Binomial Approximation

There are times when it is exceptionally hard to numerically calculate probabilities for a binomial distribution, especially when n is large. For example, say $X \sim \text{Bin}(n = 10000, p = 0.5)$ and you want to calculate $\text{P}(X > 5500)$. The correct formula is:

$$\begin{aligned}
\text{P}(X > 55) &= \sum_{i=5500}^{10000} \text{P}(X = x) \\
&= \sum_{i=5500}^{10000} \binom{10000}{i} p^i (1-p)^{10000-i}
\end{aligned}$$

That is a difficult value to calculate. Luckily there is an easier way. For deep reasons which we will cover in our section on "uncertainty theory" it turns out that a binomial distribution can be very well approximated by both Normal distributions and Poisson distributions if n is large enough.

Use the Poisson approximation when n is large (>20) and p is small (<0.05). A slight dependence between results of each experiment is ok

Use the Normal approximation when n is large (>20), and p is mid-ranged. Specifically it's considered an accurate approximation when the variance is greater then 10, in other words: $np(1-p) > 10$. There are situations where either a Poisson or a Normal can be used to approximate a Binomial. In that situation go with the Normal!

Poisson Approximation

When defining the Poisson we proved that a Binomial in the limit as $n \to \infty$ and $p = \lambda/n$ is a Poisson. That same logic can be used to show that a Poisson is a great approximation for a Binomial when the Binomial has extreme values of n and p. A Poisson random variable approximates Binomial where n is large, p is small, and $\lambda = np$ is “moderate”. Interestingly, to calculate the things we care about (PMF, expectation, variance) we no longer need to know n and p. We only need to provide λ which we call the rate. When approximating a Poisson with a Binomial, always choose $\lambda = n \cdot p$.

There are different interpretations of "moderate". The accepted ranges are $n > 20$ and $p < 0.05$ or $n > 100$ and $p < 0.1$.

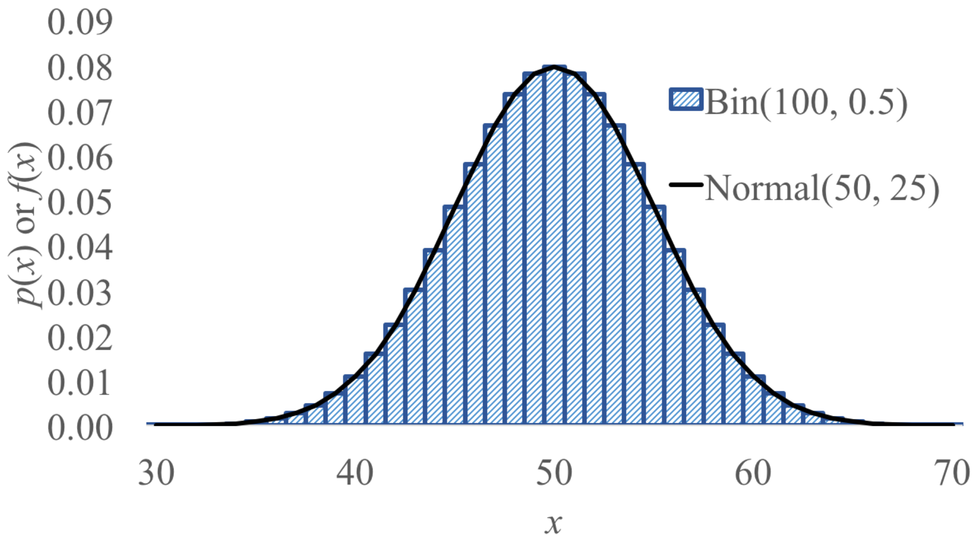
Let's say you want to send a bit string of length $n = 10^4$ where each bit is independently corrupted with $p = 10^{-6}$. What is the probability that the message will arrive uncorrupted? You can solve this using a Poisson with $\lambda = np = 10^4 10^{-6} = 0.01$. Semantically, $\lambda = 0.01$ means that we expect 0.01 corrupt bits per string, assuming bits are continuous. Let $X \sim Poi(0.01)$ be the number of corrupted bits. Using the PMF for Poisson:

$$\begin{aligned}
P(X = 0) &= \frac{\lambda^i}{i!} e^{-\lambda} \\
&= \frac{0.01^0}{0!} e^{-0.01} \\
&\sim 0.9900498
\end{aligned}$$

We could have also modelled X as a binomial such that $X \sim Bin(10^4, 10^{-6})$. That would have been impossible to calculate on a computer but would have resulted in the same number (up to the millionth decimal).

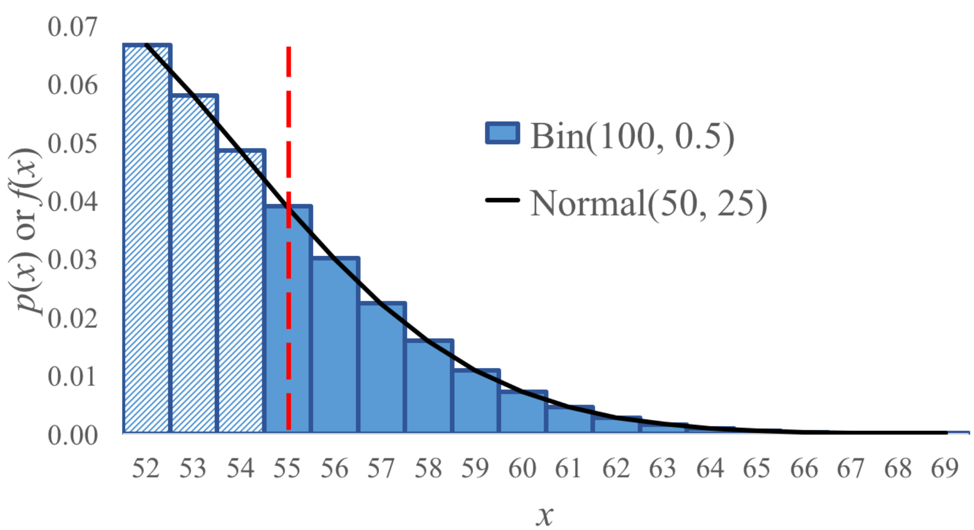
Normal Approximation

For a Binomial where n is large and p is mid-ranged, a Normal can be used to approximate the Binomial. Let's take a side by side view of a normal and a binomial:



Lets say our binomial is a random variable $X \sim \text{Bin}(100, 0.5)$ and we want to calculate $P(X \geq 55)$. We could cheat by using the closest fit normal (in this case $Y \sim N(50, 25)$). How did we choose that particular Normal? Simply select one with a mean and variance that matches the Binomial expectation and variance. The binomial expectation is $np = 100 \cdot 0.5 = 50$. The Binomial variance is $np(1-p) = 100 \cdot 0.5 \cdot 0.5 = 25$.

You can use a Normal distribution to approximate a Binomial $X \sim \text{Bin}(n, p)$. To do so define a normal $Y \sim (E[X], Var(X))$. Using the Binomial formulas for expectation and variance, $Y \sim (np, np(1-p))$. This approximation holds for large n and moderate p. That gets you very close. However since a Normal is continuous and Binomial is discrete we have to use a continuity correction to discretize the Normal.



$$P(X = k) \sim P\left(k - \frac{1}{2} < Y < k + \frac{1}{2}\right) = \Phi\left(\frac{k - np + 0.5}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - np - 0.5}{\sqrt{np(1-p)}}\right)$$

You should get comfortable deciding what continuity correction to use. Here are a few examples of discrete probability questions and the continuity correction:

Discrete (Binomial) probability question	Equivalent continuous probability question
$P(X = 6)$	$P(5.5 < X < 6.5)$
$P(X \geq 6)$	$P(X > 5.5)$
$P(X > 6)$	$P(X > 6.5)$
$P(X < 6)$	$P(X < 5.5)$
$P(X \leq 6)$	$P(X < 6.5)$

Example: 100 visitors to your website are given a new design. Let X = # of people who were given the new design and spend more time on your website. Your CEO will endorse the new design if $X \geq 65$. What is $P(\text{CEO endorses change}|\text{it has no effect})$?

$E[X] = np = 50$. $\text{Var}(X) = np(1-p) = 25$. $\sigma = \sqrt{\text{Var}(X)} = 5$. We can thus use a Normal approximation: $Y \sim \mathcal{N}(\mu = 50, \sigma^2 = 25)$.

$$P(X \geq 65) \approx P(Y > 64.5) = P\left(\frac{Y-50}{5} > \frac{64.5-50}{5}\right) = 1 - \Phi(2.9) = 0.0019$$

Example: Stanford accepts 2480 students and each student has a 68% chance of attending. Let X = # students who will attend. $X \sim \text{Bin}(2480, 0.68)$. What is $P(X > 1745)$?

$E[X] = np = 1686.4$. $\text{Var}(X) = np(1-p) = 539.7$. $\sigma = \sqrt{\text{Var}(X)} = 23.23$. We can thus use a Normal approximation: $Y \sim \mathcal{N}(\mu = 1686.4, \sigma^2 = 539.7)$.

$$\begin{aligned} P(X > 1745) &\approx P(Y > 1745.5) \\ &\approx P\left(\frac{Y-1686.4}{23.23} > \frac{1745.5-1686.4}{23.23}\right) \\ &\approx 1 - \Phi(2.54) = 0.0055 \end{aligned}$$