Grades are Not Normal

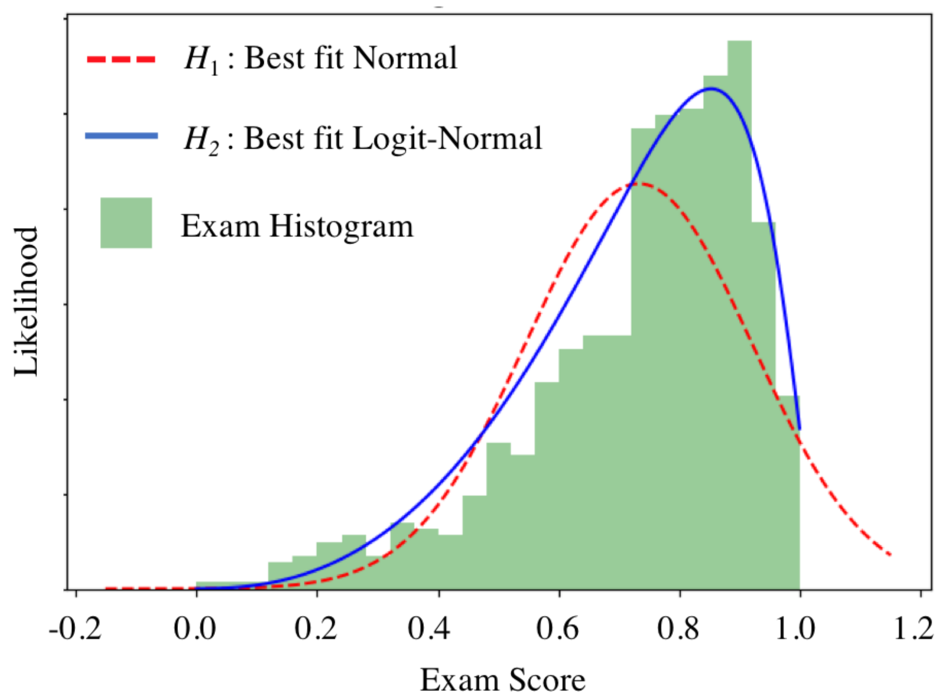
Sometimes you just feel like squashing normals:

Logit Normal
The logit normal is the continuous distribution that results from applying a special "squashing" function to a Normally distributed random variable. The squashing function maps all values the normal could take on onto the range 0 to 1. If $X \sim \text{LogitNormal}(\mu, \sigma^2)$ it has:

$$\begin{aligned} \text{PDF:} \quad & f_X(x) = \begin{cases} \frac{1}{\sigma(\sqrt{2\pi})x(1-x)} e^{-\frac{(\text{logit}(x)-\mu)^2}{2\sigma^2}} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \\ \text{CDF:} \quad & F_X(x) = \Phi\left(\frac{\text{logit}(x) - \mu}{\sigma}\right) \\ \text{Where:} \quad & \text{logit}(x) = \log\left(\frac{x}{1-x}\right) \end{aligned}$$

A new theory shows that the Logit Normal better fits exam score distributions than the traditionally used Normal. Let's test it out! We have some set of exam scores for a test with min possible score 0 and max possible score 1, and we are trying to decide between two hypotheses:

H_1: our grade scores are distributed according to $X \sim \text{Normal}(\mu = 0.7, \sigma^2 = 0.2^2)$.
H_2: our grade scores are distributed according to $X \sim \text{LogitNormal}(\mu = 1.0, \sigma^2 = 0.9^2)$.



Under the normal assumption, H_1, what is $P(0.9 < X < 1.0)$? Provide a numerical answer to two decimal places.

$$P(0.9 < X < 1.0) = \Phi\left(\frac{1.0 - 0.7}{0.2}\right) - \Phi\left(\frac{0.9 - 0.7}{0.2}\right) = \Phi(1.5) - \Phi(1.0) = 0.9332 - 0.8413 = 0.0$$

Under the logit-normal assumption, H_2, what is $P(0.9 < X < 1.0)$?

$$F_X(1.0) - F_X(0.9) = \Phi\left(\frac{\text{logit}(1.0) - 1.0}{0.9}\right) - \Phi\left(\frac{\text{logit}(0.9) - 1.0}{0.9}\right)$$

Which we can solve for numerically:

$$\Phi\left(\frac{\text{logit}(1.0) - 1.0}{0.9}\right) - \Phi\left(\frac{\text{logit}(0.9) - 1.0}{0.9}\right) = 1 - \Phi(1.33) \approx 0.91$$

Under the normal assumption, H_1, what is the maximum value that X can take on?

$$\infty$$

Before observing any test scores, you assume that (a) one of your two hypotheses is correct and (b) that initially, each hypothesis is equally likely to be correct, $P(H_1) = P(H_2) = \frac{1}{2}$. You then observe a single test score, $X = 0.9$. What is your updated probability that the Logit-Normal hypothesis is correct?

$$\begin{aligned} P(H_2|X = 0.9) &= \frac{f(X = 0.9|H_2)P(H_2)}{f(X = 0.9|H_2)P(H_2) + f(X = 0.9|H_1)P(H_1)} \\ &= \frac{f(X = 0.9|H_2)}{f(X = 0.9|H_2) + f(X = 0.9|H_1)} \\ &= \frac{\frac{1}{\sigma(\sqrt{2\pi})0.9*(1-0.9)} e^{-\frac{(\text{logit}(0.9)-1.0)^2}{2*0.9^2}}}{\frac{1}{\sigma(\sqrt{2\pi})0.9*(1-0.9)} e^{-\frac{(\text{logit}(0.9)-1.0)^2}{2*0.9^2}} + \frac{1}{0.2\sqrt{2\pi}} e^{-\frac{(0.9-0.7)^2}{2*0.2^2}}} \end{aligned}$$