# Fourier Analysis of Aperture Functions to Model Interference and Spatial Filtering in a 4f-Optical Setup

David Caro and Harel Cohen
L2 Laboratory Skills and Electronics, Fourier optics-Ph218 (Friday)

Fourier analysis is the method of deconstructing a waveform into a continuous intensity of frequencies in the Fourier-frequency space. Due to the properties of a lens, it performs an idealised Fourier transform on an aperture function at a position one focal length away; this feature can be studied and modelled, as well as provide insight into the modification of images and improvement in the resolution of lenses. In this paper the properties of the mathematics of Fourier transforms are discussed and modelled to experimental data, as well as demonstrations of spatial filtering and the effects of the Fourier transform of the pupil function of a lens known as the point-spread function.

## 1. Introduction

Since the inception of wave optics through the pioneering work of Thomas Young and Augustin-Jean Fresnel in the 18th and early 19th century [1] it has provided useful theory in describing the propagation of light and it's properties of diffraction, interference and polarisation. These effects were modelled as the superposition of curved waves described through the Huygens-Fresnel principle, which describes each point on a wavefront as a wavelet that acts as a new point source. The work of Fresnel was incapsulated in his Fresnel diffraction integral [2], which described the field incident on an aperture and integrates that field over the surface to find the transmitted field at a point downstream. However, this was a difficult analytic problem to solve, until simplifications were made through the work of Fraunhofer. At the focal point of a lens, it imprints a phase onto the field of $e^{-ik\frac{x'^2+y'^2}{2f}}$ [2], which is similarly replicated in the far field of an aperture where $\frac{x'^2}{2z}=\frac{y'^2}{2z}\to 0$. This method however directly cancels with the variables in the Fresnel integral to result in,

$$E^{(f)}=\frac{E_0 e^{ik\left(f+\frac{x^2+y^2}{2f}\right)}}{i\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')e^{-ik\frac{xx'+yy'}{2f}}dx'dy'$$

where $E_0$ is field intensity, $\lambda$ is wavelength, $f$ is the coordinate of the focal point, and $x'\,\&\,y'$ are the coordinates in the object plane $(z=0)$ and $x\,\&\,y$ the coordinates in the image plane $(z=z)$. The aperture function, $f(x',y')$, is a description of the fraction of the field that is propagated through the aperture plane at each point in space. It is immediately clear that the integral is of the form of a Fourier transform, where the aperture function is transformed into the new variables, $u=\frac{x}{\lambda f}$, $v=\frac{y}{\lambda f}$, referred to as the spatial frequencies. Spatial frequency is the number of waves per unit length, the density, determined by the angle to the optical axis of the planar wave at the aperture [3]; so the Fourier space conjugate to frequency for a transformation from position in real space. The Fourier transform describes a superposition of plane wavs propagating at different angles, with different spatial frequencies that form the angular spectrum of the light. In this paper this is the mathematical basis used in Fourier optics and uses the properties of the transforms to derive the field in a focal plane.

The Fourier transform of the aperture function,

$$F(u)=\mathcal{F}[f(x')](u)=\int_{-\infty}^{\infty} f(x')e^{-i2\pi ux'}dx'$$

produces the function of the spatial frequencies in the aperture plane. The field $E^{(0)}=E_0 f(x')$ must be propagated downstream in Fourier space through the hedgehog equation [2], to form the field at $z=f$, written in Fourier optics as

$$E^{(f)}=\frac{E_0 e^{ikr}}{i\lambda f}\mathcal{F}[f(x')](u)$$

which is proportional to the Fourier transform of $f(x')$ hence the intensity function is,

$$I^{(f)}=\left|E^{(f)}\right|^2=\frac{I_0}{\lambda^2 f^2}\left|\mathcal{F}[f(x')](u)\right|^2$$

This effective result means that Fourier transforms of the aperture function can be used to derive the field and hence intensity at the focal plane of the lens and at far field positions. Properties of Fourier transforms immediately inform us of how to model intensity patterns, such as the central coordinate theorem, linearity, translation scaling and convolution. Importantly translation, linearity and convolution theorem respectively are described as

$$\mathcal{F}[f(x-d)](u)=F(u)e^{-i2\pi ud}$$

$$\mathcal{F}[g(x)+h(x)](u)=G(u)+H(u)$$

$$\mathcal{F}[g*h](u)=G(u)*H(u)$$

Analogous to the aperture function is the pupil function $P(x,y)$ [4], which describes the transmission of the field through a lens, hence the field incident on the lens is

$$f_i(x,y)=f(x,y)P(x,y)$$

The resulting field at the focal point downstream is therefore the convolution of the functions.

$$E^{(f)}=\frac{E_0 e^{ikr}}{i\lambda f}\left(\mathcal{F}[f(x,y)]*\mathcal{F}[P(x,y)]\right)(u,v)$$

Where $psf(u,v)=\mathcal{F}[P(x,y)]$, known as the point-spread function of the lens. For standard lens this is as the $\mathrm{jinc(u,v)}$ function [4], which causes the distortion seen in images, and the size of which determines the resolution of the lens given by,

$$R=1.22\frac{\lambda}{D}$$

Once in the Fourier space the fields may be modified in a technique known as spatial filtering. The process whereby apertures are used in the Fourier plane to block certain parts of the Fourier transforms so their images are not propagated through the optical set up. A common type of filters are known as high and low pass filters that remove spatial frequencies from the Fourier function, therefore the image produced consist of only the spatial frequencies that made up certain parts of the aperture function. This consists of the body of the aperture in low pass filters and the edges in high pass filters [3].

The properties and known transform pairs of Fourier transforms provides intuition on the shape of diffraction patterns, the image from combining apertures in the object plane and effective apertures used for spatial filtering.

## 2. Method

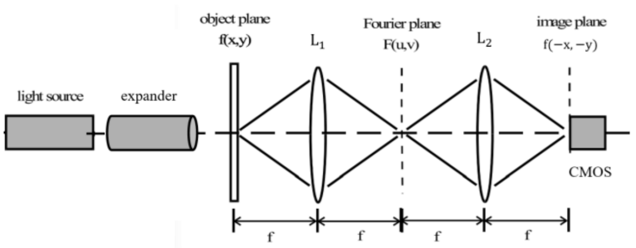


Fig. 1. The 4f-spatial filtering experiment setup. The light source used is a multivariable diode laser set to green light, the CMOS imaging sensor is a high-resolution camera from THOR labs. In practice mirrors must be used at one end due to the width of the workbench

To set up the experiment the focal length of the lens must be measured. With the lens placed in the object plane, the CMOS is held on a stand against a ruler to remain parallel to the system, in the approximate point of the focal length. A filter is used on the lens and exposure reduced so that only a bright spot is visible with no diffraction or optical effects to be seen. The CMOS is moved against the ruler until the point is most concentrated and the distance measured. The lens is replaced with the aperture stand and the rest of the apparatus is set up as seen in Fig.1. The object plane must remain one focal length before the first lens to ensure the field after the Fourier plane is symmetric to the field before the plane, the same reasoning is used for the second lens being one focal length from the Fourier plane.

Image measurements can now be made. The first part of the experiment is to capture the diffraction patterns of apertures. A double slit aperture is placed in the object plane, and the CMOS in the Fourier plane to image the intensity. Importantly when capturing he images, they must be made without saturating the camera, so the exposure is modified, or a lens used. The diffraction patterns are captured of both a diffraction grating, mesh grating (two perpendicular diffraction gratings of the same ruling density) and an arbitrary letter. Returning the CMOS to the image plane, a mesh grating is placed in the object plane with a vertical slit placed in the Fourier plane, and an image is taken in the image plane of the resulting affect. Different masks of: a horizontal slit, a diagonal slit, a cross and pin hole; are placed in the Fourier plane, with images captured. The two gratings are replaced by the letter, and images are taken using a high pass (an opaque circle) and low pass (a hole) filter. Then a combination of a diffraction grating and letter is placed in the object plane and first a pin hole is used in the Fourier plane to filter out the grating and then a horizontal series of appropriately spaced holes is placed to filter out the letter. The grating is then removed from the object plane and placed in the fourier plane and again an image is taken, this concludes the fourier and spatial filtering aspect of the experiment.

Looking into the point spread function and the minimisation of this effect, the set-up is reduced to a 2f set up. Here there is only a single lens and the CMOS is placed in the fourier plane. Any apertures in the object plane are removed and an iris aperture of a set diameter is placed immediately before the lens, with the resulting point spread function imaged without saturation. Two sets of forward and backward apodization filters are

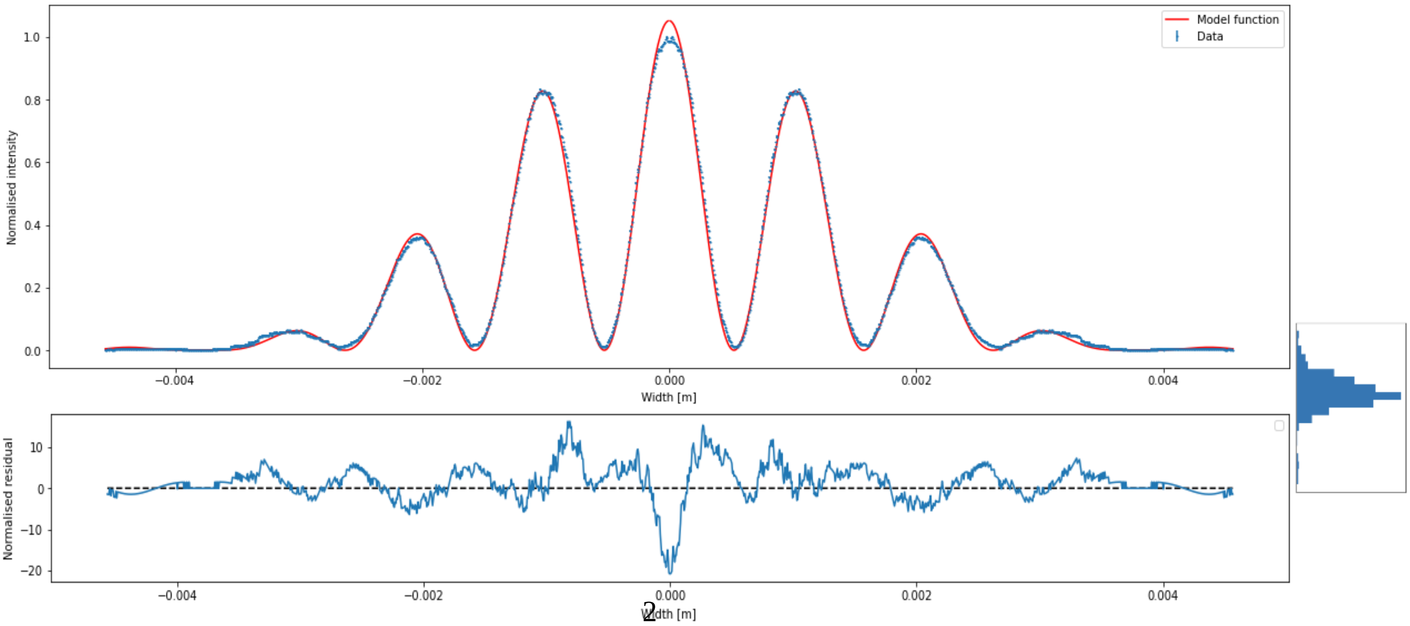




Fig. 2. (Top) The plot of the cross section of the double slit normalised intensity pattern, modelled against the normalised intensity pattern of a double slit aperture. (Bottom, left) the plot of the residuals between the model and the data, (right) a histogram of the position of the residuals.

placed on the CMOS and the effect on the airy function is measured. Error measurement of the apparatus are noted of the ruler, travelling microscope and CMOS specifications. The travelling microscope being used for measurements of the slit width and separation of the double slit and diffraction grating as well as size of iris.

## 3. Results

| Variable | Value | Error |
|---|---|---|
| Slit Width(a) [μm] | 157.00 | 0.05 |
| Slit seperation(d) [μm] | 579.00 | 0.04 |
| Intensity(Io) [Wm^-2] | 6.52 | 0.003 |
| reduced chi^2 | 21.00 | N/A |

Tab. 1 the optimised parameter values of the intensity function for a finite width double slit pattern to the data collected from the CMOS

The intesntiy from the double slit pattern was modeled with the exact equation of it's fourier transform described by

$$I^z = \frac{4I_0}{\lambda f} a^2 \mathrm{sinc}^2(\pi a u) \cos^2(\pi d u)$$

The model parameters from scipy optimization are shown in Tab. 1. The true values of the double slit used were measured via a travelling microscope and shown in Tab. 2. The parameters disagree with the true values outside the error range. The residual fit is normalised through the uncertainly of the intensity given by th CMOS position and slit width, shown in. But as can be seen from Fig.2, the model function closely matches the intensity while overestimating the peak intensity. The intensity is seen to be overestimated in the centre, likely due to error from imperfections in capturing the images and errors from the focal length used in the set up and rotation of the image.

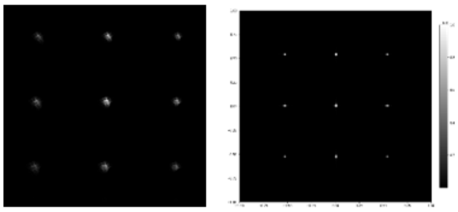

Fig. 3 (left) Diffraction pattern for a mesh grating, (right) Fourier transform of a mesh grating aperture function, known as a comb function.

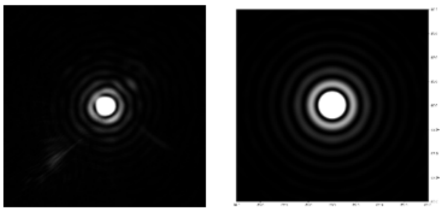

Fig. 4 (left) Airy pattern produced by the iris aperture, (right) Fourier transform of a circ function producing the point-spread function.

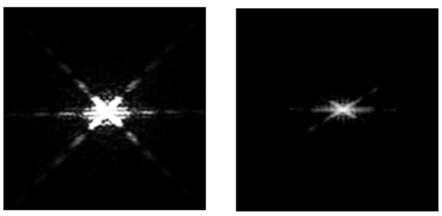

Fig. 5 (left) Diffraction pattern of the letter k, (right) Fourier transform of the unaltered image of k taken in the image plane.

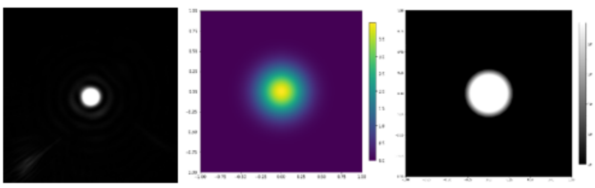

Fig. 6 (left) Image of airy pattern with backward anodization filter on lens, (middle) Gaussian function, (right) effect of gaussian function on circ function to produce forward apodization.

Code using a Fourier transform directly from the scale of the aperture was also used but unoptimized. Contour plots were produced and are compared to experimental images. The results and can be seen to accurately predict the diffraction patterns, Fig.5 shows the point spread function of a lens known as the airy pattern with Fig.4 showing that the transforms of more complex aperture functions can also be replicated. It is possible to replicate the pictures in the image planes from the experiment through a transformation from the Fourier space to produce the original aperture with a polarity change. Seen in Fig.6 is the model of a gaussian anodization filter with the experimental and modelled intensity patterns shown in Fig.6

The results of various masks can be seen in Fig.7 – Fig.10; A pin hole mask is used in the mesh grating filtering out both vertical and horizontal gratings leaving a blurred image of the beam due to effects of low pass filtering, the cross mask shows the form of the mesh but with each point an airy function due to the convolution with the point spread function. The results of high and low pass filters can be seen in Fig.8. Fig. 9 shows the use of filtering out either the diffraction grating or the letter

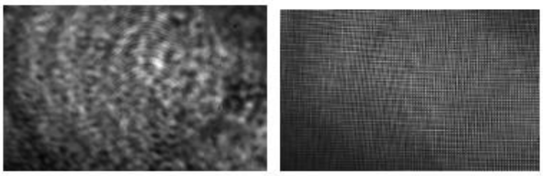

Fig. 7 (left) Pin hole aperture in Fourier plane on mesh grating shown in Fig. 3, (right) cross grating

Fig. 8 (left) high pass filter on letter J, (right) low pass filter on letter J, notice the mirror image of the letter

Fig. 9 combination of K aperture and diffraction grating in object plane with (left) pin hole used (right) series of holes in fourier plane

Fig. 10 convolution of aperture K in the object plane and diffraction grating in the fourier plane

K, as a series of holes in paper was used at the mask it proved to be quite ineffective. Fig.10 is the convolution of the Fourier transform of the letter k and the aperture function of a diffraction grating. Producing the effect of the image of the inverse K at the points of the diffraction grating Fourier transform.

## 4. Discussion

While the data taken from the experiment was generally more imprecise than desired, even with extensive care on the set up, the results clearly show a solid representation of the theory of Fourier optics. Which can be modelled and fitted onto the data with good accuracy. The data retrieved for Fig.1 needed to be heavily cleaned due to inaccuracies in the set up which resulting rotation in the intensity function and predominantly the interruption of the true intensity of the Fourier transform due to the ruler tool captured in the image. The image was first rotated clockwise by four degrees revealing tertiary and quaternary peaks that were obscured due to the effect of the tool in the initial image. This aligned more accurately with he Fourier transform using the measured width to separation ratio of the double slit. The Fourier transforms of the mesh gratings, pupil function and letter k were replicated from their aperture functions in python. It's seen that the represented images, model the experimental data closely.

Masks have the effect of preventing parts of the Fourier transform through the system, this can be used to prevent parts of the aperture function from being imaged. Masks that prevent certain spatial frequencies are shown in Fig. 8. The mask used for Fig.9 had limited precision as the holes uses to transmit the Fourier function of the diffraction grating had to be made by hand in paper due to lack of time and resources, so was merely left to proof of concept. Convolutions of the aperture function of the letter k in the object plane and the aperture function of a diffraction grating in the Fourier plane can be seen in Fig.10, described by

$$\mathcal{F}\left[\mathcal{F}(k) * Ш_{\mathrm{d}}^{(N)}\right] = -k * \mathcal{F}\left[Ш_{\mathrm{d}}^{(N)}\right]$$

where $\mathcal{F}(k)$ is the fourier transform of K and Ш is known as the comb function (aperture function of grating [4]). The point spread function of the lens was a negligible error as shown by (x) as the resolving power of the lens was more than suitable when performing the practical, however Fig.4 shows the effects of the point spread function of the lens. Where the effect of the airy pattern seen in Fig.4 is produced in each point in the image. The aperture function becomes convoluted with the point spread function of the lens, which removes higher spatial frequency terms having the effect of making the image more blurred [2]

$$\mathcal{F}[f(x',y')P(x,y)] = \mathcal{F}[f(x',y')] * \mathcal{F}[P(x,y)]$$

While not performed in the experiments, a reduction of this effect can be done through apodization where the aperture function of the lens is multiplied by the apodization function seen in Fig.6. The effect of the resulting Fourier transform reduces the intensity of subsidiary maxima, seen on Fig.6, while also creating a blurring effect on the image. Note that in Fig.6 a backwards anodization filter was used as it was placed on the lens, whereas experimental image shows the effects of a forward apodization due to it's multiplication with the point spread function of the lens. Application of the apodization techniques studies in this paper would have made limited effects as mentioned previously.

Due to the complication in the set up and the precision on the CMOS, and computational complexity of optimising values of a Fourier transform. Hence only qualitative assessments of the Fourier transform models could be made. Improvements of the experiment could have been two fold, importantly pictures qualities would need to improve with the removal of imaging tools, as well as the apertures made horizontal so an accurate cross-section can be taken. Diagonalisation of points were captured on preliminary images due to ineffective focusing. Out of focus areas in images captured in the image plane are due to the beam not normally incident on the detector, meaning one side of the beam had a longer path length than the other resulting in an out of focus image in that section.

## 5. Conclusion

The experiment, including shortcomings, gave effective basis for the use of Fourier analysis in determining and predicting images produced, and methods in modifying their image. The model fit of the double slit diffraction was precisely optimised, however disagreed with the true values of the slit, likely due to poor image quality and an inaccurate cross section of the image taken. Improvements could have been made to the precision of the set up to record more accurate images that were more suitable to analyse, as well as more efficient code to optimize parameters. Effective point spread functions were demonstrated and reduced theoretically and experimentally, as well as spatial filtering of certain apertures.

## References

[1] *Physical optics*, Wikipedia, [online] Available at: https://en.wikipedia.org/wiki/Physical_optics

[2] Charles S. Adams and Ifan G. Hughes, *Optics f2f*, Oxford university press, Oxford (2019), ch. 5, 6, 9, 19

[3] Montgomery, P.C and Perrin Stephanie, *Fourier Optics: Basis Concepts*, (2018)

[4] Visser, Taco, Agrawal, Govind and Milonni Peter, *Fourier Processing with partial coherent fields* (2017)

[5] I.G. Hughes and T. P. A Hase, *Measurements and their Uncertainties*, Oxford University Press, Oxford (2010), p. 28-30, 37-40

## Error Appendix

The resolution of the lens was given by,

$$R = 1.22\frac{\lambda}{D}$$

with the error given by simple propagation do to the error in the ruler, as the error in wavelength provided a negligible contribution as it was factory made to precise specifications

$$\alpha_R = 1.22\alpha_D$$

where $\alpha_D = \sqrt{2}p_R$, where $p$ is the precision of the ruler. The error in the intensity function was given by the error in it's amplitude, the prefactor to the function given by the functional

$$\alpha_A = A\sqrt{\left(2\frac{\alpha_a}{a}\right)^2 + \left(\frac{\alpha_f}{f}\right)^2}$$

where $\alpha_a = \sqrt{2}p_m$, where $p$ is the precision of the travelling microscope, and $\alpha_f$ the same as the error on the diameter. This provide the error on the y-values, after normalisation to allow chi-squared minimum to be calculated.

| Variable | Value | Error |
|---|---|---|
| slit width(a) [μm] | 60 | 0.7 |
| slit seperation)(d) [μm] | 260 | 0.7 |
| resolution(R) [arcsecs] | 3 | 176.0 |
| focal length(f) [cm] | 55 | 0.7 |
| lens diameter(D) [mm] | 48 | 0.7 |

To analyse the model fit, the normalised residuals are first calculated, given by the equation.

$$R = \frac{y - y(x)}{\alpha}$$

where chi-squared is calculated using this by summing the squared of each residuals, given by the equation.

$$\sum R_i^2$$

Reduced chi-squared, is given by dividing $\chi^2$ at the optimised parameters by the number of degrees of freedom of the data via

$$\chi_v^2 = \frac{\chi_{\min}}{v}$$

Covariance matrix is, in the case here, a 3x3 matrix, which is the inverse of the curvature matrix $A$, a matrix that described the curvature of the $\chi^2$ contour.

$$[\mathrm{C}] = [\mathrm{A}]^{-1}$$

The elements of the matrix quantify the statistical errors on the best fit parameters, providing that the errors are normally distributed. Here the errors in each parameter is the square root of the diagonal elements of the matrix, given by

$$\alpha_j = \sqrt{\mathrm{C}_{jj}}$$

**Scientific Summary for a General Audience**

The experiment first starts with wave optics and the work of physics named Youngs and Fresnel, in the 18th and early 19th century. The work they focused on brough about the start of the understanding of wave optics, where the movement of light is thought about using the properties of waves, this work saw many advantages over ray optics where light is thought of as packets of energy. This work was improved upon by many other, namely Fraunhofer where it progressed into modelling these waves as the combination of many planar (flat) waves, which started the foundations of Fourier optics, which uses a mathematical process developed by Josef Fouerier to model the intensity of light  downstream from the source through a Fourier transformation. Understanding this and what is known as an aperture function, which describes how much light is transmitted through an aperture at a certain position, can allow for the intensity to be modelled at certain points downstream. Using objects called masks in these downstream positions, can block parts of the Fourier transform to change the final image in the optical system, this is what spatial filtering done,

The experiment is termed a 4f set up, where there is a single focal length of the lens used between each piece of equipment. This means that an aperture one focal length from a lens is fourier transformed one focal length past this lens. An aperture is placed in the first position known as the object plane and and image is taken of it two focal lengths away in the fourier plane. This is repeated for many different types of apertures, Computer models of these apertures are made and then code called a fast fourier transform are taken of them to determine the intensity pattern produced, these are then compared to the intensity patterns taken from the images and compared for accuracy. The camera is them placed two focal lengths belong the fourier plane and different makes are placed in the fourier p[lane. The effect of these masks are tjen images on the camera and described using the mathematical properties of Fourier transforms