Hw2_613

Exercise 1

```
set.seed(100)
#Construct x1
X1<-vector()
X1<-runif(10000,min=1,max=3)
#Construct x2
X2<-vector()
X2<-rgamma(10000, 3, scale = 2)
#Construct x3
X3<-vector()
X3<-rbinom(10000,1, 0.3)
#Construct eps
eps<-vector()
eps<-rnorm(10000,mean=2,sd=1)
#Construct Y by using X1,X2,X3 and eps
Y<-0.5+ 1.2*X1 + -0.9*X2 + 0.1*X3 + eps
#Construct ydum
ydum<-vector()
for (i in Y){
  if (i>mean(Y)){
    ydum<-c(ydum,1)
  }
  else{
    ydum<-c(ydum,0)
  }
}
```

Exercise 2

Question 1

```
#Calculate the correlation between Y and X1
correlation<-cor(Y,X1)
print(correlation)
```

```
## [1] 0.2162074
```

The correlation is 0.216072, which is far away from 1.2

Question 2 & Question 3

Using OLS method

```
##Put X variables, Y variable and eps variable in the matrix form
X<-as.matrix(cbind(1,X1,X2,X3))
y<-as.matrix(Y)
eps<-as.matrix(eps)
##Calculate the coefficients on the regression by using OLS method
belta2 <-solve(t( X ) %*%X ) %*%t( X ) %*%y
##Calculate the standard errors
##Calculate the standard errors
Sderror<-(solve(t( X ) %*%X ) %*%t( X ) %*%eps)
print(Sderror)
```

```
##           [,1]
##    1.956103391
## X1 0.015799969
## X2 0.001556583
## X3 0.001876211
```

The standard error of the coefficients by using OLS method is printed above

Calculate the coefficients and their standard errors by using bootstrap

49 replications

```r
num=49
#Create a empty matrix to store samples which are re-sampled from the original data
pgn<-matrix(0,nrow=10000,ncol=4)
#Put Y and X variables in a data frame
DATA<-as.matrix(cbind(Y,X1,X2,X3))
#Construct a empty matrix to store standard errors of each sample
sdboot1<-matrix(0,nrow=num,ncol=4)
for (j in 1:num){
  #Sampling from original data with replacement
  for (i in 1:10000){
    k<-sample(1:nrow(DATA),1)
    sp<-as.matrix(DATA[k,])
    pgn[i,]<-t(sp)
  }
  #Constrcut Xloop; put new sample(X variables) in this matrix
  Xloop<-as.matrix(cbind(1,pgn[,2],pgn[,3],pgn[,4]))
  #Constrcut Yloop; put new sample(Y variable) in this matrix
  Yloop<-as.matrix(pgn[,1])
  #Calculate coefficients by using OLS method
  coeff<-solve(t( Xloop) %*%Xloop ) %*%t( Xloop ) %*%Yloop
  #Calculate residuals
  residu<-as.matrix(pgn[,1]-coeff[1]-coeff[2]*pgn[,2]-coeff[3]*pgn[,3]-coeff[4]*pgn[,
4])
  #Calculate the standard errors by using OLS method
  Sderrorboot1<-solve(t( Xloop) %*% Xloop ) %*%t( Xloop ) %*%residu
  #Plug the standard errors into matrix named sdboot1
  sdboot1[j,]<-Sderrorboot1
}
print(sdboot1)
```

```
##                [,1]          [,2]          [,3]          [,4]
##  [1,] -4.146872e-13  1.494494e-13  1.739256e-14  8.560626e-15
##  [2,] -3.347503e-13  1.547824e-13  2.153861e-15  1.321496e-14
##  [3,]  6.063702e-14 -6.765594e-14  1.357113e-14  3.028635e-14
##  [4,] -5.220668e-13  1.830336e-13  1.882257e-14  1.444794e-14
##  [5,] -7.970912e-14  2.462359e-14  6.094101e-15 -2.419885e-14
##  [6,] -3.870704e-13  1.421759e-13  1.198706e-14  2.212109e-14
##  [7,] -6.453930e-15 -2.864222e-14  2.942265e-15  1.681263e-14
##  [8,] -5.030178e-13  2.076767e-13  8.154203e-15  5.007727e-14
##  [9,] -4.172945e-13  1.533328e-13  1.462901e-14  2.516748e-14
## [10,] -5.415865e-13  1.896333e-13  2.374462e-14  2.220392e-14
## [11,] -1.995713e-13 -8.417949e-15  2.945030e-14  3.134103e-14
## [12,] -5.937649e-13  2.476301e-13  1.684162e-14  1.757633e-14
## [13,] -4.761680e-14  7.619784e-14 -1.770035e-14 -1.511521e-14
## [14,]  4.444174e-13 -6.578134e-14 -4.196044e-14 -4.016148e-14
## [15,] -5.050909e-13  2.051832e-13  1.491462e-14  6.802555e-15
## [16,]  2.680796e-14 -3.862838e-14  7.978485e-15 -1.950791e-15
## [17,] -6.853685e-13  2.284229e-13  2.528663e-14  5.356411e-14
## [18,] -3.343002e-13  1.449873e-13  1.031545e-14 -1.024029e-16
```

```
## [19,]  4.842264e-15  2.132941e-14 -3.256173e-15 -3.523656e-14
## [20,] -4.148054e-13  1.937187e-13  4.137224e-15  3.085639e-16
## [21,] -1.462696e-13  7.653786e-14 -7.347613e-15  3.634435e-15
## [22,] -1.302919e-13  1.428201e-14  2.282676e-14 -1.195996e-14
## [23,] -1.527915e-13  7.770650e-14 -3.677492e-15  1.725914e-15
## [24,] -2.001454e-14 -1.960980e-14  1.099727e-14 -1.172845e-14
## [25,]  3.023297e-14  5.376975e-14 -2.069083e-14 -1.883641e-14
## [26,] -2.265726e-13  4.933478e-14  1.691987e-14  6.439897e-15
## [27,] -1.440483e-13  1.558900e-14  1.312799e-14  7.919704e-15
## [28,] -2.261510e-13  1.409656e-13 -8.681681e-15  1.156378e-14
## [29,]  6.688003e-13 -2.876561e-13 -1.683327e-14 -2.967057e-14
## [30,] -4.169148e-14  7.541710e-14 -1.766118e-14  4.649926e-15
## [31,] -1.250862e-13  2.739904e-14  9.562838e-15  2.618911e-14
## [32,] -1.174516e-13  8.430097e-14 -1.042649e-14 -3.434495e-15
## [33,] -4.861721e-13  1.548504e-13  2.502362e-14  1.743225e-14
## [34,]  3.690218e-14  3.452594e-14 -1.447758e-14 -2.938439e-15
## [35,] -2.781678e-14 -2.003092e-14  4.295277e-15  1.636899e-14
## [36,] -1.322817e-13  5.907673e-14 -2.737541e-15 -1.056417e-14
## [37,] -2.717357e-13  1.555798e-13 -9.441222e-15  9.242345e-15
## [38,] -1.910691e-13  8.228940e-14  3.308368e-15  8.151872e-15
## [39,]  1.816811e-14 -3.120860e-14  1.026335e-14  2.428841e-14
## [40,] -2.245247e-13  2.760372e-14  1.706182e-14  3.448920e-14
## [41,]  2.345009e-13 -8.721247e-14 -6.380290e-15 -1.532426e-14
## [42,] -5.396433e-13  2.448335e-13  3.213764e-15 -6.819245e-15
## [43,]  2.895740e-13 -1.805003e-13  1.081584e-14  4.653667e-16
## [44,] -6.373793e-14  2.119848e-14  6.056040e-15 -5.555506e-15
## [45,]  3.285205e-13 -1.459777e-13 -5.090038e-15 -2.807352e-14
## [46,] -3.321217e-13  2.109772e-13 -1.773219e-14 -4.686057e-15
## [47,] -3.524534e-13  9.649914e-14  2.412234e-14  2.696930e-14
## [48,] -4.673477e-13  1.783088e-13  1.053539e-14  3.415494e-14
## [49,] -7.735316e-13  2.760566e-13  2.627171e-14  4.721845e-14
```

The standard error of the coefficients by using bootstrap (49 replications) is printed above

Using 499 replications

```r
num1=499
#Construct a empty matrix to store standard errors of each sample
sdboot2<-matrix(0,nrow=num1,ncol=4)
for (j in 1:num1){
  #Sampling from original data with replacement
  for (i in 1:10000){
    k<-sample(1:nrow(DATA),1)
    sp<-as.matrix(DATA[k,])
    pgn[i,]<-t(sp)
  }
  #Constrcut Xloop; put new sample(X variables) in this matrix
  Xloop<-as.matrix(cbind(1,pgn[,2],pgn[,3],pgn[,4]))
  #Constrcut Yloop; put new sample(Y variable) in this matrix
  Yloop<-as.matrix(pgn[,1])
  #Calculate coefficients by using OLS method
  coeff<-solve(t( Xloop) %*%Xloop ) %*%t( Xloop ) %*%Yloop
  #Calculate residuals
  residu<-as.matrix(pgn[,1]-coeff[1]-coeff[2]*pgn[,2]-coeff[3]*pgn[,3]-coeff[4]*pgn[,
4])
  #Calculate the standard errors by using OLS method
  Sderrorboot1<-solve(t( Xloop) %*% Xloop ) %*%t( Xloop ) %*%residu
  #Plug the standard errors into matrix named sdboot2
  sdboot2[j,]<-Sderrorboot1
}
print(sdboot2)
```

```
##                [,1]          [,2]          [,3]          [,4]
##   [1,] -6.817181e-13  2.819162e-13  1.038377e-14  3.542262e-14
##   [2,] -1.837455e-13  1.244300e-13 -7.561041e-15  1.461937e-14
##   [3,] -3.872366e-14  1.269029e-14  5.700923e-15  5.588473e-15
##   [4,] -4.356771e-13  1.414959e-13  1.048420e-14  3.663628e-14
##   [5,] -6.663241e-13  2.076236e-13  3.175961e-14  6.439954e-14
##   [6,] -2.923204e-13  1.749616e-13 -9.049260e-15  9.617632e-15
##   [7,] -1.770584e-13  7.630379e-14  2.764966e-15  3.917228e-14
##   [8,]  1.363405e-14 -1.096065e-14  1.384634e-15  1.468625e-14
##   [9,] -5.243194e-13  2.148483e-13  1.379449e-14  1.414038e-14
##  [10,] -6.928093e-13  1.918601e-13  3.958580e-14  7.170615e-14
##  [11,] -3.234577e-13  1.438976e-13  7.201794e-15  2.486889e-16
##  [12,] -2.524894e-13  1.226541e-13 -3.501114e-15  1.738574e-14
##  [13,] -1.538129e-13  5.757160e-14  1.906664e-15  9.449472e-15
##  [14,] -1.215427e-13  5.284303e-14  1.618125e-16  1.280732e-14
##  [15,]  1.583859e-13 -7.886310e-14 -3.372086e-15 -1.836778e-14
##  [16,] -2.846213e-13  1.581128e-13 -8.679709e-15 -2.613117e-15
##  [17,]  5.978356e-14 -5.869488e-14  1.129390e-14  1.617180e-14
##  [18,]  2.698820e-13 -1.258552e-13  6.828527e-15 -3.167250e-14
##  [19,] -8.052178e-14  4.071569e-14 -2.347196e-15  1.527203e-14
##  [20,] -3.147265e-13  1.346220e-13  4.989282e-15  1.276659e-14
##  [21,] -3.011391e-13  9.137633e-14  2.044971e-14  1.716977e-14
##  [22,] -3.016844e-13  1.438811e-13 -3.037102e-15  1.746249e-14
```

```
## [23,] -4.687071e-13  2.156991e-13  3.653995e-15  1.398306e-14
## [24,] -1.777461e-13  9.186445e-16  1.564512e-14  6.117718e-14
## [25,] -1.395721e-14 -2.608178e-14  1.133417e-14  5.520974e-15
## [26,] -4.340139e-13  2.259818e-13 -1.794178e-15  2.267229e-15
## [27,]  1.359290e-13 -8.628395e-14  1.631158e-14 -2.994024e-16
## [28,] -3.797810e-13  1.213499e-13  2.289142e-14  7.463553e-15
## [29,]  1.455038e-13 -6.245935e-14  1.362912e-15 -3.319841e-14
## [30,] -2.915842e-13  8.082787e-14  1.585108e-14  3.309495e-14
## [31,] -3.526368e-16 -7.892795e-14  2.152375e-14  1.085883e-14
## [32,] -2.434785e-13  1.028225e-13 -1.543789e-15  1.302729e-14
## [33,] -1.685391e-13  7.905474e-14  6.152512e-15  8.159895e-15
## [34,]  7.833181e-14 -1.351306e-14 -1.269258e-14 -1.233432e-16
## [35,] -7.718043e-13  3.075010e-13  1.954481e-14  2.549890e-14
## [36,] -5.289606e-14  4.061053e-14 -5.477321e-15  2.546818e-14
## [37,] -1.859516e-13  2.656996e-14  1.899576e-14  1.853543e-14
## [38,]  3.023763e-13 -1.255141e-13 -3.898452e-16 -3.725039e-14
## [39,] -2.471091e-13  8.277140e-14  1.373095e-14  7.091441e-15
## [40,] -2.600318e-14  2.984023e-14 -9.061531e-15 -2.930192e-15
## [41,] -2.926872e-13  1.703055e-13 -5.699590e-15  4.293603e-15
## [42,]  3.019434e-13 -1.007075e-13 -1.304095e-14 -1.923987e-14
## [43,]  2.251615e-13 -5.720766e-14 -1.589950e-14 -2.947252e-14
## [44,] -6.178001e-13  1.858503e-13  3.612833e-14  6.086792e-14
## [45,] -5.541545e-13  2.355564e-13  1.260737e-14  2.550390e-14
## [46,]  8.144811e-14 -2.388032e-14 -8.080532e-15  1.826620e-14
## [47,] -8.163433e-13  3.418505e-13  1.288402e-14  6.240283e-14
## [48,] -7.528417e-13  3.074991e-13  2.275967e-14  4.748682e-14
## [49,] -3.059706e-13  1.192447e-13  1.202096e-14  3.723204e-15
## [50,] -4.900003e-13  1.899872e-13  1.232657e-14  2.763103e-14
## [51,]  1.854350e-13 -5.613563e-14 -1.270478e-14 -1.372786e-14
## [52,] -3.901109e-13  1.527697e-13  9.871583e-15  2.524988e-14
## [53,] -3.911445e-13  1.459383e-13  1.520703e-14  1.011000e-14
## [54,] -2.343952e-13  6.433599e-14  1.287967e-14  1.034880e-14
## [55,] -1.165279e-12  4.173606e-13  3.796049e-14  8.378598e-14
## [56,] -2.992800e-13  1.595127e-13 -6.570850e-16 -5.393235e-15
## [57,]  2.460918e-13 -7.995528e-14 -7.177355e-15 -2.324922e-14
## [58,] -5.812543e-14 -6.830132e-14  2.616879e-14  2.234684e-14
## [59,] -8.080690e-13  2.839528e-13  3.110653e-14  5.422568e-14
## [60,]  3.721186e-13 -9.463665e-14 -2.491088e-14 -3.133588e-14
## [61,]  6.998265e-14  3.741094e-14 -2.341607e-14 -1.433662e-14
## [62,] -2.953282e-13  9.472287e-14  1.598684e-14  1.398732e-14
## [63,] -2.013627e-13  8.770729e-14  4.300852e-16  1.764707e-14
## [64,] -1.503861e-13  5.394583e-14  2.484043e-15  2.625986e-14
## [65,]  8.557217e-14 -7.763148e-14  1.749044e-14 -2.175701e-14
## [66,] -8.096854e-14  7.010461e-15  5.761581e-15  3.461656e-14
## [67,] -6.858620e-13  2.536541e-13  2.733072e-14  3.748992e-14
## [68,]  7.338062e-14  1.211314e-14 -1.342279e-14 -1.373055e-14
## [69,] -8.559993e-15 -2.316990e-14  1.255608e-14 -1.603811e-14
## [70,]  1.075051e-13 -2.151193e-14 -1.361515e-14 -1.216414e-14
## [71,] -3.735422e-13  1.111419e-13  2.606466e-14  7.076289e-15
## [72,] -5.070376e-13  1.670186e-13  1.990751e-14  1.808688e-14
```

```
##  [73,] -4.162442e-13  1.223632e-13  1.790622e-14  5.087960e-14
##  [74,] -8.185409e-13  3.368626e-13  2.181745e-14  5.957768e-14
##  [75,] -3.794608e-13  1.248850e-13  2.865217e-14  3.069723e-14
##  [76,] -5.869450e-13  2.454001e-13  1.387914e-14  3.639995e-14
##  [77,]  2.115645e-14  9.084737e-15 -6.331980e-15 -9.039719e-15
##  [78,]  3.990864e-14 -2.592115e-14 -4.570357e-16 -1.583123e-14
##  [79,] -3.395853e-13  1.924620e-13 -5.191113e-15  5.571390e-15
##  [80,] -2.743269e-13  1.757358e-13 -1.740511e-14  5.269602e-15
##  [81,] -1.058224e-13 -2.505808e-14  2.373340e-14  2.687028e-14
##  [82,]  4.633254e-14 -6.505692e-14  1.042110e-14  1.755399e-14
##  [83,] -3.730787e-13  1.671593e-13  5.371192e-15  2.030651e-14
##  [84,] -5.461685e-13  1.845182e-13  2.350403e-14  4.051066e-14
##  [85,]  5.796847e-13 -1.732824e-13 -3.363027e-14 -4.382113e-14
##  [86,] -1.633481e-13  8.808622e-14 -1.450614e-15  1.982885e-14
##  [87,] -1.991272e-13  9.858455e-14 -1.621479e-15 -2.701425e-15
##  [88,] -8.170081e-14  9.404315e-15  5.619037e-15  7.053575e-15
##  [89,] -1.877287e-13  7.197495e-14  7.420503e-15 -1.167144e-15
##  [90,] -5.007293e-13  1.890588e-13  1.313287e-14  1.231396e-15
##  [91,]  7.003169e-13 -2.944945e-13 -8.360774e-15 -2.462189e-14
##  [92,] -3.987084e-13  1.627942e-13  4.362425e-15  1.312253e-14
##  [93,]  2.354280e-14 -4.387639e-14  5.773151e-15  1.381555e-14
##  [94,] -1.331499e-13  1.170749e-13 -2.260930e-14 -1.281137e-14
##  [95,] -2.156762e-13  1.525286e-13 -1.494524e-14  3.503355e-15
##  [96,]  1.927481e-13 -6.463880e-14 -1.321263e-14 -1.825295e-15
##  [97,] -3.772773e-13  1.548794e-13  1.007754e-14  2.542083e-14
##  [98,]  4.911442e-13 -2.079056e-13 -3.263252e-15 -3.267609e-14
##  [99,]  3.331986e-14  3.358235e-14 -1.827154e-14  3.297032e-15
## [100,] -4.811199e-13  1.794398e-13  1.818429e-14  3.017335e-14
## [101,] -3.576616e-13  1.073960e-13  2.019047e-14  2.795412e-14
## [102,] -1.905241e-13  7.558372e-14  1.492828e-15  2.526748e-14
## [103,] -1.364257e-13  4.581275e-14  8.771382e-15 -1.533260e-14
## [104,] -8.614687e-13  3.617561e-13  1.498007e-14  3.075697e-14
## [105,] -2.010693e-13  9.532577e-14 -2.461207e-15  1.087514e-14
## [106,]  1.738270e-13 -9.826807e-14  1.460339e-15 -1.872096e-14
## [107,] -4.682270e-13  1.099349e-13  3.621318e-14  4.378922e-14
## [108,] -1.481011e-13  6.488907e-14  6.035787e-15 -1.216746e-15
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## [497,] -2.358215e-13  7.924816e-14  7.847442e-15  5.629503e-15
## [498,] -2.071876e-13  9.702023e-14 -4.378981e-15  3.289219e-14
## [499,] -8.712740e-13  3.845302e-13  1.363366e-14  1.626650e-14
```

The standard error of the coefficients by using bootstrap (499 replications) is printed above

Exercise 3

Question 1: Write a function that returns the likelihood of the probit

```r
#Calculate the coefficient by using OLS method
belta0 <-solve(t( X ) %*%X ) %*%t( X ) %*%ydum
belta<-belta0

#Construct a matrix to place data
X<-as.matrix(cbind(1,X1,X2,X3))
#Construct a likelihood function

likelihood_cal<-function(belta,X. = X, ydum. = ydum){
  #Calculate the cdf of X*beta
  cdf<-pnorm(X.%*%belta)
  #Check whether or not the cdf is in the range(0.00001,0.99999)
  cdf[cdf>0.99999] <- 0.99999
  cdf[cdf<0.00001] <- 0.00001
  #Calculate the likelihood of probit model
  loglikelihood<--sum(ydum.*log(cdf))-sum((1-ydum.)*log(1-cdf))
  return (loglikelihood)
}
```

Question 2: Implement the steepest ascent optimization algorithm to maximize that likelihood.

```r
steep_gradient<-function(belta,X){
  beltadif<-c(1,1,1,1)
  # Set while loop to optimize the likelihood by using gradient ascent method
  while(sqrt(sum(beltadif^2))>=0.0000001){
    likelihood<-likelihood_cal(belta,X)
    #Calculate the direction (a vector of 4*1)
    direction<-matrix(0,nrow=4,ncol=1)
    interval<-matrix(0,nrow=4,ncol=4)
    interval[1,]<-matrix(c(0.0001,0,0,0),nrow=4,ncol=1)
    interval[2,]<-matrix(c(0,0.0001,0,0),nrow=4,ncol=1)
    interval[3,]<-matrix(c(0,0,0.0001,0),nrow=4,ncol=1)
    interval[4,]<-matrix(c(0,0,0,0.0001),nrow=4,ncol=1)
    likelihood_new<-matrix(0,nrow=4,ncol=1)
    for (j in 1:4){
      transfer<-as.matrix(interval[j,])
      belta_new<-belta+transfer
      likelihood_new[j,1]<-likelihood_cal(belta_new,X)
      direction[j,1]<-(likelihood_new[j,1]-likelihood)/0.0001
    }
    #Get gradient function
    belta1<-belta-0.00001*direction
    beltadif<-as.matrix(belta1-belta)
    belta<-belta1
  }
  #Get updated likelihood function
  llh<-likelihood_cal(belta,X)
  print(belta)
  return (belta)
}
# Get optimized beta and optimized likelihood
belta_probit<-steep_gradient(belta,X)
```

```
##           [,1]
##     2.82019099
## X1  1.23985585
## X2 -0.89302926
## X3  0.04808161
```

```r
print(belta_probit)
```

```
##           [,1]
##     2.82019099
## X1  1.23985585
## X2 -0.89302926
## X3  0.04808161
```

```
llhc<-likelihood_cal(belta_probit,X)
print(llhc)
```

```
## [1] 2186.36
```

The coefficients calculated in mle method and its corresponding likelihood is printed above.

Question3: How different are the parameters from the true parameters

Compare the parameters calculated with the true parameters calculated in the exercise 4 (by using glm function) The parameters calculated are: beta0=2.82019099; beta1=1.23985585 beta2= -0.89302926 beta3=0.04808161; True parameter of probit model (Calculated in Exercise 4) are: beta0=2.81677032 beta1= 1.23905407 beta2=-0.89214080 beta3= 0.04803623. Therefore, they are very close.

Exercise 4

Question1

```
# Construct a likelihood function of probit model
probit_likelihood<-function(belta,X. = X, ydum. = ydum){
  cdf<-pnorm(X.%*%belta)
  cdf[cdf>0.99999] <- 0.99999
  cdf[cdf<0.00001] <- 0.00001
  loglikelihood<--sum(ydum.*log(cdf))-sum((1-ydum.)*log(1-cdf))
  return (loglikelihood)
}

# Construct a likelihood function of logit model
logit_likelihood<-function(belta,X. = X, ydum. = ydum){
  cdf<-exp(X.%*%belta)/(1+exp(X.%*%belta))
  cdf[cdf>0.99999] <- 0.99999
  cdf[cdf<0.00001] <- 0.00001
  loglikelihood<--sum(ydum.*log(cdf))-sum((1-ydum.)*log(1-cdf))
  return (loglikelihood)
}

belta<-solve(t( X ) %*%X ) %*%t( X ) %*%ydum
#Calculated the optimized probit_omdel and the corresponding coefficient
probit_model<-optim(par=belta,probit_likelihood)
probit_belta5<-probit_model$par
print("The summary of the probit model:")
```

```
## [1] "The summary of the probit model:"
```

```
print(probit_model)
```

```
## $par
##            [,1]
##      2.81647879
## X1  1.23927827
## X2 -0.89216027
## X3  0.04815206
##
## $value
## [1] 2186.359
##
## $counts
## function gradient
##      367       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

```
print("The beta of the probit model:")
```

```
## [1] "The beta of the probit model:"
```

```
print(probit_belta5)
```

```
##            [,1]
##      2.81647879
## X1  1.23927827
## X2 -0.89216027
## X3  0.04815206
```

```
#Calculated the optimized probit_omdel and the corresponding coefficient
logit_model<-optim(par=belta,logit_likelihood)
logit_belta5<-logit_model$par
print("The summary of the logit model:")
```

```
## [1] "The summary of the logit model:"
```

```
print(logit_model)
```

```
## $par
##            [,1]
##      5.06550636
## X1   2.23057833
## X2  -1.60572194
## X3   0.08659928
##
## $value
## [1] 2190.595
##
## $counts
## function gradient
##      229       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
```

```
print("The beta of the logit model:")
```

```
## [1] "The beta of the logit model:"
```

```
print(logit_belta5)
```

```
##            [,1]
##      5.06550636
## X1   2.23057833
## X2  -1.60572194
## X3   0.08659928
```

```
#Construct a SSR of linear model
linear_SSR<-function(belta,X.=X,ydum.=ydum){
  ydum_hat<-X%*%belta
  residu<-ydum-ydum_hat
  SSR<-sum(residu^2)
  return(SSR)
}
linear_model<-optim(par=belta,linear_SSR,hessian=TRUE)
linear_belta5<-linear_model$par
print("The summary of the linear model:")
```

```
## [1] "The summary of the linear model:"
```

```r
print(linear_model)
```

```
## $par
##            [,1]
##      0.87952298
## X1   0.15208902
## X2  -0.10554274
## X3   0.01055706
##
## $value
## [1] 1091.592
##
## $counts
## function gradient
##       97       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
##
## $hessian
##            [,1]      [,2]      [,3]     [,4]
## [1,]  20000.00  40011.82 118912.37  5922.00
## [2,]  40011.82  86666.11 237715.55 11875.14
## [3,] 118912.37 237715.55 939239.69 35238.16
## [4,]   5922.00  11875.14  35238.16  5922.00
```

```r
print("The beta of the linear model:")
```

```
## [1] "The beta of the linear model:"
```

```r
print(linear_belta5)
```

```
##            [,1]
##      0.87952298
## X1   0.15208902
## X2  -0.10554274
## X3   0.01055706
```

The model are estimated and the summaries of the models are printed above

Question2

Compare the coefficients and testify whether they are significant or not

```
#Construct a matrix and put ydum and X variables inside
DATA5<-data.frame(cbind(ydum,X1,X2,X3))
belta_probit<-glm(ydum~X1+X2+X3,family = binomial(link = "probit"),data=DATA5)
```

```
## Warning: glm.fit: fitted probabilities numerically 0 or 1 occurred
```

```
belta_linear<-lm(ydum~X1+X2+X3,data=DATA5)
belta_logit<-glm(ydum~X1+X2+X3,family = binomial,data=DATA5)
```

```
## Warning: glm.fit: fitted probabilities numerically 0 or 1 occurred
```

```
print("The summary of the probit model (by using glm function):")
```

```
## [1] "The summary of the probit model (by using glm function):"
```

```
summary(belta_probit)
```

```
##
## Call:
## glm(formula = ydum ~ X1 + X2 + X3, family = binomial(link = "probit"),
##     data = DATA5)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -3.6273  -0.1177   0.0086   0.2557   3.8444
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  2.81677    0.09726  28.962   <2e-16 ***
## X1           1.23905    0.04414  28.071   <2e-16 ***
## X2          -0.89214    0.01804 -49.457   <2e-16 ***
## X3           0.04804    0.04686   1.025    0.305
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 13721.5  on 9999  degrees of freedom
## Residual deviance:  4372.7  on 9996  degrees of freedom
## AIC: 4380.7
##
## Number of Fisher Scoring iterations: 7
```

```
print("The summary of the logit model (by using glm function):")
```

```
## [1] "The summary of the logit model (by using glm function):"
```

```
summary(belta_logit)
```

```
##
## Call:
## glm(formula = ydum ~ X1 + X2 + X3, family = binomial, data = DATA5)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -3.2817  -0.1535   0.0401   0.2656   3.4123
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  5.06602    0.18221  27.803   <2e-16 ***
## X1           2.23094    0.08252  27.037   <2e-16 ***
## X2          -1.60595    0.03612 -44.466   <2e-16 ***
## X3           0.08672    0.08425   1.029    0.303
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 13721.5  on 9999  degrees of freedom
## Residual deviance:  4381.2  on 9996  degrees of freedom
## AIC: 4389.2
##
## Number of Fisher Scoring iterations: 7
```

```
print("The summary of the linear model (by using lm function):")
```

```
## [1] "The summary of the linear model (by using lm function):"
```

```
summary(belta_linear)
```

```
##
## Call:
## lm(formula = ydum ~ X1 + X2 + X3, data = DATA5)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -0.90570 -0.26599  0.05805  0.24995  2.35722
##
## Coefficients:
##               Estimate Std. Error  t value Pr(>|t|)
## (Intercept)  0.8795230  0.0134596   65.345   <2e-16 ***
## X1           0.1520890  0.0057445   26.476   <2e-16 ***
## X2          -0.1055427  0.0009698 -108.831   <2e-16 ***
## X3           0.0105571  0.0072385    1.458    0.145
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.3305 on 9996 degrees of freedom
## Multiple R-squared:  0.5571, Adjusted R-squared:  0.557
## F-statistic:  4191 on 3 and 9996 DF,  p-value: < 2.2e-16
```

The coefficient of these three models are all significant except the coefficient of X3; The coefficients of X3 are not significant in all of three models.

Exercise 5

Question 1: Compute the marginal effect of X on Y according to the probit and logit models.

```
#Install and use the package numDeriv
#install.packages("numDeriv")
library(numDeriv)
#Construct a matrix and put X variables inside
X<-as.matrix(cbind(1,X1,X2,X3))
#Construct a function to get marginal effect of probit model
ME_probit<-function(beta.,X.){
  Y_H<-X.%*%beta.
  pdf<-dnorm(Y_H)
  ME<-pdf%*%t(beta.)
  return(ME)
}
ME_p<-ME_probit(probit_belta5,X)
write.csv(ME_p,file="marginaleffect_probit")

#Construct a function to get marginal effect of logit model
ME_logit<-function(beta.,X.){
  Y_H<-X.%*%beta.
  pdf<-(exp(Y_H)/((1+exp(Y_H))^2))
  ME<-pdf%*%t(beta.)
  return(ME)
}
ME_l<-ME_logit(logit_belta5,X)
write.csv(ME_l,file="marginaleffect_logit")
```

The output of the marginal effects are writed in the two excel forms attached

Question 2: Compute the standard deviations

By using delta method

```
#Calculate the variance-covariance matrix
myprobit<-glm(ydum~X1+X2+X3,family = binomial(link = "probit"),data=DATA5)
```

```
## Warning: glm.fit: fitted probabilities numerically 0 or 1 occurred
```

```
var_probit<-vcov(myprobit)
mylogit<-glm(ydum~X1+X2+X3,family = binomial,data=DATA5)
```

```
## Warning: glm.fit: fitted probabilities numerically 0 or 1 occurred
```

```
var_logit<-vcov(mylogit)

# Calculate the first order derivative of marginal effect in probit model (at mean le
vel)
#Calculate the marginal effect function of probit model
xmean=apply(X,2,mean)
MEmean_probit<-function(beta.,x.=xmean){
  Y_H<-x.%*%beta.
  pdf<-dnorm(Y_H)
  MEmean<-pdf%*%t(beta.)
  return(MEmean)
}
me_derivative_pro<-jacobian(MEmean_probit,probit_belta5)

#Calculate the marginal effect function of probit model
MEmean_logit<-function(beta.,x.=xmean){
  Y_H<-x.%*%beta.
  pdf<-(exp(Y_H))/((1+exp(Y_H))^2)
  MEmean<-pdf%*%t(beta.)
  return(MEmean)
}
me_derivative_log<-jacobian(MEmean_logit,logit_belta5)
#Calculate the standard errors of marginal effect of probit model
sd_me_pro<-me_derivative_pro*var_probit*t(me_derivative_pro)
print("sdandard deviation of probit model is:")
```

```
## [1] "sdandard deviation of probit model is:"
```

```
sdv_me_pro<-sqrt(diag(sd_me_pro))
print(sdv_me_pro)
```

```
## (Intercept)          X1          X2          X3
## 0.038189378 0.017365361 0.007409384 0.018693275
```

```
#Calculate the standard errors of marginal effect of logit model
sd_me_log<-me_derivative_log*var_logit*t(me_derivative_log)
sdv_me_log<-sqrt(diag(sd_me_log))
print("sdandard deviation of logit model is:")
```

```
## [1] "sdandard deviation of logit model is:"
```

```
print(sdv_me_log)
```

```
## (Intercept)          X1          X2          X3
## 0.044789888 0.020324270 0.009314116 0.021061191
```

The standard deviations (calculated by using delta method) are printed above

By using bootstrap

```r
num5=499
pgn<-matrix(0,nrow=10000,ncol=4)
DATA<-as.matrix(cbind(ydum,X1,X2,X3))
sdboot1<-matrix(0,nrow=num5,ncol=4)
Xmean<-matrix(0,nrow=num5,ncol=4)
#Calculate the marginal effect function of logit model
ME_LOGIT<-matrix(0,nrow=num5,ncol=4)
for (j in 1:num5){
  for (i in 1:10000){
    k<-sample(1:nrow(DATA),1)
    sp<-as.matrix(DATA[k,])
    pgn[i,]<-t(sp)
  }
  Xloop<-as.matrix(cbind(1,pgn[,2],pgn[,3],pgn[,4]))
  X_mean<-apply(Xloop,2,mean)
  Xmean[j,]<-as.matrix(X_mean)
  ME_p<-(exp(Xmean[j,]%*%logit_belta5)/((1+exp(Xmean[j,]%*%logit_belta5))^2))%*%t(log
it_belta5)
  ME_p<-as.matrix(ME_p)
  ME_LOGIT[j,]<-ME_p
}
#Calculate the standard errors of the logit model
sdvboot_log<-apply(ME_LOGIT,2,sd)
print("The standard errors of the logit model is (By using bootstrap):")
```

```
## [1] "The standard errors of the logit model is (By using bootstrap):"
```

```r
print(sdvboot_log)
```

```
## [1] 1.337792e-03 5.890923e-04 4.240687e-04 2.287073e-05
```

```r
#Calculate the marginal effect function of probit model
ME_PROBIT<-matrix(0,nrow=num5,ncol=4)
for (j in 1:num5){
  for (i in 1:10000){
    k<-sample(1:nrow(DATA),1)
    sp<-as.matrix(DATA[k,])
    pgn[i,]<-t(sp)
  }
  Xloop<-as.matrix(cbind(1,pgn[,2],pgn[,3],pgn[,4]))
  X_mean<-apply(Xloop,2,mean)
  Xmean[j,]<-as.matrix(X_mean)
  ME_p<-pnorm(Xmean[j,]%*%probit_belta5)%*%t(probit_belta5)
  ME_p<-as.matrix(ME_p)
  ME_PROBIT[j,]<-ME_p
}
#Calculate the standard errors of the probit model
sdvboot_pro<-apply(ME_PROBIT,2,sd)
print("The standard errors of the probit model is (By using bootstrap):")
```

```
## [1] "The standard errors of the probit model is (By using bootstrap):"
```

```r
print(sdvboot_pro)
```

```
## [1] 0.0353907610 0.0155722817 0.0112105338 0.0006050598
```