Brownian motion

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Scaled Random Walks

We start with a sequence of tosses:

$$X = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

and define $M_0 = 0$,

$$M_k = \sum_{j=1}^{k} X_j, \ \ k = 1, 2, \dots$$

The process M_k is a *symmetric random walk.*
Properties:

- Independent increments: for $k_0 < k_1 < k_2 < k_3$, the random variables:

  $$(M_{k_1} - M_{k_0}), (M_{k_2} - M_{k_1})$$

  are independent
- $\mathbb{E}[M_{k_{i+1}} - M_{k_i}] = 0$, $\mathbb{V}[M_{k_{i+1}} - M_{k_i}] = k_{i+1} - k_i$
- Martingale property: $\mathbb{E}[M_l|F_k] = \mathbb{E}[(M_l - M_k) + M_k|F_k] = M_k$

Finally, the *quadratic variation* of the symmetric random walk. By definition:

$$[M, M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k$$

Note that $\mathbb{V}(M_k)$ is computed by taking an average over all paths, while $[M, M]_k$ is computed along a single path (without any probabilities).
To approximate a Brownian motion, we speed up time and scale down the step size of a symmetric random walk. More precisely, the *scaled symmetric random walk*:

$$W^{(n)} = \frac{1}{\sqrt{n}} M_{nt}$$

with limit $n \to \infty$ we shall obtain a Brownian motion. For this process we have:

$$\mathbb{E}[W^{(n)}(t) - W^{(n)}(s)] = 0,\ \mathbb{V}[W^{(n)}(t) - W^{(n)}(s)] = t - s$$

The martingale:

$$\mathbb{E}[W^{(n)}(t)|\mathcal{F}(s)] = W^{(n)}(s)$$

The quadratic variation:

$$[W^{(n)}, W^{(n)}](t) = \sum_{j=1}^{nt} \left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right) \right]^2 = \sum_{j=1}^{nt} \left[\frac{1}{\sqrt{n}} X_j \right]^2 =$$

$$= \sum_{j=1}^{nt} \frac{1}{n} = t$$

Theorem (Central limit). *Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}$ evaluated at time t converges to the normal distribution with mean zero and variance t.*

It turns out that if we construct a price movement in Binomial model with actual probabilities $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and $d_n = 1 - \frac{\sigma}{\sqrt{n}}$, we get:

$$S_n(t) = S(0) u_n^{H_{nt}} d_n^{T_{nt}}$$

Theorem. *As $n \to \infty$, the distribution of $S_n(t)$ converges to the distribution of*

$$S(t) = S(0) exp \left\{ \sigma W(t) - \frac{1}{2}\sigma^2 t \right\},$$

where $W(t)$ is a normal random variable with mean zero and variance t.

The distribution of $S(t)$ is called *log-normal*. More generally, any random variable of the form ce^X, where c is a constant and X is normally distributed, is said to have a log normal distribution.

Brownian Motion

We obtain Brownian motion as the limit of the scaled random walks as $n \to \infty$.

Definition. *Let $(\Sigma, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and depends on ω. Then $W(t)$ is a Brownian motion if for all $0 = t_0 < t_1 < \cdots < t_m$ the increments*

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1} - W(t_i)] = 0$$
$$\mathbb{V}[W(t_{i+1} - W(t_i)] = t_{i+1} - t_i.$$

Note that one could think of ω as the Brownian motion path after a random experiment is performed. Then $W(t)$ is the value of this path at time t.
Because the increments are independent and normally distributed, the random variables $W(t_1), W(t_2), \dots, W(t_m)$ are jointly normally distributed. The joint normal distribution is determined be means and covariances. Each of the random variables $W(t_i)$ has mean zero. For any two times $0 \leq s < t$, the covariance:

$$\mathbb{E}[W(s)W(t)] = \mathbb{E}[W(s)(W(t) - W(s)) + W^2(s)] = 0 + \mathbb{V}[W(s)] = s.$$

Theorem (Alternative characterization of Brownian motion). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. The following two properties are equivalent*
(i) Increments are independent and each is normally distributed with mean and variance given above;
(ii) The random variables $W(t_1), W(t_2), \dots, W(t_m)$ are jointly normally distributed with means equal to zero and covariances determined above.

Definition. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$. A filtration for the Brownian motion is a collection of σ-algebras $\mathcal{F}(t)$, satisfying*:
(i) **(Information accumulates)** *For $0 \leq s \ < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.*
(ii) **(Adaptivity)** *For each $t \geq 0$, the Bronian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable. In other words, the information available at time t is sufficient to evaluate the Brownian motion $W(t)$ at that time.*
(iii) **(Independence of future increments)** *For $0 \leq t < u$, the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.*

Theorem. *Brownian motion is a martingale.*
Indeed, let $0 \leq s \leq t$. Then,

$$\mathbb{E}[W(t)|\mathcal{F}(s)] = \mathbb{E}[(W(t) - W(s) + W(s)|\mathcal{F}(s)] = \mathbb{E}[W(t) - W(s)] + W(s) = W(s)$$

Quadratic Variation

First-order variation

We start with a *first-order variation* which is computed as the amount of up and down oscillation (movement) of a given function f. One could compute it as:

$$FV_T(f) = \int_0^T |f'(t)|dt$$

However, in general, to compute it, we first choose a partition $\Pi = \{t_0, t_1, \dots, t_n\}$ of $[0, T]$. The maximum step size of the partition will be denoted $||\Pi|| = max_j(t_{j+1} - t_j)$. We the define:

$$FV_T(f) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|$$

By the Mean Value Theorem in each subinterval $[t_j, t_{j+1}]$ there's a point t_j^* such that:

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = f'(tj^*)$$

we see consistency of both formulas for the first-order variation.

Second-order (quadratic) variation

Definition. *Let f(t) be a function defined for* $0 \leq t \leq T$. *The quadratic variation of f up to time* T *is:*

$$[f, f](T) = \lim_{||\Pi|| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2$$

where $\Pi = \{t_0, t_1, \dots, t_n\}$.
Remark. Suppose the function f has a continuous derivative. Then

$$\sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2 = \sum_{j=0}^{n-1} |f(t_j^*)|^2 (t_{j+1} - t_j)^2 \leq ||\Pi|| \sum_{j=0}^{n-1} |f(t_j^*)|^2 (t_{j+1} - t_j),$$

and thus

$$[f, f](T) \leq \lim_{||\Pi|| \to 0} ||\Pi|| \cdot \int_0^T |f'(t)|^2 dt = 0.$$

In the last step we've assumed that the integral is finite. If it is not, than we have $0 * \infty$ situation, so the result can be anything between 0 and ∞.
The remark illustrates the reason of why one never considers quadratic variation in ordinary calculus. The paths of Brownian motion, on the other hand, cannot be differentiated with respect ot time variable.
Theorem. *Let* W *be a Brownian motion. Then* $[W, W](T) = T$ *for all* $T \geq 0$ *almost surely.*
The proof sketch: for a given partition $\Pi = \{t_0, t_1, \dots, t_n\}$ we have:

$$Q_\Pi = \sum_{j=1}^{n-1} (W(t_{j+1}) - W(t_j))^2$$

For this random variable it could be shown that it has expectation equal T and variance that converges to zero as $||\Pi|| \to 0$.
Remark. It could be seen that *Brownian motion accumulates quadratic variation at rate one per unit time.*. We denote this fact as follows:

$$dW(t)dW(t) = dt$$

Similarly, by taking the limit of $||\Pi|| \to 0$ one could see that:

$$dW(t)dt = 0, \ dtdt = 0.$$

Volatility of Geometric Brownian Motion

Let α and $\sigma > 0$ be constants, and define the geometric Brownian motion:

$$S(t) = S(0)exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right) t\right\}$$

.
Let $0 \leq T_1 < T_2$ be given with some partition $T_1 = t_0 < t_2 < \cdots < t_m = T_2$, we observe "log-returns":

$$log\frac{S(t_{j+1})}{S(t_j)} = \sigma(W(t_{j+1}) - W(t_j)) + \left(\alpha - \frac{1}{2}\sigma^2\right)(t_{j+1} - t_j)$$

The sum of the squares of the log returns is called as *realized volatility*:

$$\sum_{j=0}^{m-1} \left(log\frac{S(t_{j+1})}{S(t_j)}\right)^2 = \sigma^2 \sum_{j=0}^{m-1} (W(t_{j+1}) - W(t_j))^2 + \left(\alpha - \frac{1}{2}\sigma^2\right)^2 \sum_{j=0}^{m-1} (t_{j+1} - t_j)^2 +$$
$$+ 2\sigma\left(\alpha - \frac{1}{2}\sigma^2\right) \sum_{j=0}^{m-1} (W(t_{j+1}) - W(t_j))(t_{j+1} - t_j)$$

Assuming the maximum step size $||\Pi||$ is small, we get:

$$\frac{1}{T_2 - T_1} \sum_{j=0}^{m-1} \left(log\frac{S(t_{j+1})}{S(t_j)}\right)^2 \approx \sigma^2$$

Thus, if the asset price $S(t)$ really is geometric Brownian motion with constant volatility σ, then σ can be identified from price observations.

Other useful results

Markov property

Theorem. *Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t)$ be a filtration for this Brownian motion. Then $W(t)$ is a Markov process.*
In other words, for any f - Borel-measurable function, there is another Borel-measurable function g such that:

$$\mathbb{E}[f(W(t))|\mathcal{F}(s)] = g(W(s))$$

.

First-passage time for Brownian motion

Recall, that the first passage time is defined as:

$$\tau_m = min\{t \geq 0; W(t) = m\}$$

For $m \neq 0$, we have $\mathbb{P}\{\tau_m < \infty\} = 1$, but $\mathbb{E}\tau_m = \infty$. The random variable τ_m has density:

$$f_{\tau_m}(t) = \frac{|m|}{t\sqrt{2\pi t}}$$

.

Exercises

Exercise 1. By the property (iii), the increment $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(u_1)$. By the property (i), $\mathcal{F}(t) \subseteq \mathcal{F}(u_1)$. Thus, $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.
Exercise 2.
We have to show that $W^2(t) - t$ is a martingale given filtration $\mathcal{F}(t)$. Indeed, $W^2(t) = (W(t) - W(s))^2 + 2W(s)W(t) - 2W(s)^2 + W(s)^2 = (W(t) - W(s))^2 + 2W(s)(W(t) - W(s)) + W(s)^2 \Rightarrow \mathbb{E}_s[W^2(t) - t] = t - s + 0 + W(s)^2 - t = W(s)^2 - s$.
Exercise 4.
(i) The first order variation is $\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$. We have:

$$T_2 - T_1 = \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 \leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

As $||\Pi||$ approaches zero $|W(t_{k+1}) - W(t_k)|$ also goes to zero. Then, from

$$\lim_{||\Pi|| \to 0} \frac{T_2 - T_1}{\max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)|} \leq \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|$$

We have that the sample first variation approaches ∞.
(ii) The sample cubic variation:

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 =$$
$$= \max_{0 \leq k \leq n-1} |W(t_{k+1}) - W(t_k)| (T_2 - T_1) \to 0 \text{ as } n \to \infty$$

Exercise 5.
We have to derive the price for European call option using a price following a geometric Brownian motion:

$$S(t) = S(0) e^{(r - \frac{1}{2}\sigma^2)t - \sigma W(t)},$$

where r and *sigma* are fixed numbers. We have:

$$\mathbb{E}\left[e^{-rT}(S(T) - K)^+\right] = \int_a^{+\infty} e^{-rT} \left(S(0) e^{(r - \frac{1}{2}\sigma^2 T) + x} - K\right) \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{x^2}{2T\sigma^2}} \, dx,$$

where $a = \left(\ln \frac{K}{S(0)} - \left(r - \frac{1}{2}\sigma^2\right) T\right)$. Then, we split integral on two:

$$I_1 = \int_a^{+\infty} e^{-rT} S(0) e^{(r - \frac{1}{2}\sigma^2 T) + x} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{x^2}{2T\sigma^2}} \, dx = S(0) \int_a^{+\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{(x - T\sigma^2)^2}{2T\sigma^2}} \, dx =$$
$$= S(0) \int_b^{+\infty} \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{x^2}{2T\sigma^2}} \, dx = -\frac{S(0)}{\sigma\sqrt{T}} N(b),$$

where $b = a - T\sigma^2 = \left(\ln \frac{K}{S(0)} - \left(r + \frac{1}{2}\sigma^2\right) T\right)$.

$$I_2 = -\int_a^{+\infty} K \frac{1}{\sqrt{2\pi T}\sigma} e^{-\frac{x^2}{2T\sigma^2}} \, dx = \frac{K}{\sigma\sqrt{T}} N(a)$$

Replacing $-a$ by $d_-(T, S(0))$ and $-b$ by $d_+(T, S(0))$ we get the final formula:

$$\mathbb{E}\left[e^{-rT}(S(T) - K)^+\right] = \frac{S(0)}{\sigma\sqrt{T}} N(d_+(T, S(0))) - \frac{K}{\sigma\sqrt{T}} N(d_-(T, S(0))).$$