

In that case, the solutions are

$$u = \sum_{\lambda_n \neq \lambda} \frac{(f, \phi_n)}{\lambda_n - \lambda} \phi_n + \sum_{n=M}^N c_n \phi_n$$

where $\{c_M, \dots, c_N\}$ are arbitrary real constants.

4.11 Interior regularity

Roughly speaking, solutions of elliptic PDEs are as smooth as the data allows. For boundary value problems, it is convenient to consider the regularity of the solution in the interior of the domain and near the boundary separately. We begin by studying the interior regularity of solutions. We follow closely the presentation in [9].

To motivate the regularity theory, consider the following simple *a priori* estimate for the Laplacian. Suppose that $u \in C_c^\infty(\mathbb{R}^n)$. Then, integrating by parts twice, we get

$$\begin{aligned} \int (\Delta u)^2 dx &= \sum_{i,j=1}^n \int (\partial_{ii}^2 u)(\partial_{jj}^2 u) dx \\ &= - \sum_{i,j=1}^n \int (\partial_{ij}^3 u)(\partial_j^2 u) dx \\ &= \sum_{i,j=1}^n \int (\partial_{ij}^2 u)(\partial_{ij}^2 u) dx \\ &= \int |D^2 u|^2 dx. \end{aligned}$$

Hence, if $-\Delta u = f$, then

$$\|D^2 u\|_{L^2} = \|f\|_{L^2}.$$

Thus, we can control the L^2 -norm of all second derivatives of u by the L^2 -norm of the Laplacian of u . This estimate suggests that we should have $u \in H_{loc}^2$ if $f, u \in L^2$, as is in fact true. The above computation is, however, not justified for weak solutions that belong to H^1 ; as far as we know from previous existence theory, such solutions may not even possess second-order weak derivatives.

We will consider a PDE

$$(4.34) \quad Lu = f \quad \text{in } \Omega$$

where Ω is an open set in \mathbb{R}^n , $f \in L^2(\Omega)$, and L is a uniformly elliptic of the form

$$(4.35) \quad Lu = - \sum_{i,j=1}^n \partial_i(a_{ij} \partial_j u).$$

It is straightforward to extend the proof of the regularity theorem to uniformly elliptic operations that contain lower-order terms [9].

A function $u \in H^1(\Omega)$ is a weak solution of (4.34)–(4.35) if

$$(4.36) \quad a(u, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega),$$

References

- [9] Some Presentation