find eigenvector for $\lambda = 5$.

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{cases} -4v_1 + 4v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{cases} \Rightarrow \begin{cases} -v_1 + v_2 = 0 \\ 0 = 0 \end{cases}$$

lets say we choose $v_1 = 1 \Rightarrow v_2 = 1$
then the eigenvector is $\underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$

final - same structure.
all topics
$\approx$ 3 hours.

12.5.15

Example: Consider $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$, eigenvalues, eigenvectors?

$$P_A(t) = \det(tI - A) = \begin{vmatrix} t-2 & -1 & 0 \\ 0 & t-1 & 1 \\ 0 & -2 & t-4 \end{vmatrix} = (t-2)^2(t-3)$$

Eigenvalues then are $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 2$

eigenvector for $\lambda_1 = 3$

$$A\underline{x} = \lambda_1 \underline{x} \Rightarrow \begin{cases} 2x_1 + x_2 = 3x_1 \\ x_2 - x_3 = 3x_2 \\ 2x_2 + 4x_3 = 3x_3 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ -2x_2 - x_3 = 0 \\ 2x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ 2x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$

lets choose $x_1 = 1 \Rightarrow x_2 = 1 \quad x_3 = -2 \Rightarrow \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}}$

Eigenvectors for $\lambda_2 = 2$ and $\lambda_3 = 2$

$$A\underline{x} = 2x = \begin{cases} 2x_1 + x_2 = 2x_1 \\ x_2 - x_3 = 2x_2 \\ 2x_2 + 4x_3 = 2x_3 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ -x_2 - x_3 = 0 \\ 2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases}$$

We can choose $x_1 = 1 \Rightarrow$

The <u>only</u> eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

<u>Example</u>: Consider $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, eigenvalues, eigenvectors?

$$P_A(t) = \det(tI - A) = \begin{vmatrix} t-3 & 0 & 0 \\ 0 & t-2 & 0 \\ 0 & 0 & t-2 \end{vmatrix} = (t-2)^2(t-3)$$

$\Rightarrow$ eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 2$.

$\Rightarrow$ eigenvectors for $\lambda_2 = 2$ and $\lambda_3 = 2$?

$$Ax = 2x \Rightarrow \begin{cases} 3x_1 = 2x_1 \\ 2x_2 = 2x_2 \\ 2x_3 = 2x_3 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

We have 3 equations, 1 variable is determined, $(x_1 = 0)$, we have 2 independant variables (x_2, x_3) which we can choose arbitrarily.

$\Rightarrow \underset{\substack{x_2 = 1 \\ x_3 = 0}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{\substack{x_2 = 0 \\ x_3 = 1}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$ – we can get 2 linearly independant eigenvectors.

★ each eigenvalue can have 0 or 1 corresponding eigenvectors.

k - eigenvalues $\longrightarrow$ k corresponding eigenvectors at most.

Change of Basis

Old basis $\underline{b}_1 \dots \underline{b}_n$, new basis $\underline{d}_1 \dots \underline{d}_n$

$$\begin{pmatrix} \underline{d}_1 \\ \vdots \\ \underline{d}_n \end{pmatrix} = S \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \end{pmatrix}$$

If $\underline{v}$ are the coordinates of a vector in old basis $b_1, \dots, b_n$. $v' = (S^T)^{-1}\underline{v}$ are the coordinates of the same vector in new basis. If A is a matrix in the old basis, $A' = (S^T)^{-1} A S^T$ is the same matrix in the new basis.

<u>Theorem</u>: The characteristic polynomial of $(S^T)^{-1} A S^T$ is the same as of A.

<u>Proof</u>: $\det(tI - (S^T)^{-1} A S^T) = \det(t (S^T)^{-1} I S^T - (S^T)^{-1} A S^T)$

$= \det((S^T)^{-1}) \det(tI - A) \det(S^T) = \det(tI - A)$

$\because \det B^{-1} \cdot \det B = 1$, if B^{-1} exists.

It means that the eigenvalues <u>do not</u> change when we change the basis.

Lets assume $A\underline{v} = \lambda \underline{v}$

$$\underbrace{(S^T)^{-1} A S^T}_{A'} \cdot \underbrace{(S^T)^{-1}\underline{v}}_{v'} = (S^T)^{-1} A \underline{v} = (S^T)^{-1} \lambda v = \lambda \underbrace{(S^T)^{-1}\underline{v}}_{v'}$$

$\Rightarrow A'v' = \lambda v'$ - in the new basis.

it means that the eigenvectors of linear mapping do not change, when we change the basis, <u>only</u> coordinates change.

definition: A set of all eigenvalues of matrix $A \in \mathbb{R}^{n,n}$ is called spectrum of A.

Lets consider $A \in \mathbb{R}^{n,n}$. Lets assume that A has $\lambda_1, \dots, \lambda_n$ eigenvalues and linearly independant eigenvectors $\underline{s}_1, \dots, \underline{s}_n$.

If we consider $\underline{b}_1, \dots, \underline{b}_n$ (old basis) to be a standard basis $E_1, \dots E_n$. And $\underline{s}_1, \dots, \underline{s}_n$ as a new basis.

Then, $\begin{pmatrix} \underline{s}_1 \\ \vdots \\ \underline{s}_n \end{pmatrix} = S \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \end{pmatrix}$, $S = \begin{pmatrix} - \underline{s}_1^T \rightarrow \\ \vdots \\ - \underline{s}_n^T \rightarrow \end{pmatrix}$

A in new basis, $A' = (S^T)^{-1} A S^T$

$$A\underline{s}_1 = \lambda_1 \underline{s}_1, \; A\underline{s}_2 = \lambda_2 \underline{s}_2, \dots, A\underline{s}_n = \lambda_n \underline{s}_n$$

$$A \begin{pmatrix} | & & | \\ \underline{s}_1 & \cdots & \underline{s}_n \\ \downarrow & & \downarrow \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{\Lambda \text{ - diagonal matrix.}} \begin{pmatrix} | & & | \\ \underline{s}_1 & \cdots & \underline{s}_n \\ \downarrow & & \downarrow \end{pmatrix}$$

$AS^T = \Lambda S^T$ (multiply by $(S^T)^{-1}$ from the left.)

$$\underbrace{(S^T)^{-1} A S^T}_{A'} = \Lambda$$

If there exist n linearly independant eigenvectors of A, then A can be brought to a diagonal by change of basis.