P

① Verify whether $u = x^3 - 3xy^2$ is harmonic and find it's conjugate harmonic function V. [UQ: December 2021]

a) $u = x^3 - 3xy^2$.

$u_x = 3x^2 - 3y^2 \cdot 1 = 3x^2 - 3y^2$.

$u_{xx} = 3 \cdot 2x - 0 = 6x$.

$u_y = 0 - 3x \cdot 2y = -6xy$.

$u_{yy} = -6x \cdot 1 = -6x$.

$\therefore u_{xx} + u_{yy} = 6x + -6x = 0$.

$\Rightarrow$ 'u' is harmonic.

$\therefore u_x = V_y,\ u_y = -V_x \Rightarrow$

$3x^2 - 3y^2 = V_y,\ -6xy = -V_x$.

$\Rightarrow V_y = 3x^2 - 3y^2 \rightarrow ①,\ V_x = 6xy \rightarrow ②$

'$\int$' ① w.r.t. y (x is constant).

$\Rightarrow V = 3x^2y - \cancel{3} \cdot \frac{y^3}{\cancel{3}} + h(x)$

$\boxed{\int y^n dy = \frac{y^{n+1}}{n+1}}$

$\therefore V = 3x^2y - y^3 + h(x) \rightarrow ③$

Diff ③ partially w.r.t. x

$\therefore V_x = 3y \cdot 2x - 0 + h'(x) = 6xy + h'(x) \rightarrow ④$

$\boxed{\frac{d}{dx} h(x) = h'(x)}$

$\therefore$ From ② & ④, $6xy + h'(x) = 6xy$.

$\Rightarrow h'(x) = 6xy - 6xy = 0$.

$\therefore \int h'(x)\,dx = \int 0\,dx$. (Integrating on bothsides w.r.t. x)

(2)

$\Rightarrow h(x) = 0x + c \Rightarrow h(x) = c.$ $\boxed{\int h'(x)dx = h(x)}$

$\therefore (3) \Rightarrow v = 3x^2y - y^3 + c.$

(2) Show that $u = 2+3x-y+x^2-y^2-4xy$ is harmonic. Also find it's conjugate harmonic function 'v'.

[UQ: December 2021]

→) $u = 2+3x-y+x^2-y^2-4xy.$

$u_x = 0+3\cdot1-0+2x-0-4y\cdot1$

$= 3+2x-4y.$

$u_{xx} = 0+2\times1-0 = 2.$

$u_y = 0+0-1+0-2y-4x.$

$= -1-2y-4x.$

$u_{yy} = 0-2\cdot1-0 = -2.$

$\therefore u_{xx}+u_{yy} = 2+-2 = 2-2 = 0.$

$\Rightarrow$ 'u' is harmonic.

$\therefore u_x = v_y,\ u_y = -v_x \Rightarrow 3+2x-4y = v_y,\ -1-2y-4x = -v_x$

$\Rightarrow v_y = 3+2x-4y \rightarrow (1),\ v_x = 1+2y+4x \rightarrow (2)$

(x'ying by − on bothside)

$\int$' (1) w.r.t. y (x is constant).

$\Rightarrow v = 3y+2xy-\cancel{4}^{2}\dfrac{y^2}{\cancel{2}}+h(x).$

$\Rightarrow v = 3y+2xy-2y^2+h(x) \rightarrow (3).$

Diff (3) partially w.r.t. x.

$\therefore v_x = 0+2y\cdot1-0+h'(x)$

$v_x = 2y+h'(x) \rightarrow (4)$

$\therefore$ From (2) & (4),

$$2y + h'(x) = 1 + 2y + 4x.$$

$$\Rightarrow h'(x) = 1 + \cancel{2y} + 4x - \cancel{2y} = 1 + 4x.$$

$$\Rightarrow \int h'(x)dx = \int (1+4x)dx$$

$$= \int 1 dx + \int 4x dx$$

$$= x + \cancel{4}^{2} \frac{x^2}{\cancel{2}} + c$$

$$\boxed{\begin{array}{l} \int k dx = kx \\ \int x^n dx = \frac{x^{n+1}}{n+1} \end{array}}$$

$$\Rightarrow h(x) = x + 2x^2 + c.$$

$$(3) \Rightarrow \underline{V = 3y + 2xy - 2y^2 + x + 2x^2 + c}$$

(3) Show that the function $u = \cos x \cosh y$ is harmonic and find it's harmonic conjugate. [KU: (WP) December 2023]

Sol)

$$u = \cos x \cosh y.$$

$$u_x = \cosh y . -\sin x.$$

$$= -\cosh y \sin x.$$

$$u_{xx} = -\cosh y . \cos x.$$

$$u_y = \cos x . \sinh y$$

$$u_{yy} = \cos x . \cosh y.$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x.}$$

$$\boxed{\frac{d}{dy}(\cosh y) = \sinh y}$$

$$\boxed{\frac{d}{dy}(\sinh y) = \cosh y}$$

$$\therefore u_{xx} + u_{yy} = -\cosh y \cos x + \cos x \cosh y$$

$$= -\cosh y \cos x + \cosh y \cos x = 0.$$

$\Rightarrow u$ is harmonic.

$$\therefore u_x = V_y, u_y = -V_x \Rightarrow -\cosh y \sin x = V_y,$$

$$\cos x \sinh y = -V_x.$$

$$\Rightarrow V_y = -\cosh y \sin x \to (1), \quad V_x = -\cos x \sinh y \to (2)$$

11

'$\int$' ① w.r.t. y (x is constant).

$\Rightarrow V = -\sin x \sinh y + h(x)$ $\rightarrow$ ③. $\boxed{\int \cosh y\,dy = \sinh y}$

Diff ③ partially w.r.t. x.

$\therefore V_x = -\sinh y \cos x + h'(x) \rightarrow$ ④.

$\therefore$ From ② & ④,

$-\sinh y \cos x + h'(x) = -\cos x \sinh y$

$\Rightarrow h'(x) = -\cos x \sinh y + \sinh y \cos x.$

$= -\cos x \sinh y + \cos x \sinh y = 0.$

$\therefore \int h'(x)\,dx = \int 0\,dx$

$\Rightarrow h(x) = 0x + c = c.$

$\therefore$ ③ $\Rightarrow V = \underline{\underline{-\sin x \sinh y + c}}$

④ check whether the function xy^2 is the real part of an analytic function. [KU: (WP) December 2023].

$\rightarrow$. To check whether a given function is the real part of an analytic function, check whether the given function is harmonic.

Let $u = xy^2$.

$\therefore u_x = y^2 \cdot 1 = y^2$

$u_{xx} = 0.$

$\boxed{\frac{d}{dy}(y^n) = ny^{n-1}}$

$u_y = x \cdot 2y = 2xy.$

$u_{yy} = 2x \cdot 1 = 2x.$

$\therefore u_{xx} + u_{yy} = 0 + 2x = 2x \neq 0$

$\therefore u$ is not harmonic and hence 'u' is not a real part of an analytic function.

(5) Determine 'a' so that the function $u = e^{2x}\cos ay$ is harmonic and find the conjugate. [UQ: May 2025].

Sol) $u = e^{2x}\cos ay$.

$u_x = \cos ay \cdot e^{2x} \cdot 2$

$= 2\cos ay\, e^{2x}$.

$\boxed{\frac{d}{dx}(e^{2x}) = e^{2x} \cdot 2 = 2e^{2x}}$

$u_{xx} = 2\cos ay \cdot e^{2x} \cdot 2$

$= 4\cos ay\, e^{2x}$.

$u_y = e^{2x} \cdot -\sin ay \cdot a$

$= -a e^{2x} \sin ay$.

$\boxed{\frac{d}{dy}(\cos ay) = -\sin ay \cdot a = -a\sin ay}$

$u_{yy} = -a e^{2x} \cdot \cos ay \cdot a$

$= -a^2 e^{2x} \cos ay$.

$\boxed{\frac{d}{dy}(\sin ay) = \cos ay \cdot a = a\cos ay}$

$\therefore u_{xx} + u_{yy} = 0 \Rightarrow 4\cos ay\, e^{2x} + -a^2 e^{2x}\cos ay = 0$.

$\boxed{\because u \text{ is harmonic}}$

$\Rightarrow 4\cos ay\, e^{2x} - a^2 e^{2x}\cos ay = 0$.

$\Rightarrow e^{2x}\cos ay\,(4 - a^2) = 0$.

$\Rightarrow 4 - a^2 = 0 \Rightarrow 4 = a^2 \Rightarrow a = \sqrt{4}$

$\Rightarrow a = \pm 2$.

$\therefore u = e^{2x}\cos 2y$. $(a=2)$

$\boxed{\because \cos(2y) = \cos(-2y) = \cos 2y. \quad \cos(-\theta) = \cos\theta}$

$\because u_x = v_y,\ u_y = -v_x \Rightarrow$

$2\cos 2y\, e^{2x} = v_y,\ -2e^{2x}\sin 2y = -v_x$.

$\boxed{\because a = 2}$

$\Rightarrow v_y = 2\cos 2y\, e^{2x} \rightarrow (1),\ v_x = 2e^{2x}\sin 2y \rightarrow (2)$

(13)

$\int$ (1) w.r.t. y (x is constant)

$\Rightarrow V = \cancel{2}e^{2x} \cdot \frac{1}{\cancel{2}}\sin 2y + h(x)$ $\boxed{\int \cos 2y\,dy = \frac{1}{2}\sin 2y.}$

$V = e^{2x}\sin 2y + h(x) \rightarrow (3).$

Diff (3) partially w.r.t. x.

$V_x = \sin 2y \cdot e^{2x} \cdot 2 + h'(x)$

$V_x = 2\sin 2y\, e^{2x} + h'(x) \rightarrow (4)$

From (2) & (4),

$2\sin 2y\, e^{2x} + h'(x) = 2e^{2x}\sin 2y.$

$\therefore h'(x) = 2e^{2x}\sin 2y - 2\sin 2y\, e^{2x}$

$= 2e^{2x}\sin 2y - 2e^{2x}\sin 2y = 0$

$h'(x) = 0.$

$\therefore \int h'(x)\,dx = \int 0\,dx$

$\Rightarrow h(x) = 0 + c = c.$

$\therefore (3) \Rightarrow V = e^{2x}\sin 2y + c.$

(c) Find an analytic function whose real part is $u = \sin x \cosh y$.

$\Rightarrow$

$$f(z) = \int [u_x(z,0) - i\, u_y(z,0)]\, dz.$$

$$u = \sin x \cosh y \Rightarrow u_x = \cosh y \cdot \cos x.$$

$$u_y = \sin x \cdot \sinh y.$$

$\boxed{\text{put } x = z, y = 0 \text{ in } u_x \,\&\, u_y}$

$$\therefore u_x(z,0) = \cosh 0 \cos z$$
$$= 1 \cdot \cos z = \cos z.$$

$$u_y(z,0) = \sin z \cdot \sinh 0$$
$$= \sin z \cdot 0 = 0.$$

$\boxed{\begin{matrix}\sinh 0 = 0 \\ \cosh 0 = 1\end{matrix}}$

$$f(z) = \int [\cos z - i \cdot 0]\, dz$$

$$= \int \cos z \cdot dz$$

$\boxed{\int \cos z\, dz = \sin z}$

$$= \sin z + c.$$

$$\therefore \underline{\underline{f(z) = \sin z + c}}$$

15

Conformal mapping:

mapping under the transformation $w=\frac{1}{z}$:

$$w=\frac{1}{z} \Rightarrow z=\frac{1}{w} \Rightarrow z=\frac{1}{u+iv}$$

$$\Rightarrow x+iy=\frac{1\cdot(u-iv)}{(u+iv)(u-iv)}$$

$$=\frac{u-iv}{u^2-(iv)^2}=\frac{u-iv}{u^2-i^2v^2}$$

$$=\frac{u-iv}{u^2+v^2} \quad (i^2=-1)$$

$$=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}.$$

$$\therefore x=\frac{u}{u^2+v^2},\ y=\frac{-v}{u^2+v^2}.$$

1. Show that under the transformation $w=\frac{1}{z}$, the circle $x^2+y^2-6x=0$ is transformed in to a straight line in the w-plane. [UG, December 2020].

→). $w=\frac{1}{z} \Rightarrow x=\frac{u}{u^2+v^2},\ y=\frac{-v}{u^2+v^2}.$

$$\therefore x^2+y^2-6x=0 \Rightarrow \left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-6\cdot\frac{u}{u^2+v^2}=0$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{u^2+v^2}-\frac{6u}{u^2+v^2}=0.$$

$$\Rightarrow (u^2+v^2)-6u(u^2+v^2)=0$$ X lying by (u^2+v^2) on both sides.

$$\Rightarrow (u^2+v^2)(1-6u)=0$$

$$\Rightarrow 1-6u=0 \Rightarrow 1=6u \Rightarrow u=\frac{1}{6}.$$

which is a straight line in the w-plane.

(2) Find the image of the circle $|z-1|=1$ under the transformation $w=\frac{1}{z}$. [UQ: December 2022]

s) $x=\frac{u}{u^2+v^2}, \; y=\frac{-v}{u^2+v^2}.$

$|z-1|=1 \Rightarrow |x+iy-1|=1 \Rightarrow |(x-1)+iy|=1$

$\Rightarrow \sqrt{(x-1)^2+y^2}=1$

$\Rightarrow (x-1)^2+y^2=1^2$ $\boxed{\text{squaring on both sides}}$

$\Rightarrow x^2-2x+1^2+y^2=1$ $\boxed{(a-b)^2=a^2-2ab+b^2}$

$\Rightarrow x^2-2x+y^2=1-1=0$

$\therefore x^2-2x+y^2=0 \Rightarrow \left(\frac{u}{u^2+v^2}\right)^2-2\cdot\frac{u}{u^2+v^2}+\left(\frac{-v}{u^2+v^2}\right)^2=0$

$\Rightarrow \frac{u^2}{(u^2+v^2)^2}-\frac{2u}{u^2+v^2}+\frac{v^2}{(u^2+v^2)^2}=0$

$\Rightarrow u^2-2u(u^2+v^2)+v^2=0$ $\boxed{\text{x'ly on both by } (u^2+v^2)^2}$

$\Rightarrow (u^2+v^2)-2u(u^2+v^2)=0$

$\Rightarrow (u^2+v^2)(1-2u)=0$

$\Rightarrow 1-2u=0 \Rightarrow 1=2u$

$\Rightarrow u=\frac{1}{2}$

(3) Find image of the circle $|z-\frac{1}{2}|\leq\frac{1}{2}$ under the transformation $w=\frac{1}{z}$. [UQ: (WP) December 2023]

s). We have discussed and solved the above question in an online class. Please refer.