Automorphisms of Random Recursive Trees

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1 Introduction

Graph automorphisms are essential to understanding enumerative properties of graphs. An exact formulation of the number of rooted unlabelled trees is unknown [HP73] which emphasises the difficulty in understanding tree automorphisms. In addition, an intriguing relationship between random Fibonacci sequences and automorphism groups coming from a family of trees called random recursive trees [MA06] provides further motivation, if we needed any, to investigate automorphism groups of trees. We exploit a geometric interpretation of a graph automorphism in which subgroups of the automorphism group are associated with certain subtrees to give bounds for the expected order of the automorphism group of a random recursive tree (also known as a heap-ordered tree) of order n. We also disprove a conjecture of MacArthur [MSGA08] which states that the automorphism group can be split in a particularly pleasing way.

In Section 2 we introduce several families of increasing trees and we describe the relationship between them using several recursively defined functions. In Section 3 we give a geometric interpretation of the automorphism group and a direct product decomposition in which particular subgroups can be associated with certain subtrees. We also state MacArthur's conjecture. In Section 4 we describe two families of recursively defined functions; inductive maps and elementary differentials and describe how these may be manipulated to calculate properties of trees such as order, path length and number of leaves. In Section 5 we use inductive maps and elementary differentials to provide upper and lower bounds on the expected order of the order of random recursive trees. In Section 6 we disprove the conjecture of MacArthur and we make concluding remarks in Section 7.

2 Rooted Trees

A *rooted tree* is a triple $t = (r, V, E)$, such that (V, E) is a finite, simple connected graph without cycles with vertex set V and edge set E [Tat84]. A vertex, $r \in V$, is selected from V and called the *root*. By convention each edge is directed away from r. We denote the order of a tree, t, by $|t|$ and the set of order n rooted trees is denoted $\mathcal{R}_n$. In addition we define $R := \bigcup R_n$.

A *rooted plane tree* is a quadruple $t = (r, V, E, e)$ such that (r, V, E) is a rooted tree and e is an embedding of (r, V, E) in $\mathbb{R}^2$. The collection of order n rooted plane tres is $\mathcal{F}_n$ and we define $\mathcal{F} = \cup \mathcal{F}_n$.

A *labelled rooted tree* is a quadruple $t = (r, V, E, L)$ such that (r, V, E) is a rooted tree and

$$L : V \longrightarrow S$$

is a bijective map and S is a totally ordered set (usually we set $S = \{1, 2, \dots, |t|\}$). We denote the set of order n labelled rooted trees by $\mathcal{L}_n$ and we define $\mathcal{L} = \bigcup \mathcal{L}_n$.

For any pair of vertices u and v of a rooted tree we write $u \leq v$ if u lies on the unique shortest path from r to v. A *random recursive tree* is a quadruple $t = (r, V, E, l)$ such that t is a labelled rooted tree and l is a labelling such that if $u \leq v$ then $l(u) < l(v)$. We denote the set of order n random recursive trees by $\mathcal{T}_n$ and we define $\mathcal{T} = \bigcup \mathcal{T}_n$. Random recursive trees are also known as "heap-ordered trees".

Two rooted trees t_1 and t_2 with roots r_1 and r_2 and vertex sets $V(t_1)$ and $V(t_2)$ respectively are said to be isomorphic if there exists a bijection $f : V(t_1) \to V(t_2)$ such that vertices $u, v \in V(t_1)$ are adjacent if and only if $f(u), f(v) \in V(t_2)$ are adjacent and $f(r_1) = r_2$ [HP73]. If $V(t_1) = V(t_2)$ then f is called a rooted tree automorphism. The set of automorphisms of a tree, t, together with composition of maps form a group denoted $\mathrm{Aut}(t)$. The order of the automorphism group is

$$\sigma(t) := |\mathrm{Aut}(t)|,$$

notation taken from [But08]. Consider a map $\phi : \mathcal{L} \to R$ that simply forgets the labels of a labelled by mapping

$$(r, V, E, L) \mapsto (r, V, E)$$

The map ϕ is clearly surjective but not injective hence we define $\beta(t) = |\phi^{-1}(t)|$ to be the number of possible isomorphism classes of labellings of a rooted tree $t \in \mathcal{R}$. There are $|t|!$ possible labellings of a tree $t \in \mathcal{R}$ hence there are

$$\beta(t) = \frac{|t|!}{\sigma(t)} \tag{1}$$

isomorphism classes of labellings of t.

Since $\mathcal{T} \leq \mathcal{L}$ we denote by ψ the restriction of ϕ to random recursive trees and define

$$\alpha(t) := |\psi^{-1}(t)|$$

The notation $\alpha(t)$ is also found in (for example) [Bro00, But63, Maz04]. To provide a result analogous to Equation 1 we require several additional definitions.

Let $t_1, t_2, \dots, t_k$ be a forest of rooted trees: the rooted tree $B^+(t_1, t_2, \dots, t_k)$ is built from this forest by introducing a new vertex, r (the root of $B^+(t_1, t_2, \dots, t_k)$), and joining the roots of each tree $t_1, t_2, \dots, t_k$ to r via an edge. Since every tree t ($|t| > 1$) can be written as $B^+(t_1, t_2, \dots, t_k)$ we henceforth assume that each rooted tree $t = B^+(t_1, t_2, \dots, t_k)$ if $|t| > 1$. For convenience we will denote the rooted tree on 1 vertex by $\bullet$.

We have already seen one function $f : \mathcal{R} \to \mathbb{R}$ on rooted trees ($f(t) = |t|$), let us consider another. We define the tree factorial $t!$ recursively

$$\bullet! = 1 \tag{2}$$

$$t! = |t| \prod_{i=1}^{k} t_i! \tag{3}$$

The notation $t!$ is taken from Kreimer [Kre99] because $t!$ generalises the factorial of a number. Figure 1 provides a few examples.

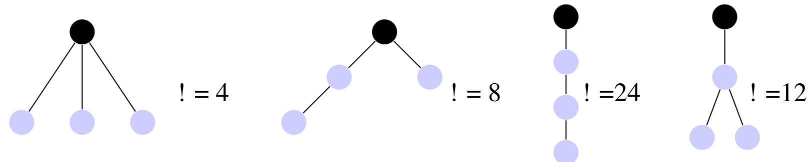


Figure 1

Let $t = (r, V, E)$ be a rooted tree. The *induced subtree*, t_v, rooted at vertex $v \in V$ is the subtree of t induced by the nodes u with $v \leq u$.

Lemma 2.1. *For a rooted tree t,*

$$\alpha(t) = \frac{|t|!}{t!\sigma(t)} \tag{4}$$

Proof. There are $n!$ ways of labelling a tree $t \in \mathcal{R}_n$, however if l is a random recursive labelling every induced subtree t_v and a totally ordered set, S, of labels there is precisely one possible label $s \in S$ for vertex v (namely $s = \min(S)$). Therefore the factor we should divide out by is precisely $t!$. In addition, to calculate the number of isomorphism classes of random recursive trees we must again divide out by $\sigma(t)$. $\square$

Alternatively see [But08] for a proof of Lemma 2.1. Finally, let $\chi : \mathcal{F} \to \mathcal{R}$ be the map that "forgets" the embedding of a rooted plane tree i.e.

$$(r, V, E, e) \mapsto (r, V, E)$$

It is clear that χ is surjective but not injective hence we define

$$\gamma(t) = |\chi^{-1}|$$

the number of isomorphism classes of embeddings of a rooted tree t. In order to obtain a third relation analogous to Equations 1 and 4 we will describe a third function $w : \mathcal{R} \to \mathbb{R}$ recursively:

$$w(\bullet) = 1 \tag{5}$$

$$w(t) = k! \prod_{i=1}^{k} w(t_i) \tag{6}$$

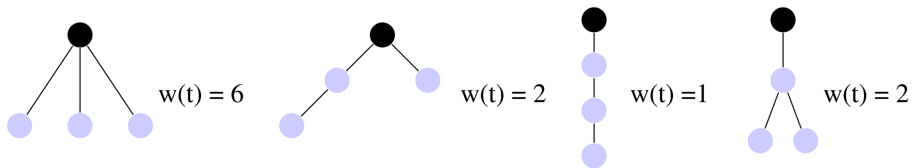


Figure 2

Let $B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$ denote a tree in which the root is incident to n_1 isomorphic copies of a tree t_1, n_2 isomorphic copies of t_2 and so forth. We remark that every rooted tree with at least 2 vertices can be written in this way.

Lemma 2.2.

$$\gamma(t) = \frac{w(t)}{\sigma(t)} \tag{7}$$

Proof. Let $B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$ denote a tree in which the root is incident to n_1 isomorphic copies of a tree t_1, n_2 isomorphic copies of t_2 and so forth. We remark that every rooted tree with at least 2 vertices can be written in this way. Let $t \in R$ and consider an induced subtree $t_v = B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$. Vertex v contributes a factor of $\left(\sum_{i=1}^k n_i\right)!$ to $w(t)$ since it has a recursive definition. The order of automorphism group, $\sigma(t)$ can also be expressed recursively:

$$\sigma(\bullet) = 1 \tag{8}$$

$$\sigma(t) = n_1! n_2! \dots n_k! \prod_{i=1}^k \sigma(t_i)^{n_i} \tag{9}$$

Therefore vertex v contribute a factor of $\prod_{i=1}^k n_i!$ to $\sigma(t)$. Finally consider, $\gamma(t)$, the number of isomorphism classes of embeddings of t. There are

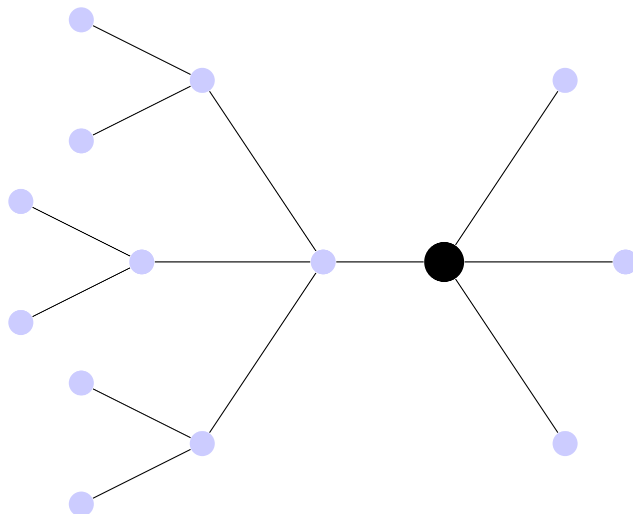
$$\frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!}$$

non-isomorphic possibilities for the ordering of the children of v. $\square$

3 Automorphisms of Trees

In this section we will describe a direct product decomposition of the automorphism group of a tree, t, in which factors of the direct product can be associated with particular induced subtrees of t.

Recall that a (rooted) tree automorphism is a permutation of vertices that preserves adjacency and the root. In [MSGA08] MacArthur, Sanchez-Garcia and Anderson showed that an automorphism group, $\mathrm{Aut}(t)$, may be decomposed by partitioning the



$$\sigma(t) = 144$$

$$t! = 3780$$

$$w(t) = 9216$$

$$\alpha(t) = 160160$$

$$\beta(t) = 605404800$$

$$\gamma(t) = 64$$

set of generators S of $\mathrm{Aut}(t)$ into support-disjoint subsets $S = S_1 \cup S_2 \cup \cdots \cup S_r$ and writing

$$\mathrm{Aut}(t) = H_1 \times H_2 \times \cdots \times H_r \tag{10}$$

where each H_i is generated by S_i. This, *geometric decomposition* is shown [MSGA08] to be unique and irreducible (each H_i cannot be written as the direct product of support-disjoint subgroups) hence the geometric decomposition is well defined. We call each H_i in the geometric decomposition of $\mathrm{Aut}(t)$ a *geometric factor*.

The reason that Equation 10 is known as a "geometric decomposition" is each geometric factor can be associated with an induced subtree. A little more rigorously (following [MSGA08]), given a tree t with automorphism group $\mathrm{Aut}(t)$ we define a *symmetric subtree* to be the induced subtree of t on the support of a geometric factor.

Example 3.1. (i) A k-star is an induced subtree consisting of a vertex adjacent to k vertices of outdegree 0. A k-star is a symmetric subtree that corresponds to a geometric factor S_k (the symmetric group on k objects).

(ii)

It is a result of Pólya that automorphism groups of trees belong to the class, $\mathcal{W}$, of permutation groups which contains the symmetric groups and is closed under taking direct and wreath products. Let t be a rooted tree and consider the following decomposition,

$$\mathrm{Aut}(t) \cong A_1 \times A_2 \times \dots A_p \times B_1 \times B_2 \times \cdots \times B_q \tag{11}$$

where each factor A_i is isomorphic to a symmetric group and each B_i is isomorphic to the wreath product of a group $G \in \mathcal{W}$.

By Pólya's Theorem the geometric decomposition (Equation 10) can be written in the form,

$$\mathrm{Aut}(t) \cong A_1 \times A_2 \times \dots A_p \times B_1 \times B_2 \times \cdots \times B_q \tag{12}$$

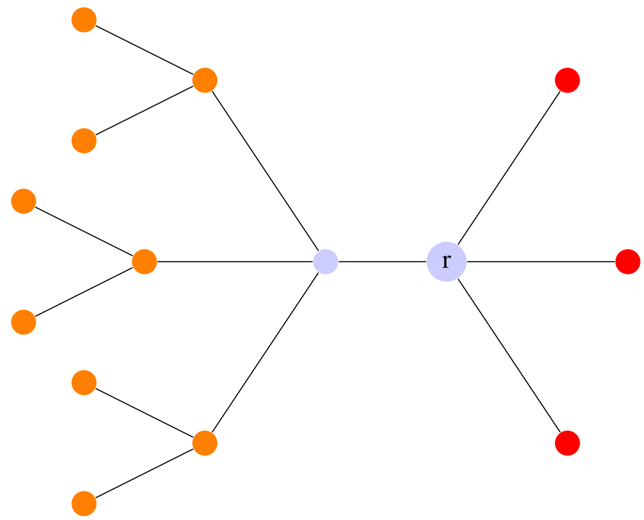


Figure 3: A tree, $t \in \mathcal{R}_n$, rooted at r such that $\mathrm{Aut}(t) \cong S_3 \times S_2 \wr S_3$. The red vertices depict a symmetric subtree (isomorphic to the leaves of a 3-star) that can be associated with the geometric factor of $S_3 \leq Aut(t)$. The orange vertices indicate a symmetric subtree that can be associated with the geometric factor $S_2 \wr S_3 \leq \mathrm{Aut}(T)$. In terms of Macarthur's decomposition $\mathcal{C}(t) \cong S_2 \wr S_3$ and $\mathcal{E}(t) \cong S_3$.

we claim that each A_i corresponds to an $(n,k) - star$ and each subgroup B_i corresponds to an extended symmetric branch of t.

There is a natural way to split the geometric decomposition of $\mathrm{Aut}(t)$ into two subgroups. We define the direct product of symmetric groups to be the *elementary subgroup*:

$$\mathcal{E}(t) = A_1 \times A_2 \times \cdots \times A_p$$

The direct product of wreath products of symmetric groups form the *complex subgroup*:

$$\mathcal{C}(t) = B_1 \times B_2 \times \cdots \times B_q$$

The order, $\sigma(t)$, of an automorphism group can also be split as follows:

$$\sigma(t) = |\mathcal{E}(t)||\mathcal{C}(t)|$$

This begs the question: does the order of either the elementary or the complex subgroup dominate the other?

We can think of the set $\mathcal{T}_n$ as a probability space by choosing each tree $t \in \mathcal{T}_n$ equiprobably. Given a random variable X on the space of order n random recursive trees we denote the expectation by $\mathbb{E}_{\mathcal{T}_n}(X)$.

MacArthur [MSGA08] made the following additional conjecture:

Conjecture 3.2. *In the limit as $n \to \infty$,*

$$\mathbb{E}_{\mathcal{T}_n}\left(|\mathcal{E}(t)|^{\frac{1}{n}}\right) = C\,,\, \mathbb{E}_{\mathcal{T}_n}\left(|\mathcal{C}(t)|^{\frac{1}{n}}\right) = 1.$$

where $C > 1$ is a constant.

In essense Conjecture 3.2 predicts that the subgroup of the automorphism group dominates the comples subgroup. Geometrically, the contribution to $\mathrm{Aut}(t)$ associated with (n,k)-stars dominates the contribution coming from extended symmertic branches.

4 Functions on Trees

In this section we will build up a tool kit of functions (called inductive maps and elementary differentials) which will be used to calculate properties of rooted trees such as path length and order.

4.1 Inductive Maps

Definition 4.1. Let $s = \{s_r\}_{r=0}^{\infty}$ be a sequence such that each $s_r \in \{0,1\}$. A *variety*, $\mathcal{V}$, of trees is a collection of random recursive trees such that each vertex is permitted to have outdegree r only if $s_r = 1$.

Remark 1. *For a more general setting see [BFS92].*

The *degree function* associated with a sequence $s = \{s_r\}_{r=0}^{\infty}$ is the exponential generating function (EGF) defined as follows:

$$\phi(w) = \sum_{r \geq 0} s_r \frac{w^r}{r!}$$

Example 4.2. The collection, $\mathcal{B}$, of increasing binary trees are the set of all random recursive trees such that each vertex has outdegree either 0,1 or 2. The degree function for increasing binary trees is

$$\phi(w) = 1 + w + \frac{w^2}{2}$$

The degree function for random recursive trees is

$$\phi(w) = \exp(w).$$

Fix a variety, $\mathcal{V}$, and define V_n to be the number of trees of order n in $\mathcal{V}$. The EGF of the variety of trees is

$$V_{\mathcal{V}}(z) = \sum_{n \geq 1} V_n \frac{z^n}{n!}$$

For example,

$$V_{\mathcal{T}}(z) = \sum_{n \geq 1} \frac{z^n}{n} \tag{13}$$

$$= \log\left(\frac{1}{1-z}\right) \tag{14}$$

since the number of random recursive trees of order n is $(n-1)!$.

Definition 4.3. Let $f = \{f_n\}_{n\geq 1}$ be a sequence of real numbers. A function $s : \mathcal{R} \to \mathbb{R}$ is called an *inductive map* if it is definable by a relation,

$$s(t) = f_{|t|} + \sum_{i=1}^{k} s(t_i)$$

where $t = B^+(t_1, t_2, \ldots, t_k)$.

Given an inductive map, s, and a variety $\mathcal{V}$ the EGF of s over $\mathcal{V}$ is

$$S(z) = \sum_{t\in\mathcal{V}} s(t)\frac{z^{|t|}}{|t|!}$$

Let $\delta_{i,j}$ be the usual Kronecker delta. In Figure 4 we list possible sequences $f = \{f_n\}_{n\geq 1}$ and give the tree parameter measured by the corresponding function s.

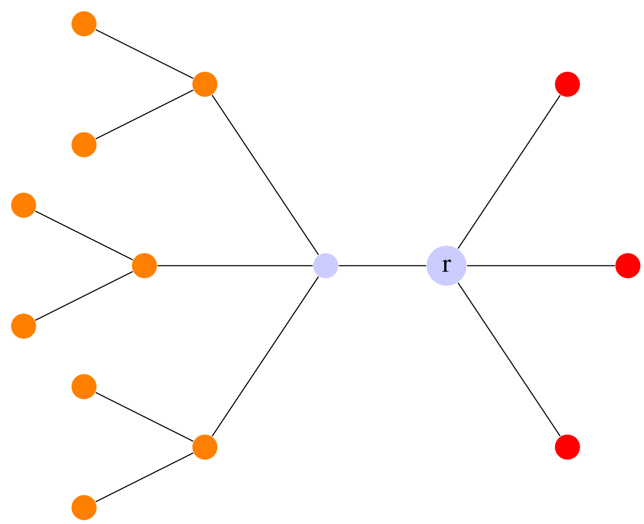


Figure 4: A tree, $t \in \mathcal{R}_n$. Let $f = \{f_n\}$ be the sequence $f_n = 1$ for each $n \geq 1$ the corresponding inductive map, s, calculates the order of t and in this case $s(t) = 14$. Let $f = \{f_n\}$ be the sequence $f_n = \delta_{n,1}$ for each $n \geq 1$, the corresponding inductive map, s, calculates the number of leaves of t in particular $s(t) = 9$.

In order to calculate $S(z)$ easily and effectively we appeal to the following Theorem of Bergeron [BFS92].

Theorem 4.4.

$$S(z) = Y'(z)\int_0^z \frac{F'(t)}{Y'(t)}dt$$

where $F(z)$ is defined from $\mathcal{V}$ and the sequence $f = \{f_n\}_{n\geq 1}$ as follows:

$$F(z) = \sum_{n\geq 0} f_n V_n \frac{z^n}{n!}$$

4.2 Butcher Series

Throughout this section let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a suitably differentiable function and consider differential equations of the form

$$\frac{d}{ds}y(s) = f(y(s)) \tag{15}$$

where $y(s_0) = y_0$. We use the notation $f'(y)$ for the derivative $\frac{d}{dy}f(y)$ and note that $f'(y)$ is a linear map (the Jacobian), the second derivative $f''(y)$ is a bilinear map and so on. We follow Butcher and make the following definition [But08].

Definition 4.5. Let $t = B^+(t_1, t_2, \ldots, t_k)$ be a rooted tree. An *elementary differential* is a map $F(t) : \mathbb{R}^n \to \mathbb{R}^n$ defined recursively by:

$$F(\bullet)(y) = f(y) \tag{16}$$
$$F(t)(y) = f^m(y)(F(t_1)(y), \ldots, F(t_k)(y)) \tag{17}$$

Example 4.6. (i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ ($n = 1$ in Equation 15) is a smooth function and that $s_0 = y_0 = 0$. These kinds of expressions have been treated in great detail in [**?**] and [But72]. Let $f(y) = \exp(y)$ so that $f^{(m)}(y) = 1$ for $m = 1, 2, 3, \ldots$ and $F(t) = 1$ for all trees t.

(ii) Again suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and that $s_0 = y_0 = 0$. Let $f(y) = \frac{1}{1-y}$ so that $f^{(m)}(y) = m!$ for $m = 1, 2, 3, \ldots$ and $F(t) = w(t)$ for all trees t.

Theorem 1. *The solution of Equation 15,*

$$y(s) = y_0 + \int_{s_0}^{s} f(y(s'))ds'$$

is

$$y(s) = y_0 + \sum_{t \in \mathcal{R}} \frac{(s - s_0)^{|t|}}{|t|!}\alpha(t)F(t)(y_0)$$

Proof. See for example [But08, Bro00] □

Example 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be as in Example 4.6(i). By Theorem 1 the solution of equation 15,

$$y(s) = \int_{0}^{s} \exp(y(s'))ds' \tag{18}$$

is

$$y(s) = \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!}\alpha(t) \tag{19}$$

We may rewrite Equation 18 as the differntial equation,

$$y'(s) = \exp(y(s))$$

which has solution

$$x(s) = -\log(1-s). \tag{20}$$

By comparing Equation 19 with 20,

$$\sum_{t \in \mathcal{R}_n} \alpha(t) = (n-1)!$$

which confirms the equation $|\mathcal{T}_n| = (n-1)!$.

5 Bounds on the expected value of $\mathrm{Aut}(t)$

5.1 A lower bound

In this section we will calculate the expected contribution k-stars make to $\sigma(t)$ which is a lower bound for the expected value of $\mathrm{Aut}(t)$. More rigorously given a tree, t, any induced subtree isomorphic to a k-star can be associated with $(k-1)!$ automorphisms $\alpha \in \mathrm{Aut}(t)$. Let $\sigma_k(t)$ denote the total number of automorphisms which correspond induced subtrees isomorphic to a k-star in t and also define

$$\sigma_*(t) = \sum_{k \geq 2} \sigma_k(t)$$

Consider the family, s^k of inductive maps over variety, $\mathcal{T}$, that are defined by the sequences $f_n^k = \delta_{k,n}$. Given a tree $t \in \mathcal{T}$ the inductive map $s^k(t)$ is the number of induced subtrees of t that have order k (see Example 4.2).

Following Theorem 4.4, to each map s^k we associate the function,

$$\begin{aligned} F_k(z) &= \sum_{n \geq 1} f_n^k V_n \frac{z^n}{n!} && (21) \\ &= \frac{z^k}{k} && (22) \end{aligned}$$

thus $F_k'(z) = z^{k-1}$. By Theorem 4.4,

$$\begin{aligned} \sum_{t \in \mathcal{T}} s^k(t) \frac{z^{|t|}}{|t|!} &= \frac{1}{1-z} \int_0^z t^{k-1}(1-t)dt && (23) \\ &= \frac{1}{1-z}\left(\frac{z^k}{k} - \frac{z^{k+1}}{k+1}\right) && (24) \\ &= \frac{z^k}{k} + \sum_{i \geq k+1} \frac{z^i}{k(k+1)} && (25) \end{aligned}$$

hence for $n \geq k+1$,

$$\sum_{t \in \mathcal{T}_n} s^k(t) = \frac{n!}{k(k+1)}.$$

Let $\hat{t}_k \in \mathcal{T}_k$ be a random recursive tree isomorphic to a $(k-1)$-star. Since $\hat{t}_k! = k$ and $\sigma(\hat{t}_k) = (k-1)!$ Lemma 2.1 states that

$$\alpha(\hat{t}_k) = 1$$

hence $\hat{t}_k$ is the only (isomorphism class of) random recursive tree(s) on k vertices isomorphic to a $(k-1)$-star. Fix $k \geq 3$ and suppose that $t = (r, V, E, l)$ is a random recursive tree and $t \in \mathcal{T}_n$ for some $n > k$. Suppose t contains an induced subtree, $t_v = (v, V_v, E_v, l_v)$ that has order k hence $v \neq r$. Given a set S and a subset $R \leq S$ we define $\bar{R}$ to be the complement of R in S. We further define the complement of t_v in t to be

$$\bar{t_v} = (r, \bar{V_v}, \bar{E_v} \backslash \{(v, f(v))\}, \bar{l_v})$$

where $f(v) \in E$ is the vertex adjacent to v with a smaller label often called the father of v. Notice that there exist $(k-1)!$ non isomorphic trees $t \in \mathcal{T}_n$ which consist of $\bar{t_v}$ and a random recursive tree of order k joined to $\bar{t_v}$ via the edge $\{(v, f(v))\}$.

Given a tree $t \in \mathcal{T}$ let $n_k(t)$ be the number of induced subtrees of t isomorphic to $\hat{t}_k$,

$$\sum_{t \in \mathcal{T}_n} n_k(t) = \frac{n!}{(k+1)!}$$

by the argument above. Since $\sigma(\hat{t_k}) = (k-1)!$,

$$\mathbb{E}_{\mathcal{T}_n}(\log(|\sigma_k|^{\frac{1}{n}}) = \frac{1}{|\mathcal{T}_n|} \sum_{t \in \mathcal{T}_n} \log((k-1)!^{\frac{n_k(t)}{n}}) \tag{26}$$

$$= \frac{1}{(n-1)!} \sum_{t \in \mathcal{T}_n} \log((k-1)!) \frac{n_k(t)}{n} \tag{27}$$

$$= \frac{\log(k-1)!}{(k+1)!} \tag{28}$$

By summing over all $k \geq 3$ we find that

$$\mathbb{E}_{\mathcal{T}_n}(\sigma_*(t)^{\frac{1}{n}}) = \sum_{k \geq 3} \frac{\log(k-1)!}{(k+1)!}$$

Finally, since $\exp$ is convex, by Jensen's inequality in the limit $n \to \infty$

$$\exp\left(\sum_{k \geq 3} \left(\frac{\log(k-1)!}{(k+1)!}\right)\right) \leq \mathbb{E}_{\mathcal{T}_n}\left(\sigma(t)^{\frac{1}{n}}\right)$$

To 5 s.f $1.0506 \leq \mathbb{E}(\sigma_*(t)^{\frac{1}{n}}$.

5.2 An upper bound

In this section we will use B-series and Theorem **??** to calculate an upper bound for the expected order $\mathbb{E}_{\mathcal{T}_n}(\sigma(t))$ of an automorphism group of a random recursive tree t.

Recall from Equation 7 that $\gamma(t) = \frac{w(t)}{\sigma(t)}$ hence

$$\mathbb{E}_{\mathcal{T},n}(\sigma(t)) = \frac{1}{|\mathcal{T}_n|} \sum_{t \in R_n} \alpha(t)\sigma(t) \tag{29}$$

$$< \frac{1}{|\mathcal{T}_n|} \sum_{t \in R_n} \alpha(t)w(t) \tag{30}$$

We define $f(s) = \frac{1}{1-s}$ as in Example 4.6(ii) so that $f^{(k)}(0) = k!$. By Theorem 1 the solution to

$$y(s) = \int_0^s = \frac{1}{1-y(s')} ds' \tag{31}$$

is

$$y(s) = \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!} \alpha(t)w(t) \tag{32}$$

We may restate Equation 31 as $y'(s) = \frac{1}{1-y(s)}$ which is a first order, non-linear differential equation with solution

$$y(s) = 1 \pm (c_1 - 2s + 1)^{\frac{1}{2}} \tag{33}$$

where c_1 is a constant to be determined. Since $x(0) = 0$ we may deduce that $c_1 = 0$. By using the binomial expansion of Equation 33 and comparing with Equation 32 we find that,

$$\sum_{|t|=n} \alpha(t)w(t) = \prod_{i=1}^{n-1} (2i-1)$$

Note that $\prod_{i=1}^{n-1}(2i-1) = (2n-3)!!$ hence using factorial identities and Stirling's approximation

$$\frac{(2n-3)!!}{(n-1)!} = \frac{(2n-3)!}{(n-1)!^2 2^{n-1}} \tag{34}$$

$$\frac{\sqrt{\pi(n-1)}}{\pi(n-1)} \left(\frac{2(n-1)}{e}\right)^{2(n-1)} \left(\frac{e}{(n-1)}\right)^{2(n-1)} \left(\frac{1}{2}\right)^{n-1} \tag{35}$$

$$= \frac{\sqrt{\pi(n-1)}}{\pi(n-1)} (2)^{n-1} \tag{36}$$

Therefore,

$$\mathbb{E}_{\mathcal{T}_n}(\sigma(t)) < 2^{n-1}$$

By Jensen's inequality,

$$\mathbb{E}_{\mathcal{T}_n}(\sigma(t)^{\frac{1}{n}}) < 2^{\frac{n-1}{n}} = \text{constant as required}$$

6 A disproof of Conjecture 3.2

In this section we will disprove Conjecture 3.2 by demonstrating that there exists a subset, $S \subset \mathcal{C}(t)$, such that the expected order of S in $\mathcal{T}_n$ grows exponentially with n. In particular, let $\tilde{t}$ be the tree on 7 vertices shown in Figure 5 and notice that every induced subtree of tree t isomorphic to $\tilde{t}$ is contained in the support of a geometric factor $H_i \leq \mathcal{C}(t)$. Let $\sigma_{\tilde{t}}(t)$ be the number of automorphisms $\alpha \in \mathrm{Aut}(t)$ that correpond to $\tilde{t}$.

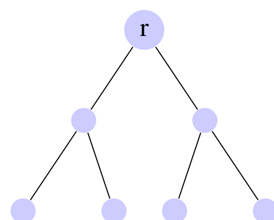


Figure 5: Tree $\tilde{t}$

Consider the inductive map, s (over the variety $\mathcal{T}$) defined by a sequence $f_n = \log(8)\delta_{n,7}$ and assume that s has EGF

$$S(z) = \sum_{t \in \mathcal{T}} s(t) \frac{z^{|t|}}{|t|!} \tag{37}$$

Since s is just the s^k defined in Section 5.1 for $n \geq 8$,

$$S(z) = \sum_{n \geq 8} \frac{z^n}{56} \tag{38}$$

$$\sum_{t \in \mathcal{T}_n} s^k(t) = \frac{n!}{56}$$

Given a random recursive tree t let $\epsilon(t)$ denote the number of induced subtrees of t that are isomorphic to $\tilde{t}$. Since $\alpha(\tilde{t}) = 10$ and $|\mathcal{T}_7| = 6!$ and $\sigma(\hat{t}) = 8$ following a similar argument to Section 5.1 we may write,

$$\sum_{t \in \mathcal{T}_n} \log(8)\epsilon(t) = \frac{10\log(8)n!}{8!}$$

Therefore the expected value,

$$\mathbb{E}_{\mathcal{T}_n}\left(\log\left(\sigma_{\tilde{t}}(t)^{\frac{1}{n}}\right)\right) = \frac{1}{|\mathcal{T}_n|} \sum_{t \in \mathcal{T}_n} \log\left(8^{\frac{\epsilon(t)}{n}}\right) \tag{39}$$

$$= \frac{1}{(n-1)!} \sum_{t \in \mathcal{T}_n} \frac{\log(8)\epsilon(t)}{n} \tag{40}$$

$$= \frac{10\log(8)}{8!} \tag{41}$$

Since exponential is a convex function,

$$1 < \exp\left(\frac{10\log(8)}{8!}\right) \leq \mathbb{E}_{\mathcal{T}_n}\left(\sigma_{\tilde{t}}(t)^{\frac{1}{n}}\right) \leq \mathbb{E}_{\mathcal{T}_n}\left(|\mathcal{C}(t)|^{\frac{1}{n}}\right)$$

by Jensen's inequality; thus disproving Conjecture 3.2.

7 Conclusions

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