

# Many Coin Flips

In this section we are going to consider the number of heads on  $n$  coin flips. This thought experiment is going to be a basis for much probability theory! It goes far beyond coin flips.

Say a coin comes up heads with probability  $p$ . Most coins are fair and as such come up heads with probability  $p = 0.5$ . There are many events for which coin flips are a great analogy that have different values of  $p$  so lets leave  $p$  as a variable. You can try simulating coins here. Note that **H** is short for Heads and **T** is short for Tails. We think of each coin as distinct:

## Coin Flip Simulator

Number of flips  $n$ :

Probability of heads  $p$ :

New  
simulation

Simulator results:

T, T, H, H, T, T, H, T, H, T

Total number of heads: 4

Let's explore a few probability questions in this domain.

## Warmups

**What is the probability that all  $n$  flips are heads?**

Lets say  $n = 10$  this question is asking what is the probability of getting:

H, H, H, H, H, H, H, H, H, H

Each coin flip is independent so we can use the rule for [probability of and with independent events](#). As such, the probability of  $n$  heads is  $p$  multiplied by itself  $n$  times:  $p^n$ . If  $n = 10$  and  $p = 0.6$  then the probability of  $n$  heads is around 0.006.

**What is the probability that all  $n$  flips are tails?**

Lets say  $n = 10$  this question is asking what is the probability of getting:

T, T, T, T, T, T, T, T, T, T

Each coin flip is independent. The probability of tails on any coin flip is  $1 - p$ . Again, since the coin flips are independent, the probability of tails  $n$  times on  $n$  flips is  $(1 - p)$  multiplied by itself  $n$  times:  $(1 - p)^n$ . If  $n = 10$  and  $p = 0.6$  then the probability of  $n$  tails is around 0.0001.

**First  $k$  heads then  $n - k$  tails**

Lets say  $n = 10$  and  $k = 4$ , this question is asking what is the probability of getting:

H, H, H, H, T, T, T, T, T, T

The coins are still independent! The first  $k$  heads occur with probability  $p^k$  the run of  $n - k$  tails occurs with probability  $(1 - p)^{n-k}$ . The probability of  $k$  heads then  $n - k$  tails is the product of those two terms:  $p^k \cdot (1 - p)^{n-k}$

## Exactly $k$ heads

Next lets try to figure out the probability of exactly  $k$  heads in the  $n$  flips. Importantly we don't care where in the  $n$  flips that we get the heads, as long as there are  $k$  of them. Note that this question is different than the question of first  $k$  heads and then  $n - k$  tails which requires that the  $k$  heads come first! That particular result does generate exactly  $k$  coin flips, but there are others.

There are many others! In fact any permutation of  $k$  heads and  $n - k$  tails will satisfy this event. Lets ask the computer to list them all for exactly  $k = 4$  heads within  $n = 10$  coin flips. The output region is scrollable:

```
(H, H, H, H, T, T, T, T, T, T)
(H, H, H, T, H, T, T, T, T, T)
(H, H, H, T, T, H, T, T, T, T)
(H, H, H, T, T, T, H, T, T, T)
(H, H, H, T, T, T, T, H, T, T)
(H, H, H, T, T, T, T, T, H, T)
(H, H, H, T, T, T, T, T, T, H)
(H, H, T, H, H, T, T, T, T, T)
(H, H, T, H, T, H, T, T, T, T)
(H, H, T, H, T, T, H, T, T, T)
(H, H, T, H, T, T, T, H, T, T)
(H, H, T, H, T, T, T, T, H, T)
(H, H, T, H, T, T, T, T, T, H)
(H, H, T, T, H, H, T, T, T, T)
(H, H, T, T, H, T, H, T, T, T)
(H, H, T, T, H, T, T, H, T, T)
(H, H, T, T, H, T, T, T, H, T)
(H, H, T, T, H, T, T, T, T, H)
```

Exactly how many outcomes are there with  $k = 4$  heads in  $n = 10$  flips? 210. The answer can be calculated using permutations of indistinct objects:

$$N = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

The probability of exactly  $k = 4$  heads is the probability of the **or** of each of these outcomes. Because we consider each coin to be unique, each of these outcomes is "mutually exclusive" and as such if  $E_i$  is the outcome from the  $i$ th row,

$$P(\text{exactly } k \text{ heads}) = \sum_{i=1}^N P(E_i)$$

The next question is, what is the probability of each of these outcomes?

Here is a arbitrarily chosen outcome which satisfies the event of exactly  $k = 4$  heads in  $n = 10$  coin flips. In fact it is the one on row 128 in the list above:

```
T, H, T, T, H, T, T, H, H, T
```

What is the probability of getting the exact sequence of heads and tails in the example above? Each coin flip is still independent, so we multiply  $p$  for each heads and  $1 - p$  for each tails. Let  $E_{128}$  be the event of this exact outcome:

$$P(E_{128}) = (1-p) \cdot p \cdot (1-p) \cdot (1-p) \cdot p \cdot (1-p) \cdot (1-p) \cdot p \cdot p \cdot (1-p)$$

If you rearrange these multiplication terms you get:

$$\begin{aligned} P(E_{128}) &= p \cdot p \cdot p \cdot p \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \cdot (1-p) \\ &= p^4 \cdot (1-p)^6 \end{aligned}$$

There is nothing too special about row 128. If you chose any row, you would get  $k$  independent heads and  $n - k$  independent tails. For any row  $i$ ,  $P(E_i) = p^k \cdot (1-p)^{n-k}$ . Now we are ready to calculate the probability of exactly  $k$  heads:

$$\begin{aligned}
P(\text{exactly } k \text{ heads}) &= \sum_{i=1}^N P(E_i) && \text{Mutual Exclusion} \\
&= \sum_{i=1}^N p^k \cdot (1-p)^{n-k} && \text{Sub in } P(E_i) \\
&= N \cdot p^k \cdot (1-p)^{n-k} && \text{Sum } N \text{ times} \\
&= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} && \text{Perm of indistinct objects}
\end{aligned}$$

## More than $k$ heads

You can use the formula for exactly  $k$  heads to compute other probabilities. For example the probability of more than  $k$  heads is:

$$\begin{aligned}
P(\text{more than } k \text{ heads}) &= \sum_{i=k+1}^n P(\text{exactly } i \text{ heads}) && \text{Mutual Exclusion} \\
&= \sum_{i=k+1}^n \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} && \text{Substitution}
\end{aligned}$$