# Characteristic Polynomial

7. 5. 15

How to find eigenvalues and eigenvectors?

Recall

① lets consider $A: V \to V$ - linear mapping.

$A$ is invertible $\Leftrightarrow$ the null space of $A$ is $\{\underline{0}\}$,

$N(A) = \{\underline{x} \in V, Ax = \underline{0}\}$

② $A$ is invertible $\Leftrightarrow$ $\det A \neq 0$.

Theorem: lets consider $A \in \mathbb{R}^{n,n}$. $\lambda$ is an eigenvalue of $A$ iff $(\lambda I - A)$ is not invertible.

Proof: "$\lambda$ - eigenvalue of $A$, $\exists \underline{v} \neq \underline{0}$ such that $A\underline{v} = \lambda \underline{v}$.

$$-\lambda \underline{v} + A\underline{v} = -(\lambda I - A)v = \underline{0}$$

$\therefore$, $(\lambda I - A)$ has non-zero vector $\underline{v}$ in its null space.

① $\Rightarrow$ $(\lambda I - A)$ is not invertible.

(⇐) $(\lambda I - A)$ is not invertible $\Rightarrow \exists \underline{v} \in N(\lambda I - A)$, $v \neq 0$

$\Rightarrow (\lambda I - A)v = 0 \Rightarrow Av = \lambda v$

definition: Consider $A \in \mathbb{R}^{n,n}$

Characteristic polynomial is defined as

$$P_A(t) = \det(tI - A)$$

example

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad P_A(t) = ?$$

$$P_A(t) = \begin{vmatrix} t-1 & 0 & 2 \\ 0 & t-1 & -1 \\ -1 & 0 & t-1 \end{vmatrix} = (t-1)^3 + 0 + 0 - 2(t-1) - 0 - 0$$

$$= (t-1)((t-1)^2 - 2)$$

$$= (t-1)(t^2 - 2t - 1)$$

**Theorem**: Consider $A \in \mathbb{R}^{n,n}$.
$\lambda$ is an eigenvalue of $A$ iff $\lambda$ is a root of characteristic polynomial $P_A(t)$.

**Proof**: ($\Rightarrow$) $\lambda$ is an eigenvalue of $A$. Then from previous theorem $(\lambda I - A)$ is non-invertible.

(2) $\Rightarrow$ $\det(\lambda I - A) = 0$

but $\det(\lambda I - A) = P_A(t=\lambda) \Rightarrow \lambda$ is a root of $P_A(t)$.

($\Leftarrow$) if $\lambda$ is a root of $P_A(t)$, then $P_A(\lambda) = \det(\lambda I - A) = 0$

(2) $\Rightarrow$ $\lambda I - A$ is not invertible, from previous theorem. $\lambda$ is an eigenvalue.

**Example**: Consider $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, find eigenvalues and eigenvectors.

$$P_A(t) = \det(tI - A) = \begin{vmatrix} t-1 & -4 \\ -2 & t-3 \end{vmatrix} = (t-1)(t-3) - 8$$
$$= t^2 - 4t - 5$$
$$= (t+1)(t-5)$$

2 roots of $P_A(t)$ are $\lambda_1 = -1$, $\lambda_2 = 5$
eigenvalues.

lets find eigenvector for $\lambda_1 = -1$

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{cases} v_1 + 4v_2 = -v_1 \\ 2v_1 + 3v_2 = -v_2 \end{cases}$$

$$\Rightarrow \begin{cases} 2v_1 + 4v_2 = 0 \\ 2v_1 + 4v_2 = 0 \end{cases} \Rightarrow \begin{cases} v_1 + 2v_2 = 0 \\ 0 = 0 \end{cases} \quad \underset{\text{we can choose.}}{v_1 = 1} \Rightarrow v_2 = -\frac{1}{2}$$

$\Rightarrow$ eigenvector is $\underline{\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}}$

find eigenvector for $\lambda = 5$.

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} -4v_1 + 4v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{cases} \Rightarrow \begin{cases} -v_1 + v_2 = 0 \\ 0 = 0 \end{cases}$$

lets say we choose $v_1 = 1 \Rightarrow v_2 = 1$
then the eigenvector is $\underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$