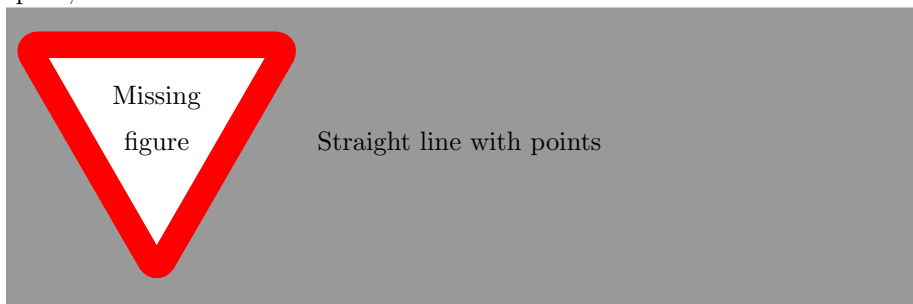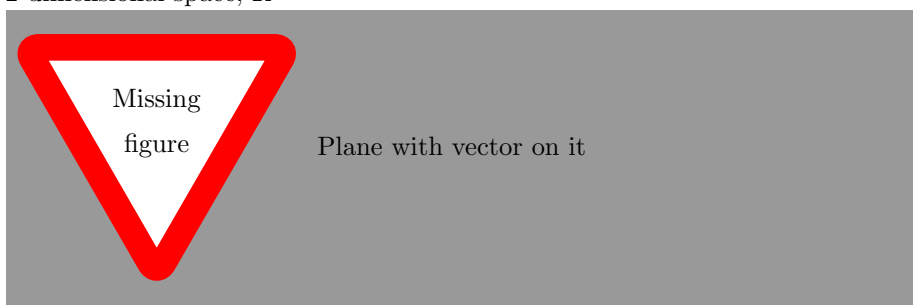# Chapter 1

Vectors

A real number can be represented by a point on a line, which is a 2-dimensional space, $\mathbb{R}$

Missing figure — Straight line with points

a pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space, $\mathbb{R}^2$

Missing figure — Plane with vector on it

a triplet od real numbers can be represented by a point in 3D space, $\mathbb{R}^3$

Missing figure — 3D drawing
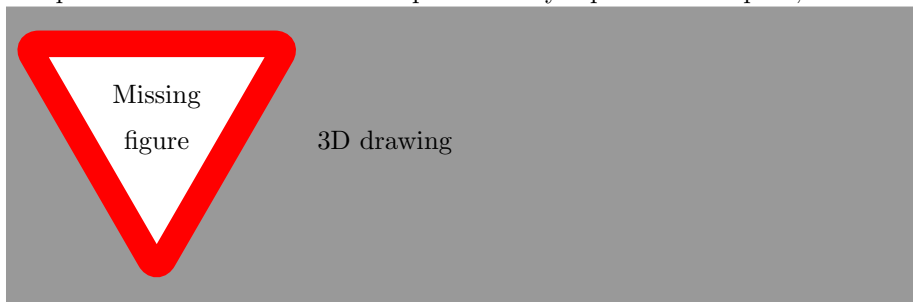
Definition

A vector is an ordered collection of n numbers

Notation

Usually vectors are given by letters, such as u, v, w. In textbooks vectors are written with bold font. In handwriting vectors are often written with a right arrow on top, such as $\overrightarrow{u}$. We will underline vectors, like so: $\underline{u}$.

$\square$

Definition

Let us consider vector $\underline{u} \in \mathbb{R}^n$. The i−th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is u_i

Example

$$\underline{u} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3 \Rightarrow u_1 = 3, u_2 = 7, u_3 = 11$$

Definition

Let us consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of $\underline{u}$ and $\underline{v}$, $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for all $i = 1, \ldots, n$

Example

1.
$$\underline{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 + (-1) \\ 5 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

2.
$$\underline{u} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

$\underline{u} + \underline{v}$ is not defined! Both vectors should have the same number of components.

Definition

1. Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, if $u_i = v_i$ for all $i = 1, \ldots, n$

2. A scalar is just another name for real number

3. Let us consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$. A product of α and $\underline{u}$ is defined as:

$$\alpha\underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot v_n \end{pmatrix}$$

Example

$$\alpha = 3, \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \Rightarrow \alpha \cdot \underline{u} \begin{pmatrix} 3 \cdot -1 \\ 3 \cdot 2 \\ 3 \cdot 5 \\ 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

Definition

Let us consider scalars α and β, and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. A sum of $\alpha\underline{u} + \beta \cdot \underline{v}$ is called a linear combination of vectors $\underline{u}$ and $\underline{v}$.

Example

1.
$$2 \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

2.
$$\underline{u} - \underline{v} = 1 \cdot \underline{u} + (-1) \cdot \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_i - v_i \end{pmatrix}$$

3.
$$\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_i - u_i \end{pmatrix} = \underline{0}$$
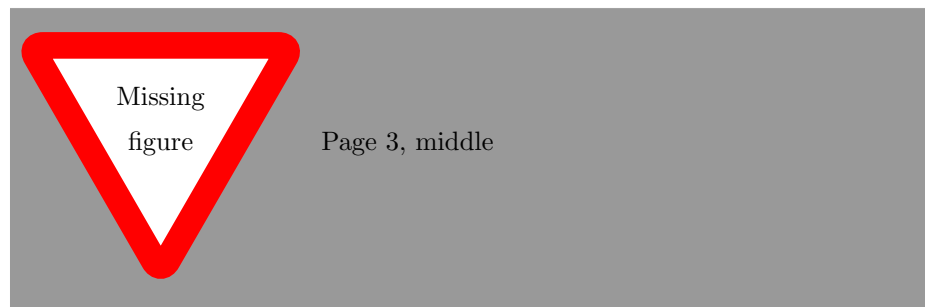

Definition

Vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1, \ldots, n$. The zero vector is often written as $\underline{0} \in \mathbb{R}^n$

## 1.1   Graphic representation of vectors and vector operations

A vector can be represented in the following way:

1. An ordered collection of numbers, $\underline{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

2. As an arrow in space

Missing figure   Page 3, middle

3. A vector is a point in space, the endpoint of a vector from the origin.

Let us consider vectors $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Missing figure   Page 3, middle
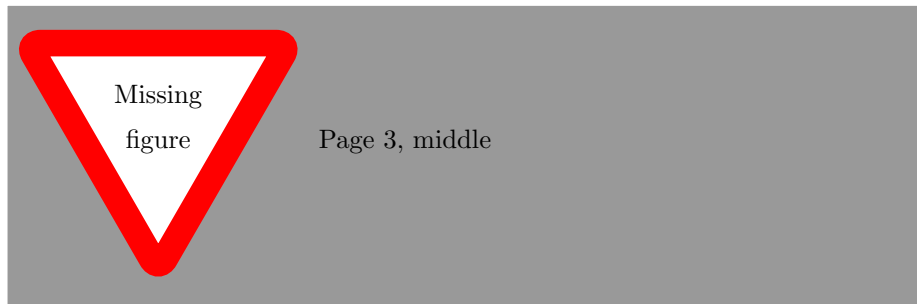
Let us consider vector $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What is $2 \cdot \underline{u}$? We can calculate as follows:

$$2 \cdot \underline{u} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We stretch vector $\underline{u}$ 2 times along the line defined by vector $\underline{u}$. What is $-\underline{u}$? Simply reverse the direction. What will be the representation of $\alpha\underline{u}$ for all possible values of α? An endless line

Missing figure

Page 3, bottom
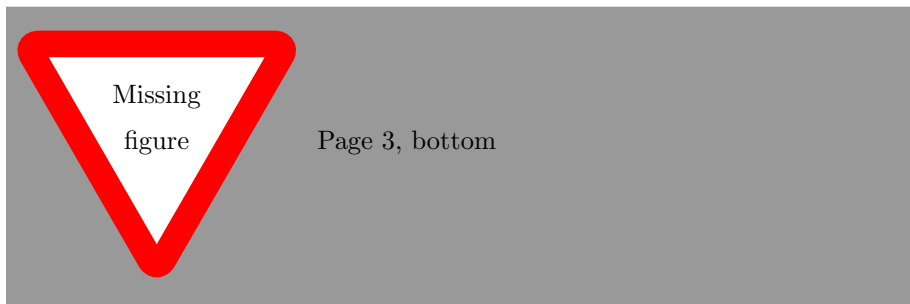
Let us consider two vectors $\underline{u} \in \mathbb{R}^2$ and $\underline{v} \in \mathbb{R}^2$. What will be the representation of all linear combinations of $\underline{u}$ and $\underline{v}$, $\alpha\underline{u} + \beta\underline{v}$

1. Plane:

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Page 4, top
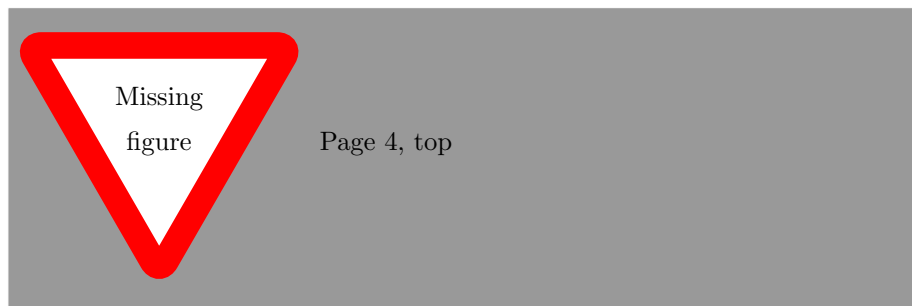
2. Line: $\underline{u}$ and $\underline{v}$ are on the same line.
   Note: Consider $\underline{u}, \underline{v} \in \mathbb{R}^n$. $\underline{u}$ and $\underline{v}$ are on the same line if there exists scalars α and β such that $\alpha\underline{u} + \beta\underline{v} = \underline{0}$, when α and $\beta \neq 0$

3. Point: if $\underline{u} = \underline{0}$ and $\underline{v} = \underline{0} \Rightarrow \alpha\underline{u} + \beta\underline{v} = \underline{0}$

Consider
vv, uv. They are on the same line if $\alpha\underline{u} + \beta\underline{v} = \underline{0}$ and $\alpha, \beta \neq 0$

1.2 Dot Product (Scalar product)

Definition

Let us consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. The dot (or scalar) product of vectors $\underline{u}$ and $\underline{v}$ is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + \underline{u}_n \underline{v}_n = \sum_{i=1}^{n} u_i v_i$$

Notation

We will use $\langle \underline{u}, \underline{v} \rangle$ to denote the dot product, but sometimes $\underline{u} \cdot \underline{v}$ is used

$\square$

Example

1.

$$\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \underline{u} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\langle \underline{u}, \underline{v} \rangle = 1 \cdot 0 + (-1) \cdot \frac{1}{2} + 3 \cdot (-1) = -3.5$$

2.

$$\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{u} = \begin{pmatrix} 0 \\ 1 \\ , \end{pmatrix} \langle \underline{u}, \underline{v} \rangle = 0$$

Let us consider $\mathbb{R}^2$. What is the set of all possible endpoints of unit vectors in $\mathbb{R}^2$, originating from the origin?

MISSING FIGURE PAGE 5, TOP

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\cos(\theta) = \frac{u_1}{\|\underline{u}\|} = u_1$$

$$\sin(\theta) = \frac{u_2}{\|\underline{u}\|} = u_2$$

$$\underline{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Now let us consider two unit vectors If $\underline{u} \neq \underline{0}$ or $\underline{v} \neq \underline{0}$ are not unit vectors

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DLE

$$\langle \underline{u}, \underline{v} \rangle = \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)$$
$$= \cos(\theta - \varphi) = \cos(\psi)$$
$$= \cos(\angle(\underline{u}, \underline{v}))$$

we can find the angle between them as follows:

Add unit vectors underbrace Page 5

$$\langle \underline{u}, \underline{v} \rangle = \left\langle \|\underline{u}\| \cdot \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \|\underline{v}\| \cdot \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle$$
$$= \|\underline{u}\|\|\underline{v}\| \left\langle \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle$$
$$= \|\underline{u}\|\|\underline{v}\| \cos\left(\angle(\underline{u}, \underline{v})\right)$$

Lemma

If $\underline{u} \neq \underline{0}, \underline{v} \neq \underline{0}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$, then

$$\cos\left(\angle(\underline{u}, \underline{v})\right) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\|\|\underline{v}\|}$$
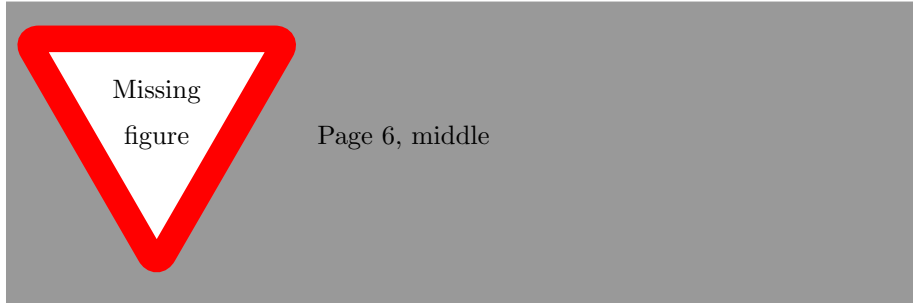
1.3   Properties of dot product

1. $\langle \alpha \cdot \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$. Proof:

$$\langle \alpha \cdot \underline{u}, \underline{v} \rangle = (\alpha u_1) \cdot v_1 + \cdots + (\alpha u_n) \cdot v_n$$
$$= \alpha \cdot (u_1 \cdot v_1 + \cdots + u_n \cdot v_n)$$
$$= \alpha \cdot \langle \underline{u}, \underline{v} \rangle$$

2. $\langle \underline{u}, \alpha \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u}, \underline{v} \in \mathbb{R}^n$

3. $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \cdot \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle, \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{vw} \in \mathbb{R}^n$

Example

Let us consider $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} . \langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \cdot 4 = 9 + 16 = 25 = 5^2$

Definition

The length of vector $\underline{u} \in \mathbb{R}^n, \|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidian norm of $\underline{u}$.

Definition

A vector with length equal to 1 is called a unit vector

If we take vector $\underline{u} \neq \underline{0}$, how to make it a unit vector? We should multiply vector $\underline{u}$ by $\frac{1}{\|\underline{u}\|}$, we will get $\frac{\underline{u}}{\|\underline{u}\|}$ = unit vector.

In our previous example: $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Unit vector is then

$$\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

We got $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cdot \cos(\angle(\underline{u}, \underline{v}))$. Let us take the absolute value of this

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \|\underline{v}\| \cdot |\cos(\angle(\underline{u}, \underline{v}))|$$

Notice that $|\cos(\angle(\underline{u}, \underline{v}))| \leq 1$

Lemma

Cauchy Schwartz Inequality: for any $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It is easy to see that Cauchy - Schwartz inequality is correct also for zero vectors.

Chapter 2

Matrices

Let us consider a linear combination of vectors

$$x_1 \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrices times the vector.

Example

1.
$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Notation

Matrices are usually written with capital letters, i.e. $A, B, c, \ldots$

A is an n by m matrix, $A \in \mathbb{R}^{n,m}$ if it has n rows and m columns.

The element of matrix A located in row i and column j is written as a_{ij} or $(A)_{ij}$.

$\square$

2.

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A \cdot \underline{x} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1) \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

For the product of matrix A with vector $\underline{x}$ to exist, matrix A should have the same number of columns as vector $\underline{x}$ components.

2.1   Matrix Operations