Example: Consider $\mathbb{R}^3$ and basis b_1, b_2, b_3.

Assume $L(b_1) = b_1 + b_2$

$L(b_2) = 5b_1 - b_2 + 3b_3$

$L(b_3) = -1b_1 + 4b_2 - 7b_3$

The matrix associated with this linear mapping is

$$\begin{pmatrix} 1 & 5 & -1 \\ 1 & -1 & 4 \\ 0 & 3 & -7 \end{pmatrix} = C^T$$

Lets say we have a vector whose coordinates in basis b_1, b_2, b_3 are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -1 \\ 1 & -1 & 4 \\ 0 & 3 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \times b_1 + 1 \times b_2 + 0 \times b_3.$$

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Change of Basis

Lets first look at how coordinates of vectors change when we change the basis.

Assume we have a vector space V. Lets also assume we have basis of V, $b_1, b_2, \cdots b_n$. and another basis. $d_1, d_2, \cdots, d_n$.

Consider $\underline{v} \in V$.

Let $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ be the coordinates of vector $\underline{v}$ with respect to basis $b_1, \cdots, b_n$ and $\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ be the coordinates of $\underline{v}$ w.r.t basis $d_1, \cdots, d_n$.

$$\underline{v} = \underline{u_1 b_1} + \cdots + \underline{u_n b_n} = (u_1, \cdots, u_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}^T \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\underline{v} = \underline{w_1 d_1} + \cdots \underline{w_n d_n}$$

$$= \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}^T \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$$

Since $\underline{b}_1, \cdots, \underline{b}_n$ is a basis we can express each vector in new basis $\underline{d}_1, \cdots, \underline{d}_n$ in terms of $\underline{b}_1, \cdots, \underline{b}_n$

$$\underline{d}_1 = \underline{s}_{11} \underline{b}_1 + \underline{s}_{12} \underline{b}_2 + \cdots + s_{1n} \underline{b}_n$$
$$\vdots$$
$$\underline{d}_n = s_{n1} \underline{b}_1 + s_{n2} \underline{b}_2 + \cdots + s_{nn} \underline{b}_n$$

$$\begin{pmatrix} \underline{d}_1 \\ \vdots \\ \underline{d}_n \end{pmatrix} = \underset{\sim}{S} \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}^T \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}^T \begin{pmatrix} \underline{d}_1 \\ \vdots \\ \underline{d}_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}^T \underset{\sim}{S} \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}^T = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}^T \underset{\sim}{S} \quad \underset{(AB)^T = B^T A^T}{\Rightarrow} \quad \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \underset{\sim}{S}^T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

coordinates of v in old basis $b_1, \cdots, b_n$

matrix S, describes $d_1, \cdots d_n$ w.r.t $b_1, \cdots, b_n$

coordinates of v in new basis $d_1, \cdots, d_n$.

Lemma: S^T is invertible.
(i.e. $(S^T)^{-1}$ exists)

we expressed $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ as $S^T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$. We could do the same procedure, but exchanging $b_1, \cdots, \underline{b}_n$ with $\underline{d}_1, \cdots, \underline{d}_n$ and we would arrive to

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = R^T \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

Now we have:

$$\left.\begin{aligned} \underline{w} &= R^T \underline{u} \overset{\underline{u} = S^T \underline{w}}{=} R^T S^T \underline{w} \Rightarrow R^T S^T = I \\ \underline{u} &= S^T w = S^T R^T \underline{u} \Rightarrow S^T R^T = I \end{aligned}\right\} \text{by def. of inverse, } R^T = (S^T)^{-1}$$

it means that $\underline{w} = (S^T)^{-1} \underline{u}$.

How matrices associated with linear mapping change when we change the basis.

Consider linear mapping $L: V \to V$.
Assume that L is represented by matrix A in basis $b_1, \cdots, b_n$, and by matrix A' in basis $d_1, \cdots, d_n$.

Consider vector $v \in V$. Then in basis $b_1, \cdots b_n$

$$L(\underline{v}) = A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

In basis $d_1, \cdots, d_n$ $\quad L(\underline{v}) = A' \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$$A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \cancel{S}^T A' \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \Rightarrow A S^T \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = S^T A' \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

Since $\underline{w}$ is an arbitrary vector,

$$\Rightarrow A S^T = S^T A' \Rightarrow A' = (S^T)^{-1} A S^T$$

A': matrix in new basis $d_1, \cdots, d_n$

A: matrix in old basis $b_1, \cdots, b_n$

The matrix associated with linear mapping changes as $A' = (S^T)^{-1} A S^T$ when we change from basis $b_1, \cdots, b_n$ to $d_1, \cdots, d_n$.

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

<u>definition</u>: Assume that $N \in \mathbb{R}^{n,n}$, N^{-1} exists.
$A' = N^{-1} A N$ is called similarity transformation.

<u>definition</u>: Matrices A' and A are called similar matrices, if $\exists N$ such that

$$A' = N^{-1} A N.$$

Example: Assume that linear mapping L is represented with matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ w.r.t. basis

$$b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider new basis $d_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $d_2 = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

How is L represented with respect to new basis?

$$\begin{cases} d_1 = 1 \times b_1 + 1 \times b_2 \\ d_2 = 1 \times b_1 - \frac{1}{2} b_2 \end{cases}$$

$$\Rightarrow S = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix} \Rightarrow S^T = \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix}$$

$$(S^T)^{-1} = -\frac{2}{3} \begin{pmatrix} -1/2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A' = (S^T)^{-1} A S^T = -\frac{2}{3} \begin{pmatrix} -1/2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

In the new basis, our linear mapping is represented in a very simple way.

Eigenvalues and Eigenvectors

5. 5. 15

Consider a vector space V and a linear mapping $A: V \to V$.

<u>definition</u>: A vector $v \in V$, $\underline{v \neq 0}$, is called an eigenvector of A, if there exists scalar λ, such that $A\underline{v} = \lambda \underline{v}$. This scalar λ is called an eigenvalue, corresponding to eigenvector $\underline{v}$.

Sometimes, eigenvectors are called characteristic vectors, and eigenvalues are called characteristic values.

<u>Example</u>: Consider $A \in \mathbb{R}^{n,n} = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$

Then,

$E_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - i$ is an eigenvector with eigenvalue a_{ii}, because $AE_i = a_{ii} E_i$

<u>Lemma</u> Consider $A: V \to V$ and $\underline{v} \neq 0$. - an eigenvector with λ - eigenvalue. Then for any scalar $\alpha \neq 0$, $(\alpha \underline{v})$ is also an eigenvector with the same eigenvalue λ.

<u>Proof</u>: $A\underline{v} = \lambda v$

$A(\alpha \underline{v}) = \alpha A v = \alpha(\lambda v)$ - by definition.

$= \underline{\lambda(\alpha v)}$

<u>Theorem</u>: Consider linear mapping $A: V \to V$ & eigenvalue λ. Assume that there exists $v_1 \cdots v_m$ eigenvectors corresponding to the eigenvalue. Then any vector from the span of $v_1, \cdots, v_m$ (any linear combination of $v_1, \cdots v_m$) is also an eigenvector of A, with the same eigenvalue λ.

Proof: Take any linear combination of $v_1, \dots, v_m$:

$$\alpha_1 v_1 + \dots + \alpha_m v_m.$$

$$A(\alpha_1 v_1 + \dots + \alpha_m v_m) = \alpha_1 A v_1 + \dots + \alpha_m A v_m$$

$$= \alpha_1 \lambda v_1 + \dots \alpha_m \lambda v_m = \lambda(\alpha_1 v_1 + \dots + \alpha_m v_m)$$

$\Rightarrow \underbrace{\alpha_1 v_1 + \dots + \alpha_m}_{}$ is indeed an eigenvector with eigenvalue λ.

Remark: It means that span of $v_1 \dots v_m$ forms a subspace in V and any non-zero vector from this subspace is an eigenvector of A with eigenvalue λ.

This subspace is called an eigenspace of A with eigenvalue λ.

Theorem: Consider $A: V \to V$ - linear mapping. Assume, that there exists eigenvectors $v_1, \dots, v_m$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$. lets also assume that all eigenvalues are distinct, $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then $v_1, \dots, v_m$ are linearly independant.

Proof: by induction on m.

$m=1$: v_1 - eigen vector, λ_1 - eigen value. by definition, $v_1 \neq 0$, therefore, v_1 is linearly independant.

$m>1$: assume that the theorem holds for any $m-1$ eigen vector and eigen value.

lets assume, $v_1, \dots, v_m$ are linearly dependant.

$$\alpha_1 v_1 + \dots + \alpha v_m = \underline{0} \qquad (*)$$

lets multiply $(*)$ by λ_m:

$\alpha_1 \lambda_m v_1 + \cdots + \alpha_m \lambda_m v_m = 0$

lets apply A to $(*)$:

$$A(\alpha_1 v_1 + \cdots + \alpha_m v_m) = \alpha_1 A v_1 + \cdots + \alpha_m A v_m$$

$$= \alpha_1 \lambda_1 v_1 + \cdots + \alpha_m \lambda_m v_m = 0$$

Subtract 1^{st} from 2^{nd}:

$$\alpha_1 (\lambda_1 - \lambda_m) v_1 + \cdots + \alpha_{m-1} (\lambda_{m-1} - \lambda_m) v_{m-1} = 0.$$

$m-1$ eigenvectors

$\Rightarrow$ they are linearly independant by i.h.

$\Rightarrow \alpha_1 (\lambda_1 \overset{\neq 0}{-} \lambda_m) = 0, \ldots, \alpha_{m-1}(\lambda_{m-1} \overset{\neq 0}{-} \lambda_m) = 0$

Since $\lambda_i \neq \lambda_j \Rightarrow \alpha_1 = 0, \cdots \alpha_{m-1} = 0$ and then

from $(*) \Rightarrow \alpha_m = 0$, since $\underline{v}_m \neq \underline{0}$.

$\Rightarrow v_1, \ldots, v_m$ are linearly independant.

<u>Remark</u>: If $A: V \to V$ is a linear mapping and V is an n-dimensional space, if we have $v_1, \ldots, v_n$ eigenvectors of A with all distinct $\lambda_1, \ldots, \lambda_n$, then $v_1, \ldots, v_n$ form a basis of V.