Homework 11

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Problem 1

Find $t(R)$ using warshall algorithm for the relation given blow.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution.

1. i=2,j=1. $M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

2. i=2,j=4. $M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

3. i=3,j=1. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

4. i=3,j=2. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

5. i=3,j=4. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

6. i=4,j=1. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

7. i=4,j=2. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

8. i=4,j=3. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

9. i=4,j=4. $M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Problem 2

Theorem 10.5.11: For a relation R on a non-empty set, prove that:

1. If R is reflexive, then $s(R)$ and $t(R)$ are both reflexive.
2. If R is transitive, then $r(R)$ is transitive. And find a counterexample to show that $s(R)$ is not transitive.

Solution.
Assume that R is a relation on set A.

1. For all $x \in A$, we have $\langle x, x\rangle \in R$ since R is reflexive. And we have $R \subseteq s(R)$ and $R \subseteq t(R)$, so for all $x \in A$, $\langle x, x\rangle \in s(R)$ and $\langle x, x\rangle \in t(R)$ still holds. So $s(R)$ and $t(R)$ are both reflexive.
2. For any $x, y, z \in A$, if $\langle x, y\rangle \in r(R)$ and $\langle y, z\rangle \in r(R)$.
   (a) If $x = y \lor y = z$, then we can easily get $\langle x, z\rangle \in r(R)$
   (b) If $x \neq y \land y \neq z$, as $r(R) = R \cup I_A$, so we have $\langle x, y\rangle \in r(R) \land x \neq y \Rightarrow \langle x, y\rangle \in R$. Similarly we have $\langle y, z\rangle \in R$, so we have
   $\langle x, y\rangle \in R \land \langle y, z\rangle \in R \Rightarrow \langle x, z\rangle \in R \Rightarrow \langle x, z\rangle \in r(R)$

   So $r(R)$ is transitive.

   Let $A = \{1, 2\}, R = \{\langle 1, 2\rangle\}$, R is transitive, but $s(R) = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$ is not transitive.

Problem 3

An equivalence closure $e(R)$ for releation R is defined by:

1. $e(R)$ is an equivalence relation.
2. For any equivalence R', if $R \subseteq R'$, then $e(R) \subseteq R'$

For a relation R on a non-empty set, prove that $tsr(R)$ (defined in theorem 10.5.12) is the equivalence closure of R.

Solution.
Assume that R is a relation on A.

Method 1:

1. Firstly we prove that $tsr(R)$ is an equivalence relation.

   According to **Theorem 10.5.11**, we know that:

   (a) R is reflexive $\Rightarrow s(R)$ is reflexive $\wedge\, t(R)$ is reflexive

   (b) R is symmetric $\Rightarrow t(R)$ is symmetric

   Then:

   (a) $r(R)$ is reflexive $\Rightarrow sr(R)$ is reflexive $\Rightarrow tsr(R)$ is reflexive

   (b) $sr(R)$ is symmetric $\Rightarrow tsr(R)$ is symmetric

   $tsr(R)$ is transitive by definition.

   So we have $tsr(R)$ is an equivalence relation.

2. Then we prove that $e(R) = tsr(R)$

   We have $sr(R) = s(R \cup I_A) = R^{-1} \cup R \cup I_A$. Since $e(R)$ is an equivalence relation and $R \subseteq e(R)$, so $R^{-1} \subseteq e(R)$ and $I_A \subseteq e(R)$. Then we have $sr(R) \subseteq e(R)$

   As $tsr(R) = t(sr(R))$ is the transitive closure of $sr(R)$, $sr(R) \subseteq e(R)$ and $e(R)$ is transitive, according to the definition of transitive closure we have $tsr(R) \subseteq e(R)$.

   As $e(R)$ is the equivalence closure of R and $tsr(R)$ is an equivalence relation, we have $e(R) \subseteq tsr(R)$.

   So we have $e(R) = tsr(R)$

   Q.E.D

Method 2:

1. Similarly we can prove that $tsr(R)$ is an equivalence relation first.

2. Then we prove that any equivalence relation R' , if $R \subseteq R'$ then $tsr(R) \subseteq R'$.

   For any $\langle x, y\rangle \in tsr(R)$, which means there's a path from x to y in $sr(R)$, or there exists $p_1, p_2, ...p_{k-1} \in A$ for certain $k \in \mathbb{N}^+$ such that $\langle x, p_1\rangle, \langle p_1, p_2\rangle, ...\langle p_{k-1}, y\rangle \in sr(R)$.

   Let $p_0 = x$ and $p_k = y$, we have $(\langle p_0, p_1\rangle \in sr(R)) \wedge (\langle p_1, p_2\rangle \in sr(R))... \wedge (\langle p_{k-1}, p_k\rangle \in sr(R)) = \bigwedge_{i=0}^{k-1}(\langle p_i, p_{i+1}\rangle \in sr(R))$

   So for any $\langle x, y\rangle \in tsr(R)$, we have:

   $\langle x, y\rangle \in tsr(R)$
   $\Rightarrow (\exists k)(k \in \mathbb{N}^+ \wedge x = p_0 \wedge y = p_k \wedge (\bigwedge_{i=0}^{k-1}\langle p_i, p_{i+1}\rangle \in sr(R))$
   $\Rightarrow (\exists k)(k \in \mathbb{N}^+ \wedge x = p_0 \wedge y = p_k \wedge (\bigwedge_{i=0}^{k-1}(\langle p_i, p_{i+1}\rangle \in r(R) \vee \langle p_{i+1}, p_i\rangle \in r(R)))$
   $\Rightarrow (\exists k)(k \in \mathbb{N}^+ \wedge x = p_0 \wedge y = p_k \wedge (\bigwedge_{i=0}^{k-1}(\langle p_i, p_{i+1}\rangle \in I_A \vee \langle p_i, p_{i+1}\rangle \in R \vee \langle p_{i+1}, p_i\rangle \in R))$
   $\Rightarrow (\exists k)(k \in \mathbb{N}^+ \wedge x = p_0 \wedge y = p_k \wedge (\bigwedge_{i=0}^{k-1}(\langle p_i, p_{i+1}\rangle \in I_A \vee \langle p_i, p_{i+1}\rangle \in R' \vee \langle p_{i+1}, p_i\rangle \in R'))$
   $\Rightarrow (\exists k)(k \in \mathbb{N}^+ \wedge x = p_0 \wedge y = p_k \wedge (\bigwedge_{i=0}^{k-1}(\langle p_i, p_{i+1}\rangle \in R'))$
   $\Rightarrow \langle x, y\rangle \in R'$

   So we have $tsr(R) \subseteq R'$.

   Q.E.D

Problem 4

Determine wether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is surjective if:

1. $f(m, n) = m + n$
2. $f(m, n) = m - n$
3. $f(m, n) = |m| - |n|$
4. $f(m, n) = m^2 + n^2$
5. $f(m, n) = m^2 - n^2$

Solution.

1. YES
2. YES
3. YES
4. NO $(m^2 + n^2 \neq -1)$
5. NO $(m^2 - n^2 \neq 2)$

Problem 5

For every function below, answer the questions:

1. Whether the function is injective, surjective or bijective. If it is bijective, write down the expression of f^{-1}
2. Write down the image of the function and the inverse image of a given set S.
3. The relation $R = \{\langle x, y\rangle | x, y \in dom(f) \land f(x) = f(y)\}$ is an equivalence relation on $dom(f)$, find this relation for the function.

All the functions:

1. $f : \mathbb{R} \to (0, \infty), f(x) = 2^x, S = [1, 2]$
2. $f : \mathbb{N} \to \mathbb{N}, f(n) = 2n + 1, S = \{2, 3\}$
3. $f : \mathbb{Z} \to \mathbb{N}, f(x) = |x|, S = \{0, 2\}$
4. $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, f(n) = \langle n, n + 1\rangle, S = \langle 2, 2\rangle$
5. $f : [0, 1] \to [0, 1], f(x) = \frac{2x+1}{4}, S = [0, \frac{1}{2}]$

Solution.

1. (a) Bijective. $f^{-1}(x) = log_2(x)$
   (b) image: $f[\mathbb{R}] = (0, \infty)$, reverse image $f^{-1}[S] = [0, 1]$
   (c) $R = I_{\mathbb{R}}$
2. (a) Injective.
   (b) image: $f[\mathbb{N}] = \{\text{positive odd numbers}\}$, reverse image $f^{-1}[S] = \{1\}$
   (c) $R = I_{\mathbb{N}}$
3. (a) Surjective.
   (b) image: $f[\mathbb{Z}] = \mathbb{N}$, reverse image $f^{-1}[S] = \{-2, 0, 2\}$
   (c) $R = I_{\mathbb{Z}} \cup \{\langle x, y\rangle | x \in \mathbb{Z} \land y \in \mathbb{Z} \land x = -y\}$
4. (a) Injective.
   (b) image: $f[\mathbb{N}] = \{\langle n, n + 1\rangle | n \in \mathbb{N}\}$, reverse image $f^{-1}[S] = \varnothing$
   (c) $R = I_{\mathbb{N}}$
5. (a) Injective.
   (b) image: $f[[0, 1]] = [\frac{1}{4}, \frac{3}{4}]$, reverse image $f^{-1}[S] = [0, \frac{1}{2}]$
   (c) $R = I_{[0,1]}$

Problem 6

Let $f, g \in A_B$, and $f \cap g \neq \varnothing$, are $f \cap g$ and $f \cup g$ are functions? If so, prove it. If not, show the counterexample.

Solution.

1. $f \cap g$ is a function.

   For any $x, y_1, y_2 \in dom(f \cap g)$:

   $\langle x, y_1 \rangle \in f \cap g \wedge \langle x, y_2 \rangle \in f \cap g$
   $\Leftrightarrow \langle x, y_1 \rangle \in f \wedge \langle x, y_1 \rangle \in g \wedge \langle x, y_2 \rangle \in f \wedge \langle x, y_2 \rangle \in g$
   $\Leftrightarrow (\langle x, y_1 \rangle \in f \wedge \langle x, y_2 \rangle \in f) \wedge (\langle x, y_1 \rangle \in g \wedge \langle x, y_2 \rangle \in g)$
   $\Rightarrow y_1 = y2$

   So $f \cap g$ is a function

2. $f \cup g$ is not a function.

   Let $A = \{1, 2\}, B = \{1, 2\}, f = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}, g = \{\langle 1, 2 \rangle, \langle 2, 2 \rangle\}$, now $f \cup g = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$ is not a function.