

# Automorphisms of Random Recursive Trees

David Matthews

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## 1 Introduction

Graph automorphisms are essential to understanding enumerative properties of graphs. An exact formulation of the number of rooted unlabelled trees is unknown which stands testament to the difficulty in understanding tree automorphisms. An intriguing relationship between random Fibonacci sequences and automorphism groups coming from a family of trees called random recursive trees provides further motivation, if we needed any, to investigate automorphism groups of trees [MA06]. We exploit a geometric interpretation of a graph automorphism in which subgroups of the automorphism group are associated with certain subtrees to give bounds for the expected order of the automorphism group of a random recursive tree (also known as a heap-ordered tree) of order  $n$ . We also disprove a conjecture of MacArthur [MSG08] which states that the automorphism group can be split in a particularly pleasing way.

In Section 2 we introduce several families of increasing trees and we describe the relationship between them using several recursively defined functions. In Section 3 we give a geometric interpretation of the automorphism group and a direct product decomposition in which particular subgroups can be associated with certain subtrees. We also state MacArthur's conjecture. In Section 4 we describe two families of recursively defined functions; inductive maps and elementary differentials, from a collection,  $\mathcal{T}$  of trees,  $f : \mathcal{T} \rightarrow \mathbb{R}$  and describe how these may be manipulated to calculate properties of trees such as order, path length and number of leaves. In Section 5 we use inductive maps and elementary differentials to provide upper and lower bounds on the expected order of the order of a family of trees called random recursive trees. In Section 6 we disprove the conjecture of MacArthur and we make concluding remarks in Section 7.

## 2 Rooted Trees

A *rooted tree* is a triple  $t = (r, V, E)$ , such that  $(V, E)$  is a finite, simple connected graph without cycles with vertex set  $V$  and edge set  $E$  [Tat84]. A vertex,  $r \in V$ , is selected from  $V$  and called the *root*. By convention each edge is directed away from  $r$ . We denote the order of a tree,  $t$ , by  $|t|$  and the set of order  $n$  rooted trees is denoted  $\mathcal{R}_n$ . In addition we define  $R := \bigcup \mathcal{R}_n$ .

A *labelled rooted tree* is a quadruple  $t = (r, V, E, L)$  such that  $(r, V, E)$  is a rooted tree and

$$L : V \longrightarrow \{1, 2, \dots, |t|\}$$

is a bijective map. We denote the set of order  $n$  labelled rooted trees by  $\mathcal{L}_n$  and we define  $\mathcal{L} = \bigcup \mathcal{L}_n$ .

For any pair of vertices  $u$  and  $v$  of a rooted tree we write  $u \leq v$  if  $u$  lies on the unique shortest path from  $r$  to  $v$ . A *random recursive tree* is a quadruple  $t = (r, V, E, l)$  such that  $t$  is a labelled rooted tree and  $l$  is a labelling such that if  $u \leq v$  then  $l(u) < l(v)$ . We denote the set of order  $n$  random recursive trees by  $\mathcal{T}_n$  and we define  $\mathcal{T} = \bigcup \mathcal{T}_n$ . Random recursive trees are also known as “heap-ordered trees”.

Two rooted trees  $t_1$  and  $t_2$  with roots  $r_1$  and  $r_2$  and vertex sets  $V(t_1)$  and  $V(t_2)$  respectively are said to be isomorphic if there exists a bijection  $f : V(t_1) \rightarrow V(t_2)$  such that vertices  $u, v \in V(t_1)$  are adjacent if and only if  $f(u), f(v) \in V(t_2)$  are adjacent and  $f(r_1) = r_2$  [HP73]. If  $V(t_1) = V(t_2)$  then  $f$  is called a rooted tree automorphism. The set of automorphisms of a tree,  $t$ , together with composition of maps forms a group denoted  $\text{Aut}(t)$ . The order of the automorphsim group is

$$\sigma(t) := |\text{Aut}(t)|,$$

notation taken from [But08]. Consider a map  $\phi : \mathcal{L} \rightarrow \mathcal{R}$  that simply forgets the labels of a labelled by mapping

$$(r, V, E, L) \mapsto (r, V, E)$$

The map  $\phi$  is clearly surjective but not injective hence we define  $\beta(t) = |\phi^{-1}(t)|$  to be the number of possible isomorphism classes of labellings of a rooted tree  $t \in \mathcal{R}$ . There are  $|t|!$  possible labellings of a tree  $t \in \mathcal{R}$  hence there are

$$\beta(t) = \frac{|t|!}{\sigma(t)} \quad (1)$$

isomorphism classes of labellings of  $t$ .

Since  $\mathcal{T} \leq \mathcal{L}$  we denote by  $\psi$  the restriction of  $\phi$  to random recursive trees and define

$$\alpha(t) := |\psi^{-1}(t)|$$

The notation  $\alpha(t)$  is also found in (for example) [Bro00, But63, Maz04]. To provide a result analagous to Equation 1 we require several additional definitions.

Let  $t_1, t_2, \dots, t_k$  be a forest of rooted trees: the rooted tree  $B^+(t_1, t_2, \dots, t_k)$  is built from this forest by introducing a new vertex,  $r$  (the root of  $B^+(t_1, t_2, \dots, t_k)$ ), and joining the roots of each tree  $t_1, t_2, \dots, t_k$  to  $r$  via an edge. Since every tree  $t$  can be written as  $B^+(t_1, t_2, \dots, t_k)$  if  $|t| > 1$  we henceforth assume that each rooted tree  $t = B^+(t_1, t_2, \dots, t_k)$  if  $|t| > 1$ . For convenience we will denote the rooted tree on 1 vertex by  $\bullet$ .

We have already seen one function  $f : \mathcal{R} \rightarrow \mathbb{R}$  on rooted trees ( $f(t) = |t|$ ), let us consider another: We define the tree factorial  $t!$  recursively

$$\bullet! = 1 \quad (2)$$

$$t! = |t| \prod_{i=1}^k t_i! \quad (3)$$

The notation  $t!$  is taken from Kreimer [Kre99] because  $t!$  generalises the factorial of a number.

**Example 2.1.**

**Lemma 2.2.** *For a rooted tree  $t$ ,*

$$\alpha(t) = \frac{|t|!}{t!\sigma(t)} \quad (4)$$

*Proof.* There are  $n!$  ways of labelling a tree  $t \in \mathcal{R}_n$ , however if  $l$  is a random recursive labelling every induced subtree  $t_v$  and a totally ordered set,  $S$ , of labels there is precisely one possible label  $s \in S$  for vertex  $v$  (namely  $s = \min(S)$ ). Therefore the factor we should divide out by is precisely  $t!$ . In addition, to calculate the number of isomorphism classes of random recursive trees we must again divide out by  $\sigma(t)$ .  $\square$

Alternatively see [But08] for a proof of Lemma 2.2. Finally, let  $\chi : \mathcal{F} \rightarrow \mathcal{R}$  be the map that “forgets” the embedding of a rooted plane tree. It is clear that  $\chi$  is surjective but not injective hence we define

$$\gamma(t) = |\chi^{-1}|$$

the number of isomorphism classes of embeddings of a rooted tree  $t$ . In order to obtain a third relation analogous to Equations 1 and 4 we will describe a third function  $w : \mathcal{R} \rightarrow \mathbb{R}$  recursively:

$$w(\bullet) = 1 \quad (5)$$

$$w(t) = k! \prod_{i=1}^k w(t_i) \quad (6)$$

**Example 2.3.** Do the same example set as before

**Lemma 2.4.**

$$\gamma(t) = \frac{w(t)}{\sigma(t)} \quad (7)$$

*Proof.* Let  $B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$  denote a tree in which the root is incident to  $n_1$  isomorphic copies of a tree  $t_1$ ,  $n_2$  isomorphic copies of  $t_2$  and so forth. We remark that every rooted tree with at least 2 vertices can be written in this way. Let  $t \in \mathcal{R}$  and consider an induced subtree  $t_v = B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$ . Vertex  $v$  contributes a factor of  $\left(\sum_{i=1}^k n_i\right)!$  to  $w(t)$  since it has a recursive definition. The order of automorphism group,  $\sigma(t)$  can also be expressed recursively:

$$\sigma(\bullet) = 1 \quad (8)$$

$$\sigma(t) = n_1! n_2! \dots n_k! \prod_{i=1}^k \sigma(t_i)^{n_i} \quad (9)$$

Therefore vertex  $v$  contribute a factor of  $\prod_{i=1}^k n_i!$  to  $\sigma(t)$ . Finally consider,  $\gamma(t)$ , the number of somorphism classes of embeddings of  $t$ . There are

$$\frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!}$$

non-isomorphic possibilities for the ordering of the children of  $v$ .  $\square$

### 3 Automorphisms of Trees

In this section we will describe a direct product decomposition of the automorphism group of a tree,  $t$ , in which factors of the direct product can be associated with particular induced subtrees of  $t$ .

Recall that a (rooted) tree automorphism is a permutation of vertices that preserves adjacency and the root. It is a result of Pólya that automorphism groups of trees belong to the class,  $\mathcal{W}$ , of permutation groups which contains the symmetric groups and is closed under taking direct and wreath products. Let  $t$  be a rooted tree and consider the following decomposition,

$$\text{Aut}(t) \cong A_1 \times A_2 \times \dots \times A_p \times B_1 \times B_2 \times \dots \times B_q \quad (10)$$

where each factor  $A_i$  is isomorphic to a symmetric group and each  $B_i$  is isomorphic to the wreath product of a group  $G \in \mathcal{W}$ . In [MSGA08] MacArthur, Sanchez-Garcia and Anderson showed that the decomposition of  $\text{Aut}(t)$  described in Equation has a geometric interpretation. An automorphism group,  $\text{Aut}(t)$  may be decomposed by partitioning the set of generators  $S$  of  $\text{Aut}(t)$  into support-disjoint subsets  $S = S_1 \cup S_2 \cup \dots \cup S_r$  and writing

$$\text{Aut}(t) = H_1 \times H_2 \times \dots \times H_r \quad (11)$$

where each  $H_i$  is generated by  $S_i$ . This, *geometric decomposition* is shown [MSGA08] to be unique and irreducible (each  $H_i$  cannot be written as the direct product of support-disjoint subgroups) hence the geometric decomposition is well defined. We call each  $H_i$  in the geometric decomposition of  $\text{Aut}(t)$  a *geometric factor*. A symmetric subtree is the induced subtree of  $t$  on the support of a geometric factor.

**Example 3.1.** (i) A  $k$ -star is an induced subtree consisting of a vertex adjacent to  $k$  vertices of outdegree 0. A  $k$ -star is a symmetric subtree that corresponds to a geometric factor  $S_k$  (the symmetric group on  $k$  objects).

(ii)

By Pólya's Theorem the geometric decomposition (Equation 11) can be written in the form,

$$\text{Aut}(t) \cong A_1 \times A_2 \times \dots \times A_p \times B_1 \times B_2 \times \dots \times B_q \quad (12)$$

such that each  $A_i$  corresponds to a symmetric subtree.

There is a natural way to split the geometric decomposition of  $\text{Aut}(t)$  into two subgroups. We define the direct product of symmetric groups to be the *elementary subgroup*:

$$\mathcal{E}(t) = A_1 \times A_2 \times \cdots \times A_p$$

The direct product of wreath products of symmetric groups form the *complex subgroup*:

$$\mathcal{C}(t) = B_1 \times B_2 \times \cdots \times B_q$$

The order,  $\sigma(t)$ , of an automorphism group can also be split as follows:

$$\sigma(t) = |\mathcal{E}(t)| |\mathcal{C}(t)|$$

This begs the question: does the order of either the elementary or the complex subgroup dominate the other? MacArthur [MSG08] made the following additional conjecture:

**Conjecture 3.2.** *Let  $\{T_t\}_{t=1}^n$  be a RRT. In the limit as  $t \rightarrow \infty$ ,  $|\mathcal{E}(T_t)|^{\frac{1}{t}} = \mathcal{V}$ , while in the limit as  $t \rightarrow \infty$ ,  $|\mathcal{C}(T_t)|^{\frac{1}{t}} = 1$ .*

We claim that the elementary subgroup captures the contribution that  $(n, k)$ -stars make to the automorphism group and the complex subgroup captures the contribution that the extended symmetric branches make to the automorphism group.

## 4 Functions on Trees

In this section we will build up a 3-part tool kit of functions  $f : \mathcal{R} \rightarrow \mathbb{R}$  which will be used to calculate properties of rooted trees such as path length and order.

### 4.1 Inductive Maps

**Definition 4.1.** Let  $s = \{s_r\}_{r=0}^{\infty}$  be a sequence such that each  $s_r \in \{0, 1\}$ . A *variety*,  $\mathcal{V}$ , of trees is a collection of random recursive trees such that each vertex is permitted to have outdegree  $r$  only if  $s_r = 1$ .

**Remark 1.** *For a more general setting see [BFS92].*

The *degree function* associated with a sequence  $s = \{s_r\}_{r=0}^{\infty}$  is the exponential generating function (EGF) defined as follows:

$$\phi(w) = \sum_{r \geq 0} s_r \frac{w^r}{r!}$$

**Example 4.2.** The collection,  $\mathcal{B}$ , of increasing binary trees are the set of all random recursive trees such that each vertex has outdegree either 0, 1 or 2. The degree function for increasing binary trees is

$$\phi(w) = 1 + w + \frac{w^2}{2}$$

The degree function for random recursive trees is

$$\phi(w) = \exp(w).$$

Fix a variety,  $\mathcal{V}$ , and define  $V_n$  to be the number of trees of order  $n$  in  $\mathcal{V}$ . The EGF of the variety of trees is

$$V_{\mathcal{V}}(z) = \sum_{n \geq 1} V_n \frac{z^n}{n!}$$

For example,

$$V_{\mathcal{T}}(z) = \sum_{n \geq 1} \frac{z^n}{n} \tag{13}$$

$$= \log \left( \frac{1}{1-z} \right) \tag{14}$$

since the number of random recursive trees of order  $n$  is  $(n-1)!$ .

**Definition 4.3.** Let  $f = \{f_n\}_{n \geq 1}$  be a sequence of real numbers. A function  $s : \mathcal{R} \rightarrow \mathbb{R}$  is called an *inductive map* if it is definable by a relation,

$$s(t) = f_{|t|} + \sum_{i=1}^k s(t_i)$$

where  $t = B^+(t_1, t_2, \dots, t_k)$ .

Given a tree function  $s$  and a variety  $\mathcal{V}$  the EGF of  $s$  over  $\mathcal{V}$  is

$$S(z) = \sum_{t \in \mathcal{V}} s(t) \frac{z^{|t|}}{|t|!}$$

**Example 4.4.** Let  $\delta_{i,j}$  be the usual kronecker delta. The following is a list of possible sequences  $f$  and the tree parameter measured by the corresponding function  $s$

- (i)  $f_n = 1$  for all  $n$  counts tree size.
- (ii)  $f_n = \delta_{n,i}$  counts the number of induced subtrees of order  $i$ .
- (iii)  $f_n = n$  counts path lengths.

In order to calculate  $S(z)$  easily and effectively we appeal to the following Theorem of Bergeron [BFS92].

**Theorem 4.5.**

$$S(z) = Y'(z) \int_0^z \frac{F'(t)}{Y'(t)} dt$$

where  $F(z)$  is defined from  $\mathcal{V}$  and the sequence  $f$

$$F(z) = \sum_{n \geq 0} f_n V_n \frac{z^n}{n!}$$

## 4.2 Butcher Series

Throughout this section let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a suitably differentiable function and consider differential equations of the form

$$\frac{d}{ds}y(s) = f(y(s))$$

where  $y(s_0) = y_0$ . We use the notation  $f'(y)$  for the derivative  $\frac{d}{dy}f(y)$  and note that  $f'(y)$  is a linear map (the Jacobian), the second derivative  $f''(y)$  is a bilinear map and so on. We follow Butcher and make the following definition [But08].

**Definition 4.6.** Let  $t = B^+(t_1, t_2, \dots, t_k)$  be a rooted tree. An *elementary differential* is a map  $F(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined recursively by:

$$F(\bullet)(y) = f(y) \tag{15}$$

$$F(t)(y) = f^m(y)(F(t_1)(y), \dots, f(t_k)(y)) \tag{16}$$

**Theorem 1.** *The solution of*

$$y(s) = y_0 + \int_{s_0}^s f(y(s'))ds'$$

is

$$y(s) = y_0 + \sum_{t \in \mathcal{R}} \frac{(s - s_0)^{|t|}}{|t|!} \alpha(t) F(t)(y_0)$$

*Proof.* See [But08, Bro00] □

**Example 4.7.** Let  $y_0 = s_0 = 0$  and consider the case  $f(s) = \frac{1}{1-s}$  so that  $f^{(k)}(0) = k!$  then  $F(t) = w(t)$ .

## 5 Bounds on the expected value of $\text{Aut}(t)$

### 5.1 A lower bound

In this section we will calculate the expected contribution the geometric factors,  $G_k$ , corresponding to  $k$ -stars which will be a lower bound for the expected value of  $\text{Aut}(t)$ . Given a tree  $t$  the number of automorphisms  $\alpha \in \text{Aut}(t)$  which correspond to a  $k$ -star for any  $k$  is written  $\sigma_*(t)$ .

Consider the family,  $s^k$  of inductive maps over variety  $\mathcal{T}$  that are defined by the sequences  $f_n^k = \delta_{k,n}$ . Given a tree  $t \in \mathcal{T}$  the map  $s^k(t)$  counts the number of induced subtrees of  $t$  that have order  $k$  (as we have seen in Example 4.2).

Following Theorem 4.5, to each map  $s^k$  we associate the function,

$$F_k(z) = \sum_{n \geq 1} f_n^k V_n \frac{z^n}{n!} \tag{17}$$

$$= \frac{z^k}{k} \tag{18}$$

thus  $F'_k(z) = z^{k-1}$ . By Theorem 4.5,

$$\sum_{t \in \mathcal{T}} s^k(t) \frac{z^{|t|_{\text{vert}}}}{|t|!} = \frac{1}{1-z} \int_0^z t^{k-1} (1-t) dt \quad (19)$$

$$= \frac{1}{1-z} \left( \frac{z^k}{k} - \frac{z^{k+1}}{k+1} \right) \quad (20)$$

$$= \frac{z^k}{k} + \sum_{i \geq k+1} \frac{z^i}{(k)(k+1)} \quad (21)$$

hence for  $n \geq k+1$ ,

$$\sum_{t \in \mathcal{T}_n} s^k(t) = \frac{n!}{k(k+1)}.$$

Let  $\hat{t}_k \in \mathcal{T}_k$  be a random recursive tree isomorphic to a  $(k-1)$ -star. Since  $\hat{t}_k! = k$  and  $\sigma(\hat{t}_k) = (k-1)!$ , by Lemma 2.2

$$\alpha(\hat{t}_k) = 1$$

i.e. there is only one isomorphism class of random recursive trees on  $k$  vertices isomorphic to  $\hat{t}_k$ . Fix  $k \geq 3$  and suppose that  $t = (r, V, E, l)$  is a random recursive tree and  $t \in \mathcal{T}_n$  for some  $n > k$ . Suppose  $t$  contains an induced subtree,  $t_v = (v, V_v, E_v, l_v)$  that has order  $k$  hence  $v \neq r$ . Given a set  $S$  and a subset  $R \leq S$  we define  $\bar{R}$  to be the complement of  $R$  in  $S$ . We further define the complement of  $t_v$  in  $t$  to be

$$\bar{t}_v = (r, \bar{V}_v, \bar{E}_v \setminus (v, f(v)), \text{bar} l_v)$$

where  $f(v) \in E$  is the vertex adjacent to  $v$  with a smaller label often called the father of  $v$ . Notice that there exist  $(k-1)!$  nonisomorphic trees  $t \in \mathcal{T}_n$  which consist of  $\bar{t}_v$  and a random recursive tree of order  $k$  joined to  $\bar{t}_v$  via the edge  $\{(v, f(v))\}$ .

Given a tree  $t \in \mathcal{T}$  let  $n_k(t)$  be the number of induced subtrees of  $t$  isomorphic to  $\hat{t}_k$ ,

$$\sum_{t \in \mathcal{T}_n} = \frac{n!}{(k+1)!}$$

by the argument above. Since  $\sigma(\hat{t}_k) = (k-1)!$ ,

$$\mathbb{E}_{\mathcal{T}_n}(\log(|G_k|^{\frac{1}{n}})) = \frac{1}{|\mathcal{T}_n|} \sum_{t \in \mathcal{T}_n} \log((k-1)!^{\frac{n_k(t)}{n}}) \quad (22)$$

$$= \frac{1}{(n-1)!} \sum_{t \in \mathcal{T}_n} \log((k-1)!^{\frac{n_k(t)}{n}}) \quad (23)$$

$$= \frac{\log(k-1)!}{(k+1)!} \quad (24)$$

By summing over all  $k \geq 3$  we find that

$$\mathbb{E}_{\mathcal{T}_n}(\sigma_*(t)^{\frac{1}{n}}) = \sum_{k \geq 3} \frac{\log(k-1)!}{(k+1)!}$$



Finally, since  $\exp$  is convex, by Jensen's inequality

$$\exp \left( \sum_{k \geq 3} \left( \frac{\log(k-1)!}{(k+1)!} \right) \right) \leq \mathbb{E}_{\mathcal{T}_n} (\sigma(t)^{\frac{1}{n}})$$

## 5.2 An upper bound

In this section we will use B-series and Theorem ?? to calculate an upper bound for the expected order  $\mathbb{E}_{\mathcal{T},n}(\sigma(t))$  of an automorphism group of a random recursive tree  $t$ .

Recall from equation 7 that  $\gamma(t) = \frac{w(t)}{\sigma(t)}$  hence

$$\mathbb{E}_{\mathcal{T},n}(\sigma(t)) = \frac{1}{|\mathcal{T}_n|} \sum_{t \in R_n} \alpha(t) \sigma(t) \quad (25)$$

$$< \frac{1}{|\mathcal{T}_n|} \sum_{t \in R_n} \alpha(t) w(t) \quad (26)$$

We define  $f(s) = \frac{1}{1-s}$  as in Example 4.7 so that  $f^{(k)}(0) = k!$ . By Theorem 1 the solution to

$$x(s) = \int_0^s \frac{1}{1-x(s')} ds' \quad (27)$$

is

$$x(s) = \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!} \alpha(t) w(t) \quad (28)$$

We may restate Equation 27 as  $x'(s) = \frac{1}{1-x(s)}$  which is a first order, non-linear differential equation with solution

$$x(s) = 1 \pm (c_1 - 2s + 1)^{\frac{1}{2}} \quad (29)$$

where  $c_1$  is a constant to be determined. Since (by Equation 27)  $x(0) = 0$  we may deduce that  $c_1 = 0$ . By using the binomial expansion of Equation 29 and comparing with Equation 28 we find that,

$$\sum_{|t|=n} \alpha(t) w(t) = \prod_{i=1}^{n-1} (2i-1)$$

Note that  $\prod_{i=1}^{n-1} (2i-1) = (2n-3)!!$  hence using factorial identities and Stirling's approximation

$$\frac{(2n-3)!!}{(n-1)!} = \frac{(2n-3)!}{(n-1)! 2^{n-1}} \quad (30)$$

$$= b \quad (31)$$

Therefore,

$$\mathbb{E}_{\mathcal{T},n}(\sigma(t)) < blah$$

By Jensen's inequality,

$$\mathbb{E}_{\mathcal{T},n}(\sigma(t)^{\frac{1}{n}}) < blah^{\frac{n-1}{n}} = \text{constant as required}$$

## 6 A disproof of Conjecture 3.2

In this section we will disprove Conjecture 3.2 by demonstrating that there exists a symmetric subtree corresponding to a geometric factor  $H$  contained in the complex subgroup of the automorphism group such that the expectation of  $|H|^{\frac{1}{n}}$  over  $\mathcal{T}_n$  is

$$\mathbb{E}_{\mathcal{T}_n}(|H|^{\frac{1}{n}}) > 1$$

The symmetric subtree,  $\hat{t}$ , reticulated in Figure ?? on 7 vertices such that  $\text{Aut}(\hat{t}) \cong S_2 \wr S_2$  hence  $\sigma(\hat{t}) = 8$ .

Consider the inductive map,  $s$  (over the variety  $\mathcal{T}$ ) defined by a sequence  $f_n = \log(8)\delta_{n,7}$  and assume that  $s$  has EGF

$$S(z) = \sum_{t \in \mathcal{T}} s(t) \frac{z^{|t|}}{|t|!} \quad (32)$$

By Theorem ??  $S(z)$  can also be expressed as follows:

$$S(z) = Y'(z) \int_0^z \frac{F'(t)}{Y'(t)} dt$$

where  $Y(z) = \frac{1}{1-z}$  and  $F(z) = \frac{\log(8)t^7}{7}$ , thus

$$S(z) = \frac{\log(8)}{(1-z)} \int_0^z (1-t)t^6 dt$$

hence, for  $n \geq 8$ ,

$$S(z) = \sum_{n \geq 8} \frac{\log(8)z^n}{56} \quad (33)$$

Let  $\delta(t)$  denote the number of induced subtrees of some tree,  $t$ , that are order exactly 7. By comparing Equation 32 and Equation 33,

$$\sum_{t \in \mathcal{T}_n} \log(8)\delta(t) = \frac{n! \log(8)}{56}$$

Given a random recursive tree  $t$  let  $\epsilon(t)$  denote the number of induced subtrees of  $t$  that are isomorphic to  $\hat{t}$ . Since  $\alpha(\hat{t}) = 10$  and  $|\mathcal{T}_7| = 6!$  we may write,

$$\sum_{t \in \mathcal{T}_n} \log(8)\epsilon(t) = \frac{10 \log(8)n!}{8!}$$

Therefore the expected value,

$$\mathbb{E}_{\mathcal{T}_n}(\log(|H|^{\frac{1}{n}})) = \frac{1}{\mathcal{T}_n} \sum_{t \in \mathcal{T}_n} \log(8^{\frac{\epsilon(t)}{n}}) \quad (34)$$

$$= \frac{1}{(n-1)!} \sum_{t \in \mathcal{T}_n} \frac{\log(8)\epsilon(t)}{n} \quad (35)$$

$$= \frac{10 \log(8)}{8!} \quad (36)$$

Since exponential is a convex function,

$$1 < \exp\left(\frac{10 \log(8)}{8!}\right) \leq \mathbb{E}_{\mathcal{T}_n}\left(|H_i|^{\frac{1}{n}}\right) \leq \mathbb{E}_{\mathcal{T}_n}\left(|\mathcal{C}(t)|^{\frac{1}{n}}\right)$$

by Jensen's inequality; thus disproving Conjecture 3.2.

## 7 Conclusions

## References

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