Stochastic Calculus

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Ito's Integral for Simple Integrands

We fix a positive number T and seek to make sense of

$$\int_0^T \Delta(t)dW(t)$$

where $W(t)$ is a Brownian motion with a filtration $\mathcal{F}(t)$ for this Brownian motion. We also let $\Delta(t)$ be an adapted stochastic process. Eventually $\Delta(t)$ be the position we take in an asset at time t, so it's quite obvious that it could depend only on the information available at that moment.
Recall that increments of the Brownian motion after time t are independent of $\mathcal{F}(t)$, and since $\Delta(t)$ is $\mathcal{F}(t)$-measurable, it must also be independent of these future Brownian increments.
The problem we face here is that Brownian motion paths cannot be differentiated with respect to time as we do it in an ordinary (Lebesgue) integral.

Construction of the Integral

Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a partition of $[0, T]$. Assume that $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$ (*simple process*). It should be noted that a path of $\Delta(t)$ depends on the same ω on which the path of the Brownian motion $W(t)$ depends. The value of $\Delta(t)$ can depend only on the information available at time t. Thus, the value of $\Delta(t)$ on the second interval $[t_1, t_2)$, can depend only on observations made during the first time interval $[0, t_1)$.
Then, if $t_k \leq t \leq t_{k+1}$

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)]$$

is the Ito integral of the simple process $\Delta(t)$, we denote it as:

$$I(t) = \int_0^t \Delta(u)dW(u).$$

Properties of the Integral

Theorem. *The Ito integral defined above is a martingale*:

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s).$$

Because $I(t)$ is a martingale and $I(0) = 0$, we have $\mathbb{E}I(t) = 0$ for all $t \geq 0$. It follows that $\mathbb{V}I(t) = \mathbb{E}I^2(t)$.
Theorem (Ito isometry). *The Ito integral satisfies:*

$$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$$

.
Proof sketch. Let $D_j = W(t_{j+1}) - W(t_j)$, then $I(t) = \sum_{j=0}^k \Delta(t_j)D_j$. We explicitly write:

$$I^2(t) = \sum_{j=0}^k \Delta^2(t_j)D_j^2 + \sum_{0\leq i<j\leq k} \Delta(t_j)\Delta(t_i)D_jD_i$$

Using the properties of the Brownian motion, we have D_j is independent of $\mathcal{F}(t_j)$. Furthermore, $\mathbb{E}D_j = 0$. We also know that $\mathbb{E}[D_j^2] = t_{j+1} - t_j$, so

$$\mathbb{E}[I^2(t)] = \sum_{j=0}^k \mathbb{E}[\Delta^2(t_j)D_j^2] = \sum_{j=0}^{k-1} \mathbb{E}\Delta^2(t_j)(t_{j+1} - t_j) + \mathbb{E}\Delta^2(t_k)(t - t_k). \square$$

Finally, we turn to the quadratic variation of the Ito integral. Brownian motion accumulates quadratic variation at rate one per unit time. However, Brownian motion is scaled in a time- and path-dependent way by the integrand $\Delta(u)$ in the Ito integral. Because increments are squared in the computation of quadratic variation, the quadratic variation of Brownian motion will be scaled by $\Delta^2(u)$:
Theorem. *The quadratic variation accumulated up to time t by the Ito integral is:*

$$[I, I](t) = \int_0^t \Delta^2(u)du$$

Proof sketch. Consider one of the subintervals $[t_j, t_{j+1}]$ on which $\Delta(u)$ is constant. For a partition $t_j = s_0 < s_1 < \cdots < s_m = t_{j+1}$, we have:

$$\sum_{i=0}^{m-1} [I(s_{i+1}) - I(s_i)]^2 = \Delta^2(t_j) \sum_{i=0}^{m-1} (W(s_{i+1}) - W(s_i))^2$$

As $m \to \infty$, $\sum_{i=0}^{m-1} (W(s_{i+1}) - W(s_i))^2$ converges to $t_{j+1} - t_j$. Thus,

$$\Delta^2(t_j)(t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \Delta^2(u)du. \square$$

From the theorems above we see that the variance of $I(t)$ is an average over all possible paths of the quadratic variation.
Finally, we recall that $dW(t)dW(t) = dt$ we interpret as the statement that Brownian motion accumulates quadratic variation at rate one per unit time. It is another way of writing $[W, W](t) = t$. The Ito integral formula $I(t) = \int_0^t \Delta(u)dW(u)$ ca be written in differential form as $dI(t) = \Delta(t)dW(t)$, and then:

$$dI(t)dI(t) = \Delta^2(t)dt,$$

so the Ito integral accumulates quadratic variation at rate $\Delta^2(t)$ per unit time.
Remark. (on notation). The notations

$$I(t) = \int_0^t \Delta(u)dW(u)$$

and

$$dI(t) = \Delta(t)dW(t)$$

mean almost the same thing. The first equation means the 'sum' meaning (see definition), the second equation has the imprecise meaning that when we move forward a little bit in time from time t, the change in the Ito integral I is $\Delta(t)$ time the change in the Brownian motion W. It also has a precise meaning:

$$I(t) = I(0) + \int_0^t \Delta(u)dW(u)$$

so the only difference between the *differential* form and the *integral* form is that the second assumes that $I(0) = 0$, whereas the first one permit $I(0)$ to be any arbitrary constant.

Ito's Integral for General Integrands

Now we want to calculate the Ito integral for integrands that are allowed to vary continuously with time and also to jump. We assume that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$. We also assume the square-integrability condition:

$$\mathbb{E} \int_0^T \Delta^2(t)dt < \infty.$$

In order to define $\int_0^T \Delta(t)dW(t)$, we approximate $\Delta(t)$ by simple processes. The approximating simple integrand is constructed by choosing a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n$, setting the approximating simple process equal to $\Delta(t_j)$ at each t_j, and then holding the simple process constant over the subintegral $[t_j, t_{j+1})$.
In general, it is possible to choose a sequence $\Delta_n(t)$ of simple processes such that as $n \to \infty$, these processes converge to the continuously varying $\Delta(t)$. By "converge" we mean that:

$$\lim_{n\to\infty} \mathbb{E}\int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0.$$

For each $\Delta_n(t)$, the corresponding Ito integral $\int_0^t \Delta_n(u)dW(u)$ has already been defined. We define the Ito integral for continuously varying integrand by the formula:

$$\int_0^t \Delta(u)dW(u) = \lim_{n\to\infty} \int_0^t \Delta_n(u)dW(u)$$

Theorem. *Let T be a positive constant and let $\Delta(t)$ be an adapted stochastic process that satisfies the square-integrability condition. Then $I(t) = \int_0^t \Delta(u)dW(u)$ defined above has the following properties:*
(i) **(Continuity)** *As a function of the upper limit of the integration t, the paths of $I(t)$ are continuous.*
(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*
(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)dW(u)$ and $J(t) = \int_0^t \Gamma(u)dW(u)$, then $I(t) \pm J(t) = \int_0^t (\Delta(u) \pm \Gamma(u))dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)dW(u)$.*
(iv) **(Martingale)** *$I(t)$ is a martingale.*
(v) **(Iso isometry)** *$\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)du$.*
(vi) **(Quadratic variation)** *$[I,I](t) = \int_0^t \Delta^2(u)du$.*

Example. Using a linear approximation, one could show that:

$$\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}[W,W](T) = \frac{1}{2}W^2(T) - \frac{1}{2}T$$

The extra term $-\frac{1}{2}T$ comes from the nonzero quadratic variation of Brownian motion and the way we've constructed the Ito integral, always evaluating the integrand at the left-hand endpoint of the subinterval. If we instead to evaluate at the midpoint, then we would not have gotten this term. However, it is inappropriate for finance. *In finance, the integrand represents a position in an asset and the integrator represents the price of that asset. We cannot decide at 9:00 a.m. which position we took at 11 a.m. Thus, we must decide the position at the beginning of each time interval, and the Ito integral is the limit of the gain achieved by that kind of trading as the time between trades approaches zero.*
The theorem above guarantees that $\int_0^t W(u)dW(u)$ is a martingale and hence has constant expectation. At $t = 0$, this martingale is 0, and hence its expectation. This is indeed the case because $\mathbb{E}W^2(t) = t$. If the term $-\frac{1}{2}t$ were not present, we would not have a martingale.

Additional example. Let's show that

$$\int_0^t W(z)dz = \int_0^t (t-z)dW(z)$$

Indeed, consider the right-hand integral:

$$\int_0^t (t-z)dW(z) = \lim_{||\Pi||\to 0} \sum_{i=0}^{k-1} (t-z_i)(W(z_{i+1}) - W(z_i)) = t(W(t) - W(0)) -$$

$$- \lim_{||\Pi||\to 0} \sum_{i=0}^{k-1} z_i(W(z_{i+1}) - W(z_i)) = \lim_{||\Pi||\to 0} \sum_{i=0}^{k-1} z_{i+1}(W(z_{i+1}) - W(z_i)) -$$

$$- \lim_{||\Pi||\to 0} \sum_{i=0}^{k-1} z_i(W(z_{i+1}) - W(z_i)) = \lim_{||\Pi||\to 0} \sum_{i=0}^{k-1} (z_{i+1} - z_i)W(t_{i+1}) = \int_0^t W(z)dz$$

Ito-Doeblin Formula

We want a rule to "differentiate" expressions of the form $f(W(t))$, where $f(t)$ is a differentiable function. Because W has nonzero quadratic variation, the result has an extra term:

$$df(W(t)) = f'(W(t))dW(t) + \frac{1}{2}f''(W(t))dt.$$

This is the *Ito-Doeblin formula in differential form.* In *integral form* we have:

$$f(W(t)) - f(W(0)) = \int_0^t f'(W(u))dW(u) + \frac{1}{2}\int_0^t f''(W(u))du.$$

In more general case we have:

Theorem (Ito-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,*

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))dt +$$
$$+ \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))dt$$

Remark. In differential form we have the sketch of the proof:

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dW(t)dW(t) +$$
$$+ f_{tx}(t, W(t))dW(t)dt + \frac{1}{2}f_{tt}(t, W(t))dtdt =$$
$$= f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt$$

Example. The Ito-Doeblin formula often simplifies the computation of Ito integrals. For example, with $f(x) = \frac{1}{2}x^2$ we have

$$\begin{aligned}\frac{1}{2}W^2(T) = f(W(T)) - f(W(0)) = \\ &= \int_0^T f'(W(t))dW(t) + \frac{1}{2}\int_0^T f''(W(t))dt = \\ &= \int_0^T W(t)dW(t) + \frac{1}{2}T,\end{aligned}$$

so we've obtained the formula for the $\int_0^T W(t)dW(t)$ without approximation by simple processes as we did it previously.

Formula for Ito Processes

Now we extend the Ito-Doeblin formula to stochastic processes more general than Brownian motion. We develop it for more general class of processes - the *Ito processes.* Almost all stochastic processes, except those that have jumps, are Ito processes.

Definition. *Let $W(t)$ be a Brownian motion, and let $\mathcal{F}(t)$ be an associated filtration. An Ito process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adapted stochastic processes.

Lemma. *The quadratic variation of the Ito process is:*

$$[X,X](t) = \int_0^t \Delta^2(u)du.$$

Remark. In differential form we have:

$$dX(t) = \Delta(t)dW(t) + \Theta(t)dt$$

and

$$dX(t)dX(t) = \cdots = \Delta^2(t)dt$$

Each time t the process X is accumulating quadratic variation at rate $\Delta^2(t)$ per unit time.

Note also that $R(t) = \int_0^t \Theta(u)du$ can be random, but it less volatile then $I(t) = \int_0^t \Delta(u)dW(U)$, and hence, doesn't affect the variation. At each time we have a good estimate of the next increment of $R(t)$:

$$R(t+h) \approx R(t) + \Theta(t)h$$

Definition. *Let $X(t)$ be an Ito process and let $\Gamma(t)$ be an adapted process. We define the integral with respect to an Ito process*

$$\int_0^t \Gamma(u)dX(u) = \int_0^t \Gamma(u)\Delta(u)dW(u) + \int_0^t \Gamma(u)\Theta(u)du.$$

Theorem (Ito-Doeblin formula for an Ito process). *Let $X(t)$ be an Ito process, let $f(t,x)$ be a function with defined and continuous partial derivatives over t, x and (xx). Then, for every T*

$$\begin{aligned} f(T, X(T)) = f(0, X(0)) &+ \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))dX(t) + \\ &+ \int_0^T f_{xx}(t, X(t))d[X,X](t) = \\ = f(0, X(0)) &+ \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\Delta(t)dW(t) + \\ &+ \int_0^T f_x(t, X(t))\theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t, X(t))\Delta^2(t)dt. \end{aligned}$$

Remark. It is easier to remember and use the differential notation:

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2} f_{xx}(t, X(t))dX(t)dX(t).$$

The guiding principle again is that we write out the Taylor series expansion of $f(t, X(t))$ with respect to all its arguments. Using the differential form of the Ito process ($dX(t) = \Delta(t)dW(t) + \Theta(t)dt$) and the accumulation of quadratic variation ($dX(t)dX(t) = \Delta^2(t)dt$), we obtain:

$$\begin{aligned} df(t, X(t)) = f_t(t, X(t))dt &+ f_x(t, X(t))\Delta(t)dW(t) + \\ &+ f_x(t, X(t))\Theta(t)dt + \frac{1}{2} f_{xx}(t, X(t))\Delta^2(t)dt \end{aligned}$$

Examples

Example (Generalized geometric Brownian motion). Let $W(t)$ be a Brownian motion, let $\mathcal{F}(t)$ be an associated filtration, and let $\alpha(t)$ and $\sigma(t)$ be adapted processes. Define the Ito process

$$X(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds$$

Then

$$dX(t) = \sigma(t)dW(t) + \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right) dt$$

and

$$dX(t)dX(t) = \sigma^2(t)dt.$$

Consider an asset price process given by

$$S(t) = S(0)e^{X(t)} = S(0)exp\left\{\int_0^t \sigma(s)dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\}$$

We may write $S(t) = f(X(t))$, where $f(x) = S(0)e^x$. According to the Ito-Doeblin formula

$$\begin{aligned}
dS(t) &= df(X(t)) \\
&= S(0)e^{X(t)}dX(t) + \frac{1}{2}S(0)e^{X(t)}dX(t)dX(t) \\
&= S(t)dX(t) + \frac{1}{2}S(t)dX(t)dX(t) \\
&= S(t)\sigma(t)dW(t) + S(t)\left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt + \frac{1}{2}S(t)\sigma^2(t)dt \\
&= \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)
\end{aligned}$$

So the asset price $S(t)$ has instantaneous mean rate of return $\alpha(t)$ and volatility $\sigma(t)$. Both of them are allowed to be random.
This example includes all possible models of an asset price process that is always positive, has no jumps, and is driven by a single Brownian motion.
In the case of constant α and σ

$$S(t) = S(0)exp\left\{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}.$$

One can incorrectly argue form this formula that since Brownian motion is a martingale, the mean rate of return for $S(t)$ must be $\alpha - \frac{1}{2}\sigma^2$. The error in this argument is that although $W(t)$ is a martingale, $S(0)e^{\sigma W(t)}$ is not. In order to correct for this, one must subtract αt in the exponential. The process $S(0)exp\left\{\sigma W(t) + \frac{1}{2}\sigma^2 t\right\}$ is a martingale (i.e. with no drift). If we add αt in the exponential, we get $S(t)$, a process with mean rate of return α.
The Ito-Doeblin formula automatically keeps track of these effects. If $\alpha = 0$, then

$$dS(t) = \sigma(t)S(t)dW(t)$$

Thus, we have the Ito integral

$$S(t) = S(0) + \int_0^t \sigma(s)S(s)dW(s)$$

which is a martingale and hence,

$$S(t) = S(0)exp\left\{\int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma^2(s)ds\right\}$$

is a martingale. When $\alpha(t)$ is nonzero, it plays a role of the mean rate of return.

Theorem (Ito integral of a deterministic integrand). *Let $W(s)$ be a Brownian motion, and let $\Delta(s)$ be a nonrandom function of time. Define $I(t) = \int_0^t \Delta(s)dW(s)$. For each $t \geq 0$, the random variable $I(t)$ is normally distributed with expected value zero and variance $\int_0^t \Delta^2(s)ds$.*
Proof sketch. Indeed, as it's an Ito integral, we have $\mathbb{E}I(t) = 0$ and $\mathbb{V}I(t) = \int_0^t \Delta^2(s)ds$. Next, we show that the moment generating function of $I(t)$ is the same as for normal distribution:

$$\begin{aligned}
&\mathbb{E}e^{uI(t)} = \exp\left\{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds\right\} \\
&\Leftrightarrow \mathbb{E}\exp\left\{uI(t) - \frac{1}{2}u^2 \int_0^t \Delta^2(s)ds\right\} = 1 \\
&\Leftrightarrow \mathbb{E}\exp\left\{\int_0^t u\Delta(s)dW(s) - \frac{1}{2}\int_0^t (u\Delta)^2(s)ds\right\} = 1
\end{aligned}$$

But the process under expectation is a martingale as we saw previously for a generalized geometric Brownian motion with zero mean rate of return and $\sigma(s) = u\Delta(s)$. Furthermore, this process takes the value 1 at $t = 1$, and hence its expectation is always 1. $\square$
Example. Consider $X(t) = x_0 + \mu t + \sigma W(t)$. Let's find the distribution of $\int_0^T X(t)dt$. Firstly, we find deterministic integral:

$$\int_0^T (x_0 + \mu t)dt = x_0 t + \frac{\mu T^2}{2}$$

Next,

$$\int_0^T W(t)dt = \int_0^T (T - z)dW(z)$$

From the theorem above we know that this integral is normally distributed with mean zero and variance $\int_0^T (T - z)^2 dz = \frac{T^3}{3}$. As result,

$$\int_0^T X(t)dt \sim N\left(x_0 t + \frac{\mu T^2}{2}, \sigma^2 \frac{T^3}{3}\right).$$

Example (Vasicek interest rate model). The Vasicek model for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

The solution can be determined in a closed form and is:

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s)$$

Indeed, if we consider:

$$f(t,x) = e^{-\beta t}R(0)\frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t}x$$

and $X(t) = \int_0^t e^{\beta s}dW(s)$. Then, using the Ito-Doeblin formula for the Ito process $X(t)$, we get that $f(t,X(t)) = R(t)$. The last theorem implies that the random variable $\int_0^t e^{\beta s}dW(s)$ has mean zero and variance

$$\int_0^t e^{2\beta s}ds = \frac{1}{2\beta}(e^{2\beta t}-1)$$

Therefore, $R(t)$ is normally distributed with mean $e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-e^{-\beta t})$ and variance $\frac{\sigma^2}{2\beta}(e^{2\beta t}-1)$. However, there is always positive probability that $R(t)$ is negative, an undesirable property for an interest rate model.
The Vasicek model has the desirable property that the interest rate is *mean-reverting.* When $R(t) = \frac{\alpha}{\beta}$, the drift term is zero. When $R(t) > \frac{\alpha}{\beta}$, the drift term is negative, which pushes $R(t)$ back toward $\frac{\alpha}{\beta}$. The same for $R(t) < \frac{\alpha}{\beta}$. Thus, if $R(0) = \frac{\alpha}{\beta}$, then $\mathbb{E}R(t) = \frac{\alpha}{\beta}$. If $R(0) \neq \frac{\alpha}{\beta}$, then $\lim_{t\to\infty}\mathbb{E}R(t) = \frac{\alpha}{\beta}$. $\square$
Example (Cox-Ingersoll-Ross (CIR) interest rate model). The Cox-Ingersoll-Ross for the interest rate process $R(t)$ is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t)$$

The CIR equation doesn't have a closed-form solution. The advantage over the Vasicek model is that the interest rate in the CIR model does not become negative. If $R(t)$ reaches zero, the term multiplying $dW(t)$ vanishes and the positive drift term αt drives the interest rate back into positive territory. The CIR model is mean-reverting.
Although one cannot derive a closed-form solution, the distribution of $R(t)$ can be determined. We instead content ourselves only with the derivation of the expectation and variance. To do this, we use $f(t,x) = e^{\beta t}x$ and the Ito-Doeblin formula

$$d(e^{\beta t}R(t)) = \cdots = \alpha e^{\beta t}dt + \sigma e^{\beta t}\sqrt{R(t)}dW(t)$$

From that we have an integral

$$e^{\beta t}R(t) = R(0) + \alpha\int_0^t e^{\beta u}du + \sigma\int_0^t e^{\beta u}\sqrt{R(u)}dW(u)$$

Taking the expectation of both sides and recalling zero expectation of an Ito integral, we get:

$$\mathbb{E}R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1-e^{-\beta t}).$$

To compute variance, we set $X(t) = e^{\beta t}R(t)$, find

$$d(X^2(t)) = 2X(t)dX(t) + dX(t)dX(t)$$

and then take the expectation of this integral to find $\mathbb{E}X^2(t)$. From this we get $\mathbb{E}R^2(t) = e^{-2\beta t}\mathbb{E}X^2(t)$ and finally receive:

$$\lim_{t\to\infty} \mathbb{V}(R(t)) = \frac{\alpha\sigma^2}{2\beta^2}.$$

Exercises

Exercise 1
Let's take some $s < t$ and consider $\mathbb{E}[I(t)|\mathcal{F}(s)]$. Then, for $t_j \geq s$ we have $\mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j)|\mathcal{F}(s)] = 0$ because of the martingale property of the Brownian motion. For j such that $t_j < s < t_{j+1}$ we have $\mathbb{E}[\Delta(t_j)(W(t_{j+1}) - W(t_j)|\mathcal{F}(s)] = \Delta(t_j)(\mathbb{E}[W(t_{j+1})|\mathcal{F}(s)] - W(t_j)) = \Delta(t_j)(W(s) - W(t_j))$. So wee see, that $I(t)$ is a martingale.

Exercise 2
(i) Following the simplification we are provided with, we need to show that $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$. Indeed, $I(t_k) - I(t_l) = \sum_l^k \Delta(t_j)(W(t_{j+1}) - W(t_j))$, which is independent of $\mathcal{F}(t_l)$ by definition of filtration (independence of future increments).
(ii) $I(t) - I(s)$ is normally distributed as the sum of normally distributed increments with $N(0, \Delta(t_j)^2(t_{j+1} - t_j)$. Thus, $\mathbb{E}[I(t) - I(s)] = 0$ and $\mathbb{V}[I(t) - I(s)] = \sum_j \Delta(t_j)^2(t_{j+1} - t_j) = int_s^t \Delta^2(u)du$.
(iii) Indeed, $\mathbb{E}[I(t)|\mathcal{F}(s)] = \mathbb{E}[I(t) - I(s) + I(s)|\mathcal{F}(s)] = \mathbb{E}[I(t) - I(s)|\mathcal{F}(s)] + I(s) = 0$, where the last equality follows from (i) and (ii).
(iv) $I^2(t) = (I(t) - I(s))^2 + 2(I(t) - I(s))I(s) + I^2(s) \Rightarrow \mathbb{E}[I^2(t)|\mathcal{F}(s)] = \int_s^t \Delta^2(u)du + 0 + I^2(s) - \int_0^t \Delta^2(u)du = I^2(s) - \int_0^s \Delta^2(u)du$.

Exercise 5

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t)$$

Let's take $f(x) = \log(x)$ and find $d(\log S(t))$ using the Ito-Doeblin formula. We have $f_t(x) = 0$, $f_x(x) = \frac{1}{x}$ and $f_{xx}(x) = -\frac{1}{x^2}$. Thus,

$$\begin{aligned} d(\log S(t)) &= \frac{1}{S(t)}dS(t) - \frac{1}{2S^2(t)}dS(t)dS(t) = \\ &= \alpha(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma^2(t)dt = \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t) \end{aligned}$$

In the integral notation we have:

$$\log S(t) = \tilde{S}(0) + \int_0^t \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt + \int_0^t \sigma(t)dW(t)$$

Taking the exponent:

$$S(t) = S(0) \exp \left\{ \int_0^t \sigma(s) dW(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds \right\}.$$

Exercise 6
Let $S(t) = S(0) \exp \left\{ \sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2 \right) t \right\}$
Consider $f(x) = x^p(t)$, $p > 0$. Then, $f_t(x) = 0$, $f_x(x) = px^{p-1} = p\frac{f(x)}{x}$ and $f_{xx}(x) = p(p-1)x^{p-2} = p(p-1)\frac{f(x)}{x^2}$.
Thus, by the Ito-Doeblin formula:

$$\begin{aligned} d(S^p(t)) &= pS(t)(\alpha dt + \sigma dW(t)) + \frac{p(p-1)}{2}\sigma^2 S^p(t) dW(t) = \\ &= S^p(t)\left(p\alpha - \frac{p(p-1)}{2}\sigma^2 \right) + p\sigma S^p(t) dW(t) \end{aligned}$$

We see that $S^p(t)$ also follows the Brownian motion process, but with different constants.
Exercise 7
(i) We have $f(x) = x^4$. Thus,

$$dW^4(t) = 4W^3(t)dW(t) + 6W^2(t)dt$$

(ii) In the integral notation:

$$W^4(t) = W^4(0) + 6\int_0^t W^2(s)ds + 4\int_0^t W^3(s)dW(s)$$

The last integral is an Ito integral which is a martingale. Hence,

$$\mathbb{E}W^4(T) = 0 + 6\int_0^T \mathbb{E}W^2(s)ds + 0 = 6\int_0^T sds = 3T^2$$

Exercise 8
The Vasicek interest rate stochastic differential equation is

$$dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t)$$

Consider $f(t,x) = e^{\beta t}x$. Hence, $f_t(t,x) = \beta e^{\beta t}x$, $f_x(t,x) = e^{\beta t}$, $f_{xx}(t,x) = 0$. By the Ito-Doeblin formula:

$$\begin{aligned} d(e^{\beta t}R(t) &= \beta e^{\beta t}R(t)dt + e^{\beta t}(\alpha - \beta R(t))dt + \sigma e^{\beta t}dW(t) = \\ &= \alpha e^{\beta t}dt + \sigma e^{\beta t}dW(t) \end{aligned}$$

In the integral notation:

$$e^{\beta t}R(t) = R(0) + \int_0^t \alpha e^{\beta s}ds + \int_0^t \sigma e^{\beta s}dW(s)$$

Hence,

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}dW(s).$$

Note that by the theorem about the Ito integral over a nonrandom integrand, $R(t)$ is distributed normally.

Black-Scholes-Merton Equation

Idea is the same as in the binomial model, which is to derivation the initial capital required to perfectly hedge a short position in the option.
Consider a portfolio valued at $X(t)$. It could be invested in a money market at a constant rate r and in a stock $S(t)$ that follows GBM:

$$dS(t) = \alpha S(t) + \sigma S(t)dW(t)$$

Suppose at each time the investor holds $\Delta(t)$ shares of stock. This position should be adapted to the filtration associated with the Brownian motion $W(t)$, the remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market. Then

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt = \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

We shall often consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value, $e^{-rt}X(t)$. According to the Ito-Doeblin formula with $f(x) = e^{-rt}x$

$$\begin{aligned} d(e^{-rt}S(t)) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) = \\ &= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t) \end{aligned}$$

and the differential of the discounted portfolio value is:

$$\begin{aligned} d(e^{-rt}X(t)) &= -re^{-rt}X(t)dt + e^{-rt}dX(t) = \\ &= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) = \\ &= \Delta(t)d(e^{-rt}S(t)) \end{aligned}$$

Evolution of Option Value

Consider a European call option that pays $(S(T) - K)^+$ at time T. Denoting $c(t, S(t))$ as the price of this call, consider:

$$\begin{aligned} dc(t, S(t)) &= c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t) = \\ &= \left[c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^t(t)c_{xx}(t, S(t))\right]dt + \sigma S(t)c_x(t, S(t))dW(t) \end{aligned}$$

Next, we take the discounted option price $e^{-rt}c(t, S(t))$

$$
\begin{aligned}
&d(e^{-rt}c(t,S(t))) = -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)) = \\
&= e^{-rt}\left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt \\
&+ e^{-rt}\sigma S(t)c_x(t,S(t))dW(t)
\end{aligned}
$$

Equating the Evolutions

A (short option) hedging portfolio should satisfy $e^{-rt}X(t) = e^{-rt}c(t, S(t))$, or

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t))), \text{ for all } t \in [0,T)$$

and $X(0) = c(0, S(0))$. Thus,

$$
\begin{aligned}
&\Delta(t)(\alpha - r)S(t)dt + \Delta(t)S(t)dW(t) = \\
&= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt \\
&+ e^{-rt}\sigma S(t)c_x(t,S(t))dW(t).
\end{aligned}
$$

We first equate $dW(t)$ terms, which gives:

$$\Delta(t) = c_x(t,S(t)) \text{ for all } t \in [0,T)$$

This is the *delta hedging rule.* We next equate the dt terms and obtain:

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \text{ for all } t \in [0,T)$$

In conclusion, we should seek a continuous function $c(t, x)$ that is a solution to the *Black-Scholes-Merton partial differential equation*

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \text{ for all } t \in [0,T), x \geq 0$$

and that satisfies the *terminal condition*

$$c(T,x) = (x-K)^+.$$

The hedge ratio $\Delta(T) = c_x(t, S(t))$ covers (or hedges) our short option position (i.e. the hedge shows how much to invest in the underlying stock).

Solution to the Black-Scholes-Merton Equation

This equation is of the type called *backward parabolic.* In addition to the terminal condition above, one needs boundary conditions at $x = 0$ and $x = \infty$. The solution to the Black-Scholes-Merton equation with these conditions is:

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),\ 0 \le t < T, x > 0$$

where

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

We sometimes use the notation $BSM(\tau, x, K, r, \sigma)$.

The Greeks

The derivatives of the function $c(t,x)$ with respect to various variables are called the *Greeks*.
If at t the stock price is x, then the short option hedge a position, whose value is $xc_x = xN(d_+)$. The hedging portfolio value is $c = xN(d_+) - Ke^{-r(T-t)}N(d_-)$. Then, the amount invested in the money market must be

$$c(t,x) - xc_x(t,x) = -Ke^{-r(T-t)}N(d_-)$$

Thus, to hedge a short position in a call option, one must borrow money.
Because delta and gamma are positive, the function $c(t,x)$ is increasing and convex in the variable x.

Put-call Parity

A *forward* contract with delivery price K obligates its holder to buy one share of the stock at expiration time T in exchange for payment K. At expiration, the value of the forward contract is $S(T) - K$. Let $f(t,x)$ denote the value of the forward contract at earlier times t if the stock price at time t is $S(t) = x$.
We argue that the value of a forward contract (NOT forward spot price) is given by

$$f(t,x) = x - e^{-r(T-t)}K$$

Indeed, if at $t = 0$ we buy a stock and borrow $e^{-r(T-t)}K$ from the money market, then we will receive $S(T) - K$ at time T.
The *forward price* of a stock at time t is defined to be the value of K that causes the forward contract at time t to have value zero. Hence, we see that in a model with a constant interest rate, the forward price at time t is

$$For(t) = e^{r(T-t)}S(t).$$

Note again that *the forward price is not the price (or value) of a forward contract*. The forward price at time t is the price one can lock at time t or the purchase of one share of stock at time T, paying the price (*settling*) at time T.
Finally, let us consider a *European put*, which pays off $(K - S(T))^+$ at time T. We observe that

$$x - K = (x-K)^+ - (K-x)^+.$$

Thus, denoting by $p(t,x)$ the value of the European put at time t we have

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T))$$

Since that, these value must agree at all previous times

$$f(t,x) = c(t,x) - p(t,x),\ x \geq 0, 0 \leq t \leq T.$$

If this were not the case, there's an arbitrage opportunity. If we make an assumption if a constant interest rate r, we have a price for the forward contract. If we make the additional assumption that the stock is geometric Brownian motion with constant volatility $\sigma > 0$, the we have the Black-Scholes-Merton call formula. Thus, we can obtain the Black-Scholes-Merton put formula

$$p(t,x) = x(N(d_+(T-t,x)) - 1) - Ke^{-r(T-t)}(N(d_-(T-t,x)) - 1) =$$
$$= Ke^{-r(T-t)}N(-d_-(T-t,x)) - xN(-d_+(T-t,x)).$$

Multivariate Stochastic Calculus

Definition. *A d-dimensional Brownian motion is a process*

$$W(t) = (W_1(t), \ldots, W_d(t))$$

with the following properties
(i) Each $W_i(t)$ is a one-dimensional Brownian motion.
(ii) If $i \neq j$, then the processes $W_i(t)$ and $W_j(t)$ are independent.
Associated with a d-dimensional Brownian motion, we have a filtration $\mathcal{F}(t)$, such that the following holds
(iii) **(Information accumulates)** *For $0 \leq s < t$, every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$.*
(iv) **(Adaptivity)** *For each $t > 0$, $W(t)$ is $\mathcal{F}(t)$-measurable.*
(v) **(Independence of future increments.)** *For $0 \leq s < u$, the vector of increments $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.*
As previously, the quadratic variation $[W_i, W_i](t) = t$, which we write informally as

$$dW_i(t)dW_i(t) = dt$$

However, if $i \neq j$, independence of W_i and W_j implies $[W_i, W_j](t) = 0$, which we informally write as

$$dW_i(t)dW_j(t) = 0,\ i \neq j$$

To justify that one should consider a time partition Π and for $i \neq j$ define the *sampled cross variation* of W_i and W_j to be

$$C_\Pi = \sum_{k=0}^{n-1}[W_i(t_{k+1}) - W_i(t_k)][W_j(t_{k+1}) - W_j(t_k)]$$

Next, using the independence of the increments, it could be shown that $\mathbb{E}[C_\Pi] = 0$ and $\mathbb{V}[C_\Pi] \to 0$ as $||\Pi|| \to 0$. That's why, C_Π converges to the constant $\mathbb{E}[C_\Pi] = 0$.

Ito-Doeblin Formula for Multiple Processes

Let's consider a two-dimensional Brownian motion. Let $X(t)$ and $Y(t)$ be Ito processes, which means

$$X(t) = X(0) + \int_0^t \Theta_1(u)du + \int_0^t \sigma_{11}(u)dW_1(u) + \int_0^t \sigma_{12}(u)dW_2(u)$$
$$Y(t) = Y(0) + \int_0^t \Theta_2(u)du + \int_0^t \sigma_{21}(u)dW_1(u) + \int_0^t \sigma_{22}(u)dW_2(u)$$

As we see, the process $X(t)$ accumulates quadratic variation at rate $\sigma_{11}^2(t) + \sigma_{12}^2(t)$ per unit time:

$$[X, X](t) = \int_0^t (\sigma_{11}^2(u) + \sigma_{12}^2(u))du$$

Or, in differential form

$$dX(t)dX(t) = (\sigma_{11}^2(t) + \sigma_{12}^2(t))dt$$

One can also derive it informally from $dtdt = 0$, $dtdW_i(t) = 0$, $dW_i(t)dW_i(t) = t$ and $dW_i(t)dW_j(t) = 0$ for $i \neq j$.
In similar way, we have

$$dY(t)dY(t) = (\sigma_{21}^2(t) + \sigma_{22}^2(t))dt$$
$$dX(t)dY(t) = (\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t))dt$$

where the last equation says that

$$[X, Y](T) = \int_0^T (\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t))dt.$$

Theorem (Two-dimensional Ito-Doeblin formula) *Let $f(t, x, y)$ be a function with defined and continuous 1st and 2nd partial derivatives. Let $X(t)$ and $Y(t)$ be Ito processes as discussed above. The two-dimensional Ito-Doeblin formula is*

$$
\begin{aligned}
df(t,X(t),Y(t)) &= f_t(t,X(t),Y(t))dt + f_x(t,X(t),Y(t))dX(t)+\\
&+ f_y(t,X(t),Y(t))dY(t) + \frac{1}{2}f_{xx}(t,X(t),Y(t))dX(t)dX(t)+\\
&+ f_{xy}(t,X(t),Y(t))dX(t)dY(t) + \frac{1}{2}f_{yy}(t,X(t),Y(t))dY(t)dY(t)
\end{aligned}
$$

Corollary (Ito product rule). *Let $X(t)$ and $Y(t)$ be Ito processes. Then,*

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

Proof: take $f(t,x,y) = xy$ and check.

Recognizing a Brownian Motion

Theorem (Levy, one dimension). Let $M(t)$ be a martingale relative to a filtration $\mathcal{F}(t)$. Assume that $M(0) = 0$, $M(t)$ has continuous paths, and $[M,M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.
Theorem (Levy, two dimensions). *Let $M_1(t)$ and $M_2(t)$ be martingales relative to a filtration $\mathcal{F}(t)$. Assume that $M_i(0) = 0$, $M_i(t)$ has continuous paths and $[M_i,M_i](t) = t$ for all $t \geq 0$. If, in addition, $[M_1,M_2](t) = 0$ for all $t \geq 0$, then $M_1(t)$ and $M_2(t)$ are independent Brownian motions.*
Example (Correlated stock prices). Suppose

$$
\begin{aligned}
\frac{dS_1(t)}{S_1(t)} &= \alpha_1 dt + \sigma_1 dW_1(t)\\
\frac{dS_2(t)}{S_2(t)} &= \alpha_2 dt + \sigma_2[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)],
\end{aligned}
$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions and $\sigma_i > 0$ and $-1 \leq \rho \leq 1$ are constant. To analyze the second stock price process, we define

$$W_3(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$$

Then $W_3(t)$ is a continuous martingale with $W_3(0) = 0$ and

$$dW_3(t)dW_3(t) = \cdots = \rho^2 dt + (1-\rho^2)dt = dt$$

In other words, $[W_3,W_3](t) = t$. According to the one-dimensional Levy Theorem, $W_3(t)$ is a Brownian motion.
Thus, we can write

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2 dW_3(t)$$

we see that $S_2(t)$ is a geometric Brownian motion with mean rate of return α_2 and volatility σ_2.

The Brownian motions $W_1(t)$ and $W_3(t)$ are correlated. According to Ito's product rule

$$d(W_1(t)W_3(t)) = W_1(t)dW_3(t) + W_3(t)dW_1(t) + \rho dt$$

Integrating, we obtain

$$W_1(t)W_3(t) = \int_0^t W_1(s)dW_3(s) + \int_0^t W_3(s)dW_1(s) + \rho t$$

The Ito integrals on the right-hand side have expectation zero (as they are martingales). That's why, the covariance of $W_1(t)$ and $W_3(t)$ is

$$\mathbb{E}[W_1(t)W_3(t)] = \rho t$$

Because both $W_1(t)$ and $W_3(t)$ have standard variation $\sqrt{t}$, the number ρ is the correlation between the processes.

Brownian Bridge

This is a stochastic process that is like a Brownian motion except that with probability one it reaches a specified point at a specified positive time.

Gaussian Processes

Definition. *A Gaussian process $X(t)$, $t \geq 0$ is a stochastic process that has the property that, for arbitrary times $0 < t_1 < t_2 < \cdots < t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.*
We denote

$$m(t) = \mathbb{E}X(t),\ c(s,t) = \mathbb{E}[(X(s) - m(s))(X(t) - m(t))]$$

Example (Brownian motion.) Brownian motion $W(t)$ is a Gaussian process. We have normally distributed and independent increments, so

$$W(t_1) = I_1,\ W(t_2) = I_1 + I_2, \ldots, W(t_n) = \sum_{j=1}^{m} I_j$$

are jointly normally distributed. These random variables are not independent. They have zero mean and the covariance for $s < t$ is given by

$$\begin{aligned} c(s,t) = \mathbb{E}[W(s)W(t)] = \mathbb{E}[W(s)(W(t) - W(s) + W(s))] = \\ = \mathbb{E}[W(s)(W(t) - W(s))] + \mathbb{E}[W(s)^2] = s \end{aligned}$$

In general $c(s,t) = min(s,t)$.
Example (Ito integral if a deterministic integrand). Let $\Delta(t)$ be a non-random function of time, and define

$$I(t) = \int_0^t \Delta(s)dW(s)$$

Then $I(t)$ is a Gaussian process. It could be shown by obtaining the moment generation function formula:

$$\mathbb{E}e^{uI(t)} = e^{\frac{1}{2}u^2 \int_0^t \Delta^2(s)ds}$$

where the right-hand side is the moment generating function for a normal random variable with mean zero and variance $\int_0^t \Delta^2(s)ds$.

Brownian Bridge as a Gaussian Process

Definition. *Let $W(t)$ be a Brownian motion. Fix $T > 0$. We define the Brownian bridge from 0 to 0 on $[0, T]$ to be the process*

$$X(t) = W(t) - \frac{t}{T}W(T),\ 0 \leq t \leq T$$

We see that $X(t)$ satisfies $X(0) = X(T) = 0$. Because $W(T)$ enters the definition of $X(t)$, the Brownian bridge $X(t)$ is not adapted to the filtration $\mathcal{F}(t)$ generated by $W(t)$. We shall later obtain a different process that has the same distribution as the process $X(t)$ but is adapted to this filtration.
The random variables

$$X(t_1) = W(t_1) - \frac{t_1}{T}W(T), \ldots, X(t_n) = W(t_n) - \frac{t_n}{T}W(T)$$

are jointly normal as $W(t_1), \ldots, W(t_n), W(T)$ are jointly normal. One could see that $m(t) = 0$ and the covariance function $c(s,t) = \min(s,t) - \frac{st}{T}$.
Definition. *Let $W(t)$ be a Brownian motion. Fix $T > 0$, $a \in \mathbb{R}$, and $b \in \mathbb{R}$. We define the Brownian bridge from a to b on $[0, T]$ to be the process*

$$X^{a \to b}(t) = a + \frac{(b-a)t}{T} + X(t),\ 0 \leq t \leq T$$

where $X(t) = X^{0 \to 0}$ is the Brownian bridge from 0 to 0.

Brownian Bridge as a Scaled Stochastic Integral

We cannot write the Brownian bridge as a stochastic integral of a deterministic integrand because the variance of the Brownian bridge

$$\mathbb{E}X^2(t) = c(t,t) = t - \frac{t^2}{T}$$

increases for $0 \leq t \leq \frac{T}{2}$ and the decreases for $\frac{T}{2} \leq t \leq T$. In case of deterministic integrands we have a nondecreasing variance $\int_0^t \Delta^2(u)du$.
Theorem. *Define the process*

$$Y(t) = \begin{cases} (T-t)\int_0^t \frac{1}{T-u} dW(u) & ,0 \le t < T \\ 0 & ,t = T \end{cases}$$

The $Y(t)$ is a continuous Gaussian process on $[0,T]$ and has mean and covariance functions

$$m^Y(t) = 0,\ c^Y(s,t) = \min(s,t) - \frac{st}{T}$$

In particular, the process $Y(t)$ has the same distribution as the Brownian bridge from 0 to 0 on $[0,T]$.
Proof sketch. By the theorem above $Y(t)$ is a Gaussian process. Next, it could be shown that $\mathbb{E}Y(t) = 0$, $c(s,t) = \min(s,t) - \frac{st}{T}\square$.
We note that the process $Y(t)$ is adapted to the filtration generated by the Brownian motion $W(t)$. It is interesting to compute the stochastic differential of $Y(t)$, which is

$$\begin{aligned} dY(t) &= \int_0^t \frac{1}{T-u} dW(u) \cdot d(T-t) + (T-t) \cdot d\int_0^t \frac{1}{T-u} dW(u) = \\ &= -\int_0^t \frac{1}{T-u} dW(u) \cdot dt + dW(t) = \frac{Y(t)}{T-t} dt + dW(t) \end{aligned}$$

The drift term $\frac{Y(t)}{T-t}dt$ drives $Y(t)$ towards zero as t approaches T.

Appendix

There's one interesting theorem from the "Mathematical modelling and computational finance" by Oostrlee and Grzelak.
Theorem (Martingale Representation Theorem). *Let $W(t)$ with $W(t_0) = W_0$ be a Brownian motion with the filtration $\mathcal{F}(t)$. For any $X(t)$ - a martingale relative to this filtration, there's an adapted process $g(t)$, such that:*

$$dX(t) = g(t)dW(t), \text{ or } X(t) = X(0) + \int_0^t g(t)dW(t).$$

Exercises

Exercise 4.9
(i) We have to verify $Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$.

Indeed,

$$N'(d_+) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2\tau}\left(log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right)^2\right\} =$$
$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2\tau}\left(log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right)^2\right\} \cdot \exp\left\{\log\frac{x}{K} + \tau r\right\} =$$
$$= \frac{x}{K} \cdot e^{r(T-t)} \cdot N'(d_-)$$

that verifies our equality.
(ii) We have $c_x(t,x) = N(d_+) + N'(d_+) - \frac{K}{x}e^{-r(T-t)}N'(d_-)$. Recall the result from (i). Thus, we see that $c_x = N'(d_+)$.
(iii) We again use the fact from (i): $c_t = xN'(d_+)\frac{1}{\sigma\tau^{3/2}}\log\frac{x}{K} - rKe^{-r(T-t)}N(d_-) - Ke^{-r(T-t)}N'(d_-)\left(\frac{1}{\sigma\tau^{3/2}}\log\frac{x}{k} + \frac{\sigma}{2\sqrt{T-t}}\right) = -rKe^{-r(T-t)}N(d_-) - \frac{x\sigma}{2\sqrt{T-t}}N'(d_+)$.
(iv) Firstly, $c_{xx} = N'(d+) \cdot \frac{1}{x\sigma\sqrt{\tau}}$. Thus,

$$c_t + rxc_x + \frac{1}{2}\sigma^2x^2c_{xx} = -rKe^{-r(T-t)}N(d_-) - \frac{x\sigma}{2\sqrt{T-t}}N'(d_+) + rxN(d_+) +$$
$$+ \frac{\sigma x}{2\sqrt{T-t}}N'(d_+) = r(xN(d_+) - Ke^{-r(T-t)}N(d_-)) = rc$$

(v) Indeed, if $x > K$ then $\log\frac{x}{K} > 0$, if $x < K$ then $\log\frac{x}{K} < 0$, which implies the result.
(vi) It follows from the $\log\frac{x}{K}$ term. The boundary condition follows from the properties of normal distribution.
(vii) Quite simple.

Exercise 4.11
Let's use the Ito formula for $f(x,t) = e^{-rt}x$ for the portfolio process $X(t)$. We have $f_t(t,x) = -re^{-rt}x$, $f_x(t,x) = e^{-rt}$ and $f_{xx}(t,x) = 0$. Thus,

$$d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t)$$

Substituting $dX(t)$ here (with subsequent elimination of the term $rX(t)$ in $dX(t)$) and recalling that actual price movement of our option follows:

$$d(c(t,S(t))) = c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}\sigma_2^2S^2(t)c_{xx}(t,S(t))dt$$

we have:

$$d(e^{-rt}X(t)) = -rc(t,S(t)) + c_t(t,S(t)) + rS(t)c_x(t,S(t))dt + \frac{1}{2}\sigma_1^2S^2(t)c_{xx}(t)dt$$

Finally, note that the price $c(t,S(t))$ we observe suits the BSM SDE with volatility σ_1^2:

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))$$

From that we obtain $d(e^{-rt}X(t)) = 0$, which gives us zero portfolio value $X(t) = 0$.

Exercise 4.12

(i) By the put-call parity $c(t,x) - p(t,x) = S(t) + Ke^{-r(T-t)}$ we have: $p_x(t,x) = c_x(t,x) - 1$, $p_{xx}(t,x) = c_{xx}(t,x)$ and $p_t(t,x) = c_t(t,x) - rKe^{-r(T-t)}$.

(ii) Indeed, for a long put position we have $p_x = c_x - 1 \leq 0$. Thus, for a short put position $p_x \geq 0$, so we need to hold a short position in the asset. In other words, in order to replicate the final payment of $(K - S(t))^+$ we have to hold a long position in money market and a short position in the stock.

(iii) $f_t(t,x) = -re^{-r(T-t)}K$, $f_x(t,x) = 1$, $f_{xx}(t,x) = 0$. Thus,

$$f_t(t,x) + rxf_x(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) = r(x - Ke^{-r(T-t)}) = rf(t,x)$$

Finally, as $p(t,x) = c(t,x) - f(t,x)$, it's also a solution of the PDE.

Exercise 4.13

Indeed, $dW_1(t) = dB_1(t)$ and $dW_2(t) = \frac{1}{\sqrt{1-\rho^2(s)}}(dB_2(t) - \rho dB_1(t))$. Thus,

$$dW_1(t)dW_2(t) = \frac{1}{\sqrt{1-\rho^2(s)}}(\rho dt - \rho dt) = 0$$

which implies that $W_1(t)$ and $W_2(t)$ are independent.

Exercise 4.15

(i) Indeed, as it's a linear combination of Brownian motions it obviously has independent increments that are normally distributed with mean zero and:

$$dB_i(t)dB_i(t) = \frac{\sum_{j=1}^{d} \sigma_{ij}^2(t)}{\sigma_i(t)} dt = dt$$

Thus, by Ito isometry we have $\mathbb{V}[B_i(t_1) - B_i(t_2)] = t_1 - t_2$.

(ii) Obviously follows from the definition.

Exercise 4.16

Could be proven by contradiction.

Exercise 4.17

(i) As $B_1(t_0+\epsilon)B_2(t_0+\epsilon) = \int_0^{t_0+\epsilon} d(B_1(s)B_2(s)) = B_1(t_0)B_2(t_0) + \int_{t_0}^{t_0+\epsilon} d(B_1(s)B_2(s))$, we find:

$$d(B_1(t)B_2(t)) = dB_1(t)B_2(t) + B_1(t)dB_2(t) + dB_1(t)dB_2(t)$$

Thus,

$$\int_{t_0}^{t_0+\epsilon} d(B_1(t)B_2(t)) = \int_{t_0}^{t_0+\epsilon} B_2(t)dB_1(t) + \int_{t_0}^{t_0+\epsilon} B_1(t)dB_2(t) + \rho\epsilon$$

As Ito integral is a martingale, then for $I(t_0+\epsilon) = \int_{t_0}^{t_0+\epsilon} B_1(t)dB_2(t)$ we have $\mathbb{E}[I(t_0+\epsilon)|\mathcal{F}(t_0)] = I(t_0) = 0$, then

$$\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} d(B_1(t)B_2(t))\Big|\mathcal{F}(t_0)\right] = \rho\epsilon$$

Finally, using the Brownian motion martingale property:

$$\mathbb{E}[(B_1(t_0+\epsilon)-B_1(t_0))(B_2(t_0+\epsilon)-B_2(t_0))|\mathcal{F}(t_0)] = \mathbb{E}[B_1(t_0+\epsilon)B_2(t_0+\epsilon)|\mathcal{F}(t_0)]+ \\ + 2B_1(t_0)B_2(t_0) - B_1(t_0)B_2(t_0) = B_1(t_0)B_2(t_0) + \rho\epsilon - B_1(t_0)B_2(t_0) = \rho\epsilon$$

(ii) By the definition

$$X_i(t_0+\epsilon) - X_i(t_0) = \theta_i\epsilon + \sigma_i(B_i(t_0+\epsilon) - B_i(t_0))$$

Thus, using the martingale property:

$$M_i(\epsilon) = \theta_i\epsilon$$

Next,

$$V_i(\epsilon) = \mathbb{E}[\Theta_i^2\epsilon^2 + 2\Theta_i\epsilon\sigma_i(B_i(t_0+\epsilon) - B_i(t_0)) + \sigma_i^2(B_i(t_0+\epsilon) - B_i(t_0))^2|\mathcal{F}(t_0)] - \\ - \Theta_i^2\epsilon^2 = 0 + 0 + \sigma_i^2\epsilon = \sigma_i^2\epsilon$$

And finally,

$$C(\epsilon) = \mathbb{E}[\Theta_1\Theta_2\epsilon^2 + \Theta_1\epsilon\sigma_2(B_2(t_0+\epsilon) - B_2(t_0)) + \Theta_2\epsilon\sigma_1(B_1(t_0+\epsilon) - B_1(t_0)) + \\ + \sigma_1\sigma_2(B_1(t_0+\epsilon)-B_1(t_0))(B_2(t_0+\epsilon)-B_2(t_0))|\mathcal{F}(t_0)] - \Theta_1\Theta_2\epsilon^2 = 0+0+0+\sigma_1\sigma_2\rho\epsilon = \\ = \sigma_1\sigma_2\rho\epsilon.$$

(iii) $X_i(t_0+\epsilon) - X_i(t_0) = \int_{t_0}^{t_0+\epsilon} \Theta_i(u)du + \int_{t_0}^{t_0+\epsilon} \sigma_i(u)dB_i(u)$
By the martingale property and the fact that $|\Theta_i(t)| \leq M$ we have:

$$\mathbb{E}[X_i(t_0+\epsilon) - X_i(t_0)|\mathcal{F}(t_0)] = \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \Theta_i(u)du\Big|\mathcal{F}(t_0)\right]$$

Finally, using the Dominated Convergence Theorem, we obtain:

$$M_i(\epsilon) \to \Theta_i(t_0)\epsilon, \text{ or } \lim_{\epsilon\to 0}\frac{1}{\epsilon}M_i(\epsilon) = \Theta_i(t_0).$$

(iv) We follow the suggested procedure:

$$
\begin{aligned}
D_{ij} = \mathbb{E}[(Y_i(t_0+\epsilon) - Y_i(t_0) + \int_{t_0}^{t_0+\epsilon} \Theta_i(u)du)(Y_j(t_0+\epsilon) - Y_j(t_0)+ \\
+ \int_{t_0}^{t_0+\epsilon} \Theta_j(u)du)|\mathcal{F}(t_0)] - M_i(\epsilon)M_j(\epsilon) = \mathbb{E}[(Y_i(t_0+\epsilon)- \\
- Y_i(t_0))(Y_j(t_0+\epsilon) - Y_j(t_0))|\mathcal{F}(t_0)] + 0 + 0 + M_i(\epsilon)M_j(\epsilon) - M_i(\epsilon)M_j(\epsilon) = \\
= \mathbb{E}[(Y_i(t_0+\epsilon) - Y_i(t_0))(Y_j(t_0+\epsilon) - Y_j(t_0))|\mathcal{F}(t_0)]
\end{aligned}
$$

For $Y_i(t_0+\epsilon)Y_j(t_0+\epsilon) = Y_i(t_0)Y_j(t_0) + \int_{t_0}^{t_0+\epsilon} d(Y_i(t)Y_j(t))$ we use Ito's product rule again:

$$
d(Y_i(t)Y_j(t)) = Y_i(t)\sigma_j(t)dB_j(t) + Y_j(t)\sigma_i(t)dB_i(t) + \sigma_i(t)\sigma_j(t)\rho_{it}(t)dt
$$

Thus,

$$
\begin{aligned}
\mathbb{E}[(Y_i(t_0+\epsilon)-Y_i(t_0))(Y_j(t_0+\epsilon)-Y_j(t_0))|\mathcal{F}(t_0)] = \mathbb{E}[Y_i(t_0+\epsilon)Y_j(t_0+\epsilon)|\mathcal{F}(t_0)]- \\
-2Y_i(t_0)Y_j(t_0)+Y_i(t_0)Y_j(t_0) = Y_i(t_0)Y_j(t_0)+\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \rho_{ij}(u)\sigma_i(u)\sigma_j(u)du \Big| \mathcal{F}(t_0)\right] - \\
- Y_i(t_0)Y_j(t_0)
\end{aligned}
$$

Finally, by the Dominated Convergence Theorem we have

$$
D_{ij}(\epsilon) = \rho_{ij}(t)\sigma_i(t)\sigma_j(t)\epsilon + o(\epsilon).
$$

(v)

$$
\begin{aligned}
V_i(\epsilon) = \mathbb{E}[(X_i(t_0+\epsilon) - X_i(t_0))^2|\mathcal{F}(t_0)] - M_i^2(\epsilon) = \\
= \mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon} \Theta_i(u)du + \int_{t_0}^{t_0+\epsilon} \sigma_i(u)dB_i(u)\right)^2 \Big| \mathcal{F}(t_0)\right] - M_i^2(\epsilon) = \\
= M_i^2(\epsilon) + 0 + \sigma_i^2(t_0)\epsilon,
\end{aligned}
$$

where we've used the martingale property and the Dominated Convergence Theorem.
As for $C(\epsilon)$, it equals $D_{12}(\epsilon)$, which was found above.
(vi) We see that $C(\epsilon) = \rho(t_0)\sqrt{V_1(\epsilon)V_2(\epsilon)} + o(\epsilon)$. Thus,

$$
\lim_{\epsilon \to 0} \frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}} = \rho(\epsilon).
$$

Exercise 4.18
(i) Consider a process $Y(t, W(t)) = \int_0^t \left(r + \frac{1}{2}\theta^2\right) du + \int_0^t \theta dW(u)$. Using the Ito formula for $f(x) = e^{-x}$ ($f_x(x) = -f(x)$, $f_{xx}(x) = f(x)$, $f_t(x) = 0$), we obtain:

$$
\begin{aligned}
d\zeta(t) &= -\zeta(t)dY(t) + \frac{1}{2}\zeta(t)dY(t)dY(t) = \\
&= -\theta\zeta(t)dW(t) - \left(r + \frac{1}{2}\theta^2\right) dt + \frac{1}{2}\theta^2(t)\zeta(t)dt = -\theta\zeta(t)dW(t) - r\zeta(t)dt.
\end{aligned}
$$

(ii) By Ito's product rule:

$$
\begin{aligned}
d(\zeta(t)X(t)) = X(t)d\zeta(t) + \zeta(t)dX(t) + d\zeta(t)dX(t) = -\theta\zeta(t)X(t)dW(t) - \\
- r\zeta(t)X(t)dt + r\zeta(t)X(t)dt + \zeta(t)\Delta(t)(\alpha - r)S(t)dt + \zeta(t)\Delta(t)\sigma S(t)dW(t) - \\
- \theta\zeta(t)\Delta(t)\sigma S(t)dt
\end{aligned}
$$

Next, recall that $\theta = \frac{\alpha - r}{\sigma}$, so

$$
d(\zeta(t)X(t)) = -\theta\zeta(t)X(t)dW(t) + \zeta(t)\Delta(t)\sigma S(t)dW(t)
$$

As the resulting differential equation has no dt term, $\zeta(t)X(t)$ is a martingale.
(iii) Suppose we've constructed a replicating portfolio $X(t)$ (with the adapted portfolio process $\Delta(t)$) such that $X(T) = V(T)$ (we assume that it can be done for any process $V(t)$). Next, we've shown above the process $\zeta(t)X(t)$ is a martingale, so

$$
\mathbb{E}[\zeta(t)X(t)|\mathcal{F}(s)] = \zeta(s)X(s).
$$

Finally, from $\zeta(0) = 1$ it follows that:

$$
X(0) = \zeta(0)X(0) = \mathbb{E}[\zeta(T)X(T)|\mathcal{F}(0)] = \mathbb{E}[\zeta(T)V(T)|\mathcal{F}(0)].
$$