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Module-I:

Partial Differential Equations (p.d.e)

Direct Integration method.

In a p.d.e, if the dependent variable (z or u) occurs only in the form of partial derivatives then the p.d.e can be solved by direct integration method.

① Solve $\frac{\partial^2 z}{\partial x^2} = xy$. [UQ: Dec 2021].

S) We have discussed and solved in the online class, please refer the class.

② Solve $\frac{\partial^2 u}{\partial x \partial y} = \cos x . \cos y$ given that $\frac{\partial u}{\partial y} = e^{-y}$ when $x=0$ and $u=0$ when $y=0$. [UQ: Dec 2023].

S). $\frac{\partial^2 u}{\partial x \partial y} = \cos x . \cos y$.

'$\int$' above w.r.t. x (y is constant)

$\therefore \frac{\partial u}{\partial y} = \cos y . \sin x + f(y)$ $\boxed{\because \int \cos x = \sin x}$

$\frac{\partial u}{\partial y} = e^{-y}$, when $x=0 \Rightarrow e^{-y} = \cos y . \sin 0 + f(y)$

$\Rightarrow e^{-y} = 0 + f(y) \Rightarrow f(y) = e^{-y}$.

$\therefore \frac{\partial u}{\partial y} = \cos y \sin x + e^{-y}$.

'$\int$' above w.r.t. y (x is constant).

$\therefore u = \sin x . \sin y + \frac{1}{-1} e^{-y} + g(x)$ $\boxed{\because \int e^{-y} dy = \frac{1}{-1} e^{-y}}$

$= \sin x \sin y - e^{-y} + g(x)$.

$\therefore u=0$, when $y=0 \Rightarrow 0 = \sin x . \sin 0 - e^{0} + g(x)$

$\Rightarrow 0 = 0 - 1 + g(x) \Rightarrow g(x) = 1$.

$\therefore \underline{u = \sin x \sin y - e^{-y} + 1}$

(3) Solve by direct Integration $\frac{\partial^2 z}{\partial x \partial y} = \sin(3x+4y)$.

[CU: (NP) Dec 2023]

$\Rightarrow$ $$\frac{\partial^2 z}{\partial x \partial y} = \sin(3x+4y).$$

$\therefore$ '$\int$' the above w.r.t. x (y is constant).

$$\therefore \frac{\partial z}{\partial y} = \frac{-1}{3}\cos(3x+4y) + f(y).$$

$$\boxed{\begin{aligned}&\int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b)\\ &\therefore \int \sin(\overset{a}{3}x+\overset{b}{4y})\,dx = \frac{-1}{3}\cos(3x+4y)\end{aligned}}$$

'$\int$' w.r.t. y (x is constant).

$$\therefore z = -\frac{1}{3}\cdot\frac{1}{4}\sin(3x+4y) + \int f(y)\,dy + g(x)$$

$$= \frac{-1}{12}\sin(3x+4y) + \int f(y)\,dy + g(x).$$

$$\boxed{\begin{aligned}&\int \cos(ay+b)\,dy = \frac{1}{a}\sin(ay+b)\\ &\therefore \int \cos(\overset{b}{3x}+\overset{a}{4y})\,dy = \frac{1}{4}\sin(3x+4y)\end{aligned}}$$

(4) Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(3x+4y)$. [CU: Dec 2022].

$\Rightarrow$. '$\int$' w.r.t. x (y is constant).

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{3}\sin(3x+4y) + f(y).$$

$$\boxed{\therefore \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)}$$

'$\int$' above w.r.t. x (y is constant).

$$\therefore \frac{\partial z}{\partial y} = \frac{1}{3}\cdot\frac{-1}{3}\cos(3x+4y) + f(y)\cdot x + g(y)$$

$\therefore \frac{\partial z}{\partial y} = \frac{-1}{9}\cos(3x+4y) + f(y)\cdot x + g(y).$

$$\boxed{\begin{aligned} &\because \int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b) \\ &\int f(y)\,dx = f(y)x \end{aligned}}$$

'$\int$' above w.r.t. y (x is constant).

$\therefore z = \frac{-1}{9}\cdot\frac{1}{4}\sin(3x+4y) + x\int f(y)\,dy + \int g(y)\,dy + h(x)$

$= \frac{-1}{36}\sin(3x+4y) + x\int f(y)\,dy + \int g(y)\,dy + h(x)$

$$\boxed{\begin{aligned} &\because \int \cos(ay+b)\,dy = \frac{1}{a}\sin(ay+b) \\ &\int f(y)x\,dy = x\int f(y)\,dy \end{aligned}}$$

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5. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$. [UQ: (NP) Dec. 2023].

so. We have discussed and solved in the online class.

6. Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$. [UQ: Dec 2020].

so. Solved in the online class.

<u>Method of seperation of Variables.</u>

Here we assume $z = XY$ or $u = XY$ or $u = XT$ and then substitute it in the given p.d.e.

1. Solve by the method of seperation of Variables $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-3x}$.

so. Let $u = XY$.

$\therefore \frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$.

(4)

$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \Rightarrow X'Y+XY'=0 \Rightarrow X'Y=-XY'$

$\Rightarrow \frac{X'}{X}=-\frac{Y'}{Y}=k.$

$\therefore \frac{X'}{X}=k \rightarrow (1), \quad -\frac{Y'}{Y}=k \Rightarrow \frac{Y'}{Y}=-k \rightarrow (2).$

$(1) \Rightarrow \int \frac{X'}{X}dx=\int k\,dx.$ (Integrating on both sides w.r.t. x)

$\boxed{\int k\,dx = kx}$

$\Rightarrow \log X = kx+\log c_1$

$\Rightarrow \log X-\log c_1 = kx \Rightarrow \log\left(\frac{X}{c_1}\right)=kx$

$\boxed{\log a-\log b=\log\left(\frac{a}{b}\right)}$

$\Rightarrow \frac{X}{c_1}=e^{kx} \Rightarrow X=c_1e^{kx}.$

$(2) \Rightarrow \frac{Y'}{Y}=-k \Rightarrow \int\frac{Y'}{Y}dy=\int -k\,dy$

$\Rightarrow \int\frac{Y'}{Y}dy=-\int k\,dy$

$\Rightarrow \log Y=-ky+\log c_2$

$\Rightarrow \log Y-\log c_2=-ky$

$\Rightarrow \log\left(\frac{Y}{c_2}\right)=-ky \Rightarrow \frac{Y}{c_2}=e^{-ky}$

$\Rightarrow Y=c_2e^{-ky}.$

$\therefore u=XY \Rightarrow u=c_1e^{kx}\cdot c_2e^{-ky}$

$=c_1c_2e^{kx-ky}$

$\therefore u \quad = c'e^{k(x-y)}$

$\boxed{\begin{array}{l} c_1\cdot c_2=c' \\ x^m\cdot x^n=x^{m+n} \end{array}}$

$\therefore u(x,0)=4e^{-3x} \Rightarrow 4e^{-3x}=c'e^{k(x-0)}$

$\Rightarrow 4e^{-3x}=c'e^{kx} \Rightarrow c'=4, k=-3.$

$\therefore \underline{u=4e^{-3(x-y)}}$

(2) Solve $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u$ with initial condition $u(x,0) = e^{2x}$ by applying method of seperation of variables.

S) Let $u = XY$.

$\therefore \frac{\partial u}{\partial x} = X'Y, \frac{\partial u}{\partial y} = XY'$.

$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=u \Rightarrow X'Y+XY'=XY$.

$\Rightarrow X'Y = XY - XY'$

$\Rightarrow X'Y = X(Y-Y')$

$\Rightarrow \frac{X'}{X} = \frac{Y-Y'}{Y} = K$.

$\therefore \frac{X'}{X} = K \rightarrow (1), \quad \frac{Y-Y'}{Y} = K \rightarrow (2)$.

$\therefore \frac{X'}{X} = K \Rightarrow \int \frac{X'}{X}dx = \int K\,dx$

$\Rightarrow \log X = Kx + \log c_1$.

$\Rightarrow \log X - \log c_1 = Kx \Rightarrow \log\left(\frac{X}{c_1}\right) = Kx$

$\Rightarrow \frac{X}{c_1} = e^{Kx} \Rightarrow X = c_1 e^{Kx}$.

(2) $\Rightarrow \frac{Y-Y'}{Y} = K \Rightarrow \frac{Y}{Y} - \frac{Y'}{Y} = K \Rightarrow 1 - \frac{Y'}{Y} = K$.

$\Rightarrow -\frac{Y'}{Y} = K-1 \Rightarrow \frac{Y'}{Y} = -(K-1)$

$\therefore \int \frac{Y'}{Y}dy = \int -(K-1)\,dy \Rightarrow \int \frac{Y'}{Y}dy = -(K-1)\int dy$

$\boxed{\int dy = y}$

$\Rightarrow \log Y = -(K-1)y + \log c_2$

$\Rightarrow \log Y - \log c_2 = -(K-1)y \Rightarrow \log\left(\frac{Y}{c_2}\right) = -(K-1)y$

$\Rightarrow \frac{Y}{c_2} = e^{-(K-1)y} \Rightarrow Y = c_2 e^{-(K-1)y}$

$\therefore u = XY \Rightarrow u = c_1 e^{kx} \cdot c_2 e^{-(k-1)y}$

$= c_1 c_2 e^{kx-(k-1)y}$.

$u = c' e^{kx-(k-1)y}$.

$\therefore u(x,0) = e^{2x} \Rightarrow e^{2x} = c' e^{kx-(k-1)0}$

$\Rightarrow e^{2x} = c' e^{kx}$

$\therefore c' = 1,\ k = 2$.

$\therefore u = e^{2x-(2-1)y} \Rightarrow u = e^{2x-y}$

(3) Solve by the method of seperation of variables

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0,\ u(x,0) = 4e^{-3x}$.

Sol. Let $u = XT$.

$\frac{\partial u}{\partial x} = X'T,\ \frac{\partial u}{\partial t} = XT'$.

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0 \Rightarrow X'T + XT' = 0 \Rightarrow X'T = -XT'$

$\Rightarrow \frac{X'}{X} = -\frac{T'}{T} = k$.

$\therefore \frac{X'}{X} = k \rightarrow (1),\ -\frac{T'}{T} = k \rightarrow (2)$.

(1) $\Rightarrow \frac{X'}{X} = k \Rightarrow \int \frac{X'}{X} dx = \int k\,dx \Rightarrow \log X = kx + \log c_1$

$\Rightarrow \log X - \log c_1 = kx \Rightarrow \log\left(\frac{X}{c_1}\right) = kx$

$\Rightarrow X = c_1 e^{kx}$.

(2) $\Rightarrow \frac{T'}{T} = -k \Rightarrow \int \frac{T'}{T} dt = \int -k\,dt \Rightarrow \log T = -kt + \log c_2$

$\Rightarrow \int \frac{T'}{T} dt = -\int k\,dt \Rightarrow \log\left(\frac{T}{c_2}\right) = -kt$

$\Rightarrow \log T - \log c_2 = -kt \Rightarrow \frac{T}{c_2} = e^{-kt} \Rightarrow T = c_2 e^{-kt}$.

$\therefore u = XT \Rightarrow u = c_1 e^{kx} \cdot c_2 e^{-kt} \Rightarrow u = c_1 c_2 e^{kx-kt}$

$\Rightarrow u = c' e^{k(x-t)}$ $\boxed{c_1 c_2 = c'}$

(7)

$u(x,0) = 4e^{-3x} \Rightarrow 4e^{-3x} = c' e^{k(x-0)}$

$\Rightarrow 4e^{-3x} = c' e^{kx} \Rightarrow c' = 4, \; k = -3.$

$\therefore u = 4e^{-3(x-t)}$

(4) Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 3e^{-5x}$ by the method of seperation of Variables. [UQ: DEC 2022]

Ans. Let $u = XT$.

$\therefore \frac{\partial u}{\partial x} = X'T, \; \frac{\partial u}{\partial t} = XT'.$

$\therefore \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \Rightarrow X'T = 2XT' + XT.$

$\Rightarrow X'T = X(2T' + T)$

$\Rightarrow \frac{X'}{X} = \frac{2T' + T}{T} = k.$

$\therefore \frac{X'}{X} = k \to (1), \; \frac{2T' + T}{T} = k \to (2)$

$(1) \Rightarrow \int \frac{X'}{X} dx = \int k\,dx \Rightarrow \log X = kx + \log c_1$

$\Rightarrow \log\left(\frac{X}{c_1}\right) = kx \Rightarrow \frac{X}{c_1} = e^{kx}$

$\Rightarrow X = c_1 e^{kx}.$

$(2) \Rightarrow \frac{2T'}{T} + \frac{T}{T} = k \Rightarrow \frac{2T'}{T} + 1 = k$

$\Rightarrow \frac{2T'}{T} = k - 1 \Rightarrow \frac{T'}{T} = \frac{k-1}{2}$

$\Rightarrow \int \frac{T'}{T} dt = \frac{k-1}{2} \int dt$

$\therefore \int \frac{T'}{T} dt = \int \frac{k-1}{2} dt$

$\Rightarrow \log T = \frac{k-1}{2} t + \log c_2$

$\Rightarrow \log T - \log c_2 = \frac{k-1}{2} t \Rightarrow \log\left(\frac{T}{c_2}\right) = \frac{k-1}{2} t$

$\Rightarrow T = c_2 e^{\frac{k-1}{2} t}.$

$\therefore u = XT \Rightarrow u = c_1 e^{kx} \cdot c_2 e^{\frac{k-1}{2} t} \Rightarrow u = c_1 c_2 e^{kx + \frac{k-1}{2} t}$

$\Rightarrow u = c' e^{kx + \frac{k-1}{2} t}.$

$u(x,0) = 3e^{5x} \Rightarrow 3e^{5x} = c' e^{kx + \frac{k-1}{2} \times 0}$

$\Rightarrow 3e^{5x} = c' e^{kx} \Rightarrow c' = 3, k = 5.$

$\therefore u = 3e^{5x + \frac{5-1}{2}t} = 3e^{5x+2t}$

(5) Solve by Method of seperation of Variables $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$. [UU: May 2025].

Sol. Solved in the online class, please refer.

Lagrange's equation.

A p.d.e, which is of the form $Pp + Qq = R$, where P, Q and R are function of x, y, z and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Here to solve the given p.d.e, first we consider the subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and then we solve by the method of grouping or by method of multipliers giving $u = c_1$, and $V = c_2$ as it's solutions. given by $u = f(v)$

* Solve $2z = xp + yq$ [UU: Dec 2020].

Sol. $\underset{P}{x}p + \underset{Q}{y}q = \underset{R}{2z}$.

$\therefore \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{2z}$.

$\therefore \frac{dx}{x} = \frac{dy}{y} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$

$\boxed{\because \int \frac{dx}{x} = \log x \ \& \ \int \frac{dy}{y} = \log y}$

$\Rightarrow \log x = \log y + \log c_1$

$\Rightarrow \log x - \log y = \log c_1$

$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1 \Rightarrow \frac{x}{y} = c_1 = u.$

$$\frac{dx}{x} = \frac{dz}{2z} \Rightarrow \int \frac{dx}{x} = \int \frac{dz}{2z}$$

$$\Rightarrow \int \frac{dx}{x} = \frac{1}{2}\int \frac{dz}{z} \Rightarrow 2\int \frac{dx}{x} = \int \frac{dz}{z}$$

$$\Rightarrow 2\log x = \log z + \log c_2$$

$$\Rightarrow 2\log x - \log z = \log c_2 \Rightarrow \log(x^2) - \log z = \log c_2.$$

$\boxed{\because n\log a = \log(a^n)}$

$$\Rightarrow \log\left(\frac{x^2}{z}\right) = \log c_2 \Rightarrow \frac{x^2}{z} = c_2 = v.$$

$$\therefore u = f(v) \Rightarrow \frac{x}{y} = f\left(\frac{x^2}{z}\right) \Rightarrow x = y f\left(\frac{x^2}{z}\right)$$

(2) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. [UQ: DEC 2020]

$\Rightarrow$ discussed and solved in the online class, please refer the video.

(3) Solve $y^2p - xyq = xz$ [UQ: DEC 2021].

$\Rightarrow$ $\underbrace{y^2}_{P}p \underbrace{- xy}_{Q}q = \underbrace{xz}_{R}$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz}$$

$$\frac{dx}{y\cancel{^2}} = \frac{dy}{-x\cancel{y}} \Rightarrow \frac{dx}{y} = \frac{dy}{-x} \Rightarrow \frac{dx}{y} = -\frac{dy}{x}$$

$$\Rightarrow x\,dx = -y\,dy \Rightarrow \int x\,dx = -\int y\,dy$$

[Integrating on both side]

$\left[\int x^n dx = \frac{x^{n+1}}{n+1}\right]$

$$\Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + c_1$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c_1 \Rightarrow x^2 + y^2 = 2c_1 = c_4$$

[multiplying by 2 on both sides]

$$\frac{dy}{-xy} = \frac{dz}{xz} \Rightarrow \frac{dy}{-xy} = \frac{dz}{xz} \Rightarrow \frac{dy}{-y} = \frac{dz}{z}.$$

$$\Rightarrow \frac{dy}{y} = -\frac{dz}{z}$$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dz}{z} \Rightarrow \log y = -\log z + \log c_2$$

$$\boxed{\int \frac{dy}{y} = \log y}$$

$$\Rightarrow \log y + \log z = \log c_2 \qquad \boxed{\log a + \log b = \log(ab)}$$

$$\Rightarrow \log(yz) = \log c_2$$

$$\Rightarrow yz = c_2.$$

$$\therefore u = f(v) \Rightarrow \underline{x^2+y^2 = f(yz)}.$$

④ Solve $xp + yq = z$. [UQ: DEC 2023].

:). $\underset{P}{x}p + \underset{Q}{y}q = \underset{R}{z}.$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$\Rightarrow \log x - \log y = \log c_1 \Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$$

$$\Rightarrow \frac{x}{y} = c_1 = u.$$

$$\frac{dx}{x}=\frac{dz}{z} \Rightarrow \int\frac{dx}{x}=\int\frac{dz}{z}$$

$$\Rightarrow \log x=\log z+\log c_2$$

$$\Rightarrow \log x-\log z=\log c_2$$

$$\Rightarrow \log\left(\frac{x}{z}\right)=\log c_2 \Rightarrow \frac{x}{z}=c_2=v.$$

$$\therefore u=f(v) \Rightarrow \frac{x}{y}=f\left(\frac{x}{z}\right) \Rightarrow x=yf\left(\frac{x}{z}\right)$$

5. Solve $(y^2+z^2)p-xyq+xz=0$. [UQ: Dec 2023].

→. $P=y^2+z^2,\ Q=-xy,\ R=-xz.$

$$\boxed{\therefore \underbrace{(y^2+z^2)}_{P}p\underbrace{-xy}_{Q}q=\underbrace{-xz}_{R}.}$$

$$\therefore \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{y^2+z^2}=\frac{dy}{-xy}=\frac{dz}{-xz}.$$

$$\therefore \frac{dy}{-xy}=\frac{dz}{-xz} \Rightarrow \frac{dy}{\cancel{-x}y}=\frac{dz}{\cancel{-x}z}$$

$$\Rightarrow \frac{dy}{y}=\frac{dz}{z} \Rightarrow \int\frac{dy}{y}=\int\frac{dz}{z}$$

$$\Rightarrow \log y=\log z+\log c_1 \Rightarrow \log y-\log z=\log c_1$$

$$\Rightarrow \log\left(\frac{y}{z}\right)=\log c_1 \Rightarrow \frac{y}{z}=c_1=u.$$

$$\frac{xdx+ydy+zdz}{x(y^2+z^2)+yx-xy+zx-xz}=\frac{xdx+ydy+zdz}{\cancel{xy^2}+\cancel{xz^2}-\cancel{xy^2}-\cancel{xz^2}}$$

$$=\frac{xdx+ydy+zdz}{0}=K.$$

$$\Rightarrow xdx+ydy+zdz=0 \Rightarrow \int(xdx+ydy+zdz)=\int 0$$

$$\Rightarrow \int xdx+\int ydy+\int zdz=0$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=0+c_2=c_2$$

$$\Rightarrow x^2+y^2+z^2=2c_2=V.$$

$$\Rightarrow x^2+y^2+z^2$$

$$\therefore u=f(v) \Rightarrow \frac{y}{z}=f(x^2+y^2+z^2) \Rightarrow \underline{y=zf(x^2+y^2+z^2)}$$

(6) Solve $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

[CU: Dec 2023].

n.
$$\underbrace{x(y^2-z^2)}_{P}p+\underbrace{y(z^2-x^2)}_{Q}q=\underbrace{z(x^2-y^2)}_{R}$$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{x(y^2-z^2)}=\frac{dz}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$$

$$\therefore \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{\cancel{x}}\cancel{x}(y^2-z^2)+\frac{1}{\cancel{y}}\cancel{y}(z^2-x^2)+\frac{1}{\cancel{z}}\cancel{z}(x^2-y^2)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y^2}-\cancel{z^2}+\cancel{z^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}$$

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=K.$$

$$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

$$\Rightarrow \int\frac{1}{x}dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=\int 0$$

$$\Rightarrow \log x+\log y+\log z=0+\log c_1=\log c_1$$

$$\Rightarrow \log(xyz)=\log c_1$$

$$\boxed{\log a+\log b+\log c=\log(abc)}$$

$$\Rightarrow xyz=c_1.$$

$$\frac{xdx+ydy+zdz}{x\cdot x(y^2-z^2)+y\cdot y(z^2-x^2)+z\cdot z(x^2-y^2)}=\frac{xdx+ydy+zdz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$=\frac{xdx+ydy+zdz}{\cancel{x^2y^2}-\cancel{x^2z^2}+\cancel{y^2z^2}-\cancel{y^2x^2}+\cancel{z^2x^2}-\cancel{z^2y^2}}=\frac{xdx+ydy+zdz}{0}$$

$$=K.$$

$\Rightarrow xdx+ydy+zdz=0$

$\Rightarrow \int xdx+\int ydy+\int zdz=\int 0$

$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=0+c_2=c_2$

$\Rightarrow x^2+y^2+z^2=2c_2=V.$

$\therefore u=f(v) \Rightarrow \underline{\underline{xyz=f(x^2+y^2+z^2)}}$.

(7) Solve $(y^2+z^2)p-xyq=xz$ [VQ:(NP) Dec 2023].

S:) Same as in Question No. 5.

(8) Solve $x(y-z)p+y(z-x)q=z(x-y)$.

[VQ: May 2025].

S). $P=x(y-z)$, $Q=y(z-x)$, $R=z(x-y)$.

$\therefore \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$.

$\therefore \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{dx+dy+dz}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$

$=\frac{dx+dy+dz}{0}=K.$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow \int dx+\int dy+\int dz=\int 0$

$\Rightarrow x+y+z=0+c_1=c_1=u$

$\boxed{\because \int dx=x, \int dy=y, \int dz=z}$

$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{\cancel{x}}\cdot\cancel{x}(y-z)+\frac{1}{\cancel{y}}\cdot\cancel{y}(z-x)+\frac{1}{\cancel{z}}\cdot\cancel{z}(x-y)}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$

$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=K.$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} = K$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \frac{1}{z}dz = \int 0$$

$$\Rightarrow \log x + \log y + \log z = 0 + \log c_2 = \log c_2$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow xyz = c_2 = v.$$

$$\because u = f(v) \Rightarrow \underline{\underline{x+y+z = f(xyz)}}$$