Many Coin Flips

In this section we are going to consider the number of heads on n coin flips. This thought experiment is going to be a basis for much probability theory! It goes far beyond coin flips.

Say a coin comes up heads with probability p. Most coins are fair and as such come up heads with probability $p = 0.5$. There are many events for which coin flips are a great analogy that have different values of p so lets leave p as a variable. You can try simulating coins here. Note that **H** is short for Heads and T is short for Tails. We think of each coin as distinct:

Coin Flip Simulator

Number of flips n: 10 Probability of heads p: 0.60 New simulation

Simulator results:

H, H, H, H, T, H, H, H, H, T

Total number of heads: 8

Let's explore a few probability questions in this domain.

Warmups

What is the probability that all n flips are heads?

Lets say $n = 10$ this question is asking what is the probability of getting:

H, H, H, H, H, H, H, H, H, H

Each coin flip is independent so we can use the rule for probability of **and** with independent events. As such, the probability of n heads is p multiplied by itself n times: p^n. If $n = 10$ and $p = 0.6$ then the probability of n heads is around 0.006.

What is the probability that all n flips are tails?

Lets say $n = 10$ this question is asking what is the probability of getting:

T, T, T, T, T, T, T, T, T, T

Each coin flip is independent. The probability of tails on any coin flip is $1 - p$. Again, since the coin flips are independent, the probability of tails n times on n flips is $(1 - p)$ multiplied by itself n times: $(1 - p)^n$. If $n = 10$ and $p = 0.6$ then the probability of n tails is around 0.0001.

First k heads then $n - k$ tails

Lets say $n = 10$ and $k = 4$, this question is asking what is the probability of getting:

H, H, H, H, T, T, T, T, T, T

The coins are still independent! The first k heads occur with probability p^k the run of $n - k$ tails occurs with probability $(1 - p)^{n-k}$. The probability of k heads then $n - k$ tails is the product of those two terms: $p^k \cdot (1 - p)^{n-k}$

Exactly k heads

Next lets try to figure out the probability of exactly k heads in the n flips. Importantly we don't care where in the n flips that we get the heads, as long as there are k of them. Note that this question is different than the question of first k heads and then $n-k$ tails which requires that the k heads come first! That particular result does generate exactly k coin flips, but there are others.

There are many others! In fact any permutation of k heads and $n-k$ tails will satisfy this event. Lets ask the computer to list them all for exactly $k=4$ heads within $n=10$ coin flips. The output region is scrollable:

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(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
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Exactly how many outcomes are there with $k=4$ heads in $n=10$ flips? 210. The answer can be calculated using permutations of indistinct objects:

$$N = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

The probability of exactly $k=4$ heads is the probability of the **or** of each of these outcomes. Because we consider each coin to be unique, each of these outcomes is "mutually exclusive" and as such if E_i is the outcome from the ith row,

$$\text{P}(\text{exactly } k \text{ heads}) = \sum_{i=1}^{N} \text{P}(E_i)$$

The next question is, what is the probability of each of these outcomes?

Here is a arbitrarily chosen outcome which satisfies the event of exactly $k=4$ heads in $n=10$ coin flips. In fact it is the one on row 128 in the list above:

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T, H, T, T, H, T, T, H, H, T
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What is the probability of getting the exact sequence of heads and tails in the example above? Each coin flip is still independent, so we multiply p for each heads and $1-p$ for each tails. Let E_{128} be the event of this exact outcome:

$$\text{P}(E_{128}) = (1-p) \cdot p \cdot (1-p) \cdot (1-p) \cdot p \cdot (1-p) \cdot (1-p) \cdot p \cdot p \cdot (1-p)$$

If you rearrange these multiplication terms you get:

$$\begin{aligned}\text{P}(E_{128}) &= p \cdot p \cdot p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \\ &= p^4 \cdot (1-p)^6\end{aligned}$$

There is nothing too special about row 128. If you chose any row, you would get k independent heads and $n-k$ independent tails. For any row i, $\text{P}(E_i) = p^k \cdot (1-p)^{n-k}$. Now we are ready to calculate the probability of exactly k heads:

$$
\begin{aligned}
\mathrm{P}(\text{exactly } k \text{ heads}) &= \sum_{i=1}^{N} \mathrm{P}(E_i) && \text{Mutual Exclusion} \\
&= \sum_{i=1}^{N} p^k \cdot (1-p)^{n-k} && \text{Sub in } \mathrm{P}(E_i) \\
&= N \cdot p^k \cdot (1-p)^{n-k} && \text{Sum } N \text{ times} \\
&= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} && \text{Perm of indistinct objects}
\end{aligned}
$$

More than k heads

You can use the formula for exactly k heads to compute other probabilities. For example the probability of more than k heads is:

$$
\begin{aligned}
\mathrm{P}(\text{more than } k \text{ heads}) &= \sum_{i=k+1}^{n} \mathrm{P}(\text{exactly } i \text{ heads}) && \text{Mutual Exclusion} \\
&= \sum_{i=k+1}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} && \text{Substitution}
\end{aligned}
$$