Determination of Planck's Constant Through Bremsstrahlung Radiation

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Bremsstrahlung radiation is a continuous spectrum of X-ray radiation that is produced when electrons are rapidly decelerated as they pass by the nuclei of matter. By using a NaCl crystal as an energy filter we were able to use Bragg's Law of diffraction to find the minimum wavelength of the spectrum for different tube high voltages. These two variables are related by a constant that depends on Planck's constant. Knowing this relation allowed us to find an experimentally determined value for Planck's constant which was $h = 6.49 \times 10^{-34} \pm 0.03$ J Hz^{-1}, this value disagreed with literature.

1. Introduction

In 1895 an unknown type of radiation, X-rays, were first produced in a controlled manner by a German physicist Wilhelm Rontgen. This was done by firing high energy electrons onto an anode where the rapid deceleration of the electrons produced X-rays. Nearly 130 years later we know they are high energy electromagnetic radiation, but this is still the main way X-rays are produced. By firing electrons from a cathode onto an anode they interact with the electrons and the nuclei of the material to produce X-rays in two different ways. In this experiment we rely on Bremsstrahlung radiation, where electrons are braked through a series of interactions with the nuclei of the material and release a spectrum of different energy X-rays as a result. [1]

The wavelength of X-rays is too short to be diffracted mechanically, but the regular spacing of ions in an atomic lattice is of the appropriate magnitude to so. It has enough regularity to act as a diffraction grating, causing X-rays to be diffracted in a 3D pattern; where the scattering angle, θ (the same as the targe angle) is related to the lattice plane spacing, d, and its wavelength, λ, as given by Eqn. 1, Bragg's Law of diffraction. [1]

$$n\lambda = 2d\sin(\theta) \tag{1}$$

This law is true when X-rays are incident on a crystalline solid, and in this experiment, it is a NaCl crystal with a lattice plane spacing of $d = 282.01\,pm$. We are also measuring the X-rays when they are at their first order of diffraction, so $n = 1$.

The Bremsstrahlung spectrum in the emission spectrum has a minimum wavelength and is produced due to the most complete deceleration of the electron. This means there is a near total transfer of kinetic energy of the electron into a single X-ray.

$$E_{\text{max}} = hf_{\text{max}} \tag{2}$$

E_{max}, is the maximum energy of the X-ray, which is related to the maximum frequency measured, f_{max}, and Planck's constant, h.

$$f = \frac{c}{\lambda} \tag{3}$$

Here the frequency is related to the speed of light, c, and the wavelength, λ. We can substitute Eqn. 2 into Eqn. 3 to get

$$E_{\text{max}} = \frac{hc}{\lambda_{\text{min}}} \tag{4}$$

$$E_{\text{kin}} = eU \tag{5}$$

Eqn. 5 gives us that, E_{kin}, the kinetic energy of the electron is related to, e, is the elementary charge and, U, the tube high voltage of the X-ray tube.

By equating Eqn. 4 and Eqn. 5 and making the minimum wavelength, λ_{min}, the subject with, A, the constant; we get

$$\lambda_{min} = \frac{hc}{e}\frac{1}{U} = A\left(\frac{1}{U}\right) \tag{6}$$

2. Method

A spectrometer is set up to measure the X-rays, this is comprised of a goniometer with a NaCl crystal and a Geiger-Muller counter tube in the Bragg configuration. The crystal and counter tube are pivoted with respect to the incident X-ray in a θ - 2θ coupling, meaning the angle of the counter to the horizontal is twice the size of that of the monocrystals target angle. The software is configured for the experiment by recording the counts against wavelength by using Braggs law and the known lattice spacing to convert from the scattering angle. The parameters for the experiment were then entered into the machine. The current for the X-ray tube was set, $I = 1$mA and the angular step width at, $\Delta\beta = 0.1°$; both were kept constant throughout the experiment. Then the tube high voltage was changed to get my five sets of data as shown in Fig. 1. Along with this, the range for the target angle, $\beta\{a \leq \theta \leq b\}$, and the time frame, Δt, were both adjusted for each data set to get the most desirable measurements.

3. Results

The Fig. 1 below shows the variation of the initial section of the Bremsstrahlung radiation for molybdenum at different tube high voltages. As can be seen in the figure the spectrum begins at a lower wavelength for higher voltages and the number of counts is increased.

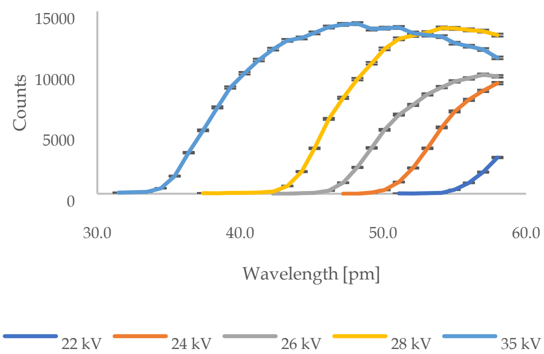

Fig. 1 the counts against wavelength, in picometers, of the Bremsstrahlung spectrum of molybdenum for the tube high voltages of; 22kV, 24kV, 26kV, 28kV and 35kV. There error bars are too small to be seen

From Fig. 1 we can identify the minimum wavelength, λ_{min}, for each tube high voltages. This is done by determining a best-fit line for the initial linear part of the spectrum and finding its intersection with the x-axis. In practice this is most easily accomplished by using the LINEST function for these data points and plotting wavelength against counts (the reverse of the variables) In this way the function will provide the y-intercept (minimum wavelength) as well as the error in the value.

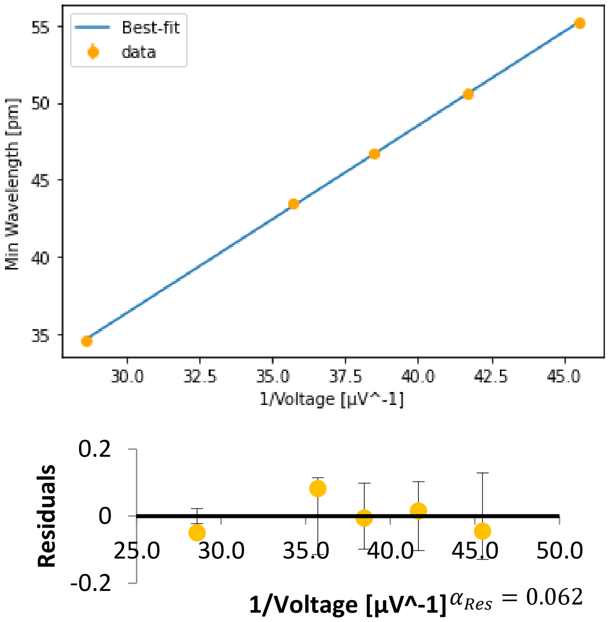

Fig. 2 the relationship between the minimum wavelength of the Bremsstrahlung spectrum and the inverse of the tube high voltages, as well as a plot of the residuals for the data points

As predicted in Eqn. 6 there is a linear relationship between the two variables, and my data can be seen to heavily support this. It shows a close fit to the relationship, shown by the plot of residuals; additionally it supports the Duane-Hunt relationship as it's y-intercept is -0.0062, passing very close to the origin. Shown in the equation is also the relationship of the gradient, A, and Planck's constant, where we can rearrange $A = \frac{hc}{e}$, into $h = \frac{Ae}{c}$. So Planck's constant is related to the gradient, A, the elementary charge of an electron, e, and the speed of light, c. From this I calculated my value of Planck's constant to be $h = 6.49 \times 10^{-34}$ ± 0.03 J Hz^{-1}.

4. Discussion

The established value of Planck's constant is 6.626 070 15 x 10^{-34} J Hz^{-1} [2]. My value, through being fairly precise, quite significantly disagrees with the true value and is quite inaccurate, being nearly five standard errors away from the accepted value. However As my fractional uncertainty is quite small, being 2.11%, it can be said that an acceptable experiment was conducted and it is important to now consider the flaws in the experiment that lead to the inaccuracy of the result.

Firstly as the majority of the experiment was conducted by a machine, it is unlikely that many errors were produced there, however as it provided all of my data it is fair to assume it may have contributed some errors. Due to the precision of the machine the most feasible explanation is a systematic error being produced. Potential possibilities for this is an offset in the true target angle being measured or a potential fault in the crystal due to grease or mechanical fractures. However I am unfamiliar with the mechanics of the machine to conclude anything further. A very likely reason for the inaccuracy is due to the human error involved in the judgement of what to data points to include in the 'linear section' at the start of the Bremsstrahlung spectrum. The variation of this selection affects what the minimum wavelength determined is. To improve on this, many more measurements could be taken across the start of the spectrum. This could be done by taking measurements for the count in smaller increments of wavelength, which may allow the linear section to be seen more clearly. The minimum wavelength is then used to plot the graph in Fig. 2, which has very few errors being included from elsewhere. However the well-fitting data points and support of the Duane-Hunt relation would potentially disagree with the possibility of inaccurate minimum wavelengths as it is consistent with the theoretical prediction. This could indicate that errors produced previously in the experiment were the cause for the inaccuracy of Planck's constant. My result is also likely to be too precise as the error on the value is only a consequence of the error in the gradient as evaluated by the LINEST function. This calculates it from the line of regression, which is based off how well the data points agree as a linear function, but does not take into account the error in the respective data points; which in this case is quite important; coming from the λ_{min} value. Finally I could have taken measurements for more values of the tube high voltage. Doing all of these would produce a more accurate final result for Planck's constant.

5. Conclusions

In conclusion I believe that this was an adequately performed experiment. I found each minimum wavelength of the Bremsstrahlung spectrum for molybdenum at different tube high voltages. Then was able to use the linear relation of Eqn. 6 to find a value of Planck's constant. The resulting value of, $h = 6.49 \times 10^{-34} \pm$ 0.03 J Hz^{-1}, was outside its error range and disagreed with the true value [2] but by a small fractional uncertainty. If the improvements were made to the experiment as in my Discussion, I may find a value for Planck's constant that is more accurate to the accepted value.

References

[1] Young, H.D, Freedman R.A, *University Physics with Modern Physics*, 15th ed, Pearson, San Francisco (2019), p. 1228-1230, 1420, 1287-1288

[2] *Fundamental Physical Constants; Planck Constant*, October 1994, The NIST Reference on Constants, Units and Uncertainties, CODATA Value: Planck constant (nist.gov) As of March 2022.

[3] I.G. Hughes and T. P. A Hase, *Measurements and their Uncertainties*, Oxford University Press, Oxford (2010), p. 28-30, 37-40

Error Appendix

The errors in Fig. 1 were calculated using the standard error for Poisson distributions given by [3]

$$\sigma_N = \sqrt{N} \tag{7}$$

So, σ, the standard distribution (and error) of the number of counts is the square root of the number of counts recorded in that time frame, N. The detector itself records the count rate of the X-rays, which is averaged over the time frame, but it is ideal to convert this to counts by

$$N = R \cdot \Delta \mathrm{t} \tag{8}$$

Where, R, is the count rate and, Δt, is the time frame. This is beneficial as it reduces the fractional uncertainty of the error.

This means that the error bars in Fig. 1 are too small to be seen. It is also assumed that there is negligeable random error from the wavelength as it is provided by the machine.

The minimum wavelength is found by extrapolating the x intercept of the initial linear part of the Bremsstrahlung radiation in Fig. 1. These values are provided by the LINEST function, which also provides the error in the intercept, which has some flaws evaluating this as mentioned. This would therefore produce a smaller error than what is accurate, as it does not account for the increased range of values from the error in the count. Also there is the unquantifiable human error that goes into the judgement in determining which data points are linear.

The minimum wavelength is then plotted against the inverse of the tube high voltage as shown in Fig. 2. Once again, we allow the LINEST function to calculate the gradient, A, and its error as well as the constant, C, and its error. As can be seen from the residual plot there is a good fit to the experimental prediction, therefore the errors for both the values is low.

We then find Planck's constant through the use of A, and Eqn. 6. Then to find the error in this value we use the single variable functional approach [3]

$$\alpha_h = \left| f(\bar{A} \pm \alpha_A) - f(\bar{A}) \right| \tag{9}$$

Where, α_h, is the error on Planck's constant (which is symmetric) and, $\bar{A}$, is the value of the gradient given by the line of best fit and, $\alpha_{\bar{A}}$, is its error. As the values of the speed of light and elementary charge are given as established values, their errors are negligible and it can be treated as only a single variable function with a constant.

Scientific Summary for a General Audience

X-rays are a type of electromagnetic radiation or light. Light is made out of little particles of energy (or quanta) called photons which are both a wave and a particle, as it exhibits the behaviours of both of these objects. X-rays are a type of photon that has a relatively very high energy, which is directly related to Planck’s constant and its frequency. This model describes how much energy light can carry and knowing the value of Planck’s constant, which is a universal value, allows us to understand how this energy is transferred in little packets and how much of it is. Therefore Planck’s constant has improved our understanding about light and matter on the atomic scale. This is vital and allows us to produce many important things like lasers and LEDs and stuff in electronics like semi-conductors and transistors that make up computer chips.

The experiment is designed so that we can find a value of Planck’s constant. To do this we look at the X-ray emission that is produced by an X-ray tube, and in particular a certain type of this emission spectra that is produced when electrons quickly decelerate around matter and transfer their energy into light as X-rays. By looking at the highest energy X-rays (those with the highest frequency) that are produced by an X-ray tube at different amounts of power. We can find a value for Planck’s constant. In science we can never know a value for certain and so we have to show the range that we believe the true value lies within our experimental value in a quantifiable way (a number). So we give the uncertainty about our value (a + and – amount) to show how close we think we are to the true value. Of course any badly conducted experiment, or those with many flaws or errors, may produce a value that is far off the true value, even though we may have a low uncertainty amount. We can never know this is the case and so we have to conduct many experiments of many different types to find a value that is accurate and precise to the true value.