Module - IV.

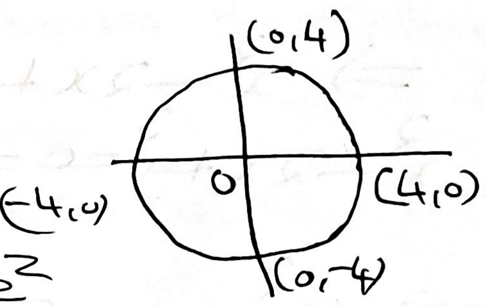
Cauchy's Integral theorem.

If $f(z)$ is analytic inside and on a simple closed path 'C', then $\int_C f(z)\,dz = 0$ [UQ: Dec 21

(1) Evaluate $\oint_C \frac{e^z}{z-5}\,dz$, where 'C' is the circle $|z| = 4$.

→. $\oint_C \frac{e^z}{z-5}\,dz$. or

$z-5=0 \Rightarrow z=5$.

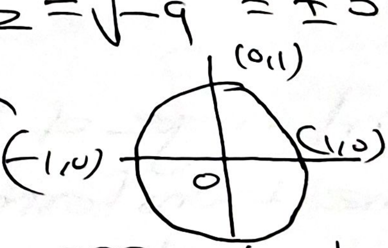


Here $z=5$ lies outside 'C'. Here $f(z) = \frac{e^z}{z-5}$ is analytic inside and on 'C'. Hence by Cauchy's Integral theorem

$$\int_C \frac{e^z}{z-5}\,dz = 0.$$

(2) Evaluate $\oint_C \frac{\cos z}{z^2+9}\,dz$, where 'C' is $|z|=1$. [UQ: (CWP) Dec 2023]

→. $z^2+9=0 \Rightarrow z^2=-9 \Rightarrow z=\sqrt{-9} = \pm 3i$.



Here $z=3i, z=-3i$ both lie outside 'C'. Hence by Cauchy's Integral theorem

$$\int_C \frac{\cos z}{z^2+9}\,dz = 0$$

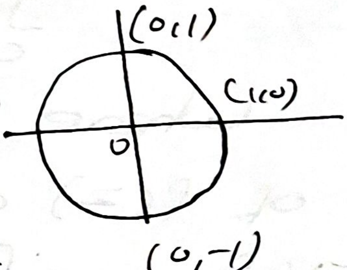
(3) Evaluate $\int_C \tan z \, dz$, where 'C' is the circle $|z|=1$. [UQ: May 2025].

A). $\tan z = \frac{\sin z}{\cos z}$.

$\therefore \cos z = 0 \Rightarrow z = (2n+1)\frac{\pi}{2}$ [odd multiple of $\frac{\pi}{2}$]
$, n = 0, \pm 1, \pm 2, \ldots$

ie, $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

All these points of 'z' clearly lie outside 'C'. Hence $f(z) = \tan z = \frac{\sin z}{\cos z}$ is analytic inside and on 'C'. Hence by Cauchy's Integral theorem



$\pi = 3.14$.
$\therefore \frac{\pi}{2} = \frac{3.14}{2} = 1.57$.

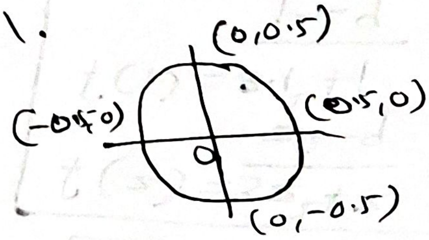
$$\oint_C \tan z \, dz = 0.$$

(4) Evaluate $\int_C \frac{dz}{(z^2+1)(z-1)}$ where $C: |z| = 0.5$ [UQ: Dec 2023].

A) $(z^2+1)(z-1) = 0 \Rightarrow z^2+1 = 0$ or $z-1=0$

$z^2+1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \sqrt{-1} = \pm i$

$z-1 = 0 \Rightarrow z = 1$.



Here $z = i$, $z = -i$ and $z = 1$, all these points lie outside 'C'. Hence by Cauchy's Integral theorem

$i \to (0,1)$
$-i \to (0,-1)$

$$\int_C \frac{dz}{(z^2+1)(z-1)} dz = 0.$$

(3)

Cauchy's Integral formula.

If $f(z)$ is analytic inside and on a simple closed path 'C' and $z=z_0$ is a point lie interior to 'C', then

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

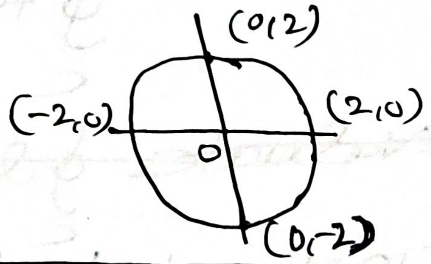
Also
$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0).$$

where $f^{n-1}(z_0)$ denote the $(n-1)^{th}$ derivative of $f(z)$ at $z=z_0$.

(1) Evaluate $\oint_C \frac{e^z}{(z-1)(z-4)}\,dz$ where 'C' is $|z|=2$ using Cauchy's Integral formula. [KU: Dec. 2021].

A).
$$(z-1)(z-4)=0 \Rightarrow z-1=0 \text{ or } z-4=0$$
$$\Rightarrow z=1 \text{ or } z=4.$$

Here $z=1$ lies inside 'C' and $z=4$ lies outside 'C'.



$$\therefore f(z) = \frac{e^z}{z-4}$$

$\boxed{z=4 \text{ lies outside} \Rightarrow z-4 \text{ is taken}}$

$$\therefore \oint_C \frac{e^z}{(z-1)(z-4)}\,dz = \int \frac{f(z)}{z-1}\,dz.$$

$\boxed{z=1 \text{ lies inside} \Rightarrow z-1 \text{ is taken as denominator}}$

$$= 2\pi i\, f(1)$$

$$= 2\pi i \cdot \frac{e^1}{-3}$$

$\boxed{f(1) = \frac{e^1}{1-4} = \frac{e^1}{-3}}$

$$= \frac{-2\pi i e^1}{3}$$

② Evaluate $\int_C \frac{3z^2+z}{(z^2-9)(z-1)}\,dz$ where 'C' is the circle $|z-1|=1$.

Sol.

$(z^2-9)(z-1)=0 \Rightarrow z^2-9=0$ or $z-1=0$

$z^2-9=0 \Rightarrow z^2=9 \Rightarrow z=\sqrt{9}=\pm 3.$

$z-1=0 \Rightarrow z=1.$

$z=3 \Rightarrow |z-1|=|3-1|=|2|=2>1.$

$\therefore z=3$ lies outside 'C'.

$z=-3 \Rightarrow |z-1|=|-3-1|=|-4|=4>1.$

$z=-3$ lies outside 'C'.

$z=1 \Rightarrow |z-1|=|1-1|=|0|=0<1.$

$z=1$ lies inside 'C'.

$\therefore z=3,-3$ both lie outside 'C' and $z=1$ lies inside 'C'.

$$\therefore f(z)=\frac{3z^2+z}{z^2-9}.$$

$\because z=3$ & $z=-3$

$\Rightarrow z-3$ & $z+3$.

$\Rightarrow (z-3)(z+3)$

$\Rightarrow (z^2-3^2)$

$\Rightarrow (z^2-9)$

$$\therefore \int_C \frac{3z^2+z}{(z^2-9)(z-1)}\,dz = \int_C \frac{f(z)}{z-1}\,dz$$

$\because z=1$ lies inside 'C'

$$=2\pi i\, f(1)$$

$$=\cancel{2}\pi i \cdot \frac{-1}{\cancel{2}}$$

$$=-\pi i$$

$$f(z)=\frac{3z^2+z}{z^2-9}$$

$$f(1)=\frac{3\cdot 1^2+1}{1^2-9}$$

$$=\frac{3+1}{1-9}=\frac{4}{-8}$$

$$=\frac{1}{-2}=\frac{-1}{2}$$

(3) Using Cauchy's Integral formula, evaluate $\int \frac{2z+3}{z^2} dz$, where 'C' is a circle $|z-i|=2$ counterclockwise. [UQ: (WP) Dec. 2023].

Sol.
$$z^2=0 \Rightarrow z=0.$$
$$|z-i|=2.$$
$$\therefore z=0 \Rightarrow |z-i| = |0-i| = |-i|$$
$$= \sqrt{0^2+(-1)^2} = \sqrt{0+1}$$
$$= \sqrt{1} = 1<2.$$

Hence $z=0$ lies inside 'C'.

$$\therefore f(z) = 2z+3.$$

$$\therefore \int_C \frac{2z+3}{z^2} dz = \int_C \frac{f(z)}{z^2} dz$$

$\because z=0$ lies inside

$$= \frac{2\pi i}{(2-1)!} f'(0)$$

$$= \frac{2\pi i}{1!} f'(0).$$

$f(z) = 2z+3.$
$f'(z) = 2\times1+0$
$= 2.$
$\therefore f'(0) = 2.$

$$= \frac{2\pi i}{1} \times 2 = \underline{\underline{4\pi i}}$$

(4) Evaluate $\int_C \frac{e^{3z}}{(z+1)^4} dz$ where 'C' is the circle $|z-1|=3$. [UQ: (WP) Dec 2023].

Sol.
$$(z+1)^4=0 \Rightarrow z+1=0$$
$$\Rightarrow z=-1.$$

⑥

$c : |z-1| = 3.$

$\because z=-1 \Rightarrow |z-1| = |-1-1| = |-2| = 2 < 3.$

$\therefore z=-1$ lies inside 'c'.

$\therefore f(z) = e^{3z}.$

$$\therefore \int_c \frac{e^{3z}}{(z+1)^4}\,dz = \int_c \frac{f(z)}{(z+1)^4}\,dz.$$

$$= \frac{2\pi i}{(4-1)!} f'''(z_0)$$

$$= \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{3!} \times 27e^{-3}$$

$$= \frac{\cancel{2}\pi i}{\cancel{6}} \times \cancel{27}^{\,9} e^{-3}$$

$$= 9\pi i e^{-3}$$

$\because z=-1$ lies inside 'c'.
& $z=-1$ is obtained from $(z+1)^4=0$.
$\therefore (z+1)^4$ is taken as denominator.

$\boxed{z_0 = -1}$

$f(z) = e^{3z}$
$f'(z) = e^{3z}.3 = 3e^{3z}$
$f''(z) = 3e^{3z}.3 = 9e^{3z}$
$f'''(z) = 9e^{3z}3 = 27e^{3z}$
$f'''(-1) = 27e^{3\times -1} = 27e^{-3}$

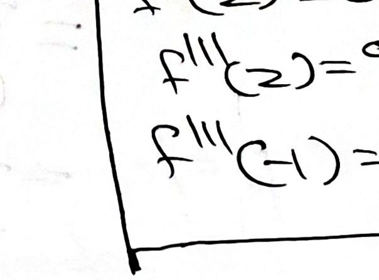
⑤ Integrate counterclockwise around the unit circle $\oint_c \frac{\sin 2z}{z^4}dz$. [UQ: Dec 2020].

s).

$z^4 = 0 \Rightarrow z = 0.$

Here $c : |z| = 1$ [unit circle].

Here $z=0$ lies inside 'C'.



$\therefore f(z) = \sin 2z.$

$$\therefore \oint_c \frac{\sin 2z}{z^4}dz = \oint_c \frac{f(z)}{z^4}dz$$

$z=0$ is obtained from $z^4=0$

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$$= \frac{2\pi i}{(4-1)!} f'''(0)$$ [z=0 lies inside 'C']

$$= \frac{2\pi i}{3!} \cdot -8$$

$$= \frac{\cancel{2}\pi i}{\cancel{6}_3} \times -8$$

$$= -\frac{8\pi i}{3}$$

$f(z) = \sin 2z$
$f'(z) = \cos 2z \cdot 2$
$= 2\cos 2z$
$f''(z) = 2 \cdot -\sin 2z \cdot 2$
$= -4\sin 2z$
$f'''(z) = -4\cos 2z \cdot 2$
$= -8\cos 2z$
$f'''(0) = -8\cos(2\times 0)$
$= -8\cos 0 = -8$

6) Evaluate $\oint_C \frac{\log z}{(z-4)^2} dz$ counterclockwise around the circle $|z-3| = 2$. [UQ: Dec 2020]

Sol). Here $\log z$ is not analytic when $z=0$.

Also $(z-4)^2 = 0 \Rightarrow z-4 = 0 \Rightarrow z = 4$.

$C: |z-3| = 2$.

$\therefore z = 0 \Rightarrow |z-3| = |0-3| = |-3| = 3 > 2$.

$\therefore z=0$ lies outside 'C'.

$z = 4 \Rightarrow |z-3| = |4-3| = |1| = 1 < 2$

$\therefore z=4$ lies inside 'C'.

$\therefore f(z) = \log z$. ∵ $\log z$ is analytic in 'C' as $z=0$ lies outside 'C'.

$$\therefore \oint_C \frac{\log z}{(z-4)^2} dz = \oint_C \frac{f(z)}{(z-4)^2} dz$$

$z=4$ lies inside 'C' and it comes from $(z-4)^2$

$$= \frac{2\pi i}{(2-1)!} f'(4).$$

$f(z) = \log z$
$f'(z) = \frac{1}{z}$
$f'(4) = \frac{1}{4}$

$$= \frac{\cancel{2}\pi i}{1!} \times \frac{1}{\cancel{4}_2} = \frac{\pi i}{2}$$

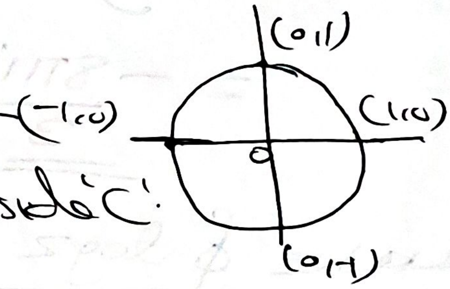
7. Evaluate $\int_C \frac{2z+1}{z^2-2z}\,dz$ where $|z|=1$. [UQ: MP Dec 2023]

→ $z^2-2z=0 \Rightarrow z(z-2)=0$

$\Rightarrow z=0$ or $z-2=0$

$\Rightarrow z=0$ or $z=2$.

Here $z=0$ lies inside 'C' and $z=2$ lies outside 'C'.



$\therefore f(z) = \frac{2z+1}{z-2}$

$\therefore \int_C \frac{2z+1}{z^2-2z}\,dz = \int_C \frac{f(z)}{z}\,dz$

$= 2\pi i\, f(z_0)$

$= 2\pi i\, f(0)$

$= \cancel{2}\pi i \cdot \frac{-1}{\cancel{2}}$

$= -\pi i$

$\therefore \frac{2z+1}{z^2-2z} = \frac{2z+1}{z(z-2)}$.
$z=2$ lies outside 'C'.
$\therefore f(z) = \frac{2z+1}{z-2}$
and $z=0$ lies inside 'C'.

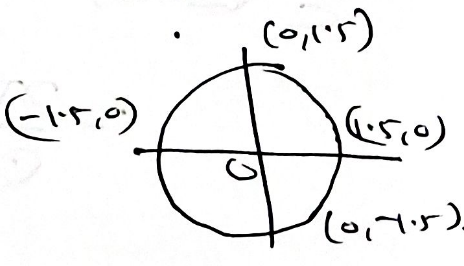
$\boxed{z_0=0}$

$f(z) = \frac{2z+1}{z-2}$
$f(0) = \frac{2\times0+1}{0-2} = \frac{1}{-2}$
$= \frac{-1}{2}$

8. Evaluate $\int \frac{3z^2+7z}{z+1}\,dz$ over (i) $|z|=1.5$ (ii) $|z+i|=1$
[UQ: Dec 2021].

→ $z+1=0 \Rightarrow z=-1$.

(i) $|z|=1.5$
Here $z=-1$ lies inside 'C'.



$\because f(z) = 3z^2 + 7z$

$$\therefore \int_C \frac{3z^2+7z}{z+1}\,dz = \int \frac{f(z)}{z+1}\,dz$$

$\because z=-1$ lies inside 'C'

$\therefore z+1$ is taken as denominator

$= 2\pi i \cdot f(-1)$

$= 2\pi i \cdot -4$

$= \underline{\underline{-8\pi i}}$

$f(-1) = 3\times(-1)^2 + 7\times -1$
$= 3 - 7 = -4$

(ii) $|z+i| = 1$.

$z=-1 \Rightarrow |z+i| = |-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{1+1}$
$= \sqrt{2} > 1$.

Here $z=-1$ lies out side the path 'C'. Hence $f(z) = \frac{3z^2+7z}{z+1}$ is analytic inside and on 'C'.

Hence by Cauchy's Integral theorem

$$\int_C \frac{3z^2+7z}{z+1}\,dz = 0.$$

Taylor & Maclaurin Series.

Taylor series expansion of an analytic function $f(z)$ about $z=z_0$ is given by

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $a_n = \frac{1}{n!} f^n(z_0)$, $n = 0,1,2,3,\ldots$

$f^n(z_0)$ denotes the nth derivative of $f(z)$ at $z=z_0$.

In the above expansion if $z_0 = 0$, then it is known as Maclaurin series expansion.

i.e, $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

where $a_n = \frac{1}{n!} f^n(0)$. $n = 0, 1, 2, 3, \cdots$

(1) Find the Taylor series expansion of e^z about $z = \pi$. [UQ: Dec 2021]

$\Rightarrow$ $f(z) = e^{z - \pi + \pi}$ $\qquad \boxed{z - z_0 = z - \pi}$

$= e^{z-\pi} \cdot e^{\pi}$ $\qquad \boxed{x^{m+n} = x^m \cdot x^n}$

$= e^{\pi} \cdot \left[1 + \frac{z-\pi}{1!} + \frac{(z-\pi)^2}{2!} + \frac{(z-\pi)^3}{3!} + \cdots \right]$

$\boxed{e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$

$= e^{\pi} \left[1 + z - \pi + \frac{(z-\pi)^2}{2!} + \frac{(z-\pi)^3}{3!} + \cdots \right]$

(2) Find the Taylor series expansion of $f(z) = \sin 2z$ about $z = \frac{\pi}{2}$. [UQ: (NP) Dec 2023].

$\Rightarrow$ $\therefore f(z) = \sin 2z$ $\qquad \boxed{z - z_0 = z - \frac{\pi}{2}}$

$= \sin 2\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right)$

$= \sin\left(2\left(z - \frac{\pi}{2}\right) + \cancel{2} \times \frac{\pi}{\cancel{2}}\right)$

$= \sin\left(\overset{A}{2\left(z - \frac{\pi}{2}\right)} + \overset{B}{\pi}\right)$.

$= \sin\left(2\left(z - \frac{\pi}{2}\right)\right) \cos\pi + \cos\left(2\left(z - \frac{\pi}{2}\right)\right) \sin\pi$

$\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$

$= \sin\left(2\left(z - \frac{\pi}{2}\right)\right) \cdot -1 + 0$ $\qquad \boxed{\sin\pi = 0, \cos\pi = -1}$

$= -\sin\left(2\left(z - \frac{\pi}{2}\right)\right)$

$$= 2\left(z-\frac{\pi}{2}\right) - \frac{\left(2\left(z-\frac{\pi}{2}\right)\right)^3}{3!} + \frac{\left(2\left(z-\frac{\pi}{2}\right)\right)^5}{5!} \cdots$$

$$\boxed{\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$= 2\left(z-\frac{\pi}{2}\right) - \frac{2^3\left(z-\frac{\pi}{2}\right)^3}{3!} + \frac{2^5\left(z-\frac{\pi}{2}\right)^5}{5!} - \cdots$$

$$= 2\left(z-\frac{\pi}{2}\right) - \frac{8}{3!}\left(z-\frac{\pi}{2}\right)^3 + \frac{32}{5!}\left(z-\frac{\pi}{2}\right)^3 - \cdots$$

(3) Find the Taylor series of $\frac{1}{z+1}$ $\left(\text{or } \frac{1}{1+z}\right)$ about the centre $z_0 = i$. [UQ: Dec 2020]

Soln)

$$f(z) = \frac{1}{1+z} \qquad \boxed{z - z_0 = z - i}$$

$$= \frac{1}{1+z-i+i} = \frac{1}{(1+i)+z-i}$$

$$= \frac{1}{(1+i)\left(1+\frac{z-i}{1+i}\right)} = \frac{1}{1+i}\left(1+\frac{z-i}{1+i}\right)^{-1}$$

$$= \frac{1}{1+i}\left[1 - \frac{z-i}{1+i} + \left(\frac{z-i}{1+i}\right)^2 - \left(\frac{z-i}{1+i}\right)^3 + \cdots\right]$$

$$\boxed{(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots}$$

$$= \frac{1}{1+i}\left[1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \frac{(z-i)^3}{1+i} + \cdots\right]$$

(4) Find the maclaurin series expansion of $f(z)=\sin z$.

s). We have discussed and solved the maclaurin series expansion of $f(z)=\sin z$ in the online class. Please refer.

(5) Find the maclaurin series expansion of $e^z \sin z$. [UQ: DEC 2023].

s). $f(z) = e^z \sin z$.

$$= \left[1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]$$

$$= z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots+\frac{z^2}{1!}-\frac{z^4}{3!}+\frac{z^6}{5!}-\cdots$$

$$+\frac{z^3}{2!}-\frac{z^5}{2!\cdot 3!}+\cdots$$

$$= z+\frac{z^3}{2!}-\frac{z^3}{3!}+\frac{z^2}{1!}-\frac{z^4}{3!}+\frac{z^5}{5!}-\frac{z^5}{2!\cdot 3!}+\cdots$$

$$= z+\frac{z^2}{1!}+\left(\frac{1}{2!}-\frac{1}{3!}\right)z^3-\frac{z^4}{3!}+\left(\frac{1}{5!}-\frac{1}{2!\cdot 3!}\right)z^5+\cdots$$

$$= z+\frac{z^2}{1!}+\left(\frac{1}{2}-\frac{1}{6}\right)z^3-\frac{z^4}{3!}+\left(\frac{1}{120}-\frac{1}{12}\right)z^5+\cdots$$

$$= z+\frac{z^2}{1!}+\frac{z^3}{3}-\frac{z^4}{6}+\cdots$$