Midterm: 16th April.

17. 3. 15

Linear Independance

definition: lets consider, vector space V and $\underline{v}_1, \dots \underline{v}_n \in V$

$\underline{v}_1, \dots, \underline{v}_n$ are linearly dependant, if there exists scalars $\alpha_1, \dots \alpha_n$ not all equal to zero, such that $\alpha_1 v_1 + \dots + \alpha_n v_n = \underline{0}$.

If no such scalars exists, the vectors $v_1, \dots, v_n$ are linearly independant.

definition: Vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are linearly independant if the following is true:

if $\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow$ all $\alpha_i = 0 \quad i = 1 \dots n$

Example: Lets consider $\mathbb{R}^n$ and vectors

$$\underline{E}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{E}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{E}_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i, \underline{E}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

$E_1 \dots E_n$ are linearly independant.

Proof: Assume that

$$\alpha_1 \underline{E}_1 + \dots \alpha_n \underline{E}_n = 0 \Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \overset{\text{by assumption}}{=} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ then all $\alpha_i = 0$ for $i = 1 \dots n$, then

based on the definition, $\underline{E}_1, \dots, \underline{E}_n$ are linearly independant.

17.3.15

Example: lets consider $\mathbb{R}^2$, $\underline{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\underline{u}_2 = \begin{pmatrix}3\\1\end{pmatrix}$
- are they linearly independant?

Proof: lets consider $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0} \Rightarrow \alpha_1 \begin{pmatrix}1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\begin{cases}\alpha_1 + 3\alpha_2 = 0\\ \alpha_1 + \alpha_2 = 0\end{cases} \to \begin{cases}2\alpha_2 = 0\\ \alpha_1 + \alpha_2 = 0\end{cases} \to \begin{cases}\alpha_2 = 0\\ \alpha_1 = 0.\end{cases}$$

if we assume this, we have to show that all α_i are zeroes $\Rightarrow$ vectors are linearly independant

Example: lets consider $\mathbb{R}^2$, $\underline{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}2\\2\end{pmatrix}$

lets assume that $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 = \underline{0}$

$$\Rightarrow \alpha_1 \begin{pmatrix}1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}\alpha_1 + 2\alpha_2 = 0\\ \alpha_1 + 2\alpha_2 = 0\end{cases} \quad \begin{cases}\alpha_1 + 2\alpha_2 = 0\\ 0 = 0\end{cases}$$ $\to$ one of the solutions:

$\alpha_1 = -2$

$\alpha_2 = 1$

linearly dependant!

Recap: If we consider vectors $\underline{v}_1, \dots, \underline{v}_n \in V$, then the span $\{\underline{v}_1, \dots, \underline{v}_n\} = \{\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n \mid \text{for all possible } \alpha_1 \cdots \alpha_n \in \mathbb{R}\}$

definition: If vector space V is generated by $\{\underline{v}_1, \dots, \underline{v}_n\}$ (in other words, $V = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$) and $\underline{v}_1, \dots, \underline{v}_n$ are linearly independant, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ is called basis of V

17.3.15

Example: Lets consider $\mathbb{R}^n$ and $\underline{E}_1 \dots \underline{E}_n$. They form basis of $\mathbb{R}^n$.

Proof: ① "V is generated by $v_1 \cdots v_n$"

lets consider any vector $u \in \mathbb{R}^n$,

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \text{we have}$$

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = u_1 E_1 + \dots + u_n \underline{E}_n \rightarrow \mathbb{R}^n = \operatorname{span}\{\underline{E}_1, \dots, \underline{E}_n\}$$

② "linear independance"

already proven before.

Example: lets consider $\mathbb{R}^2$ and $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, is it a basis?

① Is $\mathbb{R}^2 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$?

lets consider arb. vector $\underline{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{R}^2$

we should check that $\exists$ scalars α_1, α_2 such that

$$\underline{V} = \alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2 \rightarrow \underline{V} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 3\alpha_2 = V_1 \\ \alpha_1 + \alpha_2 = V_2 \end{cases} \rightarrow \begin{cases} 2\alpha_2 = V_1 - V_2 \\ \alpha_1 + \alpha_2 = V_2 \end{cases} \rightarrow \begin{cases} \alpha_2 = \dfrac{V_1 - V_2}{2} \\ \alpha_1 = V_2 - \dfrac{V_1 - V_2}{2} = \dfrac{3V_2 - V_1}{2} \end{cases}$$

② $\underline{u}_1, \underline{u}_2$ - linearly independant

↳ we showed it before.

17.3.15

definition: lets consider vector space V and vectors $\underline{v}_1, \cdots, \underline{v}_n$ that form basis of V. If vector $\underline{x} \in V$ can be written as $\underline{x} = x_1 \underline{v}_1 + \cdots + x_n \underline{v}_n$ then $(x_1, \cdots, x_n)$ are called the coordinates of $\underline{x}$ with respect to basis $\{v_1, \cdots, v_n\}$.

Theorem: Lets consider, vector space V and $v_1 \dots v_n$ that are linearly independant.

lets assume that $\underline{x} = \alpha_1 v_1 + \cdots \alpha_n v_n$ and

$\underline{x} = \beta_1 \underline{v}_1 + \cdots + \beta_n \underline{v}_n$, then

$$\alpha_i = \beta_i \quad \forall i = 1 \cdots n.$$

Proof: we have $x = \alpha_1 \underline{v}_1 + \cdots \alpha_n \underline{v}_n = \beta_1 \underline{v}_1 + \cdots + \beta_n \underline{v}_n$

$\rightarrow (\alpha_1 - \beta_1) \underline{v}_1 + \cdots (\alpha_n - \beta_n) \underline{v}_n = \underline{0}$

Since $\underline{v}_1, \dots, \underline{v}_n$ are linearly independant

$\Rightarrow \alpha_i = \beta_i \quad \forall i = 1 \cdots n.$

Remark: The coordinates of any vector $\underline{x}$ with respect to given basis $\{\underline{v}_1, \cdots, \underline{v}_n\}$ are unique.

Theorem: Lets consider vector space V. The number of vectors in any basis of V is always the same.

Proof: we'll skip it.

Remark: The number of vectors in the basis of vector space V is called the dimension of vector space V.

17.3.15

Rank of matrix

definition: The row rank of matrix A is a maximum number of linearly independant rows of matrix A.

definition: The column rank of matrix A is a maximum number of linearly independant columns of matrix A

Remark: For any matrix $A \in \mathbb{R}^{m,n}$, the row rank is equal to the column rank.

(we will prove it later). Therefore, the row rank and column rank are sometimes called rank of matrix A, rank (A).

Example:

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \rightarrow \text{we show that } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ are}$$

linearly independant.

$$\Rightarrow \text{rank}(A) = 2.$$

Example:

$$A = \begin{pmatrix} 1 & 0 \\ 7 & 0 \\ 3 & 0 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{4,2}, \quad \text{rank}(A) = 1.$$