Module-III

Complex Analysis.

Continuity.

To check the continuity of a function $f(z)$, we substitute $x = r\cos\theta$, $y = r\sin\theta$ in the Cartesian form of $f(z)$ [i.e, simplifying $f(z)$ by substituting $z = x+iy$, $|z| = \sqrt{x^2+y^2}$, $\bar{z} = x-iy$]. Then

$$\lim_{z\to 0} f(z) = \lim_{r\to 0} f(r,\theta).$$

If the limit of $f(r,\theta)$ when $r \to 0$ is zero, then we can conclude that limit of $f(z)$ exists and hence $f(z)$ is continuous at $z=0$. Now if the limit is not equal to zero [i.e, a function of θ] then the limit does not exist and hence $f(z)$ is not continuous at $z=0$.

1. Test the continuity at $z=0$ of $f(z) = \begin{cases} \dfrac{\operatorname{Im}(z)}{|z|}, & z \neq 0 \\ 0, & z=0 \end{cases}$

[U.Q. December 2021]

a) $$f(z) = \frac{\operatorname{Im}(z)}{|z|}.$$

$$z = x+iy.$$

$$\therefore \operatorname{Im}(z) = y, \quad |z| = \sqrt{x^2+y^2}.$$

$$\therefore f(z) = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \frac{r\sin\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}}$$

$$= \frac{r\sin\theta}{r} = \sin\theta. \qquad \because \sin^2\theta+\cos^2\theta = 1$$

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0} \sin\theta = \sin\theta.$$

$\therefore$ limit does not exist.

$\Rightarrow f(z)$ is not continuous at $z=0$.

(2)

2) Check whether the function $f(z)=\begin{cases}\frac{Re(z)}{1-|z|}, & z\neq 0\\ 0, & z=0\end{cases}$

is continuous at $z=0$. [UQ: December 2023].

→) $$f(z) = \frac{Re\, z}{1-|z|}.$$

$$z = x+iy.$$
$$\therefore Re\, z = x, \; |z| = \sqrt{x^2+y^2} = r.$$

$$\therefore f(z) = \frac{x}{1-\sqrt{x^2+y^2}} = \frac{r\cos\theta}{1-r}$$

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0}\frac{r\cos\theta}{1-r} = \frac{0}{1-0} = \frac{0}{1}$$
$$= 0.$$

$\therefore$ limit exists and hence $f(z)$ is continuous at $z=0$.

③ Test the continuity of $f(z)$ at the point $z=0$, where

$f(z) = \begin{cases}\frac{z}{|z|}, & z\neq 0\\ 0, & z=0.\end{cases}$ [UQ: May 2025 (S,FE)]

s.) $$z = x+iy, \; |z| = \sqrt{x^2+y^2} = r.$$

$$\therefore f(z) = \frac{z}{|z|} = \frac{x+iy}{\sqrt{x^2+y^2}} = \frac{r\cos\theta + ir\sin\theta}{r}$$

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0}\frac{r\cos\theta + ir\sin\theta}{r}$$

$$= \lim_{r\to 0}\frac{\cancel{r}(\cos\theta + i\sin\theta)}{\cancel{r}} = \cos\theta + i\sin\theta.$$

$\therefore$ limit does not exist and hence $f(z)$ is not continuous at $z=0$.

Analytic functions.

$f(z)$ is analytic $\Rightarrow$ $f(z)$ is defined and differentiable.

C-R equations. (Cauchy-Reimann equations).

To check the analyticity of a given function $f(z)$, we can also use the C-R equations.

$$\begin{array}{cc} u_x & u_y \\ v_x & v_y \end{array} \qquad u_x = v_y,\ u_y = -v_x.$$

Note:-

① An analytic function $f(z)$ is independent of $\bar{z}$ [i.e, it should not contain $\bar{z}$].

② If $f(z) = u+iv$ is analytic then $f'(z) = u_x + iv_x$.

1. Check whether $f(z) = \bar{z}$ is an analytic function? [UQ: December 2023] [UQ: December 2021]

Sol) We know that an analytic function $f(z)$ is independent of $\bar{z}$. Here $f(z)$ contains the term $\bar{z}$, so $f(z)$ is not analytic.

2. Check whether $f(z) = z - \bar{z}$ is analytic or not. [UQ: (WP) December 2023]

Sol). Same answer as above.

3. Show that $f(z) = e^z$ is analytic for all z in the complex plane and find it's derivative. [UQ: (WP) December 2023]

Sol).
$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$$
$$= e^x[\cos y + i\sin y] \qquad \Big| e^{iy} = \cos y + i\sin y$$
$$= \underbrace{e^x\cos y}_{} + i\,\underbrace{e^x\sin y}_{v}.$$

$u = e^x\cos y,\ v = e^x\sin y.$

$u_x = \cos y \cdot e^x,\ u_y = e^x \cdot -\sin y = -e^x\sin y.$

$$v_x = \sin y \cdot e^x, \quad v_y = e^x \cdot \cos y.$$

$$\therefore u_x = v_y, \quad u_y = -v_x.$$

$\Rightarrow$ C-R equations are satisfied.

Hence $f(z)$ is analytic.

$$\therefore f'(z) = u_x + i v_x$$

$$= \cos y \, e^x + i \sin y \, e^x$$

$$= \underline{\underline{e^x (\cos y + i \sin y)}}$$

4) Find all points where the Cauchy-Reimann equations are satisfied for the function $f(z) = xy$ and where it is analytic. Give reasoning for your answers.
[Uo. December 2023].

s).
$$f(z) = \underbrace{xy}_{u} = xy + i\underbrace{0}_{v}$$

$$u = xy, \; v = 0.$$

$$u_x = y \cdot 1 = y, \quad u_y = x \cdot 1 = x.$$

$$v_x = 0, \quad v_y = 0.$$

$$\therefore u_x = v_y \Rightarrow y = 0 \text{ and } u_y = -v_x$$

$$\Rightarrow x = 0$$

$\therefore$ C-R equations are satisfied only when $x = 0, y = 0$ or $z = 0$.

$\therefore f(z)$ is analytic when $z = 0$. $(x = 0, y = 0)$

5. Show that the function $f(z) = \frac{1}{z}$ is analytic except at $z = 0$. [

s)
$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)}$$

$$= \frac{x - iy}{x^2 - (iy)^2} = \frac{x - iy}{x^2 - i^2 y^2} = \frac{x - iy}{x^2 + y^2} \quad \Big| \; i^2 = -1.$$

$$= \underbrace{\frac{x}{x^2 + y^2}}_{u} - i \underbrace{\frac{y}{x^2 + y^2}}_{v}.$$

(5)

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}.$$

$$u_x = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$

$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$

$$u_y = x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y = \frac{-2xy}{(x^2+y^2)^2}.$$

$$v_x = -y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = -\left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right] = -\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$\therefore u_x = v_y, \; u_y = -v_x$ except when $(x^2+y^2)^2 = 0$

$\Rightarrow$ C-R equations are satisfied except when $x^2+y^2=0 \Rightarrow$ except when $x=0, y=0$. (ie, $z=0$)

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$.

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or

$f(z) = \frac{1}{z}$ is defined for all values in the complex plane except at $z=0$. (ie, $x=0, y=0$)

$\frac{d}{dz}\left(\frac{1}{z^n}\right) = \frac{-n}{z^{n+1}}$

Also $f'(z) = \frac{-1}{z^2}$

which exists for all values except $z=0$.

Hence we conclude that $f(z)$ is defined and differentiable for all values of z except $z=0$.

$\therefore f(z)$ is analytic for all values except $z=0$

6. Determine whether $w=\cos z$ is analytic.

S). $w=\cos z=\cos(x+iy)$ [December 2022]

$=\cos x\cos(iy)-\sin x\sin(iy)$

$\boxed{\cos(A+B)=\cos A\cos B-\sin A\sin B}$

$=\cos x\cdot\cosh y-\sin x\, i\sinh y$

$\boxed{\begin{array}{l}\cos(iy)=\cosh y\\ \sin(iy)=i\sinh y\end{array}}$

$=\cos x\cosh y-i\sin x\sinh y$

$u=\cos x\cosh y,\ v=-\sin x\sinh y.$

$u_x=\cosh y\cdot -\sin x=-\cosh y\sin x.$

$u_y=\cos x\cdot\sinh y.$

$v_x=-\sinh y\cdot\cos x,\ v_y=-\sin x\cdot\cosh y.$

$\therefore u_x=v_y,\ u_y=-v_x\Rightarrow$ C-R equations are satisfied.

$\Rightarrow f(z)$ is analytic.

or

$w=\cos z$ is defined for all values of 'z' in the complex plane.

Also $\frac{dw}{dz}=-\sin z$ which exists for all values of z. Hence $w=\cos z$ is defined and differentiable for all z. Hence $w=\cos z$ is analytic. ~~[December 2022]~~

7. If $f(z)=u(x,y)+iv(x,y)$ is analytic and $uv=2023$, then show that $f(z)$ is constant.

S). $uv=2023\longrightarrow(1)$.

(5)

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}.$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$

$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$

$$u_x = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = x \cdot \frac{-1}{(x^2+y^2)^2}\cdot 2y = \frac{-2xy}{(x^2+y^2)^2}.$$

$$v_x = -y\cdot\frac{-1}{(x^2+y^2)^2}\cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = -\left[\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}\right] = -\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$\therefore u_x = v_y,\ u_y = -v_x$ except when $(x^2+y^2)^2 = 0$

$\Rightarrow$ C-R equations are satisfied except when $x^2+y^2=0 \Rightarrow$ except when $x=0, y=0$. (ie, $z=0$)

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$.

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or

$f(z) = \frac{1}{z}$ is defined for all values in the complex plane except at $z=0$. (ie, $x=0, y=0$)

$\frac{d}{dz}\left(\frac{1}{z^n}\right) = \frac{-n}{z^{n+1}}$

Also $f'(z) = \frac{-1}{z^2}$

which exists for all values except $z=0$.

7

Diff ① partially w.r.t. x, we get

$$u\,v_x + v.u_x = 0 \rightarrow ② \qquad [d(uv) = uv' + vu']$$

$$u\,v_x = -v\,u_x \rightarrow ②$$

Diff ① partially w.r.t. y.

$$u.v_y + v.u_y = 0$$

$$\Rightarrow u.v_y = -v\,u_y \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{\cancel{u}v_x}{\cancel{u}.v_y} = \frac{-\cancel{v}.u_x}{-\cancel{v}.u_y} \Rightarrow \frac{v_x}{v_y} = \frac{u_x}{u_y}$$

$$\Rightarrow u_y.v_x = u_x v_y \Rightarrow u_x.v_y = u_y.v_x$$

$$[\because u_x = v_y,\ u_y = -v_x]$$

$$\Rightarrow u_x.u_x = -v_x.v_x$$

$$\Rightarrow (u_x)^2 = -(v_x)^2$$

$$\Rightarrow (u_x)^2 + (v_x)^2 = 0.$$

$$\Rightarrow u_x = 0,\ v_x = 0.$$

$$-u_y = 0$$

C-R equations $\Rightarrow v_y = 0,$

$$\Rightarrow v_y = 0,\ u_y = 0.$$

$$\therefore u_x = 0,\ u_y = 0,\ v_x = v_y = 0.$$

$\therefore$ u & v are both constants.

$\therefore$ $f(z) = u + iv = $ a constant.

Harmonic functions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0 \quad (\text{or } u_{xx} + u_{yy} = 0) \Rightarrow \text{'}u\text{' is}$$

harmonic.

Note: The real and imaginary parts of an analytic function are harmonic.