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Linear Algebra

Class Notes

Based on Professor Pivkin's Material

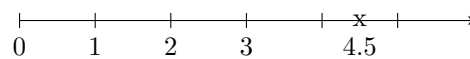
SPRING 2015

Università della Svizzera Italiana
Version 0.1.2
Generated on April 23, 2015

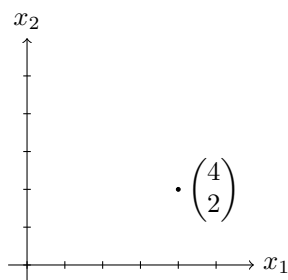
Chapter 1

Vectors

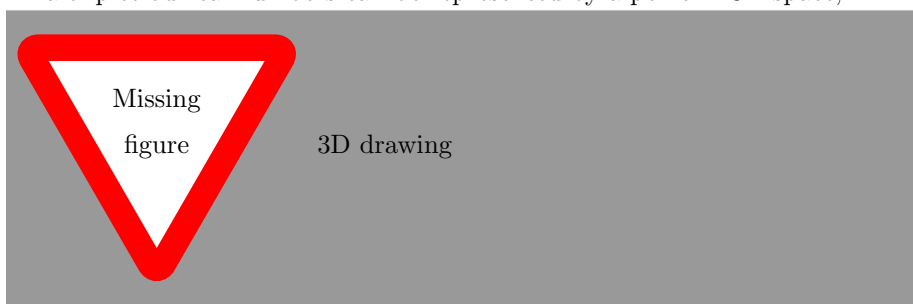
A real number can be represented by a point on a line, which is a 2-dimensional space, \mathbb{R}



a pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space, \mathbb{R}^2



a triplet of real numbers can be represented by a point in 3D space, \mathbb{R}^3



Definition

A vector is an ordered collection of n numbers

Notation

Usually vectors are given by letters, such as u, v, w . In textbooks vectors are written with bold font. In handwriting vectors are often written with a right arrow on top, such as \vec{u} . We will underline vectors, like so: \underline{u} .

□

Definition

Let us consider vector $\underline{u} \in \mathbb{R}^n$. The i -th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is u_i

Example

$$\underline{u} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3 \Rightarrow u_1 = 3, u_2 = 7, u_3 = 11$$

Definition

Let us consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of \underline{u} and \underline{v} , $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for all $i = 1, \dots, n$

Example

1.

$$\underline{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 + (-1) \\ 5 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

2.

$$\underline{u} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

$\underline{u} + \underline{v}$ is not defined! Both vectors should have the same number of components.

Definition

1. Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, if $u_i = v_i$ for all $i = 1, \dots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$. A product of α and \underline{u} is defined as:

$$\alpha \underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot v_n \end{pmatrix}$$

Example

$$\alpha = 3, \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \Rightarrow \alpha \cdot \underline{u} = \begin{pmatrix} 3 \cdot -1 \\ 3 \cdot 2 \\ 3 \cdot 5 \\ 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

Definition

Let us consider scalars α and β , and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. A sum of $\alpha \underline{u} + \beta \cdot \underline{v}$ is called a linear combination of vectors \underline{u} and \underline{v} .

Example

1.

$$2 \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

2.

$$\underline{u} - \underline{v} = 1 \cdot \underline{u} + (-1) \cdot \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_i - v_i \end{pmatrix}$$

3.

$$\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_i - u_i \end{pmatrix} = \underline{0}$$

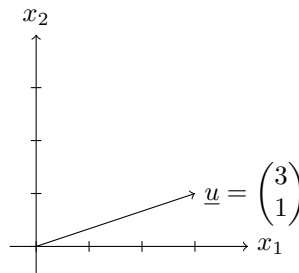
Definition

Vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1, \dots, n$. The zero vector is often written as $\underline{0} \in \mathbb{R}^n$

1.1 Graphic representation of vectors and vector operations

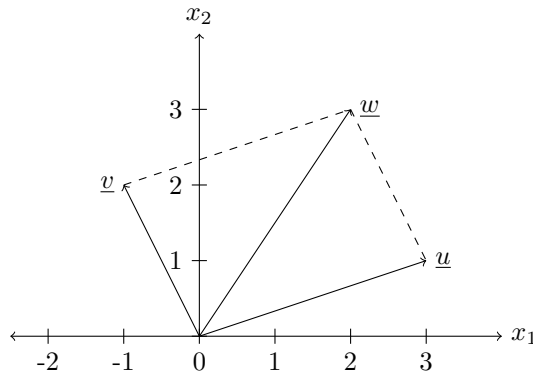
A vector can be represented in the following way:

1. An ordered collection of numbers, $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
2. As an arrow in space



3. A vector is a point in space, the endpoint of a vector from the origin.

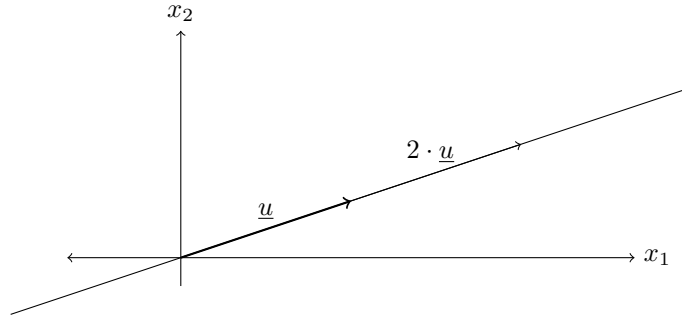
Let us consider vectors $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$



Let us consider vector $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What is $2 \cdot \underline{u}$? We can calculate as follows:

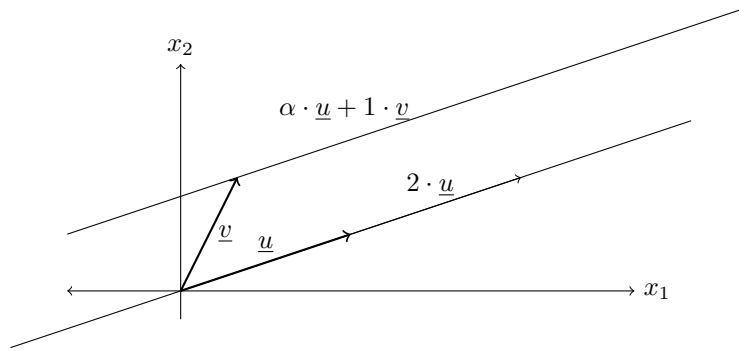
$$2 \cdot \underline{u} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We stretch vector \underline{u} 2 times along the line defined by vector \underline{u} . What is $-\underline{u}$? Simply reverse the direction. What will be the representation of $\alpha \underline{u}$ for all possible values of α ? An endless line



Let us consider two vectors $\underline{u} \in \mathbb{R}^2$ and $\underline{v} \in \mathbb{R}^2$. What will be the representation of all linear combinations of \underline{u} and \underline{v} , $\alpha\underline{u} + \beta\underline{v}$

1. Plane:



2. Line: \underline{u} and \underline{v} are on the same line.

Note: Consider $\underline{u}, \underline{v} \in \mathbb{R}^n$. \underline{u} and \underline{v} are on the same line if there exists scalars α and β such that $\alpha\underline{u} + \beta\underline{v} = \underline{0}$, when α and $\beta \neq 0$

3. Point: if $\underline{u} = \underline{0}$ and $\underline{v} = \underline{0} \Rightarrow \alpha\underline{u} + \beta\underline{v} = \underline{0}$

Consider $\underline{v}, \underline{u}$. They are on the same line if $\alpha\underline{u} + \beta\underline{v} = \underline{0}$ and $\alpha, \beta \neq 0$

1.2 Dot Product (Scalar product)

Definition

Let us consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. The dot (or scalar) product of vectors \underline{u} and \underline{v} is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^n u_i v_i$$

Notation

We will use $\langle \underline{u}, \underline{v} \rangle$ to denote the dot product, but sometimes $\underline{u} \cdot \underline{v}$ is used

□

Example

1.

$$\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

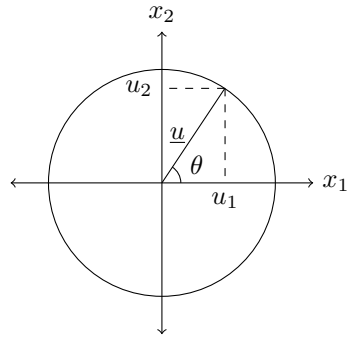
$$\langle \underline{u}, \underline{v} \rangle = 1 \cdot 0 + (-1) \cdot \frac{1}{2} + 3 \cdot (-1) = -3.5$$

2.

$$\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \langle \underline{u}, \underline{v} \rangle = 0$$

Let us consider \mathbb{R}^2 . What is the set of all possible endpoints of unit vectors in \mathbb{R}^2 , originating from the origin?

Fix positioning problem



$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

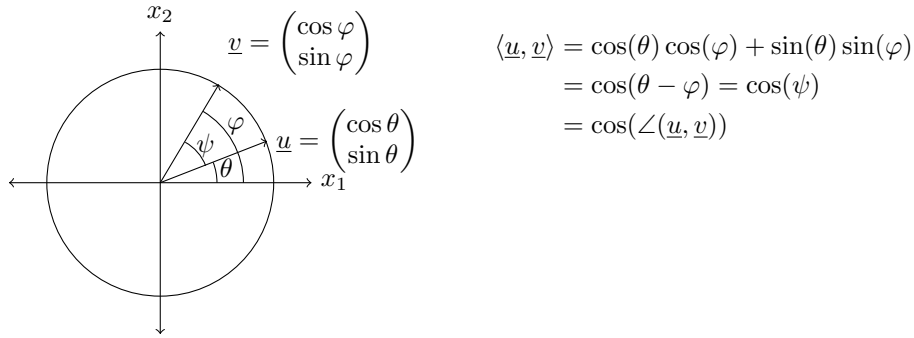
$$\cos(\theta) = \frac{u_1}{\|\underline{u}\|} = u_1$$

$$\sin(\theta) = \frac{u_2}{\|\underline{u}\|} = u_2$$

$$\underline{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Now let us consider two unit vectors. If $\underline{u} \neq \underline{0}$ or $\underline{v} \neq \underline{0}$ are not unit vectors we can find the angle between them as follows:

$$\begin{aligned} \langle \underline{u}, \underline{v} \rangle &= \left\langle \|\underline{u}\| \cdot \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \|\underline{v}\| \cdot \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle \\ &= \|\underline{u}\| \|\underline{v}\| \underbrace{\left\langle \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle}_{\text{Unit Vectors}} \\ &= \|\underline{u}\| \|\underline{v}\| \cos(\angle(\underline{u}, \underline{v})) \end{aligned}$$

**Lemma**

If $\underline{u} \neq \underline{0}, \underline{v} \neq \underline{0}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$, then

$$\cos(\angle(\underline{u}, \underline{v})) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|}$$

1.3 Properties of dot product

1. $\langle \alpha \cdot \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$. Proof:

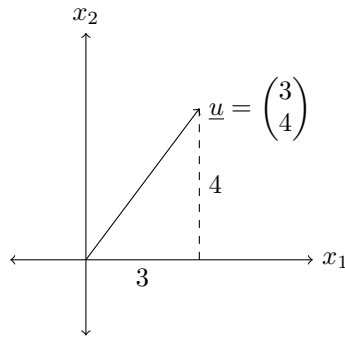
$$\begin{aligned} \langle \alpha \cdot \underline{u}, \underline{v} \rangle &= (\alpha u_1) \cdot v_1 + \cdots + (\alpha u_n) \cdot v_n \\ &= \alpha \cdot (u_1 \cdot v_1 + \cdots + u_n \cdot v_n) \\ &= \alpha \cdot \langle \underline{u}, \underline{v} \rangle \end{aligned}$$

2. $\langle \underline{u}, \alpha \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}, \underline{u}, \underline{v} \in \mathbb{R}^n$

3. $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \cdot \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle, \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$

Example

Let us consider $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. $\langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \cdot 4 = 9 + 16 = 25 = 5^2$



Definition

The length of vector $\underline{u} \in \mathbb{R}^n$, $\|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidian norm of \underline{u} .

Definition

A vector with length equal to 1 is called a unit vector

If we take vector $\underline{u} \neq \underline{0}$, how to make it a unit vector? We should multiply vector \underline{u} by $\frac{1}{\|\underline{u}\|}$, we will get $\frac{\underline{u}}{\|\underline{u}\|} = \text{unit vector}$.

In our previous example: $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Unit vector is then

$$\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

We got $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cdot \cos(\angle(\underline{u}, \underline{v}))$. Let us take the absolute value of this

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \|\underline{v}\| \cdot |\cos(\angle(\underline{u}, \underline{v}))|$$

Notice that $|\cos(\angle(\underline{u}, \underline{v}))| \leq 1$

Lemma

Cauchy Schwartz Inequality: for any $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It is easy to see that Cauchy - Schwartz inequality is correct also for zero vectors.

Chapter 2

Matrices

Let us consider a linear combination of vectors

$$x_1 \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrices times the vector.

Example

1.

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Notation

Matrices are usually written with capital letters, i.e. A, B, C, \dots

A is an n by m matrix, $A \in \mathbb{R}^{n,m}$ if it has n rows and m columns.

The element of matrix A located in row i and column j is written as a_{ij} or $(A)_{ij}$.

□

2.

$$\begin{aligned}
A &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
A \cdot \underline{x} &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (-1) \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}
\end{aligned}$$

For the product of matrix A with vector \underline{x} to exist, matrix A should have the same number of columns as vector \underline{x} components.

2.1 Matrix Operations

Definition

Let us consider matrices $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$ where n = rows, m = columns. Matrix $C \in \mathbb{R}^{n,m}$ is a sum of A and B , $C = A + B$ if $C_{ij} = A_{ij} + B_{ij}$ for all $i = 1, \dots, n$, $j = 1, \dots, m$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, C = A + B = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 4 & 5 \end{pmatrix}$$

Definition

A product of a scalar α and a matrix $A \in \mathbb{R}^{n,m}$ is defined as $(\alpha A)_{ij} = \alpha \cdot A_{ij}$, $\forall i = 1, \dots, n; j = 1, \dots, m$.

Example

$$\alpha = 3, A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \alpha \cdot A = \begin{pmatrix} 0 & 0 & 3 \\ 6 & 9 & 15 \end{pmatrix}$$

Properties

- $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$: $A + B = B + A$

Proof:

$$\begin{cases} (A + B)_{ij} = A_{ij} + B_{ij} \\ (B + A)_{ij} = B_{ij} + A_{ij} \end{cases}$$

- $A, B, C \in \mathbb{R}^{n,m}$: $(A + B) + C = A + (B + C)$
- $\alpha \cdot (A + B) = \alpha A + \alpha B$ for $\forall \alpha \in \mathbb{R}, A, B \in \mathbb{R}^{n,m}$

2.2 Matrix - Matrix multiplication

Definition

Let us consider matrix $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{m,l}$. Then $C = A \cdot B$ is an n by l matrix, $C \in \mathbb{R}^{n,l}$ such that

$$C_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Example

1.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \in \mathbb{R}^{3,2}, B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2,4} \\ C &= A \cdot B \in \mathbb{R}^{3,4} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 10 & 4 & 3 \end{pmatrix} \end{aligned}$$

2.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}; AB = \text{Not defined}$$

Properties

1. AB is not always equal to BA . (most often, is the case).

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

2. $C(A + B) = CA + CB$

3. $(A + B)C = AC + BC$

4. $\alpha(AB) = A(\alpha B)$, $A \in \mathbb{R}^{n,m}$, $B \in \mathbb{R}^{m,l}$. Proof:

$$(\alpha(AB))_{ij} = \alpha \sum_{k=1}^m a_{ik} b_{kj} = \sum_{k=1}^m a_{ik} (\alpha b_{kj}) = A(\alpha B)$$

5. $(AB)C = A(BC)$

Theorem

Let us consider matrices $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,n}$, such that A^{-1} and B^{-1} exist. Then,

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

Proof

$$\left. \begin{aligned} (AB)(B^{-1}A^{-1}) &= I \\ (B^{-1}A^{-1})(AB) &= I \end{aligned} \right\} \text{ Prove this}$$

$$(AB)(B^{-1}A^{-1}) = A \underbrace{BB^{-1}}_I A^{-1} = A \cdot I \cdot A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1} \underbrace{A^{-1}A}_I B = B^{-1} \cdot I \cdot B = I$$

\Rightarrow According to the definition $B^{-1}A^{-1}$ is the inverse of AB

□

Lemma

$A, B, C \in \mathbb{R}^{n,n}, \exists A^{-1}, \exists B^{-1}, \exists C^{-1}$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Theorem

Let us consider $A \in \mathbb{R}^{n,n}$. Let us consider that $B \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,n}$ are both inverses of A . Then $B = C$. (The inverse is unique)

Proof

$$AB = BA = I$$

$$AC = CA = I$$

$$\begin{array}{ccc} \underbrace{BA \times C = I \times C} & & \underbrace{B \times AC = B \times I} \\ & \searrow \quad \swarrow & \\ & \underline{\underline{C = B}} & \end{array}$$

□

2.3 Linear system of equations

Let us consider the following system of equations

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_2 + 2x_3 = 3 \\ 4x_3 = -1 \end{cases}$$

Find x_1, x_2, x_3 . We can write this system in matrix form.

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \in \mathbb{R}^{3,3}, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \Rightarrow A\underline{x} = \underline{b}$$

A is an upper triangular matrix. We can use backward substitution to find the solution:

$$\begin{aligned} 1. \quad x_3 &= -\frac{1}{4} = \frac{b_3}{a_{33}} \\ 2. \quad x_2 &= \frac{3-2x_3}{1} = \frac{3-2 \cdot (-\frac{1}{4})}{1} = 3.5 = \frac{b_2 - a_{23} \cdot x_3}{a_{22}} \\ 3. \quad x_1 &= \frac{2-4x_3-2x_2}{2} = -2 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}} \end{aligned}$$

In general, if $A \in \mathbb{R}^{n,n}$ is an upper triangular with $a_{ii} \neq 0, i = 1, \dots, n$ then the backward substitution works as:

$$\begin{aligned} 1. \quad x_n &= \frac{b_n}{a_{nn}} \\ 2. \quad x_{n-1} &= \frac{b_{n-1} - a_{n-1, n} x_n}{a_{n-1, n-1}} \quad \dots \quad x_i = \frac{b_i - a_{in}x_n - \dots - a_{i, i+1}x_{i+1}}{a_{ii}}, \quad i = 1, \dots, n \end{aligned}$$

2.4 Inverse Matrix

Definition

Let us consider a matrix $A \in \mathbb{R}^{n,n}$ (square matrix). Matrix $B \in \mathbb{R}^{n,n}$ is called an inverse of A , if

$$A \cdot B = I \text{ AND } B \cdot A = I$$

(Both conditions are vital)

Notation

Usually, the inverse of A is written as A^{-1}

□

Note

Not all matrices have an inverse! In most cases, it is quite difficult to find an inverse matrix. But in some cases, the inverse is easy to find.

Example

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}, a_{ii} \neq 0, \forall i = 1, \dots, n$$

Then

$$\begin{aligned} A &= \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ A \cdot A^{-1} &= \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \\ A \cdot A^{-1} &= I \end{aligned}$$

2.5 Special Matrices

- Let us consider $A \in \mathbb{R}^{n,n}$ matrix. A is called the zero matrix if all $a_{ij} = 0$, $\forall i = 1, \dots, n; j = 1, \dots, n$
- $D \in \mathbb{R}^{n,n}$ - square matrix is called diagonal matrix, if $d_{ij} = 0$ and if $i \neq j$
- Identity matrix:

$$I \in \mathbb{R}^{n,n}, I = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

- $L \in \mathbb{R}^{n,n}$ - lower triangular matrix, if

$$l_{ij} = 0, \forall i < j, L = \begin{pmatrix} * & \dots & 0 \\ \vdots & \ddots & \vdots \\ * & \dots & * \end{pmatrix}$$

- $U \in \mathbb{R}^{n,n}$ - upper triangular matrix, if

$$u_{ij} = 0, \forall i > j, L = \begin{pmatrix} * & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & * \end{pmatrix}$$

Remark:

If $A, B \in \mathbb{R}^{n,n}$ are both upper (lower) triangular matrices, then $C = A \cdot B$ is an upper triangular (lower).

If A is lower triangular, $A \in \mathbb{R}^{n,n}, a_{ii} \neq 0, i = 1, \dots, n$ then we can use forward substitution, i.e.:

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \\ &\vdots \\ x_i &= \frac{b_i - a_{i1}x_1 - \dots - a_{ii-1}x_{i-1}}{a_{ii}} \quad \forall i = 2, \dots, n \end{aligned}$$

2.6 Elementary Transition Matrices

Let us consider matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix}$$

and matrix

$$I_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$I_{21} \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

also

$$A \cdot I_{21} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

Definition

We can define the elementary transition matrix $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix $A \in \mathbb{R}^{n,n}$ then when calculating I_{pq} we take row q of A , put it into row p , replace everything else with 0.

We can also define:

$$\begin{aligned} E_{pq}(l) &= I + l \cdot I_{pq}, l \in \mathbb{R} - \text{scalar} \\ E_{pq}(l) \cdot A &= (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A \end{aligned}$$

We take row q of A , multiply it by l , add it to row p of A

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

If we have vector $\underline{b} \in \mathbb{R}^n$, then $I_{pq}\underline{b}$ - we take component q of \underline{b} , put it into component p , replace everything else with zeros.

$$E_{pq}(l)\underline{b} - \text{same as for matrices}$$

Chapter 3

Gaussian Elimination

Example

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}, A\underline{x} = \underline{b}$$

We can write this as a system of equations:

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases}$$

We can multiply equation 1 by $-\frac{a_{21}}{a_{11}} = -\frac{4}{2} = -2$, and add to equation 2. This is equivalent to multiplying $A\underline{x} = \underline{b}$ by $E_{21}\left(-\frac{a_{21}}{a_{11}}\right)$ on the left.

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b}$$

$$E_{21}\left(-\frac{a_{21}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ x_2 + 5x_3 = 12 \end{cases}$$

$$\Leftrightarrow E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b}$$

$$E_{31}\left(-\frac{a_{31}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are done with the first column. Let us denote the resulting matrix by $A^{(1)}$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ 4x_3 = 8 \end{cases}$$

$$\Leftrightarrow E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \\ a_{11} \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ a_{11} \end{pmatrix} \times \underbrace{Ax}_{\underline{b}}$$

$$E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

We are done with the second column, so we can denote the resulting matrix by $A^{(2)}$.

In fact, we got an upper triangular matrix. We can solve it using backward compatibility. Let us denote

$$E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \\ a_{11} \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ a_{11} \end{pmatrix} = U$$

where U is the upper triangular matrix. Then the inverse of it is

$$\begin{aligned} & \left[E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \\ a_{11} \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ a_{11} \end{pmatrix} \right]^{-1} \\ &= E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ a_{11} \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \\ a_{11} \end{pmatrix} E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix} \\ & A = \underbrace{E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ a_{11} \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{31}}{a_{11}} \\ a_{11} \end{pmatrix} E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ a_{22}^{(1)} \end{pmatrix}}_L \cdot U \end{aligned}$$

All matrices $E_{xx}(x)$ are lower triangular \rightarrow the product is also lower triangular ($A = L \cdot U$). So using Gaussian elimination, we represented A as a product of lower and upper triangular matrices

$$Ax = \underline{b} \Rightarrow LUx = \underline{b}$$

Let us denote Ux by \underline{y} , then we get

$$\begin{cases} L\underline{y} = \underline{b} & \text{Solve by forward substitution, find } \underline{y} \\ U\underline{x} = \underline{y} & \text{Solve by backward substitution} \end{cases}$$

Remark:

Gaussian elimination works if all elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \dots, a_{ii}^{(i-1)}$ are non-zero! These elements are called PIVOT elements.

Example

$$\begin{aligned}
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 8x_2 - 3x_3 = 6 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow A\underline{x} = \underline{b} \\
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow E_{21} \left(-\frac{a_{21}}{a_{11}} \right) A\underline{x} = E_{21} \left(-\frac{a_{21}}{a_{11}} \right) \underline{b} \\
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ x_2 + 5x_3 = 12 \end{cases} \Leftrightarrow E_{31} \left(-\frac{a_{31}}{a_{11}} \right) A\underline{x} = E_{31} \left(-\frac{a_{31}}{a_{11}} \right) E_{21} \left(-\frac{a_{21}}{a_{11}} \right) \underline{b}
\end{aligned}$$

We denote the resulting matrix by $A^{(1)}$. In order to proceed we need $a_{22}^{(1)} \neq 0$. Let us consider matrix P_{pq} -matrix, which you get from identity matrix by exchanging rows p and q . It is easy to show that $P_{pq} \cdot A$ is equal to matrix A with rows p and q exchanged.

Definition

Permutation matrix P is an identity matrix with rows in any order.

Remark:

$P^{-1} = P$. The product of permutation on matrices is a permutation matrix.

We want to exchange rows 2 and 3. We need to multiply by the permutation matrix P_{23}

$$\begin{aligned}
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + 5x_3 = 12 \\ x_3 = 2 \end{cases} \\
& \Leftrightarrow P_{23} \cdot E_{31} \left(-\frac{a_{31}}{a_{11}} \right) E_{21} \left(-\frac{a_{21}}{a_{11}} \right) A\underline{x} \\
& = P_{23} \cdot E_{31} \left(-\frac{a_{31}}{a_{11}} \right) E_{21} \left(-\frac{a_{21}}{a_{11}} \right) \underline{b}
\end{aligned}$$

In general, the Gaussian elimination proceeds like this:

$$E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} A\underline{x} = E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} \underline{b}$$

Turns out, that we can exchange the rows, or in other words multiply A by

$(P_{xx} \dots P_{xx})$ before doing the Gaussian elimination

$$\underbrace{(E_{xx} \dots E_{xx})}_{E} \underbrace{(P_{xx} \dots P_{xx})}_{P} A \underline{x} = (E_{xx} \dots E_{xx})(P_{xx} \dots P_{xx}) \underline{b}$$

$$EPA = U$$

$$PA = E^{-1}U = LU \leftarrow \text{Lower triangular}$$

Theorem

There exists permutation matrix P , such that $PA = LU$. The only necessary condition for that is that A^{-1} exists.

3.1 Matrix Transposition

Definition

Let us consider matrix $A \in \mathbb{R}^{m,n}$. Matrix $B \in \mathbb{R}^{n,m}$ is called the transpose of A if $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots m$

Notation

Usually the transpose of A is written as A^T

□

Example

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 10 \end{pmatrix} \in \mathbb{R}^{4,2} \Rightarrow A = \begin{pmatrix} 2 & 4 & 6 & 9 \\ 3 & 5 & 7 & 10 \end{pmatrix} \in \mathbb{R}^{2,4}$$

Properties

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T \cdot A^T$
4. $(A^T)^{-1} = (A^{-1})^T$

Proof

3.

$$\begin{aligned}
A \in \mathbb{R}^{m,n} &= \begin{pmatrix} -\text{row } 1 \rightarrow \\ \vdots \\ -\text{row } n \rightarrow \end{pmatrix}, B \in \mathbb{R}^{n,l} = \begin{pmatrix} \vdots & \cdots & \vdots \\ \text{col } 1 \downarrow & & \text{col } n \downarrow \end{pmatrix} \\
(AB)_{ij} &= \langle \text{row } i \text{ of } A, \text{column } j \text{ of } B \rangle \\
((AB)^T)_{pq} &= (AB)_{qp} = \langle \text{row } q \text{ of } A, \text{column } p \text{ of } B \rangle \\
B^T &= \begin{pmatrix} -\text{col } 1 \rightarrow \\ \vdots \\ -\text{col } n \rightarrow \end{pmatrix}, A^T = \begin{pmatrix} \vdots & \cdots & \vdots \\ \text{col } 1 \downarrow & & \text{col } n \downarrow \end{pmatrix} \\
(B^T A^T)_{pq} &= \langle \text{column } p \text{ of } B, \text{row } q \text{ of } A \rangle \\
\Rightarrow ((AB)^T)_{pq} &= (B^T A^T)_{pq}; p = 1, \dots, l; q = 1, \dots, m. \\
\Rightarrow (AB)^T &= B^T A^T
\end{aligned}$$

4. Assume that $A \in \mathbb{R}^{n,n}, \exists A^{-1}$

$$\begin{aligned}
AA^{-1} &= I \rightarrow (AA^{-1})^T = (A^{-1})^T \cdot A^T = I^T = I \\
A^{-1}A &= I \rightarrow (A^{-1}A)^T = A^T \cdot (A^{-1})^T = I^T = I \\
(A^T)^{-1} &= (A^{-1})^T
\end{aligned}$$

□

Let us consider vector $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^{n,1}$ - column vector. Then $\underline{u}^T \in \mathbb{R}^{1,n} =$

$(u_1 \dots u_n)$ - row vector. Let us also consider $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^{n,1}$. Then

$$\underline{u}^T \cdot \underline{v} = (u_1 \dots u_n) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \langle \underline{u}, \underline{v} \rangle$$

$$\underline{v} \cdot \underline{u}^T = n \times n \text{ matrix}$$

Definition

Matrix A is called symmetric if $A^t = A$. Matrix A should be a square matrix, $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$

$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \rightarrow I^T = I$$

Chapter 4

Vector Spaces

Definition

A vector space V is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to V , then the product of any scalar with this object belongs to V and the following properties are satisfied:

1. $\forall \underline{u}, \underline{v}, \underline{w} \in V; (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2. $\forall \underline{u}, \underline{v} \in V; \underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements $\underline{0} \in V$, such that $\forall \underline{u} \in V; \underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any $\underline{u} \in V, \exists!(-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall \underline{u}, \underline{v} \in V; \forall \alpha \in \mathbb{R}; \alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6. $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7. $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8. $\forall \underline{u} \in V; 1 \cdot \underline{u} = \underline{u}$ (1 is a scalar here)

Remark:

The “vectors” in the vector space, are not necessarily vectors ($\in \mathbb{R}^n$), but can be other objects, as long as the definition is satisfied.

Example

Let us consider a set of all 2×2 matrices. It is a vector space. Proof:

$$\begin{aligned} \text{If } A, B \in \mathbb{R}^{2,2} \quad (A + B) &\in \mathbb{R}^{2,2} \\ \alpha \in \mathbb{R}, A \in \mathbb{R}^{2,2} \quad \alpha A &\in \mathbb{R}^{2,2} \end{aligned}$$

$$1. A, B, C \in \mathbb{R}^{2,2}; (A + B) + C = A + (B + C)$$

$$2. \dots$$

$$3.$$

$$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2,2}, \forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A + \underline{0} = A$$

$$4.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow (-A) = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

Example

Let us consider a set consisting of a single object, $\underline{0}$. It is a vector space. Note: There is no vector space, which does not contain $\underline{0}$

4.1 Subspace of the vector space**Definition**

A subspace W of the vector space V , is a set of vectors in V , such that:

1. If $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$
2. If $\alpha \in \mathbb{R}, \underline{u} \in W$ then $\alpha \underline{u} \in W$

Definition

Let us consider a set of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$ is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$

Example

Is $\text{span}\{\underline{u}\}$ a subspace in \mathbb{R}^2 ? Proof:

$$\underline{v} = \alpha \underline{u} \in \text{span}\{\underline{u}\}$$

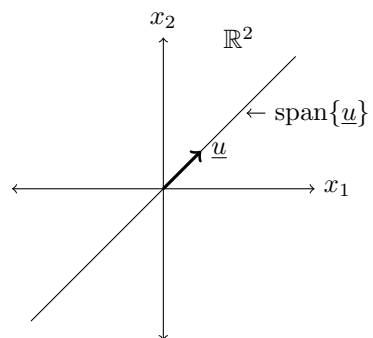
$$\underline{w} = \beta \underline{u} \in \text{span}\{\underline{u}\}$$

$$1. \underline{v} + \underline{w} = \alpha \underline{u} + \beta \underline{u} = (\alpha + \beta) \underline{u} \in \text{span}\{\underline{u}\}$$

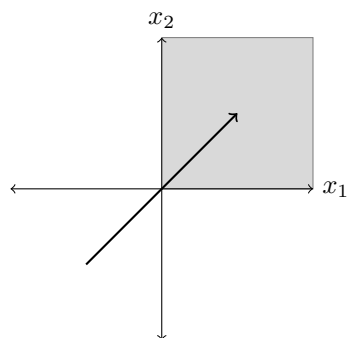
$$2. \gamma \in \mathbb{R}, \gamma \underline{v} = \gamma \cdot (\alpha \underline{u}) = (\gamma \cdot \alpha) \underline{u} \in \text{span}\{\underline{u}\}$$

Example

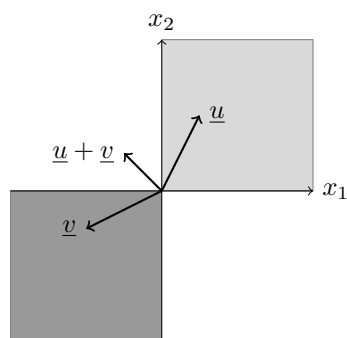
1.



2.



3.

**4.2 Linear Independence**

Definition

Let us consider vector space V and $\underline{v}_1, \dots, \underline{v}_n \in V$. $\underline{v}_1, \dots, \underline{v}_n$ are linearly dependent if there exists scalars $\alpha_1, \dots, \alpha_n$ not all equal to zero, such that $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

Definition

Vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0} \Rightarrow \text{all } \alpha_i = 0, i = 1, \dots, n$$

Example

1. Let us consider \mathbb{R}^n and vectors

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{E}_i = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$\underline{E}_1, \dots, \underline{E}_n$ are linearly independent.

2. Let us consider \mathbb{R}^2 , $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Are they linearly independent? See proof 2.

Proof

1. Assume that

$$\alpha_1 \underline{E}_1 + \dots + \alpha_n \underline{E}_n = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

\Rightarrow then all $\alpha_i = 0$ for $i = 1, \dots, n$, then based on the definition $\underline{E}_1, \dots, \underline{E}_n$ are linearly independent.

2. Let us consider $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_2 = 0 \\ \alpha_1 = 0 \end{cases}$$

If we assume $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0}$, we have to show that all α_i are zeroes \Rightarrow vectors are linearly independent.

□

Example

Let us consider \mathbb{R}^2 , $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Let us assume that

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ 0 = 0 \end{cases}$$

One possible solution:

$$\begin{cases} \alpha_1 = -2 \\ \alpha_2 = 1 \end{cases}$$

Linearly dependent.

Recap

If we consider vectors $\underline{v}_1, \dots, \underline{v}_n \in V$, then

$$\text{span}\{\underline{v}_1, \dots, \underline{v}_n\} = \{\alpha_1 \underline{v}_1, \dots, \alpha_n \underline{v}_n \mid \text{for all possible } \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$

Definition

If vector space V is generated by $\{\underline{v}_1, \dots, \underline{v}_n\}$ (in other words, $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$) and $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is called basis of V

Example

Let us consider \mathbb{R}^n and $\underline{E}_1, \dots, \underline{E}_n$. They form basis of \mathbb{R}^n .

Proof

1. “ V is generated by $\underline{v}_1, \dots, \underline{v}_n$ ”. Let us consider any vector $\underline{u} \in \mathbb{R}^n$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_n \end{pmatrix}, \text{ we have}$$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_n \end{pmatrix} = \underline{u}_1 \underline{E}_1 + \dots + \underline{u}_n \underline{E}_n \Rightarrow \mathbb{R}^n = \text{span}\{\underline{E}_1, \dots, \underline{E}_n\}$$

2. “Linear independence” already proven before.

□

Example

Let us consider \mathbb{R}^2 and $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, is it a basis?

1. Is $\mathbb{R}^2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$? Let us consider an arbitrary vector $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$. We should check that there exists scalars α_1, α_2 such that

$$\underline{v} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 \rightarrow \underline{v} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 3\alpha_2 = v_1 \\ \alpha_1 + \alpha_2 = v_2 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = v_1 - v_2 \\ \alpha_1 + \alpha_2 = v_2 \end{cases} \rightarrow \begin{cases} \alpha_2 = \frac{v_1 - v_2}{2} \\ \alpha_1 = v_2 - \frac{v_1 - v_2}{2} = \frac{3v_2 - v_1}{2} \end{cases}$$

2. $\underline{u}_1, \underline{u}_2$ = linearly independent (We showed it before).

Definition

Let us consider vector space V and vectors $\underline{v}_1, \dots, \underline{v}_n$ that form a basis of V . If vector $\underline{x} \in V$ can be written as $\underline{x} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$ then (x_1, \dots, x_n) are called the coordinates of \underline{x} with respect to basis $\{\underline{v}_1, \dots, \underline{v}_n\}$

Theorem

Let us consider vector space V and v_1, \dots, v_n that are linearly independent. Let us assume that $\underline{x} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$ and $\underline{x} = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$, then

$$\alpha_i = \beta_i \quad \forall i = 1, \dots, n$$

Proof

We have

$$\underline{x} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n \rightarrow (\alpha_1 - \beta_1) \underline{v}_1 + \dots + (\alpha_n - \beta_n) \underline{v}_n = \underline{0}$$

Since $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent $\Rightarrow \alpha_i = \beta_i, \forall i = 1, \dots, n$

□

Remark:

The coordinates of any vector \underline{x} with respect to given basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ are unique.

Theorem

Let us consider vector space V . The number of vectors in any basis of V is always the same.

Remark:

The number of vectors in the basis of vector space V is called the dimension of vector space V .

4.3 Rank of matrix

Definition

The row rank of matrix A is a maximum number of linearly independent rows of matrix A .

Definition

The column rank of matrix A is a maximum number of linearly independent columns of matrix A .

Remark:

For any matrix $A \in \mathbb{R}^{m,n}$, the row rank is equal to the column rank. Therefore the row rank and column rank are sometimes called rank of matrix A , $\text{rank}(A)$.

Example

1.

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

We have shown before that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are linearly independent, therefore $\text{rank}(A) = 2$.

2.

$$A = \begin{pmatrix} 1 & 0 \\ 7 & 0 \\ 3 & 0 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{4,2}$$

The column vectors $\begin{pmatrix} 1 \\ 7 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ are linearly dependent, thus $\text{rank}(A) =$

1 (i.e. the maximum number of linearly independent columns is 1).

Remark:

Two vectors are orthogonal if $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v} = 0$ (they basically must be perpendicular, i.e. the angle between \underline{u} and \underline{v} is 90 degrees).

Definition

Two subspaces U and W of vector space V are orthogonal, if $\forall \underline{u} \in U$ and $\forall \underline{w} \in W$, we have $\langle \underline{u}, \underline{w} \rangle = 0$

Definition

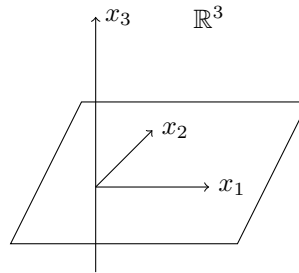
Orthogonal complement of subspace M of vector space V contains every vector orthogonal to M . This subspace is usually denoted by M^\perp

Remark:

$$\dim M + \dim M^\perp = \dim V$$

Example

Consider \mathbb{R}^3



line α plane - orthogonal subspace. Orthogonal complement of each other

Example

Not orthogonal subspace!

$$\underline{u} \neq 0$$

ADD FIGURE

$$\underline{u} \in I \text{ \& } \underline{u} \in W$$

$$\langle \underline{u} \in U, \underline{u} \in W \rangle = 0$$

Add missing figure to minipage

Note

If vector \underline{u} belongs to 2 orthogonal subspaces, this vector is necessarily a zero vector, $\underline{u} = 0$ because we should have

$$\langle \underline{u}, \underline{u} \rangle = \underline{u}^T \underline{u} = 0 \Rightarrow \underline{u} = \underline{0}$$

Chapter 5

Linear Mapping

Definition

Let us consider 2 vector spaces V and W . A function $\mathcal{L} : V \rightarrow W$ is called a linear mapping, if:

1. For any $\underline{v} \in V$ and $\underline{v}' \in V$, $\mathcal{L}(\underline{v} + \underline{v}') = \mathcal{L}(\underline{v}) + \mathcal{L}(\underline{v}')$
2. For any $\underline{v} \in V$ and any scalar α , $\mathcal{L}(\alpha \underline{v}) = \alpha \cdot \mathcal{L}(\underline{v})$

Example

Let us consider matrix $A \in \mathbb{R}^{n,m}$. We can define linear mapping \mathcal{L}_A as follows:

$$\mathcal{L}_A(\underline{v}) = A\underline{v} \quad \mathcal{L}_A : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Is \mathcal{L}_A a linear mapping? Yes!

Proof

1. $\forall \underline{v}, \underline{v}' \in \mathbb{R}^m$, we have:

$$\mathcal{L}_A(\underline{v} + \underline{v}') = A(\underline{v} + \underline{v}') = A\underline{v} + A\underline{v}' = \mathcal{L}_A(\underline{v}) + \mathcal{L}_A(\underline{v}')$$

2. $\forall \underline{v} \in \mathbb{R}^m, \forall \alpha$ (α is scalar), we have:

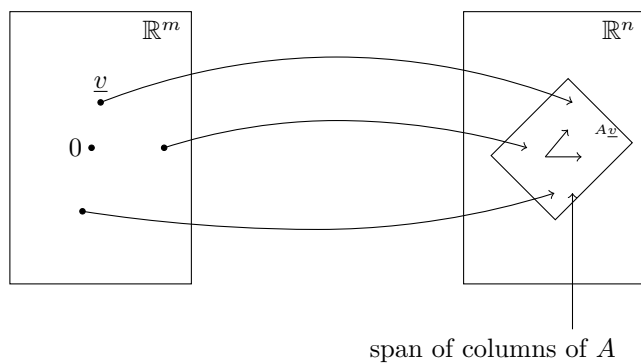
$$\mathcal{L}_A(\alpha \underline{v}) = A(\alpha \underline{v}) = \alpha \cdot A\underline{v} = \alpha \mathcal{L}_A(\underline{v})$$

□

Let us consider matrix $A \in \mathbb{R}^{n,m}$, $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$. Let us consider vector $\underline{v} \in \mathbb{R}^m$

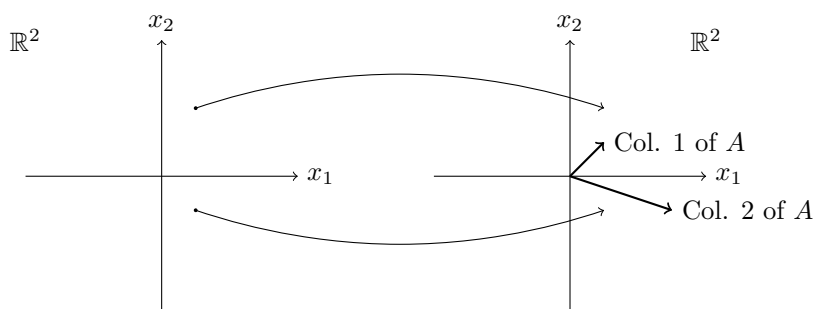
$$A\underline{v} = v_1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + v_2 \cdot \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} + \dots + v_m \cdot \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

Linear combination of columns of A

**Example**

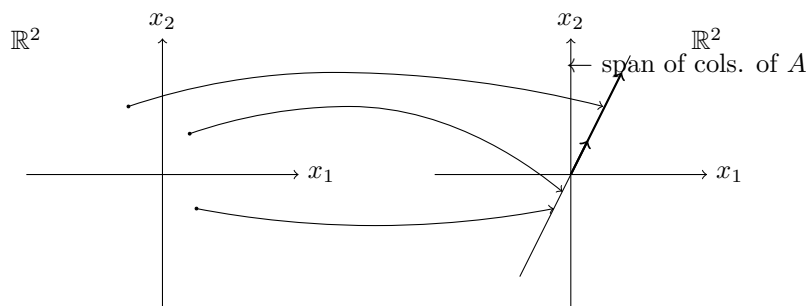
1.

$$A \in \mathbb{R}^{2,2} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$



2.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

**Note**

In order for solution of $A\underline{x} = \underline{b}$ to exist, \underline{b} should belong to a span of columns of matrix A .

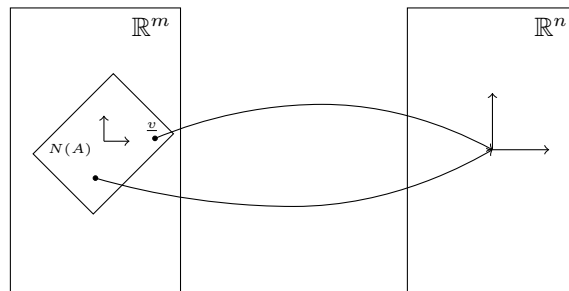
Definition

The span of columns of matrix $A \in \mathbb{R}^{n,m}$ is called a column space of A , denoted by $C(A)$, where $C(A) \subset \mathbb{R}^n$.

Definition

Let us consider matrix $A \in \mathbb{R}^{n,m}$, $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$. The null space of A is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$

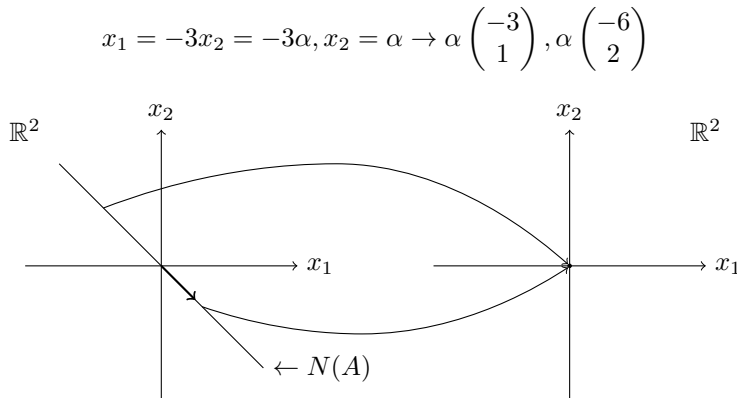
**Example**

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

What is $N(A)$? We should find all solutions of $A\underline{x} = \underline{0}$, this will give us $N(A)$.

$$\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases} \rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -3x_2 \\ 0 = 0 \end{cases}$$

The nullspace of this matrix will be a line formed by a linear combination of the vector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ ($\alpha \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, for all possible α), or in other words it will be the $\text{span}(\begin{pmatrix} -3 \\ 1 \end{pmatrix})$.



Theorem

The nullspace, $N(A)$, of $A \in \mathbb{R}^{n,m}$ is a subspace of \mathbb{R}^m .

Proof

Let us assume that $\underline{x}, \underline{x}' \in N(A)$ and α is arbitrarily scalar.

1. $A(\underline{x} + \underline{x}') = A\underline{x} + A\underline{x}' = \underline{0} + \underline{0} = \underline{0} \Rightarrow (\underline{x} + \underline{x}') \in N(A)$
2. $A(\alpha\underline{x}) = \alpha(A\underline{x}) = \alpha \cdot \underline{0} = \underline{0} \Rightarrow \alpha\underline{x} \in N(A)$

□

Theorem

The column space, $C(A)$, of $A \in \mathbb{R}^{n,m}$ is a subspace of \mathbb{R}^n .

Definition

The row space of matrix $A \in \mathbb{R}^{n,m}$ is a span of rows of A . Clearly, $R(A) = C(A^T)$ and $R(A) \subset \mathbb{R}^m$.

Definition

The left nullspace of A is defined as $N(A^T)$. $N(A^T) \subset \mathbb{R}^n$.

Theorem

$R(A)$ is a subspace of \mathbb{R}^m

Proof

Same as for the proof that $C(A)$ is a subspace of \mathbb{R}^n , but for A^T

□

Theorem

$N(A^T)$ is a subspace of \mathbb{R}^n

Proof

Same as for $N(A)$ but replace A with A^T

□

Theorem

$R(A)$ and $N(A)$ are orthogonal subspaces in \mathbb{R}^m for $A \in \mathbb{R}^{n,m}$

Proof

Let us consider $\forall \underline{x} \in N(A), A\underline{x} = \underline{0}$

$$A\underline{x} = \begin{pmatrix} - \text{row 1 of } A \rightarrow \\ \vdots \\ - \text{row } n \text{ of } A \rightarrow \end{pmatrix} \cdot \begin{pmatrix} | \\ \underline{x} \\ | \end{pmatrix} = \begin{pmatrix} < \text{row 1 of } A, \underline{x} > \\ \vdots \\ < \text{row } n \text{ of } A, \underline{x} > \end{pmatrix} \stackrel{\underline{x} \in N(A)}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

\underline{x} is orthogonal to every row of A . \underline{x} is orthogonal to every linear combination of rows of A . \underline{x} is orthogonal to $R(A)$. In fact, what we just showed is that $N(A)$ & $R(A)$ are orthogonal complements.

□

Theorem

$N(A^T)$ & $C(A) = R(A^T)$ are orthogonal complements in \mathbb{R}^n

$A \in \mathbb{R}^{n,m} : \mathbb{R}^m \rightarrow \mathbb{R}^n$. Row rank of $A = \text{rank}(A) = \dim(R(A)) = \dim(C(A))$

$$\begin{aligned} N(A) : A\underline{x} &= \underline{0} \quad \forall \underline{x} \in \mathbb{R}^m \\ C(A) : A\underline{v} &= \text{Linear combinations of columns of } A \\ &= v_1 \cdot \text{col 1 of } A + \cdots + v_n \cdot \text{col } n \text{ of } A \in \mathbb{R}^n \end{aligned}$$

Theorem

$N(A)$ is an orthogonal complement of $R(A)$ in \mathbb{R}^m ,

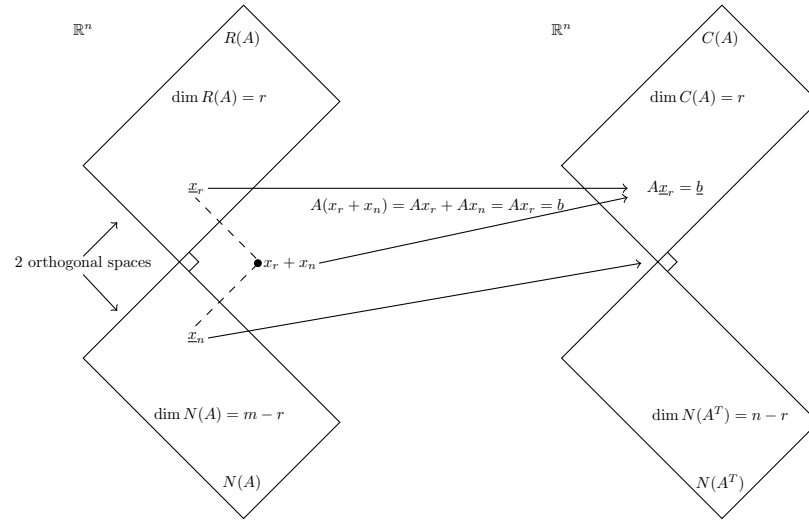
$$\dim N(A) + \underbrace{\dim R(A)}_{=\text{rank}(A)} = m$$

Theorem

$N(A^T)$ is an orthogonal complement of $R(A^T) = C(A)$ in \mathbb{R}^n ,

$$\dim N(A^T) + \underbrace{\dim C(A)}_{=\text{rank}(A)} = n$$

Let us consider $A \in \mathbb{R}^{n,m}, A : \mathbb{R}^m \rightarrow \mathbb{R}^n, \text{rank}(A) = r$



Lemma

For any vector \underline{b} in $C(A)$, there exists one and only one vector $\underline{x}_r \in R(A)$ such that $A\underline{x}_r = \underline{b}$

Proof

Let us assume that \underline{x}_r and \underline{x}'_r are in the row space, $R(A)$. Let us assume that $A\underline{x}_r = A\underline{x}'_r$. We have

$$\underline{x}_r \in R(A) - \underline{x}'_r \in R(A) \in R(A)$$

But we also have

$$A\underline{x}_r - A\underline{x}'_r = A(\underbrace{\underline{x}_r - \underline{x}'_r}_{\in N(A)}) = \underline{0}$$

It means that $(\underline{x}_r - \underline{x}'_r)$ is in $R(A)$ and $N(A)$, but they are orthogonal subspaces, therefore

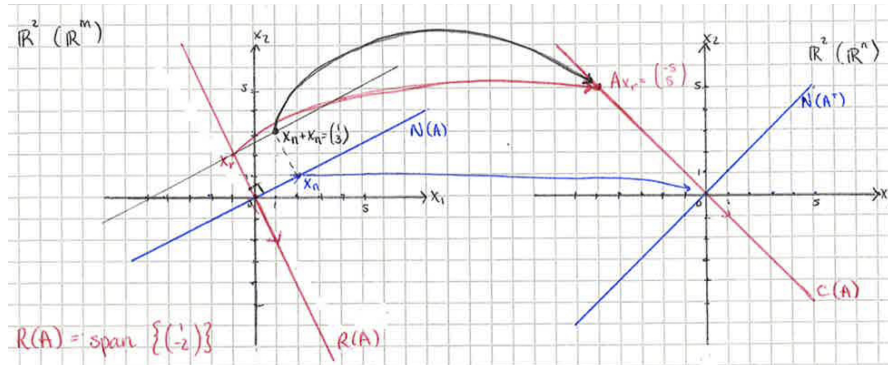
$$\underline{x}_r - \underline{x}'_r = \underline{0} \Rightarrow \underline{x}_r = \underline{x}'_r$$

□

Example

Let us consider

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \in \mathbb{R}^{2,2}$$



Row space: $\text{rank } A = 1 \Rightarrow \dim R(A) = 1$

$$R(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

Null space: $\dim N(A) = 2 - 1 = 1$

$$A\underline{x} = 0 \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow x_1 = 2x_2 \text{ (Line)}$$

Column space: $\dim C(A) = \dim R(A) = 1$

$$C(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Left Null space: $\dim N(A^T) = 2 - 1 = 1$. Consider

$$\begin{aligned} \underline{x}_r = \begin{pmatrix} -1 \\ 2 \end{pmatrix} &\Rightarrow A\underline{x}_r = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \\ \underline{x}_n = \begin{pmatrix} 2 \\ 1 \end{pmatrix} &\Rightarrow A\underline{x}_n = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

5.1 Orthogonal Basis and Gram-Schmidt process

Definition

Vectors $\underline{q}_1, \dots, \underline{q}_m$ are orthogonal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

Definition

Vectors $\underline{q}_1, \dots, \underline{q}_m$ are orthonormal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

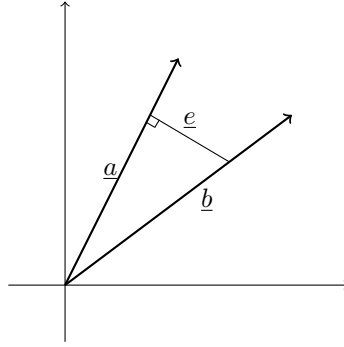
If the columns of the matrix are orthonormal vectors, then this matrix is usually denoted by Q . In this case, we have $Q^T Q = I$. If Q is not a square matrix then $Q Q^T$ is not necessarily I .

Definition

A square matrix is called orthogonal (if its columns are orthonormal vectors) if $Q^T Q = I$. In this case, since it is a square matrix, $Q Q^T = I$

5.1.1 Projection on the line

Let us assume that we have a line given by vector $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$ and vector $\underline{b} \in \mathbb{R}^n$. We want to find vector \underline{p} belonging to the line, closest to vector \underline{b} . In other words, we are looking for \underline{p} which is orthogonal projection of \underline{b} onto the line given by \underline{a}



\underline{p} is proportional to \underline{a} , $\underline{p} = \hat{x}\underline{a}$, where \hat{x} is some scalar. Let us define vector $\underline{e} = \underline{b} - \underline{p} = \underline{b} - \hat{x}\underline{a}$ (error vector). \underline{e} is orthogonal to the line, therefore

$$\begin{aligned} \langle \underline{a}, \underline{e} \rangle &= 0 \\ \langle \underline{a}, \underline{e} \rangle &= \underline{a}^T (\underline{b} - \hat{x}\underline{a}) = \underline{a}^T \underline{b} - \hat{x} \underline{a}^T \underline{a} = 0 \\ \Rightarrow \hat{x} &= \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \\ \Rightarrow \underline{p} &= \hat{x}\underline{a} = \underline{a} \hat{x} = \underline{a} \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \underbrace{\frac{\underline{a} \underline{a}^T}{\underline{a}^T \underline{a}}}_{P \in \mathbb{R}^{n,n} \text{ (projection matrix)}} \cdot \underline{b} \end{aligned}$$

Example

Let us consider $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \in \mathbb{R}^3$

$$P = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, (1 \ 2 \ 2) \right\rangle \cdot \frac{1}{9} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Let us take

$$\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{p} = P\underline{b} = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}$$

Note

$$\underline{p}^2 = \underline{p}$$

Note

$(I - P)$ — projection onto subspace orthogonal to the line given by \underline{a}

5.2 Gram-Schmidt process

Given linear independent vectors $\underline{a}, \underline{b}, \underline{c}, \dots$ we first find orthogonal vectors $\underline{a}', \underline{b}', \underline{c}', \dots$ which span the same subspace as $\underline{a}, \underline{b}, \underline{c}, \dots$ and then we normalise them,

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|}, \dots$$

So, Gram-Schmidt process allows us to construct an orthogonal basis of $\text{span}\{\underline{a}, \underline{b}, \underline{c}, \dots\} \in \mathbb{R}^n$

1. Choose $\underline{a}' = \underline{a}$
2. It is likely that \underline{b} is not orthogonal to \underline{a}' , so we need to subtract its projection on the line defined by \underline{a}'

$$\underline{b}' = \underline{b} - \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a}'$$

3. \underline{c}' is likely not orthogonal to \underline{a}' and \underline{b}' . Again, subtract its projections

$$\underline{c}' = \underline{c} - \frac{\underline{a}^T \underline{c}}{\underline{a}^T \underline{a}} \underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'} \underline{b}'$$

and so on. Finally, normalise $\underline{q}_1, \underline{q}_2, \underline{q}_3, \dots$

Example

With

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \underline{c} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

find $\underline{a}', \underline{b}', \underline{c}', \underline{q}_1, \underline{q}_2, \underline{q}_3$

- 1.

$$\underline{a}' = \underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

2.

$$\underline{b}' = \underline{b} - \frac{\underline{a}'^T \underline{b}}{\underline{a}'^T \underline{a}'} \underline{a}' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

3.

$$\underline{c}' = \underline{c} - \frac{\underline{a}'^T \underline{c}}{\underline{a}'^T \underline{a}'} \underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'} \underline{b}' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle \underline{a}', \underline{c}' \rangle = 0, \langle \underline{b}', \underline{c}' \rangle = 0$$

Finally normalise:

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

5.3 Projection onto subspace

Assume we have linearly independent vectors $a_1, \dots, a_m \in \mathbb{R}^n$. We want to project vector $\underline{b} \in \mathbb{R}^n$ onto subspace spanned by a_1, \dots, a_m . Subspace consists of all linear combinations

$$x_1 a_1 + \dots + x_m a_m = \underbrace{\begin{pmatrix} | & & | \\ a_1 & \dots & a_m \\ | & & | \end{pmatrix}}_{A \in \mathbb{R}^{n,m}} \cdot \underbrace{\hat{x}}_{\in \mathbb{R}^m}$$

We are looking for the projection \underline{p} of \underline{b} onto his subspace. We can define $\underline{e} = \underline{b} - \underline{p}$, \underline{e} should be orthogonal to all a_1, \dots, a_m

$$\left. \begin{aligned} \langle a_1, \underline{e} \rangle &= \underline{a}_1^T \cdot (\underline{b} - A\hat{x}) = 0 \\ &\vdots \\ \langle a_m, \underline{e} \rangle &= \underline{a}_m^T \cdot (\underline{b} - A\hat{x}) = 0 \end{aligned} \right\} \Rightarrow \underbrace{\begin{pmatrix} -\underline{a}_1^T \rightarrow \\ \vdots \\ -\underline{a}_m^T \rightarrow \end{pmatrix}}_{A^T} (\underline{b} - A\hat{x}) = 0$$

$$\begin{aligned} A^T(\underline{b} - A\hat{x}) &= 0 \\ A^T \underline{b} - A^T A \hat{x} &= 0 \end{aligned}$$

Theorem

A has linearly independent columns. Then $A^T A$ is:

- Square
- Symmetric

- Invertible

$$\begin{aligned}\underline{\hat{x}} &= A(A^T A)^{-1} A^T \underline{b} \\ \underline{p} &= A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{P - \text{Proj. matrix}} \cdot \underline{b} - \text{Projection vector}\end{aligned}$$