Homework 9

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December 21, 2019

Problem 1

List all the elements of relation R

1. $A = \{1, 2, 3\}, B = \{0, 2, 4\}$, and $R = \{\langle x, y\rangle | x, y \in A \cup B\}$
2. $A = \{1, 2, 3, 4, 5\}, B = \{1, 2, 3\}$, and $R = \{\langle x, y\rangle | x \in A \land y \in B \land x = y^2\}$

Solution.

1. $R = \{\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 0, 2\rangle, \langle 0, 3\rangle\ \langle 0, 4\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 3, 0\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle, \langle 3, 4\rangle, \langle 4, 0\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle\}$
2. $R = \{\langle 1, 1\rangle, \langle 4, 2\rangle\}$

Problem 2

Let R, S be relation on A, prove that:

$$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$$

Solution.

$$\begin{aligned}
&(x, y) \in R \circ (S \cup T) \\
\Leftrightarrow\ &(\exists z)((x, z) \in (S \cup T) \land (z, y) \in R) \\
\Leftrightarrow\ &(\exists z)((((x, z) \in S \lor (x, z) \in T) \land (z, y) \in R) \\
\Leftrightarrow\ &(\exists z)((((x, z) \in S \land (z, y) \in R) \lor ((x, z) \in T \land (z, y) \in R)) \\
\Leftrightarrow\ &(\exists z)((((x, z) \in S \land (z, y) \in R)) \lor (\exists z)((((x, z) \in T \land (z, y) \in R)) \\
\Leftrightarrow\ &(x, y) \in R \circ S \lor (x, y) \in R \circ T \\
\Leftrightarrow\ &(x, y) \in (R \circ S) \cup (R \circ T)
\end{aligned}$$

Problem 3

Let R_1 and R_2 be relations on a set A represented by the matrices:

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices that represents:

1. $R_1 \cup R_2$
2. $R_1 \cap R_2$
3. $R_1 \oplus R_2$
4. $R_1 \circ R_2$

Solution.

1. $R_1 \cup R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
2. $R_1 \cap R_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
3. $R_1 \oplus R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
4. $R_1 \circ R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Problem 4

For set $A = \{1, 2, 3, 4\}$, the relation R on A:

$$R = \{\langle 1, 2\rangle, \langle 4, 3\rangle, \langle 2, 2\rangle, \langle 2, 1\rangle\}$$

Construct a relation R_1 on A for each following problem, such that $R \subseteq R_1$, $R_1 \neq E_A$ and:

1. R_1 is reflexive.
2. R_1 is symmetric.

3. R_1 is transitive.

4. R_1 is an equivalence relation, then write the quotient set of R_1 on A

5. R_1 is a compatible relation, then write the complete cover of A

Solution.

1. $R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle\}$

2. $R_1 = \{\langle 1,2\rangle, \langle 4,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 2,1\rangle\}$.

3. $R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle\}$.

4. $R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 2,1\rangle\}$
   $A/R_1 = \{\{1,2\}, \{3,4\}\}$

5. $R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,3\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 2,1\rangle\}$
   $C_R(A) = \{\{1,2\}, \{3,4\}\}$

Problem 5

Let R_1 and R_2 are equivalence relations on a non-empty set A, determine whether the following relation is an equivalence relation on A. If so, prove it. If not, write a counterexample.

1. $(A \times A) - R_1$

2. R_1^2

3. $R_1 - R_2$

Solution.

1. $(A \times A) - R_1$ is not an equivalence relation.

   Let $A = \{1,2\}, R_1 = \{\langle 1,1\rangle, \langle 2,2\rangle\}$, then $(A \times A) - R_1 = \{\langle 1,2\rangle, \langle 2,1\rangle\}$. Now $\langle 1,1\rangle \notin (A \times A) - R_1$, so the reflexivity does not holds.

2. R_1^2 is an equivalence relation.

   Reflexivity: $\forall x \in A$, $\langle x,x\rangle \in R_1$ since R_1 is an equivalence relation. And we have:
   $\langle x,x\rangle \in R_1$
   $\Rightarrow \langle x,x\rangle \in R_1 \wedge \langle x,x\rangle \in R_1$
   $\Rightarrow (\exists z)(\langle x,z\rangle \in R_1 \wedge \langle z,x\rangle \in R_1)$
   $\Rightarrow \langle x,x\rangle \in R_1^2$

Symmetry: $\forall x, y \in A$

$\langle x, y\rangle \in R_1^2$
$\Leftrightarrow (\exists z)(\langle x, z\rangle \in R_1 \wedge \langle z, y\rangle \in R_1)$
$\Rightarrow (\exists z)(\langle z, x\rangle \in R_1 \wedge \langle y, z\rangle \in R_1)(R_1$ is symmetric$)$
$\Leftrightarrow \langle y, x\rangle \in R_1^2$

Transitivity: $\forall x, y, z \in A$

$\langle x, y\rangle \in R_1^2 \wedge \langle y, z\rangle \in R_1^2$
$\Leftrightarrow (\exists t_1)(\langle x, t_1\rangle \in R_1 \wedge \langle t_1, y\rangle \in R_1) \wedge (\exists t_2)(\langle y, t_2\rangle \in R_1 \wedge \langle t_2, z\rangle \in R_1)$
$\Rightarrow \langle x, y\rangle \in R_1 \wedge \langle y, z\rangle \in R_1(R_1$ is transitive$)$
$\Rightarrow (\exists t_3)(\langle x, t_3\rangle \in R_1 \wedge \langle t_3, z\rangle \in R_1)$
$\Leftrightarrow \langle x, z\rangle \in R_1^2$

3. $R_1 - R_2$ is not an equivalence relation.

   Let $R_1 = R_2 = A \times A$, then $R_1 - R_2 = \emptyset$ is not an equivalence relation.

Problem 6

R is a relation on set A, prove that:

$$S = I_A \cup R \cup R^{-1} \text{ is a compatible relation on } A$$

Solution.
Prove:

Reflexivity:
$\forall x \in A$ we have $\langle x, x\rangle \in I_A$. And $I_A \subseteq S$, so $(\forall x)(x \in A \rightarrow \langle x, x\rangle \in S)$

Symmetry: For $\langle x, y\rangle \in S$, if:

1. $\langle x, y\rangle \in I_A$, then $x = y$, so we have $\langle y, x\rangle \in S$ too.
2. $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R^{-1}$, and $R^{-1} \subseteq S$. So $\langle y, x\rangle \in S$ too.
3. $\langle x, y\rangle \in R^{-1}$, then $\langle y, x\rangle \in R$, and $R \subseteq S$. So $\langle y, x\rangle \in S$ too.

So $(\forall x)(\forall y)(\langle x, y\rangle \in S \rightarrow \langle y, x\rangle \in S)$
QED.