Core Probability Reference

Definition: Empirical Definition of Probability

The probability of any event E can be defined as:

$$\mathrm{P}(E) = \lim_{n\to\infty} \frac{\text{count}(E)}{n}$$

Where $\text{count}(E)$ is the number of times that E occured in n experiments.

Definition: Core Identities

For an event E and a sample space S

$0 \le \mathrm{P}(E) \le 1$	All probabilities are numbers between 0 and 1.
$\mathrm{P}(S) = 1$	All outcomes must be from the Sample Space.
$\mathrm{P}(E) = 1 - \mathrm{P}(E^c)$	The probability of an event from its complement.

Definition: Probability of Equally Likely Outcomes

If S is a sample space with equally likely outcomes, for an event E that is a subset of the outcomes in S:

$$\mathrm{P}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Definition: Conditional Probability.
The probability of E given that (aka conditioned on) event F already happened:

$$\mathrm{P}(E|F) = \frac{\mathrm{P}(E \text{ and } F)}{\mathrm{P}(F)}$$

Definition: Probability of **or** with Mututally Exclusive Events
If two events E and F are mutually exclusive then the probability of E ***or*** F occurring is:

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F)$$

For n events $E_1, E_2, \ldots E_n$ where each event is mutually exclusive of one another (in other words, no outcome is in more than one event). Then:

$$\mathrm{P}(E_1 \text{ or } E_2 \text{ or} \ldots \text{or } E_n) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \cdots + \mathrm{P}(E_n) = \sum_{i=1}^{n} \mathrm{P}(E_i)$$

Definition: General Probability of **or** (Inclusion-Exclusion)

For any two events E and F:

$$\mathrm{P}(E \text{ or } F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \text{ and } F)$$

For three events, E, F, and G the formula is:

$$\begin{aligned}\mathrm{P}(E \text{ or } F \text{ or } G) = \ & \mathrm{P}(E) + \mathrm{P}(F) + \mathrm{P}(G) \\ & - \mathrm{P}(E \text{ and } F) - \mathrm{P}(E \text{ and } G) - P(F \text{ and } G) \\ & + \mathrm{P}(E \text{ and } F \text{ and } G)\end{aligned}$$

For more than three events see the chapter of probability of or.

Definition: Probability of **and** for Independent Events.

If two events: E, F are independent then the probability of E ***and*** F occurring is:

$$\mathrm{P}(E \text{ and } F) = \mathrm{P}(E) \cdot \mathrm{P}(F)$$

For n events $E_1, E_2, \ldots E_n$ that are independent of one another:

$$\mathrm{P}(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_n) = \prod_{i=1}^{n} \mathrm{P}(E_i)$$

Definition: General Probability of **and** (The Chain Rule)

For any two events E and F:

$$\mathrm{P}(E \text{ and } F) = \mathrm{P}(E|F) \cdot \mathrm{P}(F)$$

For n events $E_1, E_2, \ldots E_n$:

$$\begin{aligned}\mathrm{P}(E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_n) = & \mathrm{P}(E_1) \cdot \mathrm{P}(E_2|E_1) \cdot \mathrm{P}(E_3|E_1 \text{ and } E_2) \ldots \\ & \mathrm{P}(E_n|E_1 \ldots E_{n-1})\end{aligned}$$

Definition: The Law of Total Probability

For any two events E and F:

$$\begin{aligned}\mathrm{P}(E) &= \mathrm{P}(E \text{ and } F) + \mathrm{P}(E \text{ and } F^{\mathrm{C}}) \\ &= \mathrm{P}(E|F)\,\mathrm{P}(F) + \mathrm{P}(E|F^{\mathrm{C}})\,\mathrm{P}(F^{\mathrm{C}})\end{aligned}$$

For mutually exclusive events: $B_1, B_2, \ldots B_n$ such that every outcome in the sample space falls into one of those events:

$$\begin{aligned}\mathrm{P}(E) &= \sum_{i=1}^{n} \mathrm{P}(E \text{ and } B_i) && \text{Extension of our observation} \\ &= \sum_{i=1}^{n} \mathrm{P}(E|B_i)\,\mathrm{P}(B_i) && \text{Using chain rule on each term}\end{aligned}$$

Definition: Bayes' Theorem

The most common form of Bayes' Theorem is **Bayes' Theorem Classic**:

$$\mathrm{P}(B|E) = \frac{\mathrm{P}(E|B) \cdot \mathrm{P}(B)}{\mathrm{P}(E)}$$

Bayes' Theorem combined with the Law of Total Probability:

$$\mathrm{P}(B|E) = \frac{\mathrm{P}(E|B) \cdot \mathrm{P}(B)}{\mathrm{P}(E|B) \cdot \mathrm{P}(B) + \mathrm{P}(E|B^{\mathrm{C}}) \cdot \mathrm{P}(B^{\mathrm{C}})}$$