Homework 8

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Problem 1

Let universal set $E = \{1, 2, 3, 4, 5\}$, set $A = \{1, 4\}$, $B = \{1, 2, 5\}$, $C = \{2, 4\}$, calculating the sets:

1. $A \cup -B$
2. $(A \cap B) \cup -C$
3. $(A \cup B) \oplus C$
4. $\mathcal{P}(A) - \mathcal{P}(B)$

Solution.

1. $\{1, 3, 4\}$
2. $\{1, 3, 5\}$
3. $\{1, 5\}$
4. $\{\{4\}, \{1, 4\}\}$

Problem 2

Let A, B and C be sets. Prove that:

1. $A \oplus B = (A \cup B) \cap (\overline{B} \cup \overline{A})$
2. $(A - B) - C \subseteq A - C$
3. $(B - A) \cup (C - A) = (B \cup C) - A$
4. $\bigcup(\mathcal{P}(A)) = A$
5. $\bigcup(A \cup B) = (\bigcup A) \cup (\bigcup B)$

Solution.

1. $A \oplus B = (A - B) \cup (B - A) = (A \cap \overline{B}) \cup (B \cap \overline{A})$
   $= (A \cup B) \cap (A \cup \overline{A}) \cap (\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})$
   $= (A \cup B) \cap E \cap E \cap (\overline{B} \cup \overline{A})$
   $= (A \cup B) \cap (\overline{B} \cup \overline{A})$

2. $x \in (A - B) - C$
   $\Leftrightarrow x \in (A - B) \wedge x \notin C$
   $\Leftrightarrow (x \in A \wedge x \notin B) \wedge x \notin C$
   $\Leftrightarrow x \notin B \wedge (x \in A \wedge x \notin C)$
   $\Rightarrow x \in A \wedge x \notin C$
   $\Leftrightarrow x \in A - C$

   So $(A - B) - C \subseteq A - C$

3. $(B - A) \cup (C - A) = (B \cap \overline{A}) \cup (C \cap \overline{A})$
   $= (B \cup C) \cap \overline{A}$
   $= (B \cup C) - A$

4. $x \in \bigcup(\mathcal{P}(A))$
   $\Leftrightarrow (\exists y)(x \in y \wedge y \in \mathcal{P}(A))$
   $\Leftrightarrow (\exists y)(x \in y \wedge y \subseteq A)$
   $\Leftrightarrow x \in A$

5. $x \in \bigcup(A \cup B) \Leftrightarrow (\exists y)(x \in y \wedge y \in (A \cup B))$
   $\Leftrightarrow (\exists y)(x \in y \wedge (y \in A \vee y \in B))$
   $\Leftrightarrow (\exists y)(x \in y \wedge y \in A) \vee (\exists y)(x \in y \wedge y \in B)$
   $\Leftrightarrow x \in \bigcup A \vee x \in \bigcup B \Leftrightarrow x \in (\bigcup A) \cup (\bigcup B)$

Problem 3

Definition 9.5.1 Transitive Set : A set of sets A is called transitive set if any element of A's element is an element of A, or :

$$A \text{ is a transitive set} \Leftrightarrow (\forall x)(\forall y)((x \in y \wedge y \in A) \to x \in A)$$

For example $A = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ is a transitive set. Prove that:

$$A \text{ is a transitive set} \Leftrightarrow \bigcup A \subseteq A$$

Solution.

A is a transitive set
$\Leftrightarrow (\forall x)(\forall y)(x \in y \wedge y \in A \to x \in A)$
$\Leftrightarrow (\forall x)((\forall y)(x \in y \wedge y \in A \to x \in A))$
$\Leftrightarrow (\forall x)((\exists y)(x \in y \wedge y \in A) \to x \in A)$ (according to 5.2.2)

$\Leftrightarrow (\forall x)(x \in \bigcup A \rightarrow x \in A)$
$\Leftrightarrow \bigcup A \subseteq A$

Problem 4

Let A, B and C be sets. Prove that:

1. $A \subseteq C \wedge B \subseteq C \Leftrightarrow A \cup B \subseteq C$
2. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$
3. $\mathcal{P}(A) \subseteq \mathcal{P}(B) \Leftrightarrow A \subseteq B$
4. $\mathcal{P}(A) = \mathcal{P}(B) \Leftrightarrow A = B$
5. $A \subseteq B \Rightarrow \bigcup A \subseteq \bigcup B$

Solution.

1.
$$\begin{aligned}
A \subseteq C \wedge B \subseteq C &\Leftrightarrow (\forall x)(x \in A \rightarrow x \in C) \wedge (\forall x)(x \in B \rightarrow x \in C) \\
&\Leftrightarrow (\forall x)(x \in A \rightarrow x \in C) \wedge (x \in B \rightarrow x \in C) \\
&\Leftrightarrow (\forall x)((x \in A \vee x \in B) \rightarrow x \in C) \\
&\Leftrightarrow (\forall x)(x \in A \cup B \rightarrow x \in C) \\
&\Leftrightarrow A \cup B \subseteq C
\end{aligned}$$

2. $A = B$
$\Leftrightarrow (\forall x)(x \in A \leftrightarrow x \in B)$
$\Leftrightarrow (\forall x)((x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A))$
$\Leftrightarrow (\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in A)$
$\Leftrightarrow A \subseteq B \wedge B \subseteq A$

3. $\Leftarrow$: Assume that $A \subseteq B$.

   $x \in \mathcal{P}(A) \Leftrightarrow x \subseteq A \Rightarrow x \subseteq B \Leftrightarrow x \in \mathcal{P}(B)$

   So $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

   $\Rightarrow$: Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

   $x \in A \Leftrightarrow \{x\} \subseteq A \Leftrightarrow \{x\} \in \mathcal{P}(A) \Rightarrow \{x\} \in \mathcal{P}(B) \Leftrightarrow \{x\} \subseteq B \Leftrightarrow x \in B$

   So $A \subseteq B$

   Q.E.D

4. $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A \Leftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B) \wedge \mathcal{P}(B) \subseteq \mathcal{P}(A) \Leftrightarrow \mathcal{P}(A) = \mathcal{P}(B)$

5. $x \in \bigcup A$
   $\Leftrightarrow (\exists y)(x \in y \land y \in A)$
   $\Rightarrow (\exists y)(x \in y \land y \in B)$
   $\Leftrightarrow x \in \bigcup B$
   So $\bigcup A \subseteq \bigcup B$

Problem 5

Definition 9.7.2 Singular Set: A set A is called singular set if there exists an infinite sequence of A's elements, $A_0 \in A, A_1 \in A, A_2 \in A, ..., A_n \in A, ...$(not necessarily distinct) such that:

$$... \in A_{n+1} \in A_n \in ... \in A_2 \in A_1 \in A_0$$

Prove that:

1. If $x \in x$, then $\{x\}$ is a singular set.
2. There don't exist set A and B such that $(A \in B \land B \in A)$. (Tips: Constructing a singular set based on A and B and using theorem 9.7.9)

Solution.

1. If $x \in x$, then the infinite sequence $x, x, x, x, ...$ satisfies that $... \in x \in x \in x$
2. If there exist set A, B such that $A \in B \land B \in A$, then $C = \{A, B\}$ is a valid set according to axiom of unorder pair. And the infinite sequence $A, B, A, B, A, B...$ statisfies that $...B \in A \in B \in A$, so C is a singular set. According to theorem 9.7.9, C violates axiom of regularity. Contradiction.

Problem 6

Let A be a set, prove that $\{A\}$ is a set :

1. Using the axiom of paring.
2. Without using the axiom of paring.

Solution.

1. Let $x = y = A$, then using axiom of paring, we get A is a set.
2. According to the **axiom of empty set**, $\varnothing$ is a valid set, and using the **axiom of power set**, we get $\mathcal{P}(\varnothing) = \{\varnothing\}$ is a valid set. Then we let set $t = \{\varnothing\}$ and define a predicate $P(x, y) : x = \varnothing \land y = A$. $P(x, y)$ and t satisfy the axiom of replacement. We can get $\{A\}$ is a valid set using the **axiom of replacement** on $P(x, y)$ and t .

Problem 7

Let A be a set of sets and $|A| = n (n > 0)$. Prove that:

$$|\bigcup A| > n \Rightarrow (\exists A_0)(A_0 \in A \wedge |A_0| > 1)$$

Solution.

$$|\bigcup A| > n \Rightarrow (\exists A_0)(A_0 \in A \wedge |A_0| > 1)$$
$$\equiv \neg(\exists A_0)(A_0 \in A \wedge |A_0| > 1) \Rightarrow \neg|\bigcup A| > n$$
$$\equiv (\forall A_0)(\neg(A_0 \in A \wedge |A_0| > 1)) \Rightarrow |\bigcup A| \leq n$$
$$\equiv (\forall A_0)(\neg(A_0 \in A) \vee |A_0| \leq 1) \Rightarrow |\bigcup A| \leq n$$
$$\equiv (\forall A_0)(A_0 \in A \rightarrow |A_0| \leq 1) \Rightarrow |\bigcup A| \leq n$$

From $|A \cup B| \leq |A| + |B|$ we have:$|\bigcup A| \leq \sum_{A_i \in A} |A_i|$
Assume that $(\forall A_0)(A_0 \in A \rightarrow |A_0| \leq 1)$, then we have

$$|\bigcup A| \leq \sum_{A_i \in A} |A_i| \leq \sum_{A_i \in A} 1 = |A| = n$$

Contradiction.

Q.E.D

Problem 8

Find the number of integers such that $1 \leq x \leq 2019$ and x is relatively prime to 2020.

Solution.
$2020 = 2^2 \cdot 5 \cdot 101$, so we only need to find the integer which is not devidable by 2, 5 and 101.

Let $U = \{x | 1 \leq x \leq 2019\}$, $A_2 = \{x | x \in U \wedge 2|x\}$,$A_5 = \{x | x \in U \wedge 5|x\}$, $A_{101} = \{x | x \in U \wedge 101|x\}$, $R = \{x \in U \wedge (\text{x is relatively prime to 2020})\}$

Then $|R| = |U| - |A_2 \cup A_5 \cup A_{101}|$

According to principle of inclusion and exclusion:

$|A_2 \cup A_5 \cup A_{101}|$
$= |A_2| + |A_5| + |A_{101}| - |A_2 \cap A_5| - |A_2 \cap A_{101}| - |A_5 \cap A_{101}| + |A_2 \cap A_5 \cap A_{101}|$
$= [\frac{2019}{2}] + [\frac{2019}{5}] + [\frac{2019}{101}] - [\frac{2019}{2 \cdot 5}] - [\frac{2019}{2 \cdot 101}] - [\frac{2019}{5 * 101}] + [\frac{2019}{2 \cdot 5 \cdot 101}]$
$= 1009 + 403 + 19 - 201 - 9 - 3 + 1 = 1219$

Then $|R| = 2019 - 1219 = 800$

Problem 9

Prove that:

1. $[0, 1] \approx [a, b]$, where $a < b, a \in \mathbb{R}$ and $b \in \mathbb{R}$
2. $[0, 1] \approx \mathbb{R}$

Solution.

1. Define function $f(x) = (b - a) \cdot x + a$, then we can pair $x \in [0, 1]$ and $f(x) \in [a, b]$ with out leaving any elements.
2. $|[0, 1]| \leq |\mathbb{R}|$ since $[0, 1] \subseteq \mathbb{R}$.

   We can also define a function $f(x) = \frac{arctan(x)}{\pi} + \frac{1}{2}$ from $\mathbb{R}$ to $(0, 1)$, so that we can pair $x \in \mathbb{R}$ with $f(x) \in (0, 1)$, so we have $|\mathbb{R}| = |(0, 1)| \leq |[0, 1]|$

   So $|[0, 1]| = |\mathbb{R}|$