Brownian motion

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Change of Measure

To change from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ we need to reassign probabilities in Ω using Z to tell us where in Ω we should revise probability upward ($Z > 1$) and when downward ($Z < 1$).

Theorem *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}X = \mathbb{E}[XZ]$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}Y = \tilde{\mathbb{E}}\left[\frac{Y}{Z}\right]$$

for every nonnegative random variable Y.

Definition *Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be equivalent if they agree which sets in $\mathcal{F}$ have probability zero.*

If $A \in \mathcal{F}$ is such that $\mathbb{P}(A) = 0$, then the random variable $\mathbb{I}_A Z$ is $\mathbb{P}$ almost surely zero, which implies:

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) = \int_\Omega \mathbb{I}_A(\omega) Z(\omega) d\mathbb{P}(\omega) = 0.$$

The converse is also true (assuming that $Z > 0$ almost surely): suppose $B \in \mathcal{F}$ and $\tilde{\mathbb{P}}(B) = 0$, then $\frac{1}{Z}\mathbb{I}_B = 0$ almost surely under $\tilde{\mathbb{P}}$, so

$$\tilde{\mathbb{E}}\left[\frac{1}{Z}\mathbb{I}_B\right] = 0;$$

Definition *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $\tilde{\mathbb{P}}$ *be another probability measure on* $(\Omega, \mathcal{F})$ *that is equivalent to* $\mathbb{P}$*, and let* Z *be an almost surely positive random variable that relates measures. Then* Z *is called the Radon-Nikodym derivative of* $\tilde{\mathbb{P}}$ *with respect to* $\mathbb{P}$*, and we write*:

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

Theorem (Randon-Nikodym). *Let* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *be equivalent probability measures on* $(\Omega, \mathcal{F})$*. Then there exists an almost surely positive random variable* Z *such that* $\mathbb{E}Z = 1$ *and*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for every } A \in \mathcal{F}.$$