Cherenkov Radiation

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The Cherenkov effect is a form of radiation emitted as a particle exceeds the phase velocity of light in the medium through which it travels. The original discovery of the phenomenon was by P.A. Čerenkov during his postgraduate work in 1934, investigating the luminescence of uranyl salts under the stimulation of γ emission from radium. The mathematical formulation of the effect followed, provided by his peers at the institution I.M. Frank and I.E. Tamm a few years later. All three researchers won a Nobel prize in 1958 for the discovery following the widespread, successful use of the effect in particle detectors for experimental research. Today the physics here is used in the Kamiokande experiments in Japan, detecting neutrinos, and for RICH detectors, such as those in ALICE at the LHC. This dissertation will first demonstrate its basic properties, including the characteristic angle and threshold velocity, through a geometric analysis of the Cherenkov cone. Then, using classical electrodynamic theory, we will attempt to derive the production of radiation and its spectrum, given by the Frank-Tamm formula.

1. BACKGROUND INFORMATION & CREATION

Cherenkov radiation can be most readily understood as the optical analogy of a sonic boom. It occurs when a charged particle moves through a transparent dielectric medium at a constant velocity, v, faster than the phase velocity, $v_p(\omega)$, of light of frequency ω in that medium [1]. This requirement can be defined by the equation,

$$v > \frac{c}{n(\omega)} \tag{1}$$

where $n(\omega)$ is the index of refraction. Once this occurs, there is emission of radiation with a continuous spectrum for frequencies with $v_p(\omega) < v$ and with specific angular distribution. The radiated angular distribution, θ, is given by,

$$\cos(\theta) = \frac{1}{n(\omega)\beta} \tag{2}$$

with $\beta = v/c$. The angle is defined between the velocity vector of the charged particle, $\mathbf{v}$, and wavevector of the emitted waves, $\mathbf{k}$. These results may be easily obtained using Huygens' principle, as seen in Fig.1, where at each point on the path of a rectilinear and uniformly moving charge, a spherical wave propagates outwards. Visible emission requires that the spherical surfaces have a common envelope, allowing for coherent interference of the radiation along this surface and thus, a directed wavevector. This is seen as a cone, tangent to the surfaces, with apex coinciding with the charge position and opening angle $\pi - 2\theta$. For this to occur, the distance travelled by the light during time t, $r = ct/n$, is less than $x = vt$ and hence $v > c/n$

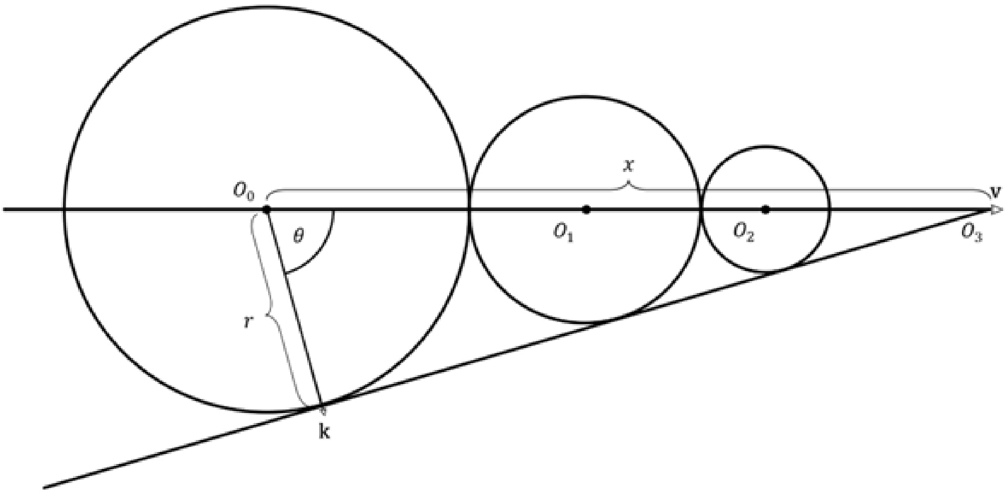

FIG. 1: Diagram for the generalisation of Cherenkov radiation. The circles demonstrate the wavefronts of points O_i and the slope the resultant wavefront.

as seen in Eq. (1). The angle can be found by constructing a triangle with hypotenuse x and adjacent length r such that, $\cos(\theta) = 1/n\beta$.

The range of angular dispersion for a particular media can be seen from Eq. (2) and ranges from $\cos(\theta) = 1$, occurring for the threshold velocity $\beta_{th} = 1/n$, to a maximum when $\beta = 1$. Given by the inequality,

$$\frac{1}{n} < \cos(\theta) < 1 \tag{3}$$

This inequality, however, is not satisfied for electromagnetic radiation with a frequency higher than that of the ultraviolet range, where $n(\omega) < 1$, and is therefore not emitted as radiation. Due to the dispersive nature of light (ω dependence of n), the angle of emission will be different between frequencies, producing a spectrum in the ring of Cherenkov radiation. The refractive index has an inverse relationship with wavelength, as given by Cauchy's formula, so it is clear from Eq. (2) that the blue end of the optical spectrum is seen in the outermost section of the ring, with a spectrum inwards to red. An example of this has been captured by work from V.P. Zrelov [2], seen as a side on perspective of the annular ring in Fig.2.

FIG. 2: Side on view of the optical spectrum of a Cherenkov ring as captured on colour film.

The analogy to a sonic Mach cone displays a notable difference as, unlike the sonic version which has a very sharp tip (due to limited dispersion of sound in air), the tip formed by Cherenkov radiation is not well defined. In addition to this, the radiation is emitted at a relatively large distance from the particle, an odd occurrence that we will formally demonstrate later in this dissertation. Though the supersonic phenomena was well known for many decades, the idea of an optical equivalent was very rarely considered in the scientific community, besides the notable physicists Heaviside and Sommerfeld [3]. Partly because particle accelerators had not been produced until the thirties, so there lacked a means to reliably create relativistic particles necessary for the effect. But predominantly as the electrodynamics of charges during this period were only considered in a vacuum, where this effect is impossible, thus the assertion that a uniformly moving charge does not radiate.

Ultimately, the first discovery of this phenomenon was by accident, seen due to the interaction of gamma rays with a sulphuric acid solvent in 1934 (in the absence of the uranyl salt solute) by the aforementioned Čerenkov. Here incident gamma rays caused the release of Compton electrons into the fluid, travelling at relativistic speeds. Indeed, before the phenomena was officially discovered, there had been a few noted observations of the effect. Namely by P. and M. Curie, in a similar experiment on the radiation of fluids by gamma rays. They, of course, were not aware that this was due to a new phenomenon and not gamma-ray induced luminescence of uranyl solutions. The deciding factor between the experiments being the new measurement techniques employed by Čerenkov's supervisor, Vavilov, allowing for sensitive observation of luminescence from solutions. This, and the insights provided by Vavilov, gives the phenomenon its alternate name: the Vavilov-Cherenkov effect.

Formally the mechanism for this effect is as follows. The atoms of a dielectric medium in the immediate vicinity of a passing charged particle are distorted by the electric field of the latter into an elongated shape with a inhomogeneous charge distribution, thus behaving as electric dipoles. For a slow particle the polarization is symmetrical, with no electromagnetic radiation is emitted. But for a fast particle with a velocity above the threshold velocity, as the fields are not given sufficient time to relax, asymmetrical polarization is produced with the resultant dipole producing emission of Cherenkov Radiation.

2. ELECTRODYNAMIC DERIVATION OF THE FRANK-TAMM FORMULA

For the derivation of the Frank-Tamm formula we will generally follow the work in Jackson [4] and Landau [5]. Firstly, in order to define our work we will state Maxwell's equations in dispersive dielectric media as follows,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon} \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \tag{4}$$

$$\boldsymbol{\nabla}\cdot\mathbf{H} = 0 \qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{\mathbf{j}}{c} + \frac{\epsilon}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{5}$$

with $\epsilon = \epsilon(\omega)$ being the frequency dependent permittivity, $\mathbf{H} = \mathbf{B}/\mu_0$ and other symbols defined in the usual way. The scalar ϕ, and vector $\mathbf{A}$, potentials can be constructed through their definition of,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{6}$$

Then the four-potential as $A^{\mu} = (\phi, c\mathbf{A})$, and four-current as $j^{\mu} = (c\rho/\epsilon, \mathbf{j})$. In the adapted Lorenz gauge for dielectric media $\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\epsilon}{c}\frac{\partial \phi}{\partial t}$, this leads to the wave equation of the four-potential as;

$$\Box A^{\mu} = -\frac{1}{c} j^{\mu} \tag{7}$$

With the d'Alembertian operator, defined in the similarly adapted way,

$$\frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box \tag{8}$$

If the charge and current density do not depend on time, then naturally the corresponding derivatives in Eq. (7) are zero and no radiation is expected. Therefore we will define the charge and current density of a rectilinearly and uniformly moving particle moving in the x direction with charge q, through the Dirac delta function δ, as,

$$\rho(\mathbf{x}, t) = q\,\delta(\mathbf{x} - \mathbf{v}t) \qquad\qquad \mathbf{j}(\mathbf{x}, t) = \mathbf{v}\,\rho(\mathbf{x}, t) \tag{9}$$

The equations for dispersive media, where $n = n(\omega)$, are most commonly solved through the use of a Fourier transform according to:

$$F(\mathbf{x}, t) = \frac{1}{(2\pi)^2}\int d^3k \int d\omega \tilde{F}(\mathbf{k}, \omega) e^{(i\mathbf{k}\cdot\mathbf{x} - i\omega t)} \tag{10}$$

This allows for the wave equations of the potentials, as defined in this gauge, to be expressed in frequency and k-space as,

$$\left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\Phi(\mathbf{k}, \omega) = \frac{4\pi}{\epsilon(\omega)}\rho(\mathbf{k}, \omega)$$
$$\left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\mathbf{A}(\mathbf{k}, \omega) = \frac{4\pi}{\epsilon(\omega)}\mathbf{j}(\mathbf{k}, \omega) \tag{11}$$

Again, the dielectric constant $\epsilon(\omega)$ appears due to the effect of microscopic electric fields in the dielectric. The corresponding charge and current density Fourier transforms are thus given as,

$$\rho(\mathbf{k}, \omega) = \frac{q}{2\pi}\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}) \qquad\qquad \mathbf{j}(\mathbf{k}, \omega) = \mathbf{v}\,\rho(\mathbf{k}, \omega) \tag{12}$$

Solving the equations for the potentials given in Eq. (11), provides the solutions,

$$\begin{aligned}\Phi(\mathbf{k},\omega) &= \frac{2q}{\epsilon(\omega)} \cdot \frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)} \\ \mathbf{A}(\mathbf{k},\omega) &= \epsilon(\omega)\frac{\mathbf{v}}{c}\Phi(\mathbf{k},\omega)\end{aligned} \tag{13}$$

Finally the Fourier transformed electromagnetic fields, given in terms of their scalar and vector potentials are,

$$\begin{aligned}\mathbf{E}(\mathbf{k},\omega) &= i\left[\frac{\omega\epsilon(\omega)}{c}\frac{\mathbf{v}}{c} - \mathbf{k}\right]\Phi(\mathbf{k},\omega) \\ \mathbf{B}(\mathbf{k},\omega) &= i\epsilon(\omega)\mathbf{k}\times\frac{v}{c}\Phi(\mathbf{k},\omega)\end{aligned} \tag{14}$$

We will only need the electric field in terms of the angular frequency and so require an integration over the 3-dimensional wavevector; this is given as,

$$\mathbf{E}(\omega) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\; \mathbf{E}(\mathbf{k},\omega)e^{ibk_2} \tag{15}$$

As the relativistic particle moves through the medium, it will interact with the electrons in an atom due to the medium's molecular polarisation. The energy lost per unit distance can then be most elegantly obtained by calculating the electromagnetic energy flow, given by the integral of the Poynting vector, through a cylinder coaxial to the direction of motion, of radius a around the path of the incident particle. With the Poynting vector given as,

$$S = \frac{c}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{c}{4\pi}\, B_3\, E_1 \tag{16}$$

where in the second step we have used the fact that the particle is moving along the x direction. The energy radiated at a perpendicular distance b from the particle thus given as,

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{1}{v}\frac{dE}{dt} = -\frac{c}{4\pi v}\oint B_3 E_1\, d\mathbf{A} \tag{17}$$

$$= -\frac{ca}{2}\int_{-\infty}^{\infty} B_3(t)E_1(t)\, dt \tag{18}$$

where we have integrated over the circle with circumference a and changed the integral through $dx = vdt$. With Parseval's theorem we can express the integral in frequency space and, as the integrand is symmetric, take the integral purely over the positive domain.

$$\left(\frac{dE}{dx}\right)_{b>a} = -ca\mathrm{Re}\left[\int_0^{\infty} B_3^*(\omega)E_1(\omega)d\omega\right] \tag{19}$$

Where the frequency dependent $\mathbf{E}(\omega)$ and $\mathbf{B}(\omega)$ fields can be found from Eq. (14) by Eq. (15). In order to evaluate this complex integral requires the use of known integrals of the modified Bessel function and defining a substitution variable λ, given as

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\epsilon(\omega) = \frac{\omega^2}{v^2}[1 - \beta^2\epsilon(\omega)] \tag{20}$$

Varying a will allow us to see how the energy is deposited throughout the medium. For Cherenkov radiation to occur, it requires the limiting case where $|\lambda a| >> 1$ such that the energy is deposited far from the path. With this limiting case we can approximate the Bessel functions in the resultant integral by their asymptotic forms. The components of each of the electric and magnetic fields required for Eq. (19), provided by Eq. (14) and integrated over k-space as shown in Eq. (15) are thus given as,

$$\begin{aligned} E_1(\omega, b) &= i\frac{q\omega}{c^2}\left(1 - \frac{1}{\beta^2\epsilon(\omega)}\right)\frac{e^{-\lambda b}}{\sqrt{\lambda b}} \\ E_2(\omega, b) &= \frac{q}{v\epsilon(\omega)}\sqrt{\frac{\lambda}{b}}e^{-\lambda b} \\ B_3(\omega, b) &= \beta\epsilon(\omega)E_2(\omega, b) \end{aligned} \tag{21}$$

Resulting in the integrand of Eq. (19) in this limit being,

$$(-caB_3^*E_1) \rightarrow \frac{q^2}{c^2}\left(-i\sqrt{\frac{\lambda^*}{\lambda}}\right)\omega\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]e^{-(\lambda+\lambda^*)a} \tag{22}$$

This integral provides the energy deposited far from the path of the particle. If λ has a positive real part, as is generally true due to the dielectric constant $\epsilon(\omega)$ often being real for all frequencies and β being small. Then the exponential factor will cause the expression to vanish rapidly at large distances, resulting in all the energy being deposited near the path. However, when λ is purely imaginary, then the exponential is unity (hence independent of a) resulting in some of the energy escaping to infinity as Cherenkov radiation.

From Eq. (20), it can be seen that for λ to be purely imaginary requires that $\epsilon(\omega)$ is real, or very close to real (hence limited absorption), and the velocity of the particle is large, ensuring that $\beta^2\epsilon(\omega) > 1$. This condition can be expressed in the familiar form,

$$v > \frac{c}{\sqrt{\epsilon(\omega)}} = \frac{c}{n} \tag{23}$$

Only, however, for a material that is purely a dielectric, i.e it has no relative permeability, so that $n = \sqrt{\epsilon(\omega)}$. Where here the common mathematical description of Cherenkov radiation is derived purely from considering the energy loss of the particle at distance. Lastly, by considering the phase of λ as $\beta^2\epsilon(\omega)$ changes from less than unity to greater than unity and, as mentioned, assuming that $\epsilon(\omega)$ has an infinitesimal positive imaginary part for $\omega > 0$. Shows that $\lambda = i|\lambda|$ for $\beta^2\epsilon(\omega) > 1$, resulting in $(\lambda^*/\lambda)^{1/2} = i$ and Eq. (19) is real and independent of a. Thus under these circumstances, the energy radiated as Cherenkov radiation per unit distance along the path of the particle is given by the Frank-Tamm formula [6],

$$\left(\frac{dE}{dx}\right)_{\text{rad}} = \frac{q^2}{c^2}\int d\omega\,\omega\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right] \tag{24}$$

The limits of integration of Eq. (24), are the frequencies over which the inequality $\epsilon(\omega) > \frac{1}{\beta^2}$ holds, which provides the frequency band of emission; while the integrand provides the, also familiar, differential spectrum in frequency. The inequality for the velocity required for radiation is thus seen to be $\frac{1}{\sqrt{\epsilon(\omega)}} < \beta < 1$, with both previous inequalities also demonstrated by our analysis in section.1. As the dielectric constant increases with wavelength, it is clear that higher frequencies of light are emitted more readily, up to the high ultraviolet range, providing an explanation for why the radiation is seen as a bright blue.

REFERENCES

[1] P. Cherenkov, Visible luminescence of pure liquids under the influence of γ-radiation, in Dokl. Akad. Nauk SSSR, Vol. 2 (1934) pp. 451–454.

[2] B. M. Bolotovskii, Vavilov–cherenkov radiation: its discovery and application, Physics-Uspekhi **52**, 1099 (2009).

[3] V. L. Ginzburg, Radiation by uniformly moving sources (vavilov–cherenkov effect, transition radiation, and other phenomena), Physics-Uspekhi **39**, 973 (1996).

[4] J. D. Jackson, Classical electrodynamics (John Wiley & Sons, 2021) pp. 631–640.

[5] L. D. Landau and E. M. Lífshíts, Electrodynamics of continuous media (Pergamon Press Oxford, 1946) pp. 399–408.

[6] I. Frank and I. Tamm, Coherent visible radiation of fast electrons passing through matter, in Selected Papers (Springer, 1991) pp. 29–35.