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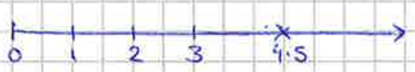
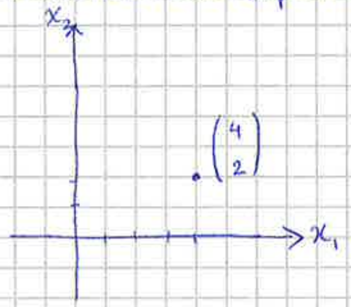
Linear Algebra

Vectors

A real number can be represented by a point on a line, which is a 1-Dimensional space, $\mathbb{R}$.



A pair of real numbers can be represented by a point on a plane, which is a 2-dimensional space, $\mathbb{R}^2$.



A triplet of real numbers can be represented by a point in 3D space, $\mathbb{R}^3$.

Definition: A vector is an ordered collection of n numbers.

Notation: Usually vectors are given by letters $u, v, w, \dots$
In textbooks, vectors are written with BOLD font.
In handwriting, vectors are often written with an arrow on top, $\vec{u}$.

We will underline vetors, $\underline{u}$.

Definition: Let us consider the vector $\underline{u} \in \mathbb{R}^n$. The i^{th} component of vector $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix}$ is u_i

eg: $\underline{u} \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix} \in \mathbb{R}^3$, $u_1 = 3$, $u_2 = 7 \dots$

Definition: Lets consider vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$. Vector $\underline{w} \in \mathbb{R}^n$ is a sum of $\underline{u}$ and $\underline{v}$, $\underline{w} = \underline{u} + \underline{v}$, if $w_i = u_i + v_i$ for ALL $i = 1, \dots n$

eg 1:

$$\underline{u} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-1 \\ 5+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

eg 2: $\underline{u} = \begin{pmatrix} 3 \\ 9 \\ 2 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$

$\underline{u} + \underline{v}$ = not defined!
Both vectors should have the same no. of components
i.e belong to the same space.

Definition: Vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$ are equal, iff $u_i = v_i$ for all $i = 1 \dots n$

Definition: A scalar is just another name for real number.

Definition: Lets consider a scalar $\alpha \in \mathbb{R}$ and vector $\underline{u} \in \mathbb{R}^n$.
A product of α and $\underline{u}$ is defined as

$$\alpha \underline{u} = \alpha \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix}$$

example: $\alpha = 3, \quad \underline{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \\ 7 \end{pmatrix}$

$$\alpha \underline{u} = \begin{pmatrix} 3 \cdot (-1) \\ 3 \times (2) \\ 3 \times (5) \\ 3 \times (7) \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 15 \\ 21 \end{pmatrix}$$

Definition: Lets consider scalars α and β and vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$.
A sum of $\alpha \underline{u} + \beta \underline{v}$ is called a linear combination of vectors $\underline{u}$ and $\underline{v}$.

Example:

$$2\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} + 3\begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 8 \end{pmatrix}$$

eg 2:

$$\underline{u} - \underline{v} = 1 \times \underline{u} + (-1) \times \underline{v} = \begin{pmatrix} u_1 - v_1 \\ \vdots \\ u_n - v_n \end{pmatrix}$$

eg 3: $\underline{u} - \underline{u} = \begin{pmatrix} u_1 - u_1 \\ \vdots \\ u_n - u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

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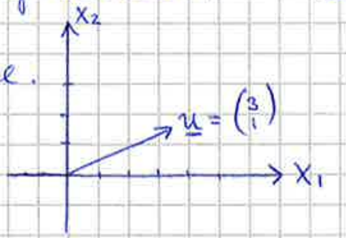
Definition: vector $\underline{u} \in \mathbb{R}^n$ is called a zero vector if all $u_i = 0$, $i = 1 \cdots n$.

zero vector is often written as $\underline{0} \in \mathbb{R}^n$.

Graphic representation of vectors and vector operations.

A vector can be represented in the following ways:

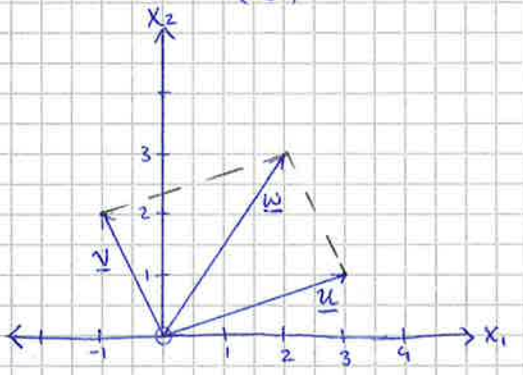
1. an ordered collection of numbers. $\underline{u} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$
2. by an arrow in space.



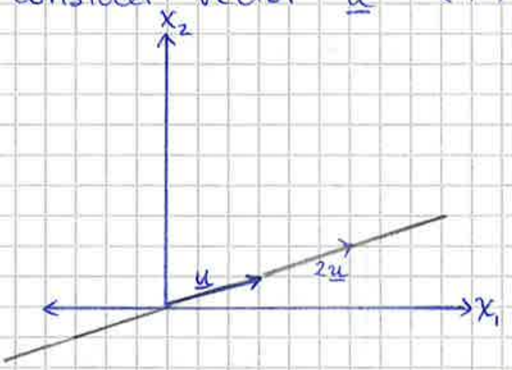
3. as a point in space, the endpoint of vector from the origin.

Lets consider vector $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\underline{w} = \underline{u} + \underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



lets consider vector $\underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



What is $2 \cdot \underline{u} = 2\binom{3}{1} = \binom{6}{2}$

we stretch vector $\underline{u}$ 2 times along the line defined by vector $\underline{u}$

what is $-\underline{u}$? - reverse the direction.

what will be representation of $\alpha \cdot \underline{u}$ for all possible values of α.

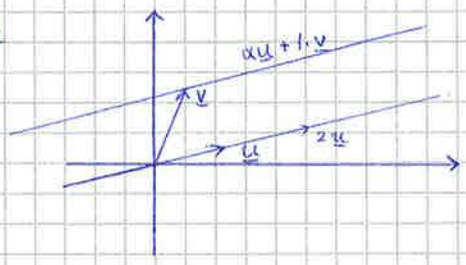
- endless line.

Lets consider 2 vectors $\underline{u} \in \mathbb{R}^2$ and $\underline{v} \in \mathbb{R}^2$.

what will be representation of all linear combinations of $\underline{u}$ and $\underline{v}$,

$\alpha \underline{u} + \beta \underline{v}$

1. plane



2. line - $\underline{u}$ and $\underline{v}$ are on the same line.

Note: Consider, $\underline{u}, \underline{v} \in \mathbb{R}^n$. $\underline{u}$ and $\underline{v}$ are on the same line if there exist scalars α and β such that $\alpha \underline{u} + \beta \underline{v} = \underline{0}$.

when α and $\beta \neq 0$.

3. point - if $\underline{u} = 0$ and $\underline{v} = 0 \Rightarrow \alpha \underline{u} + \beta \underline{v} = \underline{0}$.

Consider $\underline{v}, \underline{u}$. They're on the same line if $\alpha \underline{u} + \beta \underline{v} = \underline{0}$ and $\alpha, \beta \neq 0$

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Dot Product (scalar product)

definition: Lets consider two vectors $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$.
The dot (or scalar) product of vectors $\underline{u}$ & $\underline{v}$ is defined as

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_2 v_2 + \dots u_n v_n$$

$$= \sum_{i=1}^{n} u_i v_i$$

eg1. $\underline{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -1 \end{pmatrix}$

$$\langle \underline{u}, \underline{v} \rangle = 1 \times 0 + (-1) \times (\tfrac{1}{2}) + (3 \times (-1))$$

$$= \underline{\underline{-3.5}}$$

eg2. $\underline{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\langle \underline{u}, \underline{v} \rangle = 0$

Notation: we will use $\langle \underline{u}, \underline{v} \rangle$ to denote the dot product, but sometimes $\underline{u} * \underline{v}$ is used.

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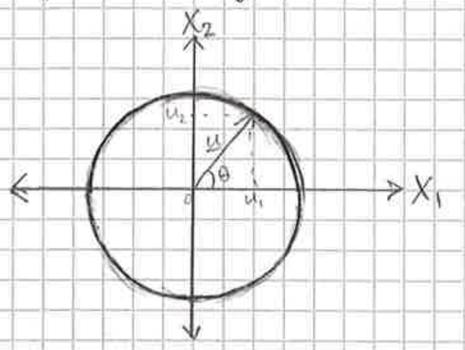
Lets consider $\mathbb{R}^2$. What is the set of all possible endpoints of unit vectors in $\mathbb{R}^2$, originating from the origin.



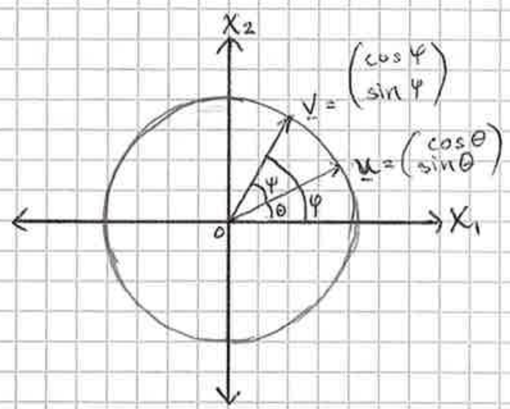
$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\cos\theta = \frac{u_1}{\underbrace{\|\underline{u}\|}_{=1}} = u_1$$

$$\sin\theta = \frac{u_2}{\underbrace{\|\underline{u}\|}_{=1}} = u_2$$

$$\therefore \underline{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Now lets consider 2 unit vectors



lets consider $\langle \underline{u}, \underline{v} \rangle$

$= \cos\theta\cos\varphi + \sin\theta\sin\varphi$

$= \cos(\varphi - \theta) = \cos\psi$

$= \cos(\underbrace{\angle}_{\text{angle between } \underline{u} \text{ and } \underline{v}}(\underline{u}, \underline{v}))$

If $\underline{u} \neq \underline{0}$ or $\underline{v} \neq \underline{0}$ are not unit vectors we can find the angle between them as follows:

$$\langle \underline{u}, \underline{v} \rangle = \left\langle \|\underline{u}\| \times \frac{1}{\|\underline{u}\|} \times \underline{u},\ \|\underline{v}\| \times \frac{1}{\|\underline{v}\|} \times \underline{v} \right\rangle$$

$$= \|\underline{u}\| \|\underline{v}\| \langle \underbrace{\frac{1}{\|\underline{u}\|} \times \underline{u}}_{\text{unit vectors}}, \underbrace{\frac{1}{\|\underline{v}\|} \times \underline{v}}_{\text{unit vectors}} \rangle$$

$$= \|\underline{u}\| \|\underline{v}\| \cos(\angle(\underline{u}, \underline{v}))$$

Lemma: If $\underline{u} \neq \underline{0}$, $\underline{v} \neq \underline{0}$, $\underline{u} \in \mathbb{R}^n$, $\underline{v} \in \mathbb{R}^n$, then

$$\cos(\angle(\underline{u}, \underline{v})) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \times \|\underline{v}\|}$$

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Properties of dot product

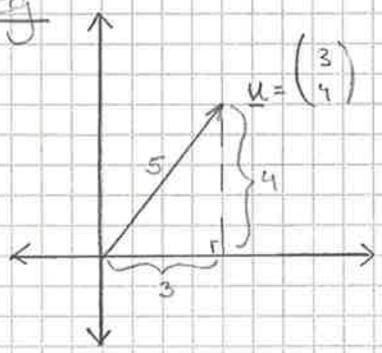
1.) $\langle \alpha \times \underline{u}, \underline{v} \rangle = \alpha \times \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}$, $\underline{u} \in \mathbb{R}^n$, $v \in \mathbb{R}^n$

Proof: $\langle \alpha \times \underline{u}, \underline{v} \rangle = (\alpha \cdot u_1) \times v_1 + (\alpha u_2) \times v_2 + \dots (\alpha u_n) \times v_n$
$= \alpha (u_1 \cdot v_1 + u_2 v_2 + \dots u_n v_2)$
$= \alpha \times \langle \underline{u}, \underline{v} \rangle$

2.) $\langle \underline{u}, \alpha \underline{v} \rangle = \alpha \langle \underline{u}, \underline{v} \rangle$ for any $\alpha \in \mathbb{R}$, $\underline{u}, \underline{v} \in \mathbb{R}^n$

3.) $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle \quad \forall \alpha \in \mathbb{R}, \forall \underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^n$.

eg



Lets consider $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\langle \underline{u}, \underline{u} \rangle = 3 \cdot 3 + 4 \times 4 = 9 + 16 = 25 = 5^2$

length of $\underline{u}$ squared.

definition: the length of vector $\underline{u} \in \mathbb{R}^n$, $\|\underline{u}\|$, is defined as $\|\underline{u}\| = \sqrt{\langle \underline{u}, \underline{u} \rangle}$. Sometimes it is also called the Euclidean norm of $\underline{u}$

definition: A vector with length equal to 1 is called a unit vector. If we take vector $\underline{u} \neq \underline{0}$, how to make it a unit vector?

We should multiply vector $\underline{u}$ by $\frac{1}{\|u\|}$, we will get $\frac{u}{\|\underline{u}\|}$ = unit vector.

In our previous example: $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

unit vector is then $\frac{1}{\|\underline{u}\|} \times \underline{u} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$

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We got $\langle \underline{u}, \underline{v} \rangle = \|\underline{u}\| \|\underline{v}\| \cos(\angle(\underline{u}, \underline{v}))$

lets take the absolute value of this.

$$|\langle \underline{u}, \underline{v} \rangle| = \|\underline{u}\| \cdot \|\underline{v}\| \cdot |\cos(\angle(\underline{u}, \underline{v}))|$$

Notice that $|\cos(\angle(\underline{u}, \underline{v}))| \leq 1$.

$\therefore$ Lemma : Cauchy-Schwartz inequality.

for any $\underline{u} \in \mathbb{R}^n$ and $\underline{v} \in \mathbb{R}^n$

$$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$$

Remark: It's easy to see that Cauchy-Schwartz inequality is correct also for zero vectors.

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Matrices

lets consider a linear combination of vectors.

$$x_1 \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + x_2 \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + x_3 \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

This can be written using matrices in the following way.

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In matrix-vector multiplication, we take dot products of rows of matrix times the vector.

eg 1:

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 2 \\ 1 & -1 & 5 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 0\cdot 0 + (-1)\cdot 1 \\ 3\cdot 1 + 1\cdot 0 + 2\times 1 \\ 1\cdot 1 + (-1)\cdot 0 + 5\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Notation: Matrices are usually written with capital letters i.e i.e $A, B, C, \dots$

A is an n by m matrix, $A \in \mathbb{R}^{n,m}$, if it has n rows and m columns.

The element of of matrix A located in row i and column j is written as a_{ij} or $(A)_{ij}$.

eg 2:

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad \underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A\underline{x} = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1\cdot 1 + 2\cdot 1 + 3\cdot 1 \\ 0\cdot 1 + 1\cdot 1 + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

For the product of matrix A with vector $\underline{x}$ to exist, matrix A should have the same number of columns as vector $\underline{x}$ components.

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Matrix operations

Definition: lets consider matrices $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$ where n = rows, m = columns.

Matrix $C \in \mathbb{R}^{n,m}$ is a sum of A and B, $C = A + B$ if

$$C_{ij} = a_{ij} + b_{ij} \text{ for all } i = 1 \dots n,\ j = 1 \cdots m$$

eg. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$, $C = A + B = \begin{pmatrix} 0 & 2 \\ 3 & 3 \\ 4 & 6 \end{pmatrix}$

Definition: A product of scalar α and a matrix $A \in \mathbb{R}^{n,m}$ is defined as $(\alpha A)_{ij} = \alpha a_{ij}$ $\forall i = 1 \cdots n,\ j = 1 \cdots m$.

Example: $\alpha = 3$, $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 5 \end{pmatrix} \Rightarrow \alpha \cdot A \begin{pmatrix} 0 & 0 & 3 \\ 6 & 9 & 15 \end{pmatrix}$

Properties

* $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{n,m}$

$$A + B = B + A$$

Proof: $(A+B)_{ij} = a_{ij} + b_{ij}$

$(B+A)_{ij} = b_{ij} + a_{ij}$ equal.

* $A, B, C \in \mathbb{R}^{n,m}$.

$$(A+B) + C = A + (B + C)$$

* $\alpha(A + B) = \alpha A + \alpha B$ for $\forall \alpha \in \mathbb{R}$

$A, B \in \mathbb{R}^{n,m}$

Matrix - Matrix multiplication

Definition: lets consider matrix $A \in \mathbb{R}^{n,m}$ and $B \in \mathbb{R}^{m,l}$
then $C = A \cdot B$, is an n by l matrix, $C \in \mathbb{R}^{n,l}$ such that:

$$C_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

eg: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \in \mathbb{R}^{3,2}$, $B = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{2,4}$

$$C = \underset{3\times 2}{A} \cdot \underset{2\times 4}{B} \in \mathbb{R}^{3,4}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \overset{1\cdot 1 + 2\cdot(-1)}{-1} & 4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}$, $\underset{3\times 2}{A}\underset{3\times 3}{B}$ = not Defined.

Properties

* AB is not always equal to BA. (most often, this is the case)

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

* $C(A+B) = CA + CB$
* $(A+B)C = AC + BC$
* $\alpha(AB) = A(\alpha B)$, $A \in \mathbb{R}^{n,m}$, $B \in \mathbb{R}^{m,l}$
  ↳ Proof: $(\alpha(AB))_{ij} = \alpha \sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{k=1}^{m} a_{ik}(\alpha \cdot b_{kj}) = A(\alpha B)$
* $(AB)C = A(BC)$

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<u>Theorem</u>: Lets consider matrices $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,n}$, such that A^{-1} and B^{-1} exist. Then,

$$(AB)^{-1} = B^{-1} \times A^{-1}$$

<u>Proof</u>: $\left.\begin{array}{l}(AB)(B^{-1}A^{-1}) = I \\ (B^{-1}A^{-1})(AB) = I\end{array}\right\}$ Prove this.

$$(AB)(B^{-1}A^{-1}) = A\underbrace{BB^{-1}}_{I}A^{-1} = A \times I \times A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{A^{-1}A}_{I}B = B^{-1}B = I$$

$\Rightarrow$ According to the definition $B^{-1}A^{-1}$ is the inverse of AB.

<u>Lemma</u>: $A, B, C \in \mathbb{R}^{n,n}$, $\exists A^{-1}, \exists B^{-1}, \exists C^{-1}$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

<u>Theorem</u>: Lets consider $A \in \mathbb{R}^{n,n}$. Lets consider that $B \in \mathbb{R}^{n,n}$ and $C \in \mathbb{R}^{n,n}$ are both inverses of A.

Then $B = C$. (The inverse is unique)

<u>Proof</u>: $AB = BA = I$ $\qquad$ $AC = CA = I$

$$\underbrace{BA}_{I} \times C = I \times C. \qquad B \times \underbrace{AC}_{I} = B \times I$$

$$\underline{\underline{C = B}}$$

Linear System of equations

Lets consider the following system of equations.

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ 1x_2 + 2x_3 = 3 \\ 4x_3 = -1 \end{cases}$$

Find x_1, x_2, x_3.

We can write this system ~~of~~ in matrix form.

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \in \mathbb{R}^{3,3}, \quad \underline{x} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \underline{b} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$\Rightarrow A\underline{x} = \underline{b}$$

A is an upper triangualar matrix.

We can use Backward Substitution to find the solution

1.) $x_3 = \frac{-1}{4} = -\frac{1}{4} = \frac{b_3}{a_{33}}$

2.) $x_2 = \frac{3 - 2x_3}{1} = \frac{3 - 2(-\frac{1}{4})}{1} = 3.5 = \frac{b_2 - a_{33}x_3}{a_{22}}$

3. $x_1 = \frac{2 - 4x_3 - 2x_2}{2} = \frac{2 - 4 \times (\frac{1}{4}) - 2 \times 3.5}{2} = -2$

$$= \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$$

In general, If $A \in \mathbb{R}^{n,n}$, upper triangular, with $a_{ii} \neq 0$, $i = 1 \cdots n$.

then the Backwords substitution works as

1) $x_n = \frac{b_n}{a_{nn}}$

2.) $x_{n-1} = \frac{b_{n-1} - a_{n-1\,n}x_n}{a_{n-1\,n-1}}$ $\cdots$ $x_i = \frac{b_i - a_{in}x_n - \cdots - a_{i\,i+1}x_{i+1}}{a_{ii}}$,

$i = n-1 \cdots 1$

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Inverse Matrix

__Definition__: Lets consider a Matrix $A \in \mathbb{R}^{n,n}$ (square matrix).
Matrix $B \in \mathbb{R}^{n,n}$ is called an inverse of A, if

$$A \times B = I \quad \underline{\text{AND}} \quad B \times A = I$$

(Both conditions are vital)

__Notation__: Usually, the inverse of A is written as A^{-1}

__Note__: Not all matrices have inverse!

In most cases, it is quite difficult to find an inverse matrix. But in some cases, the inverse is easy to find.

__Eg__: Consider $A = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots & \\ & & & a_{nn} \end{pmatrix}$, $a_{ii} \neq 0 \quad \forall\, i = 1 \dots n$.

Then, $A^{-1} = \begin{pmatrix} a_{11}^{-1} & & 0 \\ & a_{22}^{-1} & \\ 0 & & \ddots & \\ & & & a_{nn}^{-1} \end{pmatrix}$

$$AA^{-1} = \begin{pmatrix} a_{11} & \cdots & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11}^{-1} & \cdots & \cdots & 0 \\ 0 & a_{22}^{-1} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{nn}^{-1} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & 0 \\ 0 & & \cdots & 1 \end{pmatrix}}}$$

$$\underline{\underline{A^{-1}A = I}}$$

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Special Matrices

* lets consider $A \in \mathbb{R}^{n,m}$ matrix.

A is called zero matrix if all $a_{ij} = 0$ $\quad i = 1 \cdots n$, $j = 1 \cdots n$

* $D \in \mathbb{R}^{n,n}$ - square matrix is called diagonal matrix.

if $d_{ij} = 0$ if $i \neq j$

$$\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

* Identity matrix, $I \in \mathbb{R}^{n,n}$, $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

* $L \in \mathbb{R}^{n,n}$ - lower triangular matrix, if

$l_{ij} = 0 \quad \forall i < j$

$$\begin{pmatrix} * & 0 & 0 \\ & * & 0 \\ * & & * \end{pmatrix}$$

* $U \in \mathbb{R}^{n,n}$ - upper triangular matrix if

$u_{ij} = 0 \quad \forall i > j$

$$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

26. 2. 15

Remark: If $A, B \in \mathbb{R}^{n,n}$ are both upper (lower) triangular matrices, then $C = A \times B$ is an upper triangular.
(lower).

If $A \in$ is lower triangular, $A \in \mathbb{R}^{n,n}$, $a_{ii} \neq 0$, $i = 1 \dots n$

then we can use <u>forward substitution</u>:

1.) $x_1 = \dfrac{b_1}{a_{11}}$

$\vdots$

$$x_i = \frac{b_i - a_{i1}x_1 - \dots - a_{ii-1}x_{i-1}}{a_{ii}} \qquad i = 2 \dots n$$

<u>Remark</u>: Computation complexity of Backword and forward substitution is $O(n^2)$.

<u>Exercise XXX</u>:

1.) Define something.

2. Compute something.

<u>3. . . 15</u> <u>Elementary Transition Matrices</u>

Lets consider matrix $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix}$

and matrix $I_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

☆ $$I_{21} A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What happened?

$$A I_{21} = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 7 \\ 2 & -1 & 0 & 0 \\ -1 & 3 & 5 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$$

Definition: We ~~cane~~ can define elementary transition matrix $I_{pq} \in \mathbb{R}^{n,n}$,

$$\left(I_{pq}\right)_{ij} = \begin{cases} 1 & i=p,\ q=j \\ 0, & \text{otherwise} \end{cases}$$

If we take a matrix $A \in \mathbb{R}^{n,n}$ then,

$I_{pq} A$ - we take row q of A, put it into row p, replace everything else with 0.

We can also define:

$E_{pq}(l) = I + l \cdot I_{pq}$, $l \in \mathbb{R}$ - scalar.

$$E_{pq}(l) \cdot A = \left(I + l I_{pq}\right) A = A + l \times I_{pq} A.$$

We take row q of A, multiply it by l, add it to row p of A.

$$E_{pq}^{-1}(l) = E_{pq}(-l)$$

If we have vector $\underline{b} \in \mathbb{R}^n$, then $I_{pq}\underline{b}$ - we take component q of $\underline{b}$, put it into component p, replace everything else with zeros.

$E_{pq}(l)\underline{b}$ - same as for matrices.

3.3.15

Gaussian Elimination

Example:

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}, \quad A\underline{x} = b$$

we can write this as a system of equations:

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases}$$

we can multiply equation 1 by $-2 = \frac{4}{2} = -\frac{a_{21}}{a_{11}}$ and add to equation 2. This is equivalent to multiplying $A\underline{x} = \underline{b}$ by $E_{21}\left(\frac{-a_{21}}{a_{11}}\right)$ on the left.

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ \quad x_2 + x_3 = 4 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \leftrightarrow E_{21}\left(\frac{-a_{21}}{a_{11}}\right) \times A\underline{x} = E_{21}\left(\frac{-a_{21}}{a_{11}}\right)\underline{b},$$

$$E_{21}\left(\frac{-a_{21}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ \quad x_2 + x_3 = 4 \\ \quad x_2 + 5x_3 = 12 \end{cases} \leftrightarrow \overbrace{E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) A}^{A^{(1)}}\underline{x} = E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right)\underline{b},$$

$$E_{31}\left(\frac{-a_{31}}{a_{11}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We are done with the first column.

Lets denote the resulting matrix by $A^{(1)}$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ x_2 + x_3 = 4 \\ 4x_3 = 8 \end{cases} \longleftrightarrow \overbrace{E_{32}\left(\frac{-a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) A}^{A^{(2)}} \underbrace{x}_{b}$$

$$E_{32}\left(\frac{-a_{32}^{(1)}}{a_{22}^{(1)}}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

We are done with the second column, so we can denote the resulting matrix by $A^{(2)}$

In fact, we got an upper triangular matrix. We can solve it using backward compatibility.

Lets denote $E_{32}\left(\frac{-a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(-\frac{a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) = U$ upper triangular matrix.

What is $\left[E_{32}\left(\frac{-a_{32}^{(1)}}{a_{22}^{(1)}}\right) E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(-\frac{a_{21}}{a_{11}}\right)\right]^{-1}$

$$= E_{21}\left(\frac{a_{21}}{a_{11}}\right) E_{31}\left(\frac{a_{31}}{a_{11}}\right) E_{32}\left(\frac{a_{32}^{(1)}}{a_{22}^{(1)}}\right)$$

$$A = \underbrace{E_{21}\left(\frac{a_{21}}{a_{11}}\right) E_{31}\left(\frac{a_{31}}{a_{11}}\right) E_{21}\left(\frac{a_{21}}{a_{11}}\right)}_{L} U$$

All matrices $E_{xx}(x)$ are lower triangular $\rightarrow$ the product is also lower triangular.

$$A = L \times U$$

So using Gaussian elimination, we represented A as a product of lower & upper triangular matrices.

$$A\underline{x} = \underline{b} \rightarrow LU\underline{x} = \underline{b}$$

Lets denote $U\underline{x}$ by y then we get

$$\begin{cases} Ly = \underline{b} & \text{- solve by forward substitution, find } y. \\ U\underline{x} = y & \text{- solve by backward substitution.} \end{cases}$$

<u>Remark:</u> Gaussian elimination works if all elements $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \dots a_{ii}^{(i-1)}$ are non-zero!

These elements are called - <u>PIVOT</u> elements.

<u>5. 3. 15</u>

<u>Example:</u>

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 8x_2 - 3x_3 = 6 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \leftrightarrow A\underline{x} = b$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ \qquad\qquad x_3 = 2 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{cases} \longleftrightarrow E_{21}\left(\frac{-a_{21}}{a_{11}}\right) A\underline{x} = E_{21}\left(\frac{-a_{21}}{a_{11}}\right)$$

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ \qquad\qquad x_3 = 2 \\ \qquad x_2 + 5x_3 = 12 \end{cases} \Leftrightarrow E_{31}\left(\frac{-a_{31}}{a_{11}}\right) A\underline{x} = E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) \underline{b},$$

Denote resulting matrix by $A^{(1)}$.

In order to proceed we need $a_{22}^{(1)} \neq 0$.
Lets consider matrix P_{pq} - matrix, which you get from identity matrix by exchanging rows p and q.
It is easy to show that $P_{pq}A$ is equal to matrix A with rows p and q exchanged.

Definition: Permutation matrix P is an identity matrix with rows in any order.

Remark: $P^{-1} = P$.

The product of permutation matrices is a permutation matrix.

We want to exchange rows 2 and 3. We need to multiply by permutation matrix P_{23}

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = 2 \\ \qquad x_2 + 5x_3 = 12 \\ \qquad\qquad x_3 = 2 \end{cases} \longleftrightarrow P_{23}\, E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) A\underline{x} = P_{23}\, E_{31}\left(\frac{-a_{31}}{a_{11}}\right) E_{21}\left(\frac{-a_{21}}{a_{11}}\right) \underline{b}$$

In general, the gaussian elimination proceeds like this:

Pivot element is zero

$$E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} P_{xx} E_{xx} \cdots E_{xx} A\underline{x} = E_{xx} \dots E_{xx} P_{xx} E_{xx} \dots E_{xx} P_{xx} E_{xx} \cdots E_{xx} \underline{b}$$

Turns out, that we can exchange the rows, or in other words multiply A by $(P_{xx} \dots P_{xx})$ before doing the gaussian elimination.

$$\underbrace{\underbrace{(E_{xx} \cdots E_{xx})}_{E}\underbrace{(P_{xx} \dots P_{xx})}_{P} A}_{U}\,\underline{x} = (E_{xx} \dots E_{xx})(P_{xx} \dots P_{xx})\,\underline{b}$$

$$EPA = U$$

$$PA = \underset{\substack{\uparrow \\ \text{lower} \\ \text{triangular.}}}{E^{-1}} U = LU$$

<u>Theorem:</u> There exists permutation matrix P, such that $PA = LU$. The only necessary condition for that is that A^{-1} exists.

Proof: no proof.

10. 3. 15

Matrix Transposition

definition := Lets consider matrix $A \in \mathbb{R}^{m,n}$. Matrix $B \in \mathbb{R}^{n,m}$ is called transpose of A if $(B)_{ij} = (\ )_{ji}$, $i = 1 \cdots n$, $j = 1 \cdots m$

Notation: usually transpose of A is written as A^T.

Example:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 9 & 10 \end{pmatrix} \in \mathbb{R}^{4,2} \quad ; \quad A^T = \begin{pmatrix} 2 & 4 & 6 & 9 \\ 3 & 5 & 7 & 10 \end{pmatrix}$$

Properties:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(A^T)^{-1} = (A^{-1})^T$ assum: $A \ \mathbb{R}^{n,n}$, $\exists A^{-1}$

Proof that $(AB)^T = B^T A^T$

$$A \in \mathbb{R}^{m,n} = \begin{pmatrix} -\text{row } 1 \rightarrow \\ \vdots \\ -\text{row } m \rightarrow \end{pmatrix}, \quad B \in \mathbb{R}^{n,l} = \begin{pmatrix} \text{col } 1 \downarrow & \cdots & \text{col } l \downarrow \end{pmatrix}$$

$$(AB)_{ij} = \langle \text{row } i \text{ of } A, \text{ column } j \text{ of } B \rangle$$

$$((AB)^T)_{pq} = (AB)_{qp} = \langle \text{row } q \text{ of } A, \text{ col } p \text{ of } B \rangle$$

equal!

$$B^T = \begin{pmatrix} -\text{col } 1 \rightarrow \\ \vdots \\ -\text{col } l \rightarrow \end{pmatrix}, \quad A^T = \begin{pmatrix} \text{row } 1 \downarrow & & \text{row } m \downarrow \end{pmatrix}$$

$$(B^T A^T)_{pq} = \langle \text{col } p \text{ of } B, \text{ row } q \text{ of } A \rangle$$

$$\Rightarrow ((AB)^T)_{pq} = (B^T A^T)_{pq} \quad p = 1 \cdots l, \ q = 1 \cdots m.$$

$$\Rightarrow (AB)^T = B^T A^T$$

Proof that $(A^{T})^{-1} = (A^{-1})^{T}$ - assume that $A \in \mathbb{R}^{n,n}$, $\exists A^{-1}$

$$AA^{-1} = I \rightarrow (AA^{-1})^{T} = (A^{-1})^{T} \times A^{T} = I^{T} = I$$ by def of inverse.

$$A^{-1}A = I \rightarrow (A^{-1}A)^{T} = A^{T} \times (A^{-1})^{T} = I^{T} = I.$$

$$\underline{\underline{(A^{T})^{-1} = (A^{-1})^{T}}}$$

Lets consider vector $\underline{u} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^{n,1}$ - column vector.

Then $\underline{u}^{T} \in \mathbb{R}^{1,n} = (u_1 \cdots u_n)$ - row vector.

lets also consider $\underline{v} \in \mathbb{R}^{n,1} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$$\overset{1,n}{\underline{u}^{T}} \times \overset{n,1}{\underline{v}} = (u_1 \cdots u_n) \times \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 \cdots u_nv_n = \langle \underline{u}, \underline{v} \rangle$$

$$\overset{n,1}{\underline{v}} \times \overset{1 \times n}{\underline{u}^{T}} = n \times n \text{ matrix.}$$

definition: Matrix A is called <u>symmetric</u> if $A^{T} = A$.

Matrix A should be a square matrix, $A \in \mathbb{R}^{n,n}$.

eg. $A = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \rightarrow A^{T} = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow A^{T} = A.$

eg. $A = I \in \mathbb{R}^{n,n} \rightarrow I^{T} = I.$

Vector Spaces

10.3.15

Definition: a vector space V is a set of objects, such that any two objects can be added together, any object can be multiplied by a scalar,

If two objects belong to the vector space, then their sum also belongs to the vector space.

If object belongs to V, then the product of any scalar with this object belongs to V and the following properties are satisfied:

① $\forall \underline{u}, \underline{v}, \underline{w} \in V$

$$(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$$

② $\forall \underline{u}, \underline{v} \in V$

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

③ There exists unique element $\underline{0} \in V$, such that $\forall \underline{u} \in V$

$$\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$$

④ For any $\underline{u} \in V$, $\exists! (-\underline{u}) \in V$, such that $\underline{u} + (-\underline{u}) = \underline{0}$.

⑤ $\forall \underline{u}, \underline{v} \in V$, $\forall \alpha \in \mathbb{R}$.

$$\alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$$

⑥ $\forall \underline{u} \in V$, $\forall \alpha, \beta \in \mathbb{R}$

$$(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}.$$

⑦ $\forall \underline{u} \in V$, $\forall \alpha, \beta \in \mathbb{R}$

$$(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$$

⑧ $\forall \underline{u} \in V$

$1 \times \underline{u} = \underline{u}$ (1 is a scalar here).

Remark: The "vectors" in the vector space, are not necessarily vectors ($\in \mathbb{R}^n$), but can be other objects, as long as the definition is satisfied

Example: lets consider a set of ALL 2×2 matrices. It is a vector space.

Proof: If $A, B \in \mathbb{R}^{2,2}$ $\quad (A+B) \in \mathbb{R}^{2,2}$.

$\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{22}$ $\quad \alpha A \in \mathbb{R}^{2,2}$.

① $A, B, C \in \mathbb{R}^{2,2}$

$(A+B)+C = A+(B+C)$

② ...

③ $\underline{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2,2}$, $\forall A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$A + \underline{O} = \underline{A}$.

④ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\Rightarrow (-A) = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$

⋮

⑧

Example: Lets consider a set consisting of a single object, $\underline{O}$. It is a vector space.

Note: There is no vector space, which does not contain $\underline{O}$.

Subspace of the vector space.

definition: A subspace W of the vector space V, is a set of vectors in V, such that: ① if $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} \in W$

② if $\alpha \in \mathbb{R}$, $\underline{u} \in W$ then $\alpha \underline{u} \in W$

definition: lets consider a set of vectors $\{\underline{u}_1, \ldots, \underline{u}_n\}$. The span of vectors $\{\underline{u}_1, \ldots, \underline{u}_n\}$ is defined as:

$$S = \text{span}\{\underline{u}_1 \quad \underline{u}_n\} = \left\{ \alpha_1 \underline{u}_1 + \cdots \alpha_n u_n \;\middle|\; \begin{array}{l} \text{for all possible values} \\ \text{of } \alpha_1 \cdots \alpha_n \in \mathbb{R} \end{array} \right\}$$

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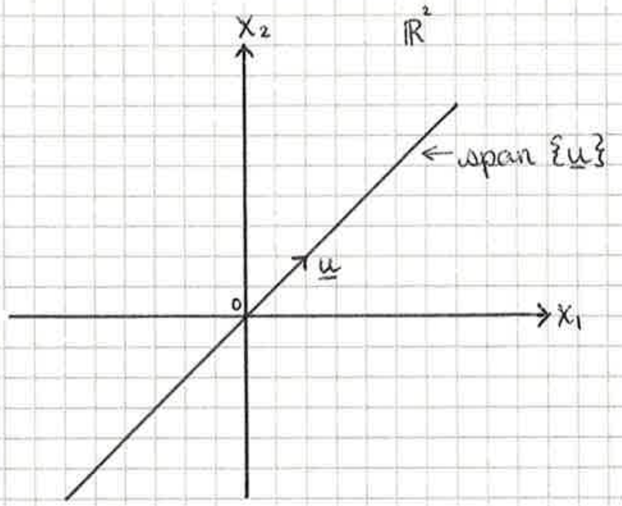
Example:



If span of

Is span {$\underline{u}$} a subspace in $\mathbb{R}^2$?

Proof: $\underline{v} = \alpha\ \underline{u} \in \text{span}\{\underline{u}\}$

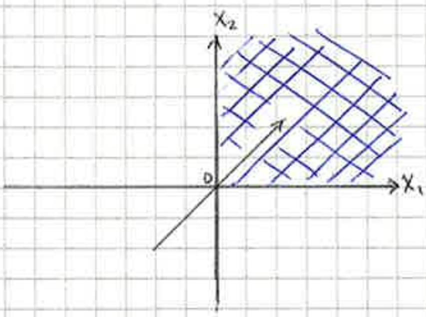
$\underline{w} = \beta\ \underline{u} \in \text{span}\{\underline{u}\}$

① $\underline{v} + \underline{w} = \alpha\underline{u} + \beta\underline{u} = (\alpha+\beta)\underline{u} \in \text{Span}\{\underline{u}\}$

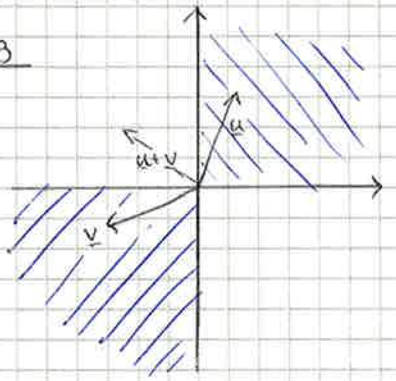
② $\gamma \in \mathbb{R}$

$\gamma \times \underline{v} = \gamma \times (\alpha\underline{u}) = (\gamma \times \alpha)\underline{u} \in \text{span}\{\underline{u}\}$

eg 2



eg 3



eg 4: $\mathbb{R}^2$ - it is a subspace of itself.