$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ 4x_3 = 8 \end{cases}$$

$$\begin{aligned} \Leftrightarrow & E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times \underbrace{A\underline{x}}_{\underline{b}} \\ & E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{aligned}$$

We are done with the second column, so we can denote the resulting matrix by $A^{(2)}$.

In fact, we got an upper triangular matrix. We can solve it using backward compatibility. Let us denote

$$E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) = U$$

where $U$ is the upper triangular matrix. Then the inverse of it is

$$\begin{aligned} & \left[E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right)\right]^{-1} \\ = & E_{21}\left(-\frac{a_{21}}{a_{11}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right) \\ A = & \underbrace{E_{21}\left(-\frac{a_{21}}{a_{11}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{32}\left(-\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)}_{L} \cdot U \end{aligned}$$

All matrices $E_{xx}(x)$ are lower triangular $\rightarrow$ the product is also lower triangular $(A = l \cdot U)$. So using Gaussian elimination, we represented $A$ as a product of lower and upper triangular matrices

$$A\underline{x} = \underline{b} \Rightarrow LU\underline{x} = \underline{b}$$

Let us denote $U\underline{x}$ by $\underline{y}$, then we get

$$\begin{cases} l\underline{y} = \underline{b} & \text{Solve by forward substitution, find } \underline{y} \\ U\underline{x} = \underline{y} & \text{Solve by backward substitution} \end{cases}$$

**<u>Remark:</u>**

Gaussian elimination works if all elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{ii}^{(i-1)}$ are non-zero! These elements are called <u>PIVOT</u> elements.

**Example**

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 8x_2 - 3x_3 = 6 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow A\underline{x} = \underline{b}$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow E_{21}\left(-\frac{a_{21}}{a_{11}}\right) A\underline{x} = E_{21}\left(-\frac{a_{21}}{a_{11}}\right)$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ x_2 + 5x_3 = 12 \end{cases} \Leftrightarrow E_{31}\left(-\frac{a_{31}}{a_{11}}\right) A\underline{x} = E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b}$$

We denote the resulting matrix by $A^{(1)}$. In order to proceed we need $a_{22}^{(1)} \neq 0$. Let us consider matrix $P_{pq}$-matrix, which you get from identity matrix by exchanging rows $p$ and $q$. It is easy to show that $P_{pq} \cdot A$ is equal to matrix $A$ with rows $p$ and $q$ exchanged.

**Definition**

Permutation matrix $P$ is an identity matrix with rows in any order.

**Remark:**

$P^{-1} = P$. The product of permutation on matrices is a permutation matrix.

We want to exchange rows 2 and 3. We need to multiply by the permutation matrix $P_{23}$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + 5x_3 = 12 \\ x_3 = 2 \end{cases}$$

$$\begin{aligned} \Leftrightarrow & P_{23} \cdot E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) A\underline{x} \\ = & P_{23} \cdot E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b} \end{aligned}$$

In general, the Gaussian elimination proceeds like this:

$$E_{xx} \ldots E_{xx} P_{xx} E_{xx} \ldots E_{xx} A\underline{x} = E_{xx} \ldots E_{xx} P_{xx} E_{xx} \ldots E_{xx} \underline{b}$$

Turns out, that we can exchange the rows, or in other words multiply $A$ by

$(P_{xx} \dots P_{xx}$ before doing the Gaussian elimination

$$\underbrace{\underbrace{(E_{xx} \dots E_{xx})}_{E} \underbrace{(P_{xx} \dots P_{xx})}_{P} A}_{U} \underline{x} = (E_{xx} \dots E_{xx})(P_{xx} \dots P_{xx}) \underline{b}$$

$$EPA = U$$

$$PA = E^{-1}U = LU \leftarrow \text{Lower triangular}$$

**Theorem**

There exists permutation matrix $P$, such that $PA = LU$. The only necessary condition for that is that $A^{-1}$ exists.

## 3.1 Matrix Transposition

**Definition**

Let us consider matrix $A \in \mathbb{R}^{m,n}$. Matrix $B \in \mathbb{R}^{n,m}$ is called the transpose of $A$ if $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots n$

**Notation**

Usually the transpose of $A$ is written as $A^T$

□

**Example**

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 10 \end{pmatrix} \in \mathbb{R}^{4,2} \Rightarrow A = \begin{pmatrix} 2 & 4 & 6 & 9 \\ 3 & 5 & 7 & 10 \end{pmatrix} \in \mathbb{R}^{2,4}$$

**Properties**

1. $\left(A^T\right)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T \cdot A^T$
4. $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

**Proof**

3.

$$A \in \mathbb{R}^{m,n} = \begin{pmatrix} -\text{row } 1 \to \\ \vdots \\ -\text{row } n \to \end{pmatrix}, B \in \mathbb{R}^{n,l} = \begin{pmatrix} \text{-col 1}\downarrow & \cdots & \text{-col n}\downarrow \end{pmatrix}$$

$$(AB)_{ij} = \langle \text{row } i \text{ of } A, \text{ column } j \text{ of } B \rangle$$

$$\left((AB)^T\right)_{pq} = (AB)_{qp} = \langle \text{row } q \text{ of } A, \text{ column } p \text{ of } B \rangle$$

$$B^T = \begin{pmatrix} -\text{col } 1 \to \\ \vdots \\ -\text{col } n \to \end{pmatrix}, A^T = \begin{pmatrix} \text{-col 1}\downarrow & \cdots & \text{-col n}\downarrow \end{pmatrix}$$

$$\left(B^T A^T\right)_{pq} = \langle \text{column } p \text{ of } B, \text{ row } q \text{ of } A \rangle$$

$$\Rightarrow \left((AB)^T\right)_{pq} = \left(B^T A^T\right)_{pq}; p = 1, \ldots, l; q = 1, \ldots, m.$$

$$\Rightarrow (AB)^T = B^T A^T$$

4. Assume that $A \in \mathbb{R}^{n,n}, \exists A^{-1}$

$$AA^{-1} = I \to (AA^{-1})^T = \left(A^{-1}\right)^T \cdot A^T = I^T = I$$

$$A^{-1}A = I \to (A^{-1}A)^T = A^T \cdot \left(A^{-1}\right)^T = I^T = I$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

□

Let us consider vector $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^{n,1}$ - column vector. Then $\underline{u}^T \in \mathbb{R}^{1,n} =$ $(u_1 \ldots u_n)$ - row vector. Let us also consider $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^{n,1}$. Then

$$\underline{u}^T \cdot \underline{v} = (u_1 \ldots u_n) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \langle \underline{u}, \underline{v} \rangle$$

$$\underline{v} \cdot \underline{u}^T = n \times n \text{ matrix}$$

**Definition**

Matrix $A$ is called symmetric if $A^t = A$. Matrix $A$ should be a square matrix, $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \to A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$

$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \to I^T = I$$

# Chapter 4

# Vector Spaces

**Definition**

A vector space $V$ is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to $V$, then the product of any scalar with this object belongs to V and the following properties are satisfied:

1. $\forall \underline{u}, \underline{v}, \underline{w} \in V$; $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2. $\forall \underline{u}, \underline{v} \in V$; $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements $\underline{0} \in V$, such that $\forall \underline{u} \in V$; $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any $\underline{u} \in V, \exists!(-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall \underline{u}, \underline{v} \in V$; $\forall \alpha \in \mathbb{R}$; $\alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8. $\forall \underline{u} \in V$; $1 \cdot \underline{u} = \underline{u}$ (1 is a scalar here)

**Remark:**

The "vectors" in the vector space, are not necessarily vectors $(\in \mathbb{R}^n)$ , but can be other objects, as long as the definition is satisfied.

**Example**

Let us consider a set of all $2 \times 2$ matrices. It is a vector space. Proof:
If $A, B \in \mathbb{R}^{2,2}$ $\quad (A+B) \in \mathbb{R}^{2,2}$
$\alpha \in \mathbb{R}, A \in \mathbb{R}^{2,2}$ $\quad \alpha A \in \mathbb{R}^{2,2}$

1. $A, B, C \in \mathbb{R}^{2,2}$; $(A+B)+C = A+(B+C)$
2. ...
3. $$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2,2}, \forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A + \underline{0} = A$$
4. $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow (-A) = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

**Example**

Let us consider a set consisting of a single object, $\underline{0}$. It is a vector space. Note: There is no vector space, which does not contain $\underline{0}$

## 4.1 Subspace of the vector space

**Definition**

A subspace $W$ of the vector space $V$, is a set of vectors in $V$, such that:

1. If $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$
2. If $\alpha \in \mathbb{R}$, $\underline{u} \in W$ then $\alpha \underline{u} \in W$

**Definition**

Let us consider a set of vectors $\{\underline{u}_1, \ldots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \ldots, \underline{u}_n\}$ is defined as

$$\mathcal{S} = \operatorname{span}\{\underline{u}_1, \ldots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \ldots \alpha_n \in \mathbb{R}\}$$

**Example**

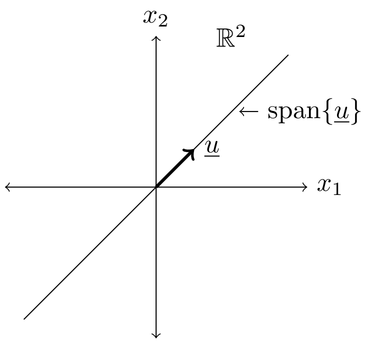
Is $\operatorname{span}\{\underline{u}\}$ a subspace in $\mathbb{R}^2$? Proof:

$$\underline{v} = \alpha \underline{u} \in \operatorname{span}\{\underline{u}\}$$
$$\underline{w} = \beta \underline{u} \in \operatorname{span}\{\underline{u}\}$$

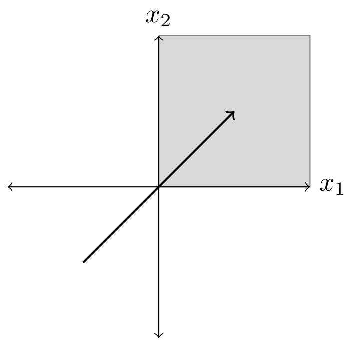
1. $\underline{v} + \underline{w} = \alpha \underline{u} + \beta \underline{u} = (\alpha + \beta)\underline{u} \in \operatorname{span}\{\underline{u}\}$
2. $\gamma \in \mathbb{R}, \gamma \underline{v} = \gamma \cdot (\alpha \underline{u}) = (\gamma \cdot \alpha)\underline{u} \in \operatorname{span}\{\underline{u}\}$

**Example**

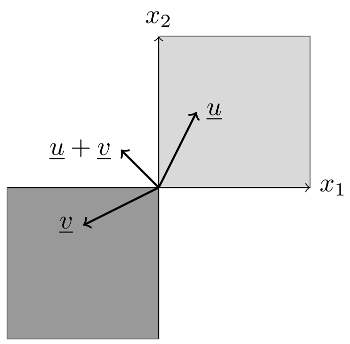
1.




2.




3.




## 4.2 Linear Independence

**Definition**

Let us consider vector space $V$ and $\underline{v}_1, \dots, \underline{v}_n \in V$. $\underline{v}_1, \dots, \underline{v}_n$ are linearly dependent if there exists scalars $\alpha_1, \dots, \alpha_n$ not all equal to zero, such that $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

**Definition**

Vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = 0 \Rightarrow \text{ all } \alpha_i = 0, i = 1, \dots, n$$

**Example**

1. Let us consider $\mathbb{R}^n$ and vectors
$$\underline{E}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{E}_i = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$
$\underline{E}_1, \dots, \underline{E}_n$ are linearly independent.

2. Let us consider $\mathbb{R}^2$, $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Are they linearly independent? See proof 2.

**Proof**

1. Assume that
$$\alpha_1 \underline{E}_1 + \dots + \alpha_n \underline{E}_n = 0 \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$
$\Rightarrow$ then all $\alpha_i = 0$ for $i = 1, \dots, n$,then based on the definition $\underline{E}_1, \dots, \underline{E}_n$ are linearly independent.

2. Let us consider $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$$\begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_2 = 0 \\ \alpha_1 = 0 \end{cases}$$
If we assume $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0}$, we have to show that all $\alpha_i$ are zeroes $\Rightarrow$ vectors are linearly independent.

$\square$

**Example**

Let us consider $\mathbb{R}^2$, $\underline{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}2\\2\end{pmatrix}$. Let us assume that

$$\alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 = \underline{0} \Rightarrow \alpha_1\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}\alpha_1 + 2\alpha_2 = 0\\ \alpha_1 + 2\alpha_2 = 0\end{cases} \rightarrow \begin{cases}\alpha_1 + 2\alpha_2 = 0\\ 0 = 0\end{cases}$$

One possible solution:

$$\begin{cases}\alpha_1 = -2\\ \alpha_2 = 1\end{cases}$$

Linearly dependent.

**Recap**

If we consider vectors $\underline{v}_1, \dots, \underline{v}_n \in V$, then

$$\textbf{span}\{\underline{v}_1, \dots, \underline{v}_n\} = \{\alpha_1\underline{v}_1, \dots, \alpha_n\underline{v}_n \mid \text{ for all possible } \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$

**Definition**

If vector space $v$ is generated by $\{\underline{v}_1, \dots, \underline{v}_n\}$ (in other words, $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$) and $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is called basis of $V$

**Example**

Let us consider $\mathbb{R}^n$ and $\underline{E}_1, \dots, \underline{E}_n$. They form basis of $\mathbb{R}^n$.

**Proof**

1. "$V$ is generated by $\underline{v}_1, \dots, \underline{v}_n$". Let us consider any vector $\underline{u} \in \mathbb{R}^n$

$$\underline{u} = \begin{pmatrix}\underline{u}_1\\ \vdots\\ \underline{u}_n\end{pmatrix}, \text{ we have}$$

$$\underline{u} = \begin{pmatrix}\underline{u}_1\\ \vdots\\ \underline{u}_n\end{pmatrix} = \underline{u}_1\underline{E}_1 + \cdots + \underline{u}_n\underline{E}_n \Rightarrow \mathbb{R}^n = \text{span}\{\underline{E}_1, \dots, \underline{E}_n\}$$

2. "Linear independence" already proven before.

□

**Example**

Let us consider $\mathbb{R}^2$ and $\underline{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}3\\1\end{pmatrix}$, is it a basis?

1. Is $\mathbb{R}^2 = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$? Let us consider an arbitrary vector $\underline{v} = \begin{pmatrix}\underline{v}_1\\\underline{v}_2\end{pmatrix} \in \mathbb{R}^2$. We should check that there exists scalars $\alpha_1, \alpha_2$ such that

$$\underline{v} = \alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 \to \underline{v} = \alpha_1\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}\underline{v}_1\\\underline{v}_2\end{pmatrix}$$

$$\begin{cases}\alpha_1 + 3\alpha_2 = \underline{v}_1\\ \alpha_1 + \alpha_2 = \underline{v}_2\end{cases} \to \begin{cases}2\alpha_2 = \underline{v}_1 - \underline{v}_2\\ \alpha_1 + \alpha_2 = \underline{v}_2\end{cases} \to \begin{cases}\alpha_2 = \frac{\underline{v}_1 - \underline{v}_2}{2}\\ \alpha_1 = \underline{v}_2 - \frac{\underline{v}_1 - \underline{v}_2}{2} = \frac{3\underline{v}_2 - \underline{v}_1}{2}\end{cases}$$

2. $\underline{u}_1, \underline{u}_2 =$ linearly independent (We showed it before).

> **Definition**
>
> Let us consider vector space $V$ and vectors $\underline{v}_1, \dots, \underline{v}_n$ that form a basis of $V$. If vector $\underline{x} \in V$ can be written as $\underline{x} = x_1\underline{v}_1 + \cdots + x_n\underline{v}_n$ then $(x_1, \dots, x_n)$ are called the coordinates of $\underline{x}$ with respect to basis $\{v_1, \dots, v_n\}$

**Theorem**

Let us consider vector space $V$ and $v_1, \dots, v_n$ that are linearly independent. Let us assume that $\underline{x} = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $\underline{x} = \beta_1\underline{v}_1 + \cdots + \beta_n\underline{v}_n$, then

$$\alpha_i = \beta_i \quad \forall i = 1, \dots, n$$

**Proof**

We have

$$x = \alpha_1\underline{v}_1 + \cdots + \alpha_n\underline{v}_n = \beta_1\underline{v}_1 + \cdots + \beta_n\underline{v}_n \to (\alpha_1 - \beta_1)\underline{v}_1 + \cdots + (\alpha_n - \beta_n)\underline{v}_n = \underline{0}$$

Since $v_1, \dots, v_n$ are linearly independent $\Rightarrow \alpha_i = \beta_i$, $\forall i = 1, \dots, n$

□

**<u>Remark:</u>**

The coordinates of any vector $\underline{x}$ with respect to given basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ are unique.

**Theorem**

Let us consider vector space $V$. The number of vectors in any basis of $V$ is always the same.

**Remark:**

The number of vectors in the basis of vector space $V$ is called the dimension of vector space $V$.

## 4.3 Rank of matrix

**Definition**

The row rank of matrix A is a maximum number of linearly independent rows of matrix $A$.

**Definition**

The column rank of matrix $A$ is a maximum number of linearly independent columns of matrix $A$.

**Remark:**

For any matrix $A \in \mathbb{R}^{m,n}$, the row rank is equal to the column rank. Therefore the row rank and column rank are sometimes called rank of matrix $A$, $\text{rank}(A)$.

**Example**

1.
$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$
We have shown before that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are linearly independent, therefore $\text{rank}(A) = 2$.

2.
$$A = \begin{pmatrix} 1 & 0 \\ 7 & 0 \\ 3 & 0 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{4,2}$$
The column vectors $\begin{pmatrix} 1 \\ 7 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ are linearly dependent, thus $\text{rank}(A) = 1$ (i.e. the maximum number of linearly independent columns is 1).

**Remark:**

Two vectors are orthogonal if $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v} = 0$ (they basically must be perpendicular, i.e. the angle between $\underline{u}$ and $\underline{v}$ is 90 degrees).

**Definition**

Two subspaces $U$ and $W$ of vector space $V$ are orthogonal, if $\forall \underline{u} \in U$ and $\forall \underline{w} \in W$, we have $\langle \underline{u}, \underline{w} \rangle = 0$

**Definition**

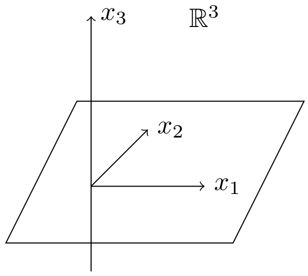
Orthogonal complement of subspace $M$ of vector space $V$ contains every vector orthogonal to $M$. This subspace is usually denoted by $M^{\perp}$

**<u>Remark:</u>**

$\dim M + \dim M^{\perp} = \dim V$

**Example**

Consider $\mathbb{R}^3$




line $\alpha$ plane - orthogonal subspace. Orthogonal complement of each other

**Example**

Not orthogonal subspace!
$\underline{u} \neq 0$ ADD FIGURE
$\underline{u} \in I \;\; \& \;\; \underline{u} \in W$
$\langle \underline{u} \in U, \underline{u} \in W \rangle = 0$

Add missing figure to minipage

**Note**

If vector $\underline{u}$ belongs to 2 orthogonal subspaces, this vector is necessarily a zero vector, $\underline{u} = 0$ because we should have

$$\langle \underline{u}, \underline{u} \rangle = \underline{u}^T \underline{u} = 0 \Rightarrow \underline{u} = \underline{0}$$

# Chapter 5

# Linear Mapping

**Definition**

Let us consider 2 vector spaces $V$ and $W$. A function $\mathcal{L} : V \to W$ is called a linear mapping, if:

1. For any $\underline{v} \in V$ and $\underline{v}' \in V$, $\mathcal{L}(\underline{v} + \underline{v}') = \mathcal{L}(\underline{v}) + \mathcal{L}(\underline{v}')$
2. For any $\underline{v} \in V$ and any scalar $\alpha$, $\mathcal{L}(\alpha\underline{v}) = \alpha \cdot \mathcal{L}(\underline{v})$

**Example**

Let us consider matrix $A \in \mathbb{R}^{n,m}$. We can define linear mapping $\mathcal{L}_A$ as follows:

$$\mathcal{L}_A(\underline{v}) = A\underline{v} \qquad \mathcal{L}_A : \mathbb{R}^m \to \mathbb{R}^n$$

Is $\mathcal{L}_A$ a linear mapping? Yes!

**Proof**

1. $\forall \underline{v}, \underline{v}' \in \mathbb{R}^m$, we have:

$$\mathcal{L}_A(\underline{v} + \underline{v}') = A(\underline{v} + \underline{v}') = A\underline{v} + A\underline{v}' = \mathcal{L}_A(\underline{v}) + \mathcal{L}_A(\underline{v}')$$

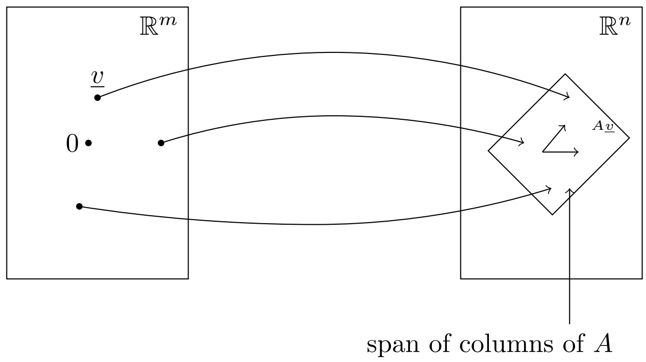
2. $\forall \underline{v} \in \mathbb{R}^m, \forall \alpha$ ($\alpha$ is scalar), we have:

$$\mathcal{L}_A(\alpha\underline{v}) = A(\alpha\underline{v}) = \alpha \cdot A\underline{v} = \alpha\mathcal{L}_A(\underline{v})$$

□

Let us consider matrix $A \in \mathbb{R}^{n,m}$, $A : \mathbb{R}^m \to \mathbb{R}^n$. Let us consider vector $\underline{v} \in \mathbb{R}^m$ .

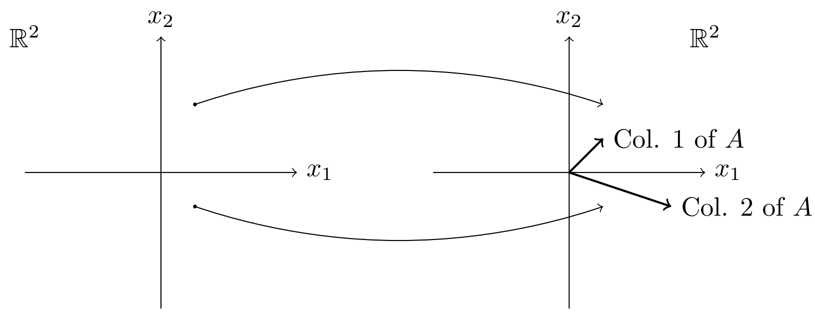
$$A\underline{v} = \underbrace{v_1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + v_2 \cdot \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} + ... + v_m \cdot \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}}_{\text{Linear combination of columns of } A}$$




**Example**

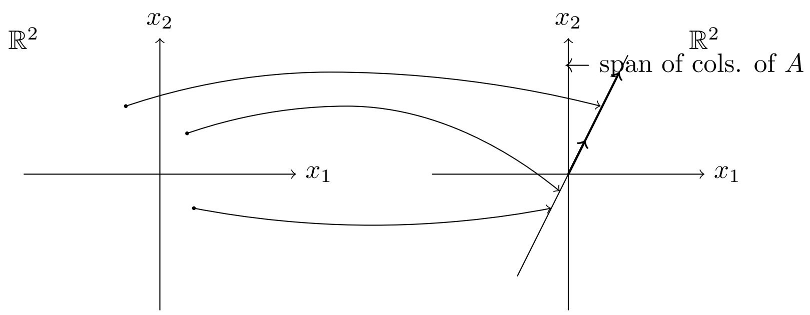
1.

$$A \in \mathbb{R}^{2,2} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$




2.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$




**Note**

In order for solution of $A\underline{x} = \underline{b}$ to exist, $\underline{b}$ should belong to a span of columns of matrix $A$.

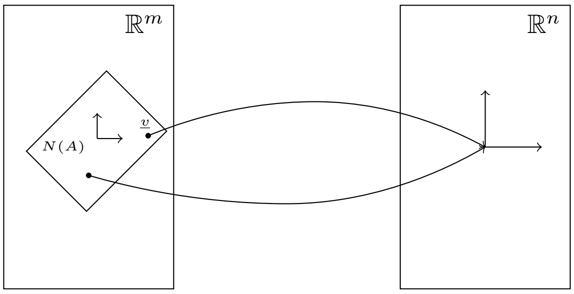
**Definition**

The span of columns of matrix $A \in \mathbb{R}^{n,m}$ is called a column space of $A$, denoted by $C(A)$, where $C(A) \subset R^n$.

**Definition**

Let us consider matrix $A \in \mathbb{R}^{n,m}, A : \mathbb{R}^m \to \mathbb{R}^n$. The null space of $A$ is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$




**Example**

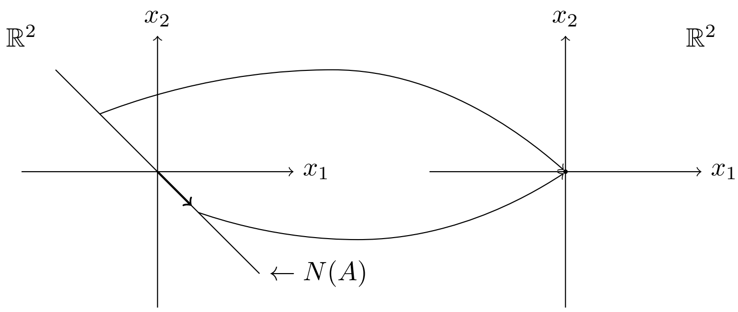
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

What is $N(A)$? We should find all solutions of $A\underline{x} = \underline{0}$, this will give us $N(A)$.

$$\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases} \to \begin{cases} x_1 + 3x_2 = 0 \\ 0 = 0 \end{cases} \to \begin{cases} x_1 = -3x_2 \\ 0 = 0 \end{cases}$$

The nullspace of this matrix will be a line formed by a linear combination of the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $(\alpha \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, for all possible $\alpha$), or in other words it will be the $span(\begin{pmatrix} -3 \\ 1 \end{pmatrix})$.

$$x_1 = -3x_2 = -3\alpha, x_2 = \alpha \to \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \alpha \begin{pmatrix} -6 \\ 2 \end{pmatrix}$$



**Theorem**

The nullspace, $N(A)$, of $A \in \mathbb{R}^{n,m}$ is a subspace of $\mathbb{R}^m$.

**Proof**

Let us assume that $\underline{x}, \underline{x}' \in N(A)$ and $\alpha$ is arbitrarily scalar.

1. $A(\underline{x} + \underline{x}') = A\underline{x} + A\underline{x}' = \underline{0} + \underline{0} = \underline{0} \Rightarrow (\underline{x} + \underline{x}') \in N(A)$

2. $A(\alpha\underline{x}) = \alpha(A\underline{x}) = \alpha \cdot \underline{0} = \underline{0} \Rightarrow \alpha\underline{x} \in N(A)$

$\square$

**Theorem**

The column space, $C(A)$, of $A \in \mathbb{R}^{n,m}$ is a subspace of $\mathbb{R}^n$.

**Definition**

The row space of matrix $A \in R^{n,m}$ is a span of rows of $A$. Clearly, $R(A) = C(A^T)$ and $R(A) \subset \mathbb{R}^m$.

**Definition**

The left nullspace of $A$ is defined as $N(A^T)$. $N(A^T) \subset \mathbb{R}^n$.

**Theorem**

$R(A)$ is a subspace of $\mathbb{R}^m$

**Proof**

Same as for the proof that $C(A)$ is a subspace of $\mathbb{R}^n$, but for $A^T$

$\square$

**Theorem**

$N(A^T)$ is a subspace of $\mathbb{R}^n$

**Proof**

Same as for $N(A)$ but replace $A$ with $A^T$

$\square$

**Theorem**

$R(A)$ and $N(A)$ are orthogonal subspaces in $\mathbb{R}^m$ for $A \in \mathbb{R}^{n,m}$

**Proof**

Let us consider $\forall \underline{x} \in N(A), A\underline{x} = \underline{0}$

$$A\underline{x} = \begin{pmatrix} - \text{ row 1 of } A \rightarrow \\ \vdots \\ - \text{ row } n \text{ of } A \rightarrow \end{pmatrix} \cdot \begin{pmatrix} | \\ x \\ \downarrow \end{pmatrix} = \begin{pmatrix} < \text{ row 1 of } A, \underline{x} > \\ \vdots \\ < \text{ row } n \text{ of } A, \underline{x} > \end{pmatrix} \overset{\underline{x} \in N(A)}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\underline{x}$ is orthogonal to every row of $A$. $\underline{x}$ is orthogonal to every linear combination of rows of $A$. $\underline{x}$ is orthogonal to $R(A)$. In fact, what we just showed is that $N(A)$ & $R(A)$ are orthogonal complements.

□

**Theorem**

$N(A^T)$ & $C(A) = R(A^T)$ are orthogonal complements in $\mathbb{R}^n$

$A \in \mathbb{R}^{n,m} : \mathbb{R}^m \rightarrow \mathbb{R}^n$. Row rank of $A = \text{rank}(A) = \dim(R(A)) = \dim(C(A))$

$$\begin{aligned} N(A) &: A\underline{x} = \underline{0} \quad \forall x \in \mathbb{R}^m \\ C(A) &: A\underline{v} = \text{ Linear combinations of columns of } A \\ &\qquad = v_1 \cdot \text{col 1 of } A + \cdots + v_n \cdot \text{col } n \text{ of } A \in \mathbb{R}^n \end{aligned}$$

**Theorem**

$N(A)$ is an orthogonal complement of $R(A)$ in $\mathbb{R}^m$,

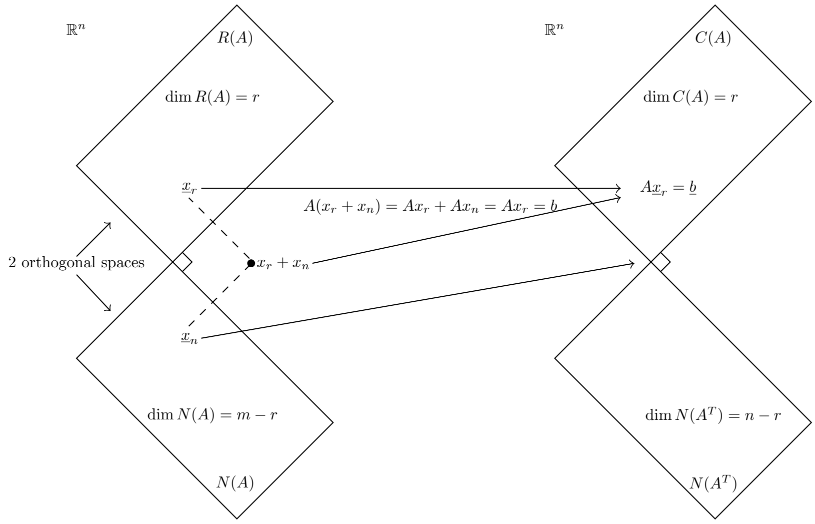
$$\dim N(A) + \underbrace{\dim R(A)}_{=\text{rank}(A)} = m$$

**Theorem**

$N(A^T)$ is an orthogonal complement of $R(A^T) = C(A)$ in $\mathbb{R}^n$,

$$\dim N(A^T) + \underbrace{\dim C(A)}_{=\text{rank}(A)} = n$$

Let us consider $A \in \mathbb{R}^{n,m}, A : \mathbb{R}^m \rightarrow \mathbb{R}^n, \text{rank}(A) = r$




**Lemma**

For any vector $\underline{b}$ in $C(A)$, there exists one and only one vector $\underline{x}_r \in R(A)$ such that $A\underline{x}_r = b$

**Proof**

Let us assume that $\underline{x}_r$ and $\underline{x}'_r$ are in the row space, $R(A)$. Let us assume that $A\underline{x}_r = A\underline{x}'_r$. We have

$$\underline{x}_r \in R(A) - \underline{x}'_r \in R(A) \in R(A)$$

But we also have

$$A\underline{x}_r - A\underline{x}'_r = \underbrace{A(\underline{x}_r - \underline{x}'_r}_{\in N(A)} = \underline{0}$$

It means that $(\underline{x}_r - \underline{x}'_r)$ is in $R(A)$ and $N(A)$, but they are orthogonal subspaces, therefore

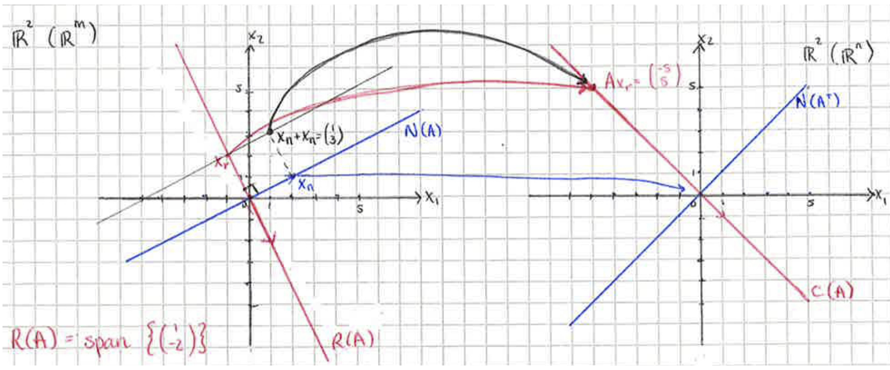
$$\underline{x}_r - \underline{x}'_r = \underline{0} \Rightarrow \underline{x}_r = \underline{x}'_r$$

$\square$

**Example**

Let us consider

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \in \mathbb{R}^{2,2}$$




Row space: $\operatorname{rank} A = 1 \Rightarrow \dim R(A) = 1$

$$R(A) = \operatorname{span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$$

Null space: $\dim N(A) = 2 - 1 = 1$

$$A\underline{x} = 0 \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow x_1 = 2x_2 \text{ (Line)}$$

Column space: $\dim C(A) = \dim R(A) =\perp$

$$C(A) = \operatorname{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\}$$

Left Null space: $\dim N(A^T) = 2 - 1 = 1$. Consider

$$\underline{x}_r = \begin{pmatrix}-1\\2\end{pmatrix} \Rightarrow A\underline{x}_r = \begin{pmatrix}1 & -2\\-1 & 2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}-5\\5\end{pmatrix}$$

$$\underline{x}_n = \begin{pmatrix}2\\1\end{pmatrix} \Rightarrow A\underline{x}_n = \begin{pmatrix}1 & -2\\-1 & 2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

## 5.1 Orthogonal Basis and Gram-Schmidt process

**Definition**

Vectors $\underline{q}_1, \ldots, \underline{q}_m$ are orthogonal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

**Definition**

Vectors $\underline{q}_1, \ldots, \underline{q}_m$ are orthonormal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

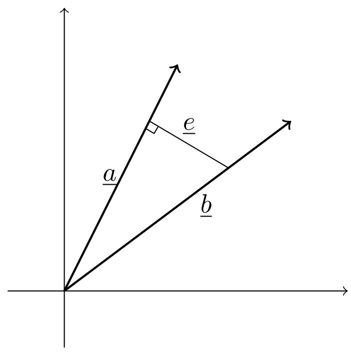
If the columns of the matrix are orthonormal vectors, then this matrix is usually denoted by $Q$, In this case, we have $Q^TQ = I$. If $Q$ is not a square matrix then $QQ^T$ is not necessarily $I$.

**Definition**

A square matrix is called orthogonal (if its columns are orthonormal vectors) if $Q^TQ = I$. In this case, since it is a square matrix, $QQ^T = I$

### 5.1.1 Projection on the line

Let us assume that we have a line given by vector $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ and vector $\underline{b} \in \mathbb{R}^n$. We want to find vector $\underline{p}$ belonging to the line, closest to vector $\underline{b}$. In other words, we are looking for $\underline{p}$ which is orthogonal projection of $\underline{b}$ onto the line given by $\underline{a}$




$\underline{p}$ is proportional to $\underline{a}$, $\underline{p} = \hat{x}\underline{a}$, where $\hat{x}$ is some scalar. Let us define vector $\underline{e} = \underline{b} - \underline{p} = \underline{b} - \hat{x}\underline{a}$ (error vector). $\underline{e}$ is orthogonal to the line, therefore

$$\begin{aligned}
&\langle \underline{a}, \underline{e} \rangle = 0 \\
&\langle \underline{a}, \underline{e} \rangle = \underline{a}^T((\underline{b}) - \hat{x}\underline{a}) = \underline{a}^T\underline{b} - \hat{x}\underline{a}^T\underline{a} = 0 \\
&\Rightarrow \hat{x} = \frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}} \\
&\Rightarrow \underline{p} = \hat{x}\underline{a} = \underline{a}\hat{x} = \underline{a}\frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}} = \underbrace{\frac{\underline{a}\underline{a}^T}{\underline{a}^T\underline{a}}}_{P\in\mathbb{R}^{n,n} \text{ (projection matrix)}} \cdot \underline{b}
\end{aligned}$$

**Example**

Let us consider $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \in \mathbb{R}^3$

$$P = \frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}} = \langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \rangle \cdot \frac{1}{9} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Let us take

$$\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{p} = P\underline{b} = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}$$

**Note**

$\underline{p}^2 = \underline{p}$

**Note**

$(I - P)-$ projection onto subspace orthogonal to the line given by $\underline{a}$

## 5.2 Gram-Schmidt process

Given linear independent vectors $\underline{a}, \underline{b}, \underline{c}, \ldots$ we first find orthogonal vectors $\underline{a}', \underline{b}', \underline{c}', \ldots$ which span the same subspace as $\underline{a}, \underline{b}, \underline{c}, \ldots$ and then we normalise them,

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|}, \ldots$$

So, Gram-Schmidt process allows us to construct an orthogonal basis of span$\{\underline{a}, \underline{b}, \underline{c}$

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1. Choose $\underline{a}' = \underline{a}$

2. It is likely that $\underline{b}$ is not orthogonal to $\underline{a}'$, so we need to subtract its projection on the line defined by $\underline{a}'$

$$\underline{b}' = \underline{b} - \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a}'$$

3. $\underline{c}'$ is likely not orthogonal to $\underline{a}'$ and $\underline{b}'$. Again, subtract its projections

$$\underline{c}' = \underline{c} - \frac{\underline{a}^T \underline{c}}{\underline{a}^T \underline{a}} \underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'} \underline{b}'$$

and so on. Finally, normalise $\underline{q}_1, \underline{q}_2, \underline{q}_3, \ldots$

**Example**

With

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \underline{c} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

find $\underline{a}', \underline{b}', \underline{c}', \underline{q}_1, \underline{q}_2, \underline{q}_3$

1. 

$$\underline{a}' = \underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

2.

$$\underline{b}' = \underline{b} - \frac{\underline{a}'^T \underline{b}}{\underline{a}'^T \underline{a}'}\underline{a}' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \rangle}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

3.

$$\underline{c}' = \underline{c} - \frac{\underline{a}^T \underline{c}}{\underline{a}^T \underline{a}}\underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'}\underline{b}' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle \underline{a}', \underline{c}' \rangle = 0, \langle \underline{b}', \underline{c}' \rangle = 0$$

Finally normalise:

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

## 5.3 Projection onto subspace

Assume we have linearly independent vectors $a_1, \dots, a_m \in \mathbb{R}^n$. We want to project vector $\underline{b} \in \mathbb{R}^n$ onto subspace spanned by $a_1, \dots, a_m$. Subspace consists of all linear combinations

$$x_1 a_1 + \cdots + x_m a_m = \underbrace{\begin{pmatrix} | & & | \\ a_1 & \dots & a_m \\ \downarrow & & \downarrow \end{pmatrix}}_{A \in \mathbb{R}^{n,m}} \cdot \underbrace{\hat{x}}_{\in \mathbb{R}^m}$$

We are looking for the projection $\underline{p}$ of $\underline{b}$ onto his subspace. We can define $\underline{e} = \underline{b} - \underline{p}$, $\underline{e}$ should be orthogonal to all $a_1, \dots, a_m$

$$\left.\begin{aligned} \langle a_1, \underline{e} \rangle = \underline{a_1}^T \cdot (\underline{b} - A\hat{x}) = 0 \\ \vdots \\ \langle a_m, \underline{e} \rangle = \underline{a_m}^T \cdot (\underline{b} - A\hat{x}) = 0 \end{aligned}\right\} \Rightarrow \underbrace{\begin{pmatrix} -\underline{a}_1^T \rightarrow \\ \vdots \\ -\underline{a}_m^T \rightarrow \end{pmatrix}}_{A^T} (\underline{b} - A\hat{x}) = 0$$

$$A^T(\underline{b} - A\hat{x} = 0$$
$$A^T\underline{b} - A^T A\hat{x} = 0$$

**Theorem**

$A$ has linearly independent columns. Then $A^T A$ is:

- Square
- Symmetric

- Invertible

$$\underline{\hat{x}} = (A^T A)^{-1} A^T \underline{b}$$
$$\underline{p} = A\hat{x} = \underbrace{A(A^T A)^{-1}}_{P\text{ - Proj. matrix}} \cdot \underline{b} \text{ - Projection vector}$$