Chapter 1

Vector Spaces

Definition

A vector space V is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar.

If two objects belong to the vector space, then their sum also belongs to the vector space.

If an object belongs to V, then the product of any scalar with this object belongs to V and the following properties are satisfied:

1. $\forall \underline{u}, \underline{v}, \underline{w} \in V$; $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
2. $\forall \underline{u}, \underline{v} \in V$; $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
3. There exists unique elements $\underline{0} \in V$, such that $\forall \underline{u} \in V$; $\underline{u}+\underline{0} = \underline{0}+\underline{u} = \underline{u}$
4. For any $\underline{u} \in V, \exists!(-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$
5. $\forall \underline{u}, \underline{v} \in V$; $\forall \alpha \in \mathbb{R}$; $\alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$
6. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$
7. $\forall \underline{u} \in V$; $\forall \alpha, \beta \in \mathbb{R}$; $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$
8. $\forall \underline{u} \in V$; $1 \cdot \underline{u} = \underline{u}$ (1 is a scalar here)

Remark:

The "vectors" in the vector space, are not necessarily vectors ($\in \mathbb{R}^n$) , but can be other objects, as long as the definition is satisfied.

Example

Let us consider a set of all 2×2 matrices. It is a vector space. Proof:
If $A, B \in \mathbb{R}^{2,2}$ $\quad (A + B) \in \mathbb{R}^{2,2}$
$\alpha \in \mathbb{R}, A \in \mathbb{R}^{2,2}$ $\quad \alpha A \in \mathbb{R}^{2,2}$

1. $A, B, C \in \mathbb{R}^{2,2}$; $(A+B)+C = A+(B+C)$
2. ...
3. $$\underline{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2,2}, \forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A + \underline{0} = A$$
4. $$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow (-A) = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

Example

Let us consider a set consisting of a single object, $\underline{0}$. It is a vector space. Note: There is no vector space, which does not contain $\underline{0}$

1.1 Subspace of the vector space

Definition

A subspace W of the vector space V, is a set of vectors in V, such that:

1. If $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$
2. If $\alpha \in \mathbb{R}$, $\underline{u} \in W$ then $\alpha \underline{u} \in W$

Definition

Let us consider a set of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \dots, \underline{u}_n\}$ is defined as

$$\mathcal{S} = \text{span}\{\underline{u}_1, \dots, \underline{u}_n\} = \{\alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n \mid \forall \alpha_1 \dots \alpha_n \in \mathbb{R}\}$$