Homework 10

123456,皮卡丘

December 12, 2019

Problem 1

Determine whether the following relations on $\{0, 1, 2, 3\}$ are partial ordering. If not, show the reason.

1. $\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\}$
2. $\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle\}$
3. $\{\langle 0, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\}$

Solution.

1. No, $\langle 2, 3\rangle, \langle 3, 2\rangle$ can not exist in a partial relation together.
2. Yes
3. Yes

Problem 2

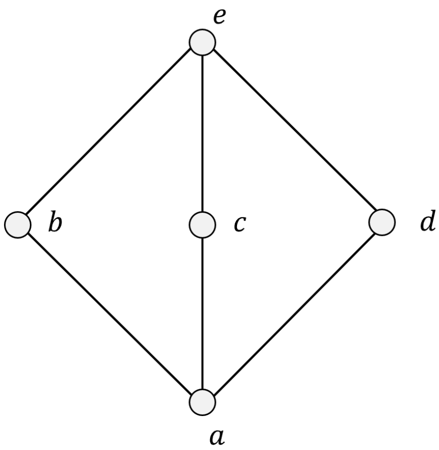
Draw the hasse diagram of poset $\langle A, R\rangle$, and write the maximal, minimal, maximum and minimum elements of A

1. $A = \{a, b, c, d, e\}$

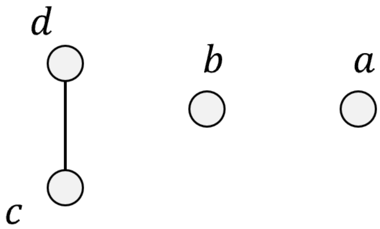
   $R = \{\langle a, d\rangle, \langle a, c\rangle, \langle a, b\rangle, \langle a, e\rangle, \langle b, e\rangle, \langle c, e\rangle, \langle d, e\rangle\} \cup I_A$
2. $A = \{a, b, c, d\}$

   $R = \{\langle c, d\rangle\} \cup I_A$

Solution.



1.



2.

Problem 3

Let R be a partial order on set A, $B \subseteq A$, prove that $R \cap (B \times B)$ is a partial order on B.

Solution.

Prove:

Reflexivity: For any $x \in B$

$x \in B \Rightarrow \langle x, x \rangle \in R \wedge \langle x, x \rangle \in (B \times B) \Leftrightarrow \langle x, x \rangle \in R \cap (B \times B)$

Anti-symmetry: For any $x, y \in B$

$\langle x, y \rangle \in R \cap (B \times B) \Leftrightarrow \langle x, y \rangle \in R \wedge \langle x, y \rangle \in (B \times B) \Rightarrow \langle x, y \rangle \in R$

So

$\langle x, y \rangle \in R \cap (B \times B) \wedge \langle y, x \rangle \in R \cap (B \times B)$

$\Rightarrow \langle x, y \rangle \in R \wedge \langle y, x \rangle \in R$

$\Rightarrow x = y$

Transitivity: For any $x, y, z \in B$

$\langle x, y \rangle \in R \cap (B \times B) \wedge \langle y, z \rangle \in R \cap (B \times B)$

$\Rightarrow \langle x, y \rangle \in R \wedge \langle y, z \rangle \in R$

$\Rightarrow \langle x, z \rangle \in R$

$\Rightarrow \langle x, z\rangle \in R \wedge \langle x, z\rangle \in (B \times B)$
$\Rightarrow \langle x, z\rangle \in R \cap (B \times B)$

Q.E.D

Problem 4

Find $r(R), s(R), t(R)$ for the relation given blow.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution.

1. $M_{r(R)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

2. $M_{s(R)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

3. $M_{t(R)} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Problem 5

Theorem 10.5.5: Let R_1, R_2 be relations on a non-empty set A, if $R_1 \subseteq R_2$ prove that (only using definition):

1. $r(R_1) \subseteq r(R_2)$

2. $s(R_1) \subseteq s(R_2)$

3. $t(R_1) \subseteq t(R_2)$

Solution.

1. $R_2 \subseteq r(R_2)$ and $R_1 \subseteq R_2$, so we have $R_1 \subseteq r(R_2)$. We know $r(R_2)$ is reflexive and according to the defination of closure, any reflexive relation that contains R_1 must contains its reflexive closure, so $r(R_1) \subseteq r(R_2)$.

2. $R_2 \subseteq s(R_2)$ and $R_1 \subseteq R_2$, so we have $R_1 \subseteq s(R_2)$. We know $s(R_2)$ is symmetric and according to the defination of closure, any symmetric relation that contains R_1 must contains its symmetric closure, so $s(R_1) \subseteq s(R_2)$.

3. $R_2 \subseteq t(R_2)$ and $R_1 \subseteq R_2$, so we have $R_1 \subseteq t(R_2)$. We know $r(R_2)$ is transitive and according to the defination of closure, any transitive relation that contains R_1 must contains its transitive closure, so $t(R_1) \subseteq t(R_2)$.

Problem 6

Theorem 10.5.6: Let R_1, R_2 be relations on a non-empty set A, prove that (only using definition):

1. $s(R_1) \cup s(R_2) = s(R_1 \cup R_2)$

2. $t(R_1) \cup t(R_2) \subseteq t(R_1 \cup R_2)$

Solution.

1. $R_1 \subseteq R_1 \cup R_2$ and $R_2 \subseteq R_1 \cup R_2$, so we have $s(R_1) \subseteq s(R_1 \cup R_2)$ and $s(R_2) \subseteq s(R_1 \cup R_2)$ according to problem 5. So $s(R_1) \cup s(R_2) \subseteq s(R_1 \cup R_2)$

   We then prove that $s(R_1) \cup s(R_2)$ is symmetric.

   $\langle x, y\rangle \in s(R_1) \cup s(R_2)$
   $\Leftrightarrow \langle x, y\rangle \in s(R_1) \vee \langle x, y\rangle \in s(R_2)$
   $\Rightarrow \langle y, x\rangle \in s(R_1) \vee \langle y, x\rangle \in s(R_2)$
   $\Leftrightarrow \langle y, x\rangle \in s(R_1) \cup s(R_2)$

   So $s(R_1) \cup s(R_2)$ is symmetric. And we have $R_1 \cup R_2 \subseteq s(R_1) \cup s(R_2)$, according to the definition of closure we have $s(R_1 \cup R_2) \subseteq s(R_1) \cup s(R_2)$.

   $s(R_1) \cup s(R_2) \subseteq s(R_1 \cup R_2) \wedge s(R_1 \cup R_2) \subseteq s(R_1) \cup s(R_2)$
   $\Rightarrow s(R_1 \cup R_2) = s(R_1) \cup s(R_2)$

2. $R_1 \subseteq R_1 \cup R_2$ and $R_2 \subseteq R_1 \cup R_2$, so we have $t(R_1) \subseteq t(R_1 \cup R_2)$ and $t(R_2) \subseteq t(R_1 \cup R_2)$ according to problem 5. So $t(R_1) \cup t(R_2) \subseteq t(R_1 \cup R_2)$