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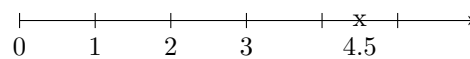
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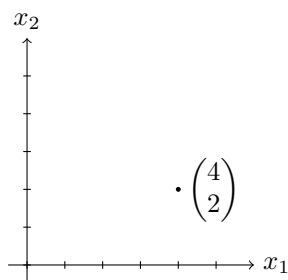
# Chapter 1

## Vectors

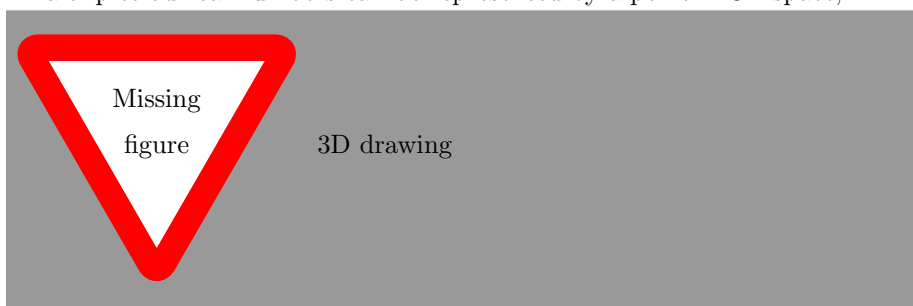
A real number can be represented by a point on a line, which is a 2-dimensional space,  $\mathbb{R}$



a pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space,  $\mathbb{R}^2$



a triplet of real numbers can be represented by a point in 3D space,  $\mathbb{R}^3$



### Definition

A vector is an ordered collection of  $n$  numbers

**Notation**

Usually vectors are given by letters, such as  $u, v, w$ . In textbooks vectors are written with bold font. In handwriting vectors are often written with a right arrow on top, such as  $\vec{u}$ . We will underline vectors, like so:  $\underline{u}$ .

□

**Definition**

Let us consider vector  $\underline{u} \in \mathbb{R}^n$ . The  $i$ -th component of vector

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is  $u_i$

**Example**

$$\underline{u} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3 \Rightarrow u_1 = 3, u_2 = 7, u_3 = 11$$

**Definition**

Let us consider vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . Vector  $\underline{w} \in \mathbb{R}^n$  is a sum of  $\underline{u}$  and  $\underline{v}$ ,  $\underline{w} = \underline{u} + \underline{v}$ , if  $w_i = u_i + v_i$  for all  $i = 1, \dots, n$

**Example**

1.

$$\underline{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 + (-1) \\ 5 + 0 \\ 1 + 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

2.

$$\underline{u} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

$\underline{u} + \underline{v}$  is not defined! Both vectors should have the same number of components.

### Definition

1. Vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$  are equal, if  $u_i = v_i$  for all  $i = 1, \dots, n$
2. A scalar is just another name for real number
3. Let us consider a scalar  $\alpha \in \mathbb{R}$  and vector  $\underline{u} \in \mathbb{R}^n$ . A product of  $\alpha$  and  $\underline{u}$  is defined as:

$$\alpha \underline{u} = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot v_n \end{pmatrix}$$

### Example

$$\alpha = 3, \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix} \Rightarrow \alpha \cdot \underline{u} = \begin{pmatrix} 3 \cdot -1 \\ 3 \cdot 2 \\ 3 \cdot 5 \\ 3 \cdot 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

### Definition

Let us consider scalars  $\alpha$  and  $\beta$ , and vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . A sum of  $\alpha \underline{u} + \beta \cdot \underline{v}$  is called a linear combination of vectors  $\underline{u}$  and  $\underline{v}$ .

### Example

1.

$$2 \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3 \cdot \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

2.

$$\underline{u} - \underline{v} = 1 \cdot \underline{u} + (-1) \cdot \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_i - v_i \end{pmatrix}$$

3.

$$\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_i - u_i \end{pmatrix} = \underline{0}$$

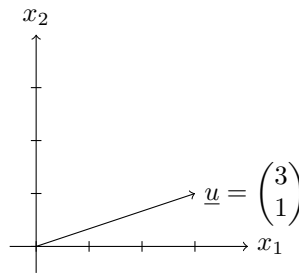
### Definition

Vector  $\underline{u} \in \mathbb{R}^n$  is called a zero vector if all  $u_i = 0$ ,  $i = 1, \dots, n$ . The zero vector is often written as  $\underline{0} \in \mathbb{R}^n$

## 1.1 Graphic representation of vectors and vector operations

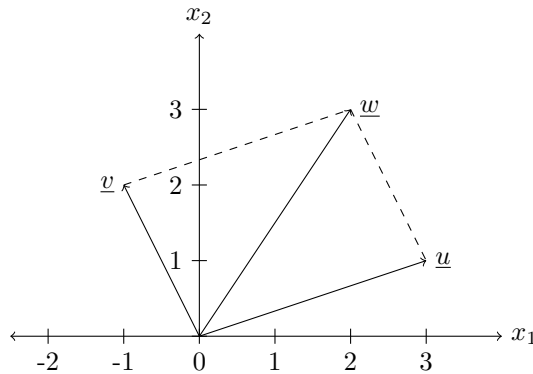
A vector can be represented in the following way:

1. An ordered collection of numbers,  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
2. As an arrow in space



3. A vector is a point in space, the endpoint of a vector from the origin.

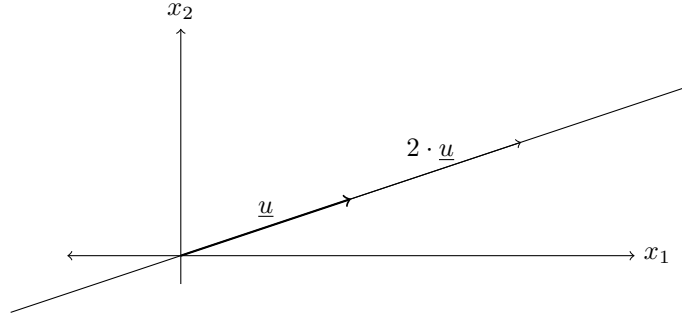
Let us consider vectors  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ,  $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$  and  $\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$



Let us consider vector  $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ . What is  $2 \cdot \underline{u}$ ? We can calculate as follows:

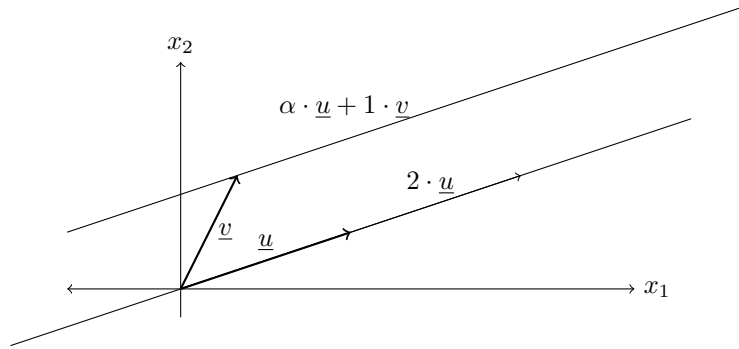
$$2 \cdot \underline{u} = 2 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

We stretch vector  $\underline{u}$  2 times along the line defined by vector  $\underline{u}$ . What is  $-\underline{u}$ ? Simply reverse the direction. What will be the representation of  $\alpha \underline{u}$  for all possible values of  $\alpha$ ? An endless line



Let us consider two vectors  $\underline{u} \in \mathbb{R}^2$  and  $\underline{v} \in \mathbb{R}^2$ . What will be the representation of all linear combinations of  $\underline{u}$  and  $\underline{v}$ ,  $\alpha\underline{u} + \beta\underline{v}$

1. Plane:



2. Line:  $\underline{u}$  and  $\underline{v}$  are on the same line.

Note: Consider  $\underline{u}, \underline{v} \in \mathbb{R}^n$ .  $\underline{u}$  and  $\underline{v}$  are on the same line if there exists scalars  $\alpha$  and  $\beta$  such that  $\alpha\underline{u} + \beta\underline{v} = \underline{0}$ , when  $\alpha$  and  $\beta \neq 0$

3. Point: if  $\underline{u} = \underline{0}$  and  $\underline{v} = \underline{0} \Rightarrow \alpha\underline{u} + \beta\underline{v} = \underline{0}$

Consider  $\underline{v}, \underline{u}$ . They are on the same line if  $\alpha\underline{u} + \beta\underline{v} = \underline{0}$  and  $\alpha, \beta \neq 0$

## 1.2 Dot Product (Scalar product)

### Definition

Let us consider two vectors  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$ . The dot (or scalar) product of vectors  $\underline{u}$  and  $\underline{v}$  is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1v_1 + u_2v_2 + \cdots + u_nv_n = \sum_{i=1}^n u_i v_i$$

### Notation

We will use  $\langle \underline{u}, \underline{v} \rangle$  to denote the dot product, but sometimes  $\underline{u} \cdot \underline{v}$  is used

□

**Example**

1.

$$\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

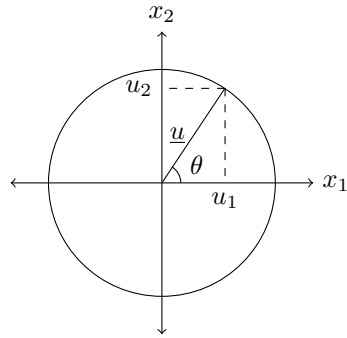
$$\langle \underline{u}, \underline{v} \rangle = 1 \cdot 0 + (-1) \cdot \frac{1}{2} + 3 \cdot (-1) = -3.5$$

2.

$$\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \langle \underline{u}, \underline{v} \rangle = 0$$

Let us consider  $\mathbb{R}^2$ . What is the set of all possible endpoints of unit vectors in  $\mathbb{R}^2$ , originating from the origin?

Fix positioning problem



$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

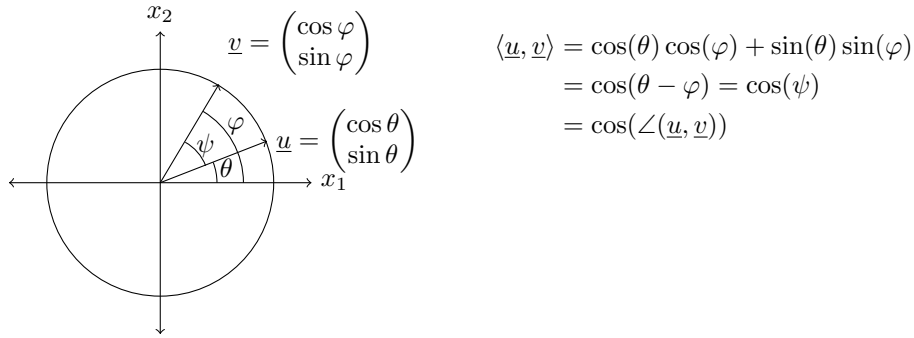
$$\cos(\theta) = \frac{u_1}{\|\underline{u}\|} = u_1$$

$$\sin(\theta) = \frac{u_2}{\|\underline{u}\|} = u_2$$

$$\underline{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

Now let us consider two unit vectors. If  $\underline{u} \neq \underline{0}$  or  $\underline{v} \neq \underline{0}$  are not unit vectors we can find the angle between them as follows:

$$\begin{aligned} \langle \underline{u}, \underline{v} \rangle &= \left\langle \|\underline{u}\| \cdot \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \|\underline{v}\| \cdot \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle \\ &= \|\underline{u}\| \|\underline{v}\| \underbrace{\left\langle \frac{1}{\|\underline{u}\|} \cdot \underline{u}, \frac{1}{\|\underline{v}\|} \cdot \underline{v} \right\rangle}_{\text{Unit Vectors}} \\ &= \|\underline{u}\| \|\underline{v}\| \cos(\angle(\underline{u}, \underline{v})) \end{aligned}$$

**Lemma**

If  $\underline{u} \neq \underline{0}, \underline{v} \neq \underline{0}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$ , then

$$\cos(\angle(\underline{u}, \underline{v})) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|}$$

**1.3 Properties of dot product**

1.  $\langle \alpha \cdot \underline{u}, \underline{v} \rangle = \alpha \cdot \langle \underline{u}, \underline{v} \rangle$  for any  $\alpha \in \mathbb{R}, \underline{u} \in \mathbb{R}^n, \underline{v} \in \mathbb{R}^n$ . Proof:

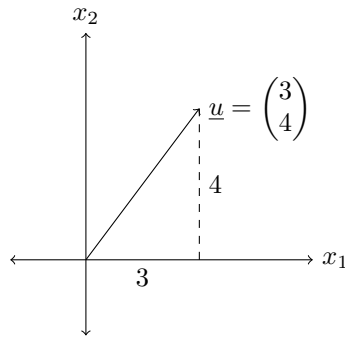
$$\begin{aligned} \langle \alpha \cdot \underline{u}, \underline{v} \rangle &= (\alpha u_1) \cdot v_1 + \cdots + (\alpha u_n) \cdot v_n \\ &= \alpha \cdot (u_1 \cdot v_1 + \cdots + u_n \cdot v_n) \\ &= \alpha \cdot \langle \underline{u}, \underline{v} \rangle \end{aligned}$$

2.  $\langle \underline{u}, \alpha \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$  for any  $\alpha \in \mathbb{R}, \underline{u}, \underline{v} \in \mathbb{R}^n$

3.  $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \cdot \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle, \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$

**Example**

Let us consider  $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .  $\langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \cdot 4 = 9 + 16 = 25 = 5^2$





**Definition**

The length of vector  $\underline{u} \in \mathbb{R}^n$ ,  $\|\underline{u}\|$ , is defined as  $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$ . Sometimes it is also called the Euclidian norm of  $\underline{u}$ .

**Definition**

A vector with length equal to 1 is called a unit vector

If we take vector  $\underline{u} \neq \underline{0}$ , how to make it a unit vector? We should multiply vector  $\underline{u}$  by  $\frac{1}{\|\underline{u}\|}$ , we will get  $\frac{\underline{u}}{\|\underline{u}\|} = \text{unit vector}$ .

In our previous example:  $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . Unit vector is then

$$\frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{5} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

We got  $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cdot \cos(\angle(\underline{u}, \underline{v}))$ . Let us take the absolute value of this

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \|\underline{v}\| \cdot |\cos(\angle(\underline{u}, \underline{v}))|$$

Notice that  $|\cos(\angle(\underline{u}, \underline{v}))| \leq 1$

**Lemma**

Cauchy Schwartz Inequality: for any  $\underline{u} \in \mathbb{R}^n$  and  $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It is easy to see that Cauchy - Schwartz inequality is correct also for zero vectors.

## Chapter 2

# Matrices

Let us consider a linear combination of vectors

$$x_1 \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way:

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrices times the vector.

### Example

1.

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 + (-1) \cdot 1 \\ 3 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 0 + 5 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

### Notation

Matrices are usually written with capital letters, i.e.  $A, B, C, \dots$

$A$  is an  $n$  by  $m$  matrix,  $A \in \mathbb{R}^{n,m}$  if it has  $n$  rows and  $m$  columns.

The element of matrix  $A$  located in row  $i$  and column  $j$  is written as  $a_{ij}$  or  $(A)_{ij}$ .

□

2.

$$\begin{aligned}
A &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
A \cdot \underline{x} &= \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (-1) \cdot 1 + 2 \cdot 1 + 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}
\end{aligned}$$

For the product of matrix  $A$  with vector  $\underline{x}$  to exist, matrix  $A$  should have the same number of columns as vector  $\underline{x}$  components.

## 2.1 Matrix Operations

### Definition

Let us consider matrices  $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{n,m}$  where  $n$  = rows,  $m$  = columns. Matrix  $C \in \mathbb{R}^{n,m}$  is a sum of  $A$  and  $B$ ,  $C = A + B$  if  $C_{ij} = A_{ij} + B_{ij}$  for all  $i = 1, \dots, n$ ,  $j = 1, \dots, m$

### Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, C = A + B = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 4 & 5 \end{pmatrix}$$

### Definition

A product of a scalar  $\alpha$  and a matrix  $A \in \mathbb{R}^{n,m}$  is defined as  $(\alpha A)_{ij} = \alpha \cdot A_{ij}$ ,  $\forall i = 1, \dots, n; j = 1, \dots, m$ .

### Example

$$\alpha = 3, A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \alpha \cdot A = \begin{pmatrix} 0 & 0 & 3 \\ 6 & 9 & 15 \end{pmatrix}$$

### Properties

- $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{n,m}$ :  $A + B = B + A$

Proof:

$$\begin{cases} (A + B)_{ij} = A_{ij} + B_{ij} \\ (B + A)_{ij} = B_{ij} + A_{ij} \end{cases}$$

- $A, B, C \in \mathbb{R}^{n,m}$ :  $(A + B) + C = A + (B + C)$
- $\alpha \cdot (A + B) = \alpha A + \alpha B$  for  $\forall \alpha \in \mathbb{R}, A, B \in \mathbb{R}^{n,m}$

## 2.2 Matrix - Matrix multiplication

### Definition

Let us consider matrix  $A \in \mathbb{R}^{n,m}$  and  $B \in \mathbb{R}^{m,l}$ . Then  $C = A \cdot B$  is an  $n$  by  $l$  matrix,  $C \in \mathbb{R}^{n,l}$  such that

$$C_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

### Example

1.

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \in \mathbb{R}^{3,2}, B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2,4} \\ C &= A \cdot B \in \mathbb{R}^{3,4} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & 10 & 4 & 3 \end{pmatrix} \end{aligned}$$

2.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}; AB = \text{Not defined}$$

### Properties

1.  $AB$  is not always equal to  $BA$ . (most often, is the case).

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

2.  $C(A + B) = CA + CB$

3.  $(A + B)C = AC + BC$

4.  $\alpha(AB) = A(\alpha B)$ ,  $A \in \mathbb{R}^{n,m}$ ,  $B \in \mathbb{R}^{m,l}$ . Proof:

$$(\alpha(AB))_{ij} = \alpha \sum_{k=1}^m a_{ik} b_{kj} = \sum_{k=1}^m a_{ik} (\alpha b_{kj}) = A(\alpha B)$$

5.  $(AB)C = A(BC)$

**Theorem**

Let us consider matrices  $A \in \mathbb{R}^{n,n}$  and  $B \in \mathbb{R}^{n,n}$ , such that  $A^{-1}$  and  $B^{-1}$  exist. Then,

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

**Proof**

$$\left. \begin{aligned} (AB)(B^{-1}A^{-1}) &= I \\ (B^{-1}A^{-1})(AB) &= I \end{aligned} \right\} \text{ Prove this}$$

$$(AB)(B^{-1}A^{-1}) = A \underbrace{BB^{-1}}_I A^{-1} = A \cdot I \cdot A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1} \underbrace{A^{-1}A}_I B = B^{-1} \cdot I \cdot B = I$$

$\Rightarrow$  According to the definition  $B^{-1}A^{-1}$  is the inverse of  $AB$

□

**Lemma**

$A, B, C \in \mathbb{R}^{n,n}, \exists A^{-1}, \exists B^{-1}, \exists C^{-1}$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

**Theorem**

Let us consider  $A \in \mathbb{R}^{n,n}$ . Let us consider that  $B \in \mathbb{R}^{n,n}$  and  $C \in \mathbb{R}^{n,n}$  are both inverses of  $A$ . Then  $B = C$ . (The inverse is unique)

**Proof**

$$AB = BA = I$$

$$AC = CA = I$$

$$\begin{array}{ccc} \underbrace{BA \times C = I \times C} & & \underbrace{B \times AC = B \times I} \\ & \searrow \quad \swarrow & \\ & \underline{\underline{C = B}} & \end{array}$$

□

## 2.3 Linear system of equations

Let us consider the following system of equations

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_2 + 2x_3 = 3 \\ 4x_3 = -1 \end{cases}$$

Find  $x_1, x_2, x_3$ . We can write this system in matrix form.

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \in \mathbb{R}^{3,3}, \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \Rightarrow A\underline{x} = \underline{b}$$

$A$  is an upper triangular matrix. We can use backward substitution to find the solution:

$$1. \ x_3 = -\frac{1}{4} = \frac{b_3}{a_{33}}$$

$$2. \ x_2 = \frac{3-2x_3}{1} = \frac{3-2 \cdot (-\frac{1}{4})}{1} = 3.5 = \frac{b_2 - a_{23} \cdot x_3}{a_{22}}$$

$$3. \ x_1 = \frac{2-4x_3-2x_2}{2} = -2 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

In general, if  $A \in \mathbb{R}^{n,n}$  is an upper triangular with  $a_{ii} \neq 0, i = 1, \dots, n$  then the backward substitution works as:

$$1. \ x_n = \frac{b_n}{a_{nn}}$$

$$2. \ x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} \quad \dots \quad x_i =$$

Cannot understand last calculation, page 12, LinearAlgebraNotes\_1

## 2.4 Inverse Matrix

### Definition

Let us consider a matrix  $A \in \mathbb{R}^{n,n}$  (square matrix). Matrix  $B \in \mathbb{R}^{n,n}$  is called an inverse of  $A$ , if

$$A \cdot B = I \text{ AND } B \cdot A = I$$

(Both conditions are vital)

### Notation

Usually, the inverse of  $A$  is written as  $A^{-1}$

□

### Note

Not all matrices have an inverse! In most cases, it is quite difficult to find an inverse matrix. But in some cases, the inverse is easy to find.

**Example**

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix}, a_{ii} \neq 0, \forall i = 1, \dots, n$$

Then

$$\begin{aligned} A &= \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ A \cdot A^{-1} &= \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & a_{nn}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \\ A \cdot A^{-1} &= I \end{aligned}$$

**2.5 Special Matrices**

- Let us consider  $A \in \mathbb{R}^{n,n}$  matrix.  $A$  is called the zero matrix if all  $a_{ij} = 0$ ,  $\forall i = 1, \dots, n; j = 1, \dots, n$
- $D \in \mathbb{R}^{n,n}$  - square matrix is called diagonal matrix, if  $d_{ij} = 0$  and if  $i \neq j$
- Identity matrix:

$$I \in \mathbb{R}^{n,n}, I = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}$$

- $L \in \mathbb{R}^{n,n}$  - lower triangular matrix, if

$$l_{ij} = 0, \forall i < j, L = \begin{pmatrix} * & \dots & 0 \\ \vdots & \ddots & \vdots \\ * & \dots & * \end{pmatrix}$$

- $U \in \mathbb{R}^{n,n}$  - upper triangular matrix, if

$$u_{ij} = 0, \forall i > j, L = \begin{pmatrix} * & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & * \end{pmatrix}$$

**Remark:**

If  $A, B \in \mathbb{R}^{n,n}$  are both upper (lower) triangular matrices, then  $C = A \cdot B$  is an upper triangular (lower).

If  $A$  is lower triangular,  $A \in \mathbb{R}^{n,n}, a_{ii} \neq 0, i = 1, \dots, n$  then we can use forward substitution, i.e.:

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} \\ &\vdots \\ x_i &= \frac{b_i - a_{i1}x_1 - \dots - a_{ii-1}x_{i-1}}{a_{ii}} \quad \forall i = 2, \dots, n \end{aligned}$$

## 2.6 Elementary Transition Matrices

Let us consider matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix}$$

and matrix

$$I_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then

$$I_{21} \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

also

$$A \cdot I_{21} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$



**Definition**

We can define the elementary transition matrix  $I_{pq} \in \mathbb{R}^{n,n}$

$$(I_{pq}) = \begin{cases} 1 & i = p, q = j \\ 0 & \text{otherwise} \end{cases}$$

If we take a matrix  $A \in \mathbb{R}^{n,n}$  then when calculating  $I_{pq}$  we take row  $q$  of  $A$ , put it into row  $p$ , replace everything else with 0.

We can also define:

$$\begin{aligned} E_{pq}(l) &= I + l \cdot I_{pq}, l \in \mathbb{R} - \text{scalar} \\ E_{pq}(l) \cdot A &= (I + lI_{pq}) \cdot A = A + l \cdot I_{pq}A \end{aligned}$$

We take row  $q$  of  $A$ , multiply it by  $l$ , add it to row  $p$  of  $A$

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

If we have vector  $\underline{b} \in \mathbb{R}^n$ , then  $I_{pq}\underline{b}$  - we take component  $q$  of  $\underline{b}$ , put it into component  $p$ , replace everything else with zeros.

$$E_{pq}(l)\underline{b} - \text{same as for matrices}$$

## Chapter 3

# Gaussian Elimination

### Example

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}, A\underline{x} = \underline{b}$$

We can write this as a system of equations:

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases}$$

We can multiply equation 1 by  $-\frac{a_{21}}{a_{11}} = -\frac{4}{2} = -2$ , and add to equation 2. This is equivalent to multiplying  $A\underline{x} = \underline{b}$  by  $E_{21}\left(-\frac{a_{21}}{a_{11}}\right)$  on the left.

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b}$$

$$E_{21}\left(-\frac{a_{21}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ x_2 + 5x_3 = 12 \end{cases}$$

$$\Leftrightarrow E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \times A\underline{x} = E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right) \underline{b}$$

$$E_{31}\left(-\frac{a_{31}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are done with the first column. Let us denote the resulting matrix by  $A^{(1)}$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ 4x_3 = 8 \end{cases}$$

$$\Leftrightarrow E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ 1 \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ 1 \\ 1 \end{pmatrix} \times \underbrace{Ax}_{\underline{b}}$$

$$E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

We are done with the second column, so we can denote the resulting matrix by  $A^{(2)}$ .

In fact, we got an upper triangular matrix. We can solve it using backward compatibility. Let us denote

$$E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ 1 \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ 1 \\ 1 \end{pmatrix} = U$$

where  $U$  is the upper triangular matrix. Then the inverse of it is

$$\begin{aligned} & \left[ E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ 1 \end{pmatrix} E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ 1 \\ 1 \end{pmatrix} \right]^{-1} \\ &= E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ 1 \\ 1 \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ 1 \end{pmatrix} E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix} \\ & A = \underbrace{E_{21} \begin{pmatrix} -\frac{a_{21}}{a_{11}} \\ 1 \\ 1 \end{pmatrix} E_{31} \begin{pmatrix} -\frac{a_{31}}{a_{11}} \\ -\frac{a_{21}}{a_{11}} \\ 1 \end{pmatrix} E_{32} \begin{pmatrix} -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} \\ -\frac{a_{31}^{(1)}}{a_{22}^{(1)}} \\ 1 \end{pmatrix}}_L \cdot U \end{aligned}$$

All matrices  $E_{xx}(x)$  are lower triangular  $\rightarrow$  the product is also lower triangular ( $A = L \cdot U$ ). So using Gaussian elimination, we represented  $A$  as a product of lower and upper triangular matrices

$$Ax = \underline{b} \Rightarrow LUx = \underline{b}$$

Let us denote  $Ux$  by  $\underline{y}$ , then we get

$$\begin{cases} L\underline{y} = \underline{b} & \text{Solve by forward substitution, find } \underline{y} \\ U\underline{x} = \underline{y} & \text{Solve by backward substitution} \end{cases}$$

**Remark:**

Gaussian elimination works if all elements  $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \dots, a_{ii}^{(i-1)}$  are non-zero! These elements are called PIVOT elements.

**Example**

$$\begin{aligned}
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 8x_2 - 3x_3 = 6 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow A\underline{x} = \underline{b} \\
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \Leftrightarrow E_{21} \left( -\frac{a_{21}}{a_{11}} \right) A\underline{x} = E_{21} \left( -\frac{a_{21}}{a_{11}} \right) \underline{b} \\
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_3 = 2 \\ x_2 + 5x_3 = 12 \end{cases} \Leftrightarrow E_{31} \left( -\frac{a_{31}}{a_{11}} \right) A\underline{x} = E_{31} \left( -\frac{a_{31}}{a_{11}} \right) E_{21} \left( -\frac{a_{21}}{a_{11}} \right) \underline{b}
\end{aligned}$$

We denote the resulting matrix by  $A^{(1)}$ . In order to proceed we need  $a_{22}^{(1)} \neq 0$ . Let us consider matrix  $P_{pq}$ -matrix, which you get from identity matrix by exchanging rows  $p$  and  $q$ . It is easy to show that  $P_{pq} \cdot A$  is equal to matrix  $A$  with rows  $p$  and  $q$  exchanged.

**Definition**

Permutation matrix  $P$  is an identity matrix with rows in any order.

**Remark:**

$P^{-1} = P$ . The product of permutation on matrices is a permutation matrix.

We want to exchange rows 2 and 3. We need to multiply by the permutation matrix  $P_{23}$

$$\begin{aligned}
& \begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + 5x_3 = 12 \\ x_3 = 2 \end{cases} \\
& \Leftrightarrow P_{23} \cdot E_{31} \left( -\frac{a_{31}}{a_{11}} \right) E_{21} \left( -\frac{a_{21}}{a_{11}} \right) A\underline{x} \\
& = P_{23} \cdot E_{31} \left( -\frac{a_{31}}{a_{11}} \right) E_{21} \left( -\frac{a_{21}}{a_{11}} \right) \underline{b}
\end{aligned}$$

In general, the Gaussian elimination proceeds like this:

$$E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} A\underline{x} = E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} \underline{b}$$

Turns out, that we can exchange the rows, or in other words multiply  $A$  by

$(P_{xx} \dots P_{xx})$  before doing the Gaussian elimination

$$\underbrace{\underbrace{(E_{xx} \dots E_{xx})}_E \underbrace{(P_{xx} \dots P_{xx})}_P}_{U} A \underline{x} = (E_{xx} \dots E_{xx})(P_{xx} \dots P_{xx}) \underline{b}$$

$$EPA = U$$

$$PA = E^{-1}U = LU \leftarrow \text{Lower triangular}$$

### Theorem

There exists permutation matrix  $P$ , such that  $PA = LU$ . The only necessary condition for that is that  $A^{-1}$  exists.

## 3.1 Matrix Transposition

### Definition

Let us consider matrix  $A \in \mathbb{R}^{m,n}$ . Matrix  $B \in \mathbb{R}^{n,m}$  is called the transpose of  $A$  if  $(B)_{ij} = (A)_{ji}, i = 1 \dots n, j = 1 \dots m$

### Notation

Usually the transpose of  $A$  is written as  $A^T$

□

### Example

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 10 \end{pmatrix} \in \mathbb{R}^{4,2} \Rightarrow A = \begin{pmatrix} 2 & 4 & 6 & 9 \\ 3 & 5 & 7 & 10 \end{pmatrix} \in \mathbb{R}^{2,4}$$

### Properties

1.  $(A^T)^T = A$
2.  $(A + B)^T = A^T + B^T$
3.  $(AB)^T = B^T \cdot A^T$
4.  $(A^T)^{-1} = (A^{-1})^T$

**Proof**

3.

$$\begin{aligned}
A \in \mathbb{R}^{m,n} &= \begin{pmatrix} -\text{row } 1 \rightarrow \\ \vdots \\ -\text{row } n \rightarrow \end{pmatrix}, B \in \mathbb{R}^{n,l} = \begin{pmatrix} \vdots & \cdots & \vdots \\ \text{col } 1 \downarrow & & \text{col } n \downarrow \end{pmatrix} \\
(AB)_{ij} &= \langle \text{row } i \text{ of } A, \text{column } j \text{ of } B \rangle \\
((AB)^T)_{pq} &= (AB)_{qp} = \langle \text{row } q \text{ of } A, \text{column } p \text{ of } B \rangle \\
B^T &= \begin{pmatrix} -\text{col } 1 \rightarrow \\ \vdots \\ -\text{col } n \rightarrow \end{pmatrix}, A^T = \begin{pmatrix} \vdots & \cdots & \vdots \\ \text{col } 1 \downarrow & & \text{col } n \downarrow \end{pmatrix} \\
(B^T A^T)_{pq} &= \langle \text{column } p \text{ of } B, \text{row } q \text{ of } A \rangle \\
\Rightarrow ((AB)^T)_{pq} &= (B^T A^T)_{pq}; p = 1, \dots, l; q = 1, \dots, m. \\
\Rightarrow (AB)^T &= B^T A^T
\end{aligned}$$

4. Assume that  $A \in \mathbb{R}^{n,n}, \exists A^{-1}$ 

$$\begin{aligned}
AA^{-1} &= I \rightarrow (AA^{-1})^T = (A^{-1})^T \cdot A^T = I^T = I \\
A^{-1}A &= I \rightarrow (A^{-1}A)^T = A^T \cdot (A^{-1})^T = I^T = I \\
(A^T)^{-1} &= (A^{-1})^T
\end{aligned}$$

□

Let us consider vector  $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^{n,1}$  - column vector. Then  $\underline{u}^T \in \mathbb{R}^{1,n} =$

$(u_1 \dots u_n)$  - row vector. Let us also consider  $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^{n,1}$ . Then

$$\underline{u}^T \cdot \underline{v} = (u_1 \dots u_n) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \langle \underline{u}, \underline{v} \rangle$$

$$\underline{v} \cdot \underline{u}^T = n \times n \text{ matrix}$$

**Definition**

Matrix  $A$  is called symmetric if  $A^t = A$ . Matrix  $A$  should be a square matrix,  $A \in \mathbb{R}^{n,n}$

$$\text{e.g. } A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^T = A$$

$$\text{e.g. } A = I \in \mathbb{R}^{n,n} \rightarrow I^T = I$$



## Chapter 4

# Vector Spaces

### Definition

A vector space  $V$  is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to  $V$ , then the product of any scalar with this object belongs to  $V$  and the following properties are satisfied:

1.  $\forall \underline{u}, \underline{v}, \underline{w} \in V; (\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2.  $\forall \underline{u}, \underline{v} \in V; \underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements  $\underline{0} \in V$ , such that  $\forall \underline{u} \in V; \underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$
4. For any  $\underline{u} \in V, \exists!(-\underline{u}) \in V$ , such that  $\underline{u} + (-\underline{u}) = \underline{0}$
5.  $\forall \underline{u}, \underline{v} \in V; \forall \alpha \in \mathbb{R}; \alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6.  $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7.  $\forall \underline{u} \in V; \forall \alpha, \beta \in \mathbb{R}; (\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8.  $\forall \underline{u} \in V; 1 \cdot \underline{u} = \underline{u}$  (1 is a scalar here)

### Remark:

The “vectors” in the vector space, are not necessarily vectors ( $\in \mathbb{R}^n$ ), but can be other objects, as long as the definition is satisfied.



**Example**

Let us consider a set of all  $2 \times 2$  matrices. It is a vector space. Proof:

$$\begin{aligned} \text{If } A, B \in \mathbb{R}^{2,2} \quad (A + B) &\in \mathbb{R}^{2,2} \\ \alpha \in \mathbb{R}, A \in \mathbb{R}^{2,2} \quad \alpha A &\in \mathbb{R}^{2,2} \end{aligned}$$

$$1. A, B, C \in \mathbb{R}^{2,2}; (A + B) + C = A + (B + C)$$

$$2. \dots$$

$$3.$$

$$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2,2}, \forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A + \underline{0} = A$$

$$4.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow (-A) = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

**Example**

Let us consider a set consisting of a single object,  $\underline{0}$ . It is a vector space. Note: There is no vector space, which does not contain  $\underline{0}$

**4.1 Subspace of the vector space****Definition**

A subspace  $W$  of the vector space  $V$ , is a set of vectors in  $V$ , such that:

1. If  $\underline{u}, \underline{v} \in W$  then  $\underline{u} + \underline{v} \in W$
2. If  $\alpha \in \mathbb{R}, \underline{u} \in W$  then  $\alpha \underline{u} \in W$

**Definition**

Let us consider a set of vectors  $\{\underline{u}_1, \dots, \underline{u}_n\}$ . The span of vectors  $\{\underline{u}_1, \dots, \underline{u}_n\}$  is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$

**Example**

Is  $\text{span}\{\underline{u}\}$  a subspace in  $\mathbb{R}^2$ ? Proof:

$$\underline{v} = \alpha \underline{u} \in \text{span}\{\underline{u}\}$$

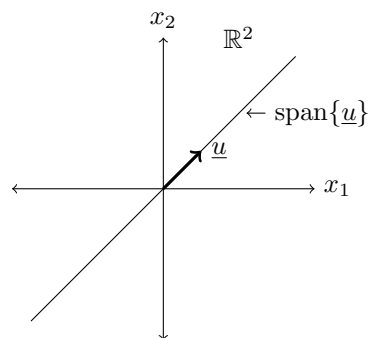
$$\underline{w} = \beta \underline{u} \in \text{span}\{\underline{u}\}$$

$$1. \underline{v} + \underline{w} = \alpha \underline{u} + \beta \underline{u} = (\alpha + \beta) \underline{u} \in \text{span}\{\underline{u}\}$$

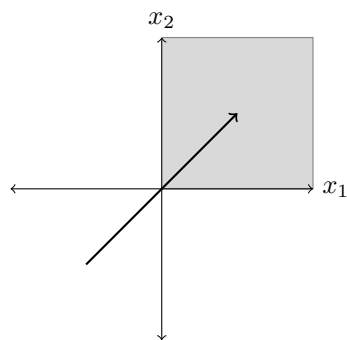
$$2. \gamma \in \mathbb{R}, \gamma \underline{v} = \gamma \cdot (\alpha \underline{u}) = (\gamma \cdot \alpha) \underline{u} \in \text{span}\{\underline{u}\}$$

**Example**

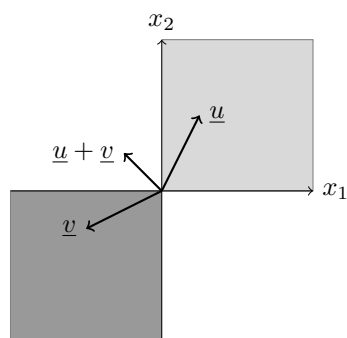
1.



2.



3.

**4.2 Linear Independence**

**Definition**

Let us consider vector space  $V$  and  $\underline{v}_1, \dots, \underline{v}_n \in V$ .  $\underline{v}_1, \dots, \underline{v}_n$  are linearly dependent if there exists scalars  $\alpha_1, \dots, \alpha_n$  not all equal to zero, such that  $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$

If no such scalars exist, the vectors  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent.

**Definition**

Vectors  $\underline{v}_1, \dots, \underline{v}_n \in V$  are linearly independent if the following is true:

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0} \Rightarrow \text{all } \alpha_i = 0, i = 1, \dots, n$$

**Example**

1. Let us consider  $\mathbb{R}^n$  and vectors

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{E}_i = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_n = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$\underline{E}_1, \dots, \underline{E}_n$  are linearly independent.

2. Let us consider  $\mathbb{R}^2$ ,  $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ . Are they linearly independent? See proof 2.

**Proof**

1. Assume that

$$\alpha_1 \underline{E}_1 + \dots + \alpha_n \underline{E}_n = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$  then all  $\alpha_i = 0$  for  $i = 1, \dots, n$ , then based on the definition  $\underline{E}_1, \dots, \underline{E}_n$  are linearly independent.

2. Let us consider  $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{cases} \alpha_1 + 3\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_2 = 0 \\ \alpha_1 = 0 \end{cases}$$

If we assume  $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0}$ , we have to show that all  $\alpha_i$  are zeroes  $\Rightarrow$  vectors are linearly independent.

□

**Example**

Let us consider  $\mathbb{R}^2$ ,  $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\underline{u}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ . Let us assume that

$$\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ \alpha_1 + 2\alpha_2 = 0 \end{cases} \rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ 0 = 0 \end{cases}$$

One possible solution:

$$\begin{cases} \alpha_1 = -2 \\ \alpha_2 = 1 \end{cases}$$

Linearly dependent.

**Recap**

If we consider vectors  $\underline{v}_1, \dots, \underline{v}_n \in V$ , then

$$\text{span}\{\underline{v}_1, \dots, \underline{v}_n\} = \{\alpha_1 \underline{v}_1, \dots, \alpha_n \underline{v}_n \mid \text{for all possible } \alpha_1, \dots, \alpha_n \in \mathbb{R}\}$$

**Definition**

If vector space  $V$  is generated by  $\{\underline{v}_1, \dots, \underline{v}_n\}$  (in other words,  $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$ ) and  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent, then  $\{\underline{v}_1, \dots, \underline{v}_n\}$  is called basis of  $V$

**Example**

Let us consider  $\mathbb{R}^n$  and  $\underline{E}_1, \dots, \underline{E}_n$ . They form basis of  $\mathbb{R}^n$ .

**Proof**

1. “ $V$  is generated by  $\underline{v}_1, \dots, \underline{v}_n$ ”. Let us consider any vector  $\underline{u} \in \mathbb{R}^n$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_n \end{pmatrix}, \text{ we have}$$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_n \end{pmatrix} = \underline{u}_1 \underline{E}_1 + \dots + \underline{u}_n \underline{E}_n \Rightarrow \mathbb{R}^n = \text{span}\{\underline{E}_1, \dots, \underline{E}_n\}$$

2. “Linear independence” already proven before.

□

**Example**

Let us consider  $\mathbb{R}^2$  and  $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ , is it a basis?

1. Is  $\mathbb{R}^2 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ ? Let us consider an arbitrary vector  $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$ . We should check that there exists scalars  $\alpha_1, \alpha_2$  such that

$$\underline{v} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 \rightarrow \underline{v} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 3\alpha_2 = v_1 \\ \alpha_1 + \alpha_2 = v_2 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = v_1 - v_2 \\ \alpha_1 + \alpha_2 = v_2 \end{cases} \rightarrow \begin{cases} \alpha_2 = \frac{v_1 - v_2}{2} \\ \alpha_1 = v_2 - \frac{v_1 - v_2}{2} = \frac{3v_2 - v_1}{2} \end{cases}$$

2.  $\underline{u}_1, \underline{u}_2$  = linearly independent (We showed it before).

**Definition**

Let us consider vector space  $V$  and vectors  $\underline{v}_1, \dots, \underline{v}_n$  that form a basis of  $V$ . If vector  $\underline{x} \in V$  can be written as  $\underline{x} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$  then  $(x_1, \dots, x_n)$  are called the coordinates of  $\underline{x}$  with respect to basis  $\{\underline{v}_1, \dots, \underline{v}_n\}$

**Theorem**

Let us consider vector space  $V$  and  $v_1, \dots, v_n$  that are linearly independent. Let us assume that  $\underline{x} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$  and  $\underline{x} = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$ , then

$$\alpha_i = \beta_i \quad \forall i = 1, \dots, n$$

**Proof**

We have

$$\underline{x} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n \rightarrow (\alpha_1 - \beta_1) \underline{v}_1 + \dots + (\alpha_n - \beta_n) \underline{v}_n = \underline{0}$$

Since  $\underline{v}_1, \dots, \underline{v}_n$  are linearly independent  $\Rightarrow \alpha_i = \beta_i, \forall i = 1, \dots, n$

□

**Remark:**

The coordinates of any vector  $\underline{x}$  with respect to given basis  $\{\underline{v}_1, \dots, \underline{v}_n\}$  are unique.

**Theorem**

Let us consider vector space  $V$ . The number of vectors in any basis of  $V$  is always the same.

**Remark:**

The number of vectors in the basis of vector space  $V$  is called the dimension of vector space  $V$ .

### 4.3 Rank of matrix

**Definition**

The row rank of matrix  $A$  is a maximum number of linearly independent rows of matrix  $A$ .

**Definition**

The column rank of matrix  $A$  is a maximum number of linearly independent columns of matrix  $A$ .

**Remark:**

For any matrix  $A \in \mathbb{R}^{m,n}$ , the row rank is equal to the column rank. Therefore the row rank and column rank are sometimes called rank of matrix  $A$ ,  $\text{rank}(A)$ .

**Example**

1.

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

We have shown before that  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$  are linearly independent, therefore  $\text{rank}(A) = 2$ .

2.

$$A = \begin{pmatrix} 1 & 0 \\ 7 & 0 \\ 3 & 0 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{4,2}$$

The column vectors  $\begin{pmatrix} 1 \\ 7 \\ 3 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$  are linearly dependent, thus  $\text{rank}(A) =$

1 (i.e. the maximum number of linearly independent columns is 1).

**Remark:**

Two vectors are orthogonal if  $\langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v} = 0$  (they basically must be perpendicular, i.e. the angle between  $\underline{u}$  and  $\underline{v}$  is 90 degrees).

**Definition**

Two subspaces  $U$  and  $W$  of vector space  $V$  are orthogonal, if  $\forall \underline{u} \in U$  and  $\forall \underline{w} \in W$ , we have  $\langle \underline{u}, \underline{w} \rangle = 0$

**Definition**

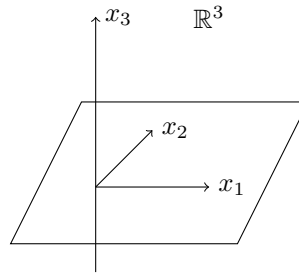
Orthogonal complement of subspace  $M$  of vector space  $V$  contains every vector orthogonal to  $M$ . This subspace is usually denoted by  $M^\perp$

**Remark:**

$$\dim M + \dim M^\perp = \dim V$$

**Example**

Consider  $\mathbb{R}^3$



line  $\alpha$  plane - orthogonal subspace. Orthogonal complement of each other

**Example**

Not orthogonal subspace!

$$\underline{u} \neq 0$$

$$\underline{u} \in I \text{ \& } \underline{u} \in W$$

$$\langle \underline{u} \in U, \underline{u} \in W \rangle = 0$$

ADD FIGURE

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**Note**

If vector  $\underline{u}$  belongs to 2 orthogonal subspaces, this vector is necessarily a zero vector,  $\underline{u} = 0$  because we should have

$$\langle \underline{u}, \underline{u} \rangle = \underline{u}^T \underline{u} = 0 \Rightarrow \underline{u} = \underline{0}$$

## Chapter 5

# Linear Mapping

### Definition

Let us consider 2 vector spaces  $V$  and  $W$ . A function  $\mathcal{L} : V \rightarrow W$  is called a linear mapping, if:

1. For any  $\underline{v} \in V$  and  $\underline{v}' \in V$ ,  $\mathcal{L}(\underline{v} + \underline{v}') = \mathcal{L}(\underline{v}) + \mathcal{L}(\underline{v}')$
2. For any  $\underline{v} \in V$  and any scalar  $\alpha$ ,  $\mathcal{L}(\alpha \underline{v}) = \alpha \cdot \mathcal{L}(\underline{v})$

### Example

Let us consider matrix  $A \in \mathbb{R}^{n,m}$ . We can define linear mapping  $\mathcal{L}_A$  as follows:

$$\mathcal{L}_A(\underline{v}) = A\underline{v} \quad \mathcal{L}_A : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Is  $\mathcal{L}_A$  a linear mapping? Yes!

### Proof

1.  $\forall \underline{v}, \underline{v}' \in \mathbb{R}^m$ , we have:

$$\mathcal{L}_A(\underline{v} + \underline{v}') = A(\underline{v} + \underline{v}') = A\underline{v} + A\underline{v}' = \mathcal{L}_A(\underline{v}) + \mathcal{L}_A(\underline{v}')$$

2.  $\forall \underline{v} \in \mathbb{R}^m, \forall \alpha$  ( $\alpha$  is scalar), we have:

$$\mathcal{L}_A(\alpha \underline{v}) = A(\alpha \underline{v}) = \alpha \cdot A\underline{v} = \alpha \mathcal{L}_A(\underline{v})$$

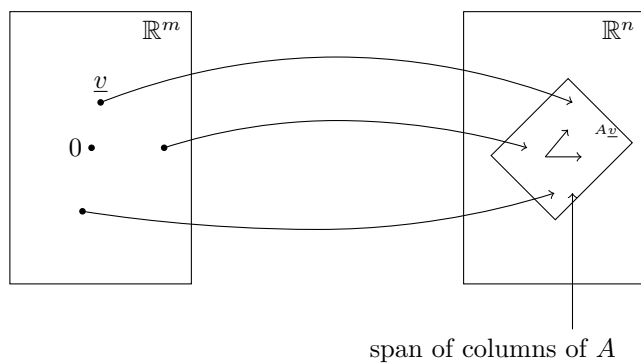
□

Let us consider matrix  $A \in \mathbb{R}^{n,m}$ ,  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let us consider vector  $\underline{v} \in \mathbb{R}^m$

$$A\underline{v} = v_1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} + v_2 \cdot \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix} + \dots + v_m \cdot \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

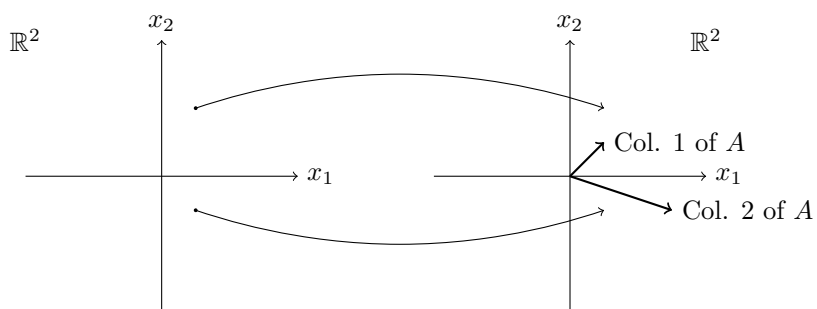
Linear combination of columns of  $A$



**Example**

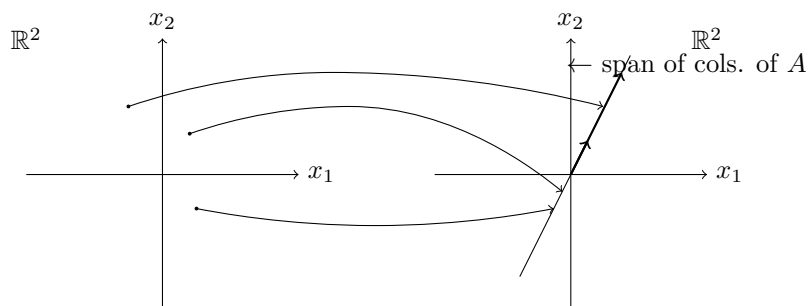
1.

$$A \in \mathbb{R}^{2,2} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$



2.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

**Note**

In order for solution of  $A\underline{x} = \underline{b}$  to exist,  $\underline{b}$  should belong to a span of columns of matrix  $A$ .

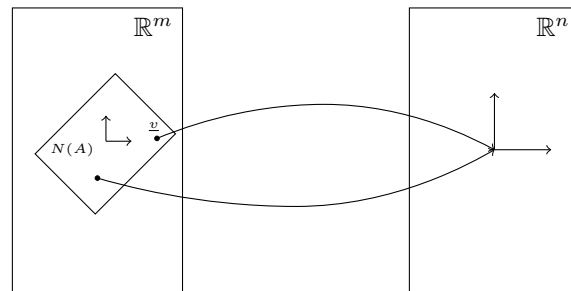
**Definition**

The span of columns of matrix  $A \in \mathbb{R}^{n,m}$  is called a column space of  $A$ , denoted by  $C(A)$ , where  $C(A) \subset \mathbb{R}^n$ .

**Definition**

Let us consider matrix  $A \in \mathbb{R}^{n,m}$ ,  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . The null space of  $A$  is defined as

$$N(A) = \{\underline{v} \in \mathbb{R}^m \mid A\underline{v} = \underline{0}\}, N(A) \subset \mathbb{R}^m$$

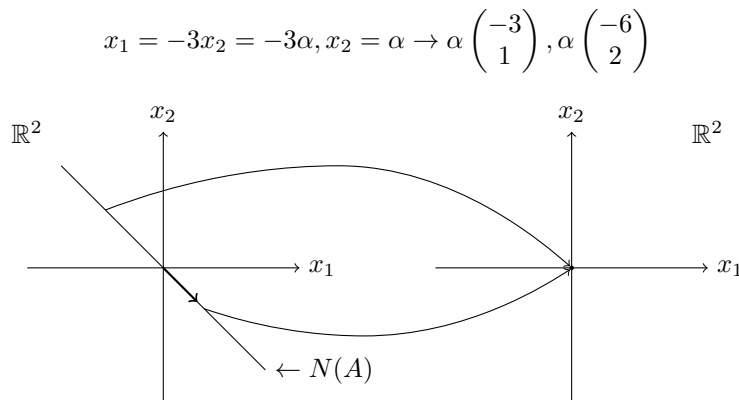
**Example**

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

What is  $N(A)$ ? We should find all solutions of  $A\underline{x} = \underline{0}$ , this will give us  $N(A)$ .

$$\begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 6x_2 = 0 \end{cases} \rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ 0 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -3x_2 \\ 0 = 0 \end{cases}$$

The nullspace of this matrix will be a line formed by a linear combination of the vector  $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$  ( $\alpha \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ , for all possible  $\alpha$ ), or in other words it will be the  $\text{span}(\begin{pmatrix} -3 \\ 1 \end{pmatrix})$ .



**Theorem**

The nullspace,  $N(A)$ , of  $A \in \mathbb{R}^{n,m}$  is a subspace of  $\mathbb{R}^m$ .

**Proof**

Let us assume that  $\underline{x}, \underline{x}' \in N(A)$  and  $\alpha$  is arbitrarily scalar.

1.  $A(\underline{x} + \underline{x}') = A\underline{x} + A\underline{x}' = \underline{0} + \underline{0} = \underline{0} \Rightarrow (\underline{x} + \underline{x}') \in N(A)$
2.  $A(\alpha\underline{x}) = \alpha(A\underline{x}) = \alpha \cdot \underline{0} = \underline{0} \Rightarrow \alpha\underline{x} \in N(A)$

□

**Theorem**

The column space,  $C(A)$ , of  $A \in \mathbb{R}^{n,m}$  is a subspace of  $\mathbb{R}^n$ .

**Definition**

The row space of matrix  $A \in \mathbb{R}^{n,m}$  is a span of rows of  $A$ . Clearly,  $R(A) = C(A^T)$  and  $R(A) \subset \mathbb{R}^m$ .

**Definition**

The left nullspace of  $A$  is defined as  $N(A^T)$ .  $N(A^T) \subset \mathbb{R}^n$ .

**Theorem**

$R(A)$  is a subspace of  $\mathbb{R}^m$

**Proof**

Same as for the proof that  $C(A)$  is a subspace of  $\mathbb{R}^n$ , but for  $A^T$

□

**Theorem**

$N(A^T)$  is a subspace of  $\mathbb{R}^n$

**Proof**

Same as for  $N(A)$  but replace  $A$  with  $A^T$

□

**Theorem**

$R(A)$  and  $N(A)$  are orthogonal subspaces in  $\mathbb{R}^m$  for  $A \in \mathbb{R}^{n,m}$

**Proof**

Let us consider  $\forall \underline{x} \in N(A), A\underline{x} = \underline{0}$

$$A\underline{x} = \begin{pmatrix} - \text{row 1 of } A \rightarrow \\ \vdots \\ - \text{row } n \text{ of } A \rightarrow \end{pmatrix} \cdot \begin{pmatrix} | \\ \underline{x} \\ | \end{pmatrix} = \begin{pmatrix} < \text{row 1 of } A, \underline{x} > \\ \vdots \\ < \text{row } n \text{ of } A, \underline{x} > \end{pmatrix} \stackrel{\underline{x} \in N(A)}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\underline{x}$  is orthogonal to every row of  $A$ .  $\underline{x}$  is orthogonal to every linear combination of rows of  $A$ .  $\underline{x}$  is orthogonal to  $R(A)$ . In fact, what we just showed is that  $N(A)$  &  $R(A)$  are orthogonal complements.

□

**Theorem**

$N(A^T)$  &  $C(A) = R(A^T)$  are orthogonal complements in  $\mathbb{R}^n$

$A \in \mathbb{R}^{n,m} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Row rank of  $A = \text{rank}(A) = \dim(R(A)) = \dim(C(A))$

$$\begin{aligned} N(A) : A\underline{x} &= \underline{0} \quad \forall \underline{x} \in \mathbb{R}^m \\ C(A) : A\underline{v} &= \text{Linear combinations of columns of } A \\ &= v_1 \cdot \text{col 1 of } A + \cdots + v_n \cdot \text{col } n \text{ of } A \in \mathbb{R}^n \end{aligned}$$

**Theorem**

$N(A)$  is an orthogonal complement of  $R(A)$  in  $\mathbb{R}^m$ ,

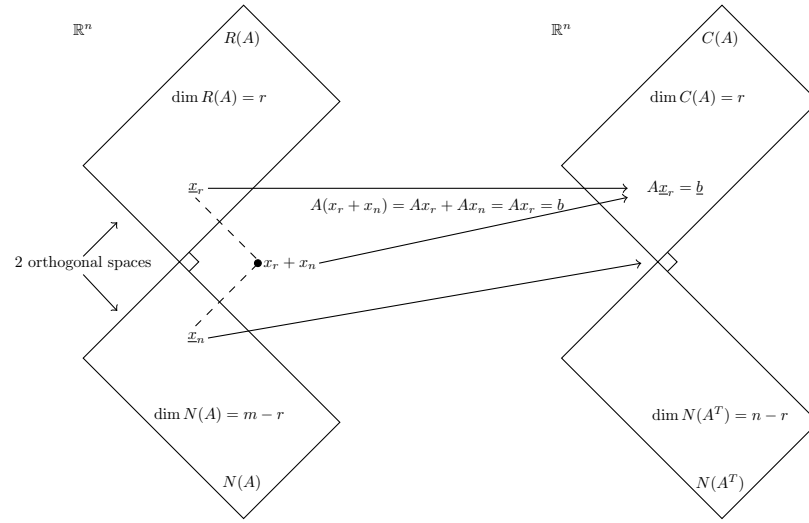
$$\dim N(A) + \underbrace{\dim R(A)}_{=\text{rank}(A)} = m$$

**Theorem**

$N(A^T)$  is an orthogonal complement of  $R(A^T) = C(A)$  in  $\mathbb{R}^n$ ,

$$\dim N(A^T) + \underbrace{\dim C(A)}_{=\text{rank}(A)} = n$$

Let us consider  $A \in \mathbb{R}^{n,m}, A : \mathbb{R}^m \rightarrow \mathbb{R}^n, \text{rank}(A) = r$



### Lemma

For any vector  $\underline{b}$  in  $C(A)$ , there exists one and only one vector  $\underline{x}_r \in R(A)$  such that  $A\underline{x}_r = \underline{b}$

### Proof

Let us assume that  $\underline{x}_r$  and  $\underline{x}'_r$  are in the row space,  $R(A)$ . Let us assume that  $A\underline{x}_r = A\underline{x}'_r$ . We have

$$\underline{x}_r \in R(A) - \underline{x}'_r \in R(A) \in R(A)$$

But we also have

$$A\underline{x}_r - A\underline{x}'_r = A(\underbrace{\underline{x}_r - \underline{x}'_r}_{\in N(A)}) = \underline{0}$$

It means that  $(\underline{x}_r - \underline{x}'_r)$  is in  $R(A)$  and  $N(A)$ , but they are orthogonal subspaces, therefore

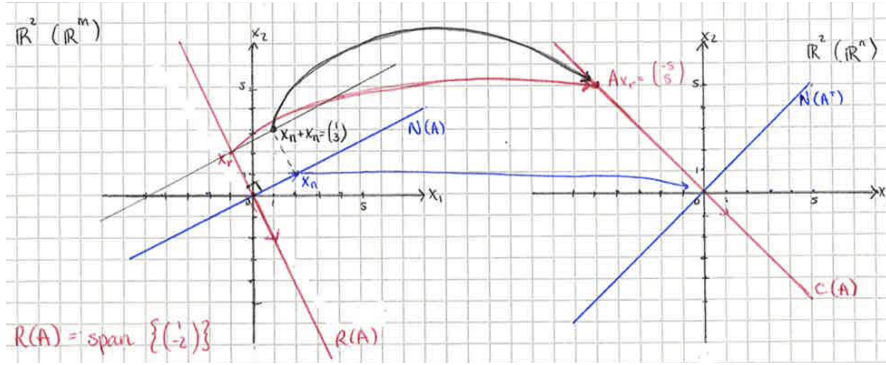
$$\underline{x}_r - \underline{x}'_r = \underline{0} \Rightarrow \underline{x}_r = \underline{x}'_r$$

□

### Example

Let us consider

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \in \mathbb{R}^{2,2}$$



Row space:  $\text{rank } A = 1 \Rightarrow \dim R(A) = 1$

$$R(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

Null space:  $\dim N(A) = 2 - 1 = 1$

$$A\underline{x} = 0 \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow x_1 = 2x_2 \text{ (Line)}$$

Column space:  $\dim C(A) = \dim R(A) = 1$

$$C(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Left Null space:  $\dim N(A^T) = 2 - 1 = 1$ . Consider

$$\begin{aligned} \underline{x}_r = \begin{pmatrix} -1 \\ 2 \end{pmatrix} &\Rightarrow A\underline{x}_r = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \\ \underline{x}_n = \begin{pmatrix} 2 \\ 1 \end{pmatrix} &\Rightarrow A\underline{x}_n = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

## 5.1 Orthogonal Basis and Gram-Schmidt process

### Definition

Vectors  $\underline{q}_1, \dots, \underline{q}_m$  are orthogonal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = 0 \quad \text{if } i \neq j$$

### Definition

Vectors  $\underline{q}_1, \dots, \underline{q}_m$  are orthonormal if:

$$\langle \underline{q}_i, \underline{q}_j \rangle = \underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

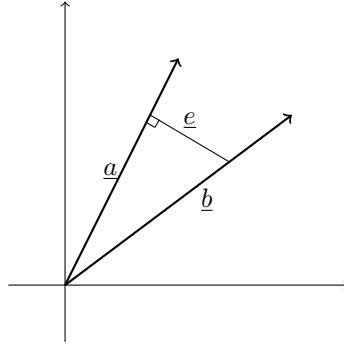
If the columns of the matrix are orthonormal vectors, then this matrix is usually denoted by  $Q$ . In this case, we have  $Q^T Q = I$ . If  $Q$  is not a square matrix then  $Q Q^T$  is not necessarily  $I$ .

**Definition**

A square matrix is called orthogonal (if its columns are orthonormal vectors) if  $Q^T Q = I$ . In this case, since it is a square matrix,  $Q Q^T = I$

### 5.1.1 Projection on the line

Let us assume that we have a line given by vector  $\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$  and vector  $\underline{b} \in \mathbb{R}^n$ . We want to find vector  $\underline{p}$  belonging to the line, closest to vector  $\underline{b}$ . In other words, we are looking for  $\underline{p}$  which is orthogonal projection of  $\underline{b}$  onto the line given by  $\underline{a}$



$\underline{p}$  is proportional to  $\underline{a}$ ,  $\underline{p} = \hat{x}\underline{a}$ , where  $\hat{x}$  is some scalar. Let us define vector  $\underline{e} = \underline{b} - \underline{p} = \underline{b} - \hat{x}\underline{a}$  (error vector).  $\underline{e}$  is orthogonal to the line, therefore

$$\begin{aligned} \langle \underline{a}, \underline{e} \rangle &= 0 \\ \langle \underline{a}, \underline{e} \rangle &= \underline{a}^T (\underline{b} - \hat{x}\underline{a}) = \underline{a}^T \underline{b} - \hat{x} \underline{a}^T \underline{a} = 0 \\ \Rightarrow \hat{x} &= \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \\ \Rightarrow \underline{p} &= \hat{x}\underline{a} = \underline{a} \hat{x} = \underline{a} \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \underbrace{\frac{\underline{a} \underline{a}^T}{\underline{a}^T \underline{a}}}_{P \in \mathbb{R}^{n,n} \text{ (projection matrix)}} \cdot \underline{b} \end{aligned}$$

**Example**

Let us consider  $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \in \mathbb{R}^3$

$$P = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \left\langle \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, (1 \ 2 \ 2) \right\rangle \cdot \frac{1}{9} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Let us take

$$\underline{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underline{p} = P\underline{b} = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}$$

**Note**

$$\underline{p}^2 = \underline{p}$$

**Note**

$(I - P)$ — projection onto subspace orthogonal to the line given by  $\underline{a}$

## 5.2 Gram-Schmidt process

Given linear independent vectors  $\underline{a}, \underline{b}, \underline{c}, \dots$  we first find orthogonal vectors  $\underline{a}', \underline{b}', \underline{c}', \dots$  which span the same subspace as  $\underline{a}, \underline{b}, \underline{c}, \dots$  and then we normalise them,

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|}, \dots$$

So, Gram-Schmidt process allows us to construct an orthogonal basis of  $\text{span}\{\underline{a}, \underline{b}, \underline{c}\}$

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1. Choose  $\underline{a}' = \underline{a}$
2. It is likely that  $\underline{b}$  is not orthogonal to  $\underline{a}'$ , so we need to subtract its projection on the line defined by  $\underline{a}'$

$$\underline{b}' = \underline{b} - \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a}'$$

3.  $\underline{c}'$  is likely not orthogonal to  $\underline{a}'$  and  $\underline{b}'$ . Again, subtract its projections

$$\underline{c}' = \underline{c} - \frac{\underline{a}^T \underline{c}}{\underline{a}^T \underline{a}} \underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'} \underline{b}'$$

and so on. Finally, normalise  $\underline{q}_1, \underline{q}_2, \underline{q}_3, \dots$

**Example**

With

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \underline{c} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

find  $\underline{a}', \underline{b}', \underline{c}', \underline{q}_1, \underline{q}_2, \underline{q}_3$

- 1.

$$\underline{a}' = \underline{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$



2.

$$\underline{b}' = \underline{b} - \frac{\underline{a}'^T \underline{b}}{\underline{a}'^T \underline{a}'} \underline{a}' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

3.

$$\underline{c}' = \underline{c} - \frac{\underline{a}'^T \underline{c}}{\underline{a}'^T \underline{a}'} \underline{a}' - \frac{\underline{b}'^T \underline{c}}{\underline{b}'^T \underline{b}'} \underline{b}' = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\langle \underline{a}', \underline{c}' \rangle = 0, \langle \underline{b}', \underline{c}' \rangle = 0$$

Finally normalise:

$$\underline{q}_1 = \frac{\underline{a}'}{\|\underline{a}'\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \underline{q}_3 = \frac{\underline{c}'}{\|\underline{c}'\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

### 5.3 Projection onto subspace

Assume we have linearly independent vectors  $a_1, \dots, a_m \in \mathbb{R}^n$ . We want to project vector  $\underline{b} \in \mathbb{R}^n$  onto subspace spanned by  $a_1, \dots, a_m$ . Subspace consists of all linear combinations

$$x_1 a_1 + \dots + x_m a_m = \underbrace{\begin{pmatrix} | & & | \\ a_1 & \dots & a_m \\ \downarrow & & \downarrow \end{pmatrix}}_{A \in \mathbb{R}^{n,m}} \cdot \underbrace{\hat{x}}_{\in \mathbb{R}^m}$$

We are looking for the projection  $\underline{p}$  of  $\underline{b}$  onto his subspace. We can define  $\underline{e} = \underline{b} - \underline{p}$ ,  $\underline{e}$  should be orthogonal to all  $a_1, \dots, a_m$

$$\left. \begin{aligned} \langle a_1, \underline{e} \rangle &= \underline{a}_1^T \cdot (\underline{b} - A\hat{x}) = 0 \\ &\vdots \\ \langle a_m, \underline{e} \rangle &= \underline{a}_m^T \cdot (\underline{b} - A\hat{x}) = 0 \end{aligned} \right\} \Rightarrow \underbrace{\begin{pmatrix} -\underline{a}_1^T \rightarrow \\ \vdots \\ -\underline{a}_m^T \rightarrow \end{pmatrix}}_{A^T} (\underline{b} - A\hat{x}) = 0$$

$$A^T(\underline{b} - A\hat{x}) = 0$$

$$A^T \underline{b} - A^T A \hat{x} = 0$$

#### Theorem

$A$  has linearly independent columns. Then  $A^T A$  is:

- Square
- Symmetric

- Invertible

$$\hat{\underline{x}} = (A^T A)^{-1} A^T \underline{b}$$

$$\underline{p} = A\hat{\underline{x}} = \underbrace{A(A^T A)^{-1}}_{P - \text{Proj. matrix}} \cdot \underline{b} - \text{Projection vector}$$