1 Introduction

2 Rooted Trees

A *rooted tree* is a pair, (t, r), such that t is a finite, simple connected graph without cycles and r is a vertex that has been selected from the set, $V(t)$, of vertices of t . By convention each edge is directed away from r. We denote the order of a tree, t, by $|t|$ and the set of order n rooted trees is denoted $\mathcal{R}_n$. In addition we define $R := \bigcup R_n$.

A *labelled rooted tree* is a triple (t, r, L) such that (t, r) is a rooted tree and

$$L : V(t) \longrightarrow \{1, 2, \ldots, |t|\}$$

is a bijective map. We denote the set of order n labelled rooted trees by $\mathcal{L}_n$ and we define $\mathcal{L} = \bigcup \mathcal{L}_n$.

Vertices u and v of a rooted tree, (t, r) we write $u \leq v$ if u lies on the unique shortest path from r to v. A *random recursive tree* is a triple (t, r, l) such that l is a labelling that satisfies if $u \leq v$ then $l(u) < l(v)$. We denote the set of order n random recursive trees by $\mathcal{T}_n$ and we define $\mathcal{T} = \bigcup \mathcal{T}_n$.

Two rooted trees, (t_1, r_1) and (t_2, r_2) are said to be isomorphic of there exists a bijection $f : V(t_1) \to V(t_2)$ such that vertices $u, v \in V(t_1)$ are adjacent if and only if $f(u), f(v) \in V(t_2)$ are adjacent and $f(r_1) = r_2$. If $V(t_1) = V(t_2)$ then f is called a rooted tree automorphism. The set of automorphisms of a tree, t, together with composition of maps forms a group denoted $\mathrm{Aut}(t)$. The order of the automorphsim group is denoted

$$\sigma(t) := |\mathrm{Aut}(t)|$$

Consider a map $\phi : \mathcal{L} \to R$ that simply forgets the labels of a labelled by mapping

$$(t, r, L) \mapsto (t, r)$$

The map ϕ is clearly surjective but not injective hence we define $\beta(t) = |\phi^{-1}(t)|$ to be the number of possible isomorphism classes of labellings of a rooted tree $t \in \mathcal{R}$. There are $|t|!$ possible labellings of a tree $t \in \mathcal{R}$ hence there are

$$\beta(t) = \frac{|t|!}{\sigma(t)} \tag{1}$$

isomorphism classes of labellings of such a tree.

Since $\mathcal{T} \leq \mathcal{L}$ we may restrict ϕ to random recursive trees, hence we define

$$\alpha(t) := |\phi^{-1} \restriction_{\mathcal{T}} (t)|$$