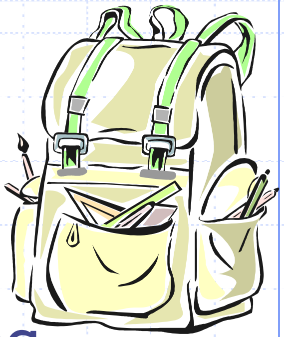
Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Dynamic Programming: 0/1 Knapsack

Civil War Knapsack. U.S. government image. Vicksburg National Military Park. Public domain.

The 0/1 Knapsack Problem

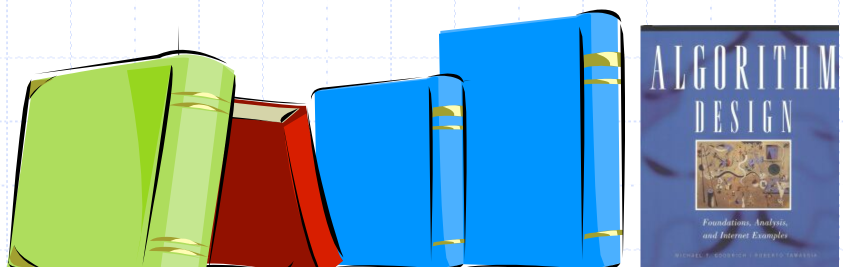


- Given: A set S of n items, with each item i having
  - w_i - a positive weight
  - b_i - a positive benefit
- Goal: Choose items with maximum total benefit but with weight at most W.
- If we are **not** allowed to take fractional amounts, then this is the **0/1 knapsack problem**.
  - In this case, we let T denote the set of items we take
  - Objective: maximize $$\sum_{i \in T} b_i$$
  - Constraint: $$\sum_{i \in T} w_i \leq W$$

Example

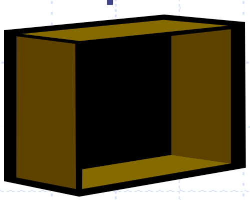
- Given: A set S of n items, with each item i having
  - b_i - a positive "benefit"
  - w_i - a positive "weight"
- Goal: Choose items with maximum total benefit but with weight at most W.

Items:



	1	2	3	4	5
Weight:	4 in	2 in	2 in	6 in	2 in
Benefit:	$20	$3	$6	$25	$80

"knapsack"



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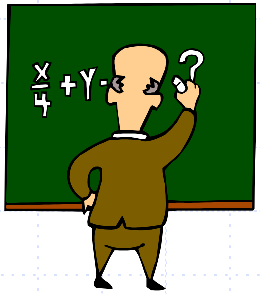
Solution:
- item 5 ($80, 2 in)
- item 3 ($6, 2in)
- item 1 ($20, 4in)

The General Dynamic Programming Technique



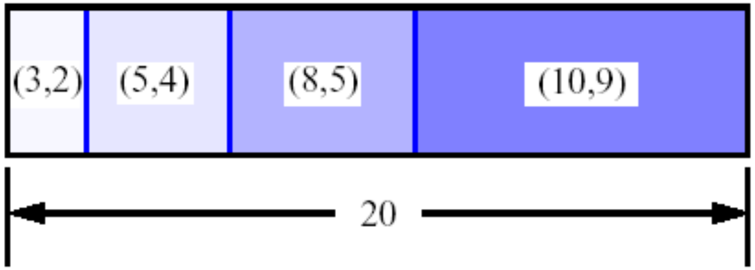
- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

A 0/1 Knapsack Algorithm, First Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k] = best selection from S_k.
- Problem: does not have subproblem optimality:
  - Consider set S={(3,2),(5,4),(8,5),(4,3),(10,9)} of (benefit, weight) pairs and total weight W = 20

Best for S_4:

Best for S_5:



A 0/1 Knapsack Algorithm, Second (Better) Attempt

- S_k: Set of items numbered 1 to k.
- Define B[k,w] to be the best selection from S_k with weight at most w
- Good news: this does have subproblem optimality.

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w],\ B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- I.e., the best subset of S_k with weight at most w is either
  - the best subset of S_{k-1} with weight at most w or
  - the best subset of S_{k-1} with weight at most $w-w_k$ plus item k

0/1 Knapsack Algorithm

$$B[k,w] = \begin{cases} B[k-1,w] & \text{if } w_k > w \\ \max\{B[k-1,w], B[k-1,w-w_k]+b_k\} & \text{else} \end{cases}$$

- Recall the definition of B[k,w]
- Since B[k,w] is defined in terms of B[k−1,*], we can use two arrays of instead of a matrix
- Running time: O(nW).
- Not a polynomial-time algorithm since W may be large
- This is a pseudo-polynomial time algorithm

Algorithm ***01Knapsack(S, W)***:

Input: set S of n items with benefit b_i and weight w_i; maximum weight W

Output: benefit of best subset of S with weight at most W

let A and B be arrays of length $W + 1$

for $w \leftarrow 0$ **to** W **do**

  $B[w] \leftarrow 0$

for $k \leftarrow 1$ **to** n **do**

  copy array B into array A

  for $w \leftarrow w_k$ **to** W **do**

    if $A[w-w_k] + b_k > A[w]$ **then**

      $B[w] \leftarrow A[w-w_k] + b_k$

return $B[W]$

Fractional Knapsack Problem

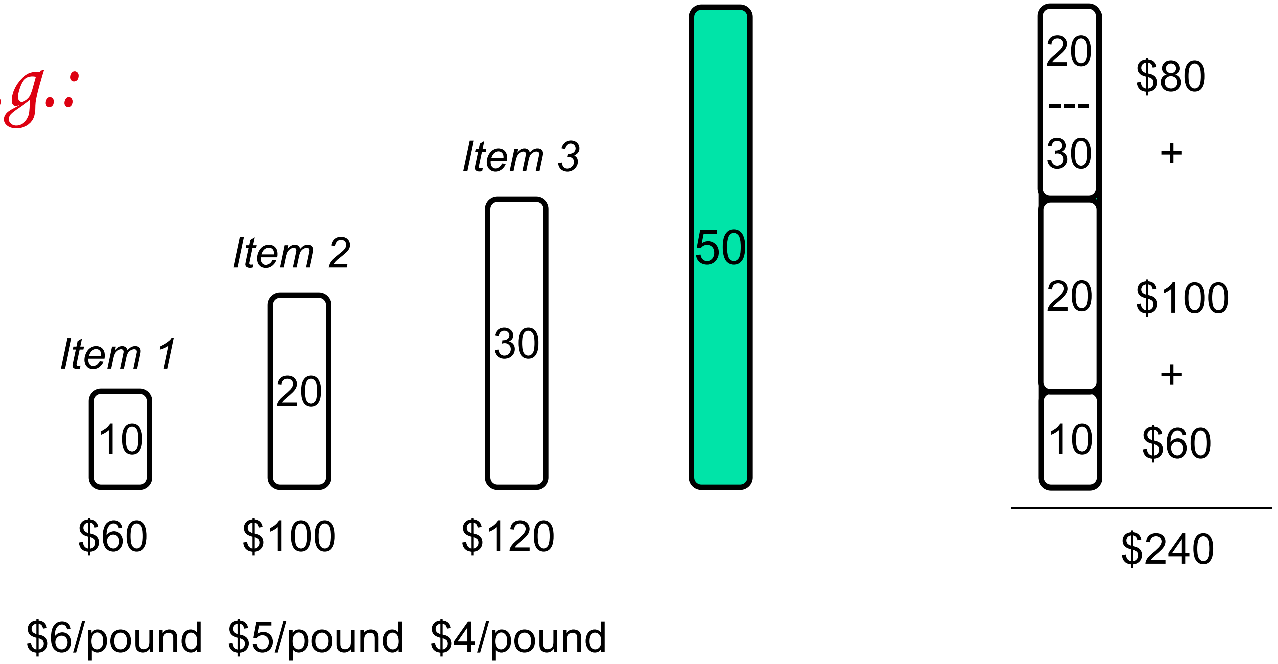
- Knapsack capacity: W
- There are n items: the i-th item has value v_i and weight w_i
- Goal:
  - find x_i such that for all $0 \le x_i \le 1$, $i = 1, 2, .., n$

    $\sum w_i x_i \le W$ and

    $\sum x_i v_i$ is maximum

Fractional Knapsack - Example

- *E.g.:*



Fractional Knapsack Problem

- Greedy strategy 1:
  - Pick the item with the maximum value
- *E.g.:*
  - $W = 1$
  - $w_1 = 100, \ v_1 = 2$
  - $w_2 = 1, v_2 = 1$
  - Taking from the item with the maximum value:

    Total value taken $= v_1/w_1 = 2/100$
  - Smaller than what the thief can take if choosing the other item

    Total value (choose item 2) $= v_2/w_2 = 1$

Fractional Knapsack Problem

Greedy strategy 2:

- Pick the item with the maximum value per pound v_i/w_i
- If the supply of that element is exhausted and the thief can carry more: take as much as possible from the item with the next greatest value per pound
- It is good to order items based on their value per pound

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \ldots \geq \frac{v_n}{w_n}$$

Fractional Knapsack Problem

Alg.: Fractional-Knapsack (W, $v[n]$, $w[n]$)

1. While $w > 0$ and as long as there are items remaining
2. pick item with maximum v_i/w_i
3. $x_i \leftarrow \min(1, w/w_i)$
4. remove item i from list
5. $w \leftarrow w - x_i w_i$

w – the amount of space remaining in the knapsack ($w = W$)

- Running time: $\Theta(n)$ if items already ordered; else $\Theta(n\lg n)$