

# 1 Introduction

## 2 Rooted Trees

A *rooted tree* is a triple  $t = (r, V, E)$ , such that  $(V, E)$  is a finite, simple connected graph without cycles with vertex set  $V$  and edge set  $E$ . A vertex,  $r \in V$ , is selected from  $V$  and called the *root*. By convention each edge is directed away from  $r$ . We denote the order of a tree,  $t$ , by  $|t|$  and the set of order  $n$  rooted trees is denoted  $\mathcal{R}_n$ . In addition we define  $R := \bigcup \mathcal{R}_n$ .

A *labelled rooted tree* is a quadruple  $t = (r, V, E, L)$  such that  $(r, V, E)$  is a rooted tree and

$$L : V \longrightarrow \{1, 2, \dots, |t|\}$$

is a bijective map. We denote the set of order  $n$  labelled rooted trees by  $\mathcal{L}_n$  and we define  $\mathcal{L} = \bigcup \mathcal{L}_n$ .

For any pair of vertices  $u$  and  $v$  of a rooted tree we write  $u \leq v$  if  $u$  lies on the unique shortest path from  $r$  to  $v$ . A *random recursive tree* is a quadruple  $t = (r, V, E, l)$  such that  $t$  is a labelled rooted tree and  $l$  is a labelling such that if  $u \leq v$  then  $l(u) < l(v)$ . We denote the set of order  $n$  random recursive trees by  $\mathcal{T}_n$  and we define  $\mathcal{T} = \bigcup \mathcal{T}_n$ .

Two rooted trees  $t_1$  and  $t_2$  with roots  $r_1$  and  $r_2$  and vertex sets  $V(t_1)$  and  $V(t_2)$  respectively are said to be isomorphic if there exists a bijection  $f : V(t_1) \rightarrow V(t_2)$  such that vertices  $u, v \in V(t_1)$  are adjacent if and only if  $f(u), f(v) \in V(t_2)$  are adjacent and  $f(r_1) = r_2$ . If  $V(t_1) = V(t_2)$  then  $f$  is called a rooted tree automorphism. The set of automorphisms of a tree,  $t$ , together with composition of maps forms a group denoted  $\text{Aut}(t)$ . The order of the automorphism group is

$$\sigma(t) := |\text{Aut}(t)|$$

Consider a map  $\phi : \mathcal{L} \rightarrow R$  that simply forgets the labels of a labelled by mapping

$$(r, V, E, L) \mapsto (r, V, E)$$

The map  $\phi$  is clearly surjective but not injective hence we define  $\beta(t) = |\phi^{-1}(t)|$  to be the number of possible isomorphism classes of labellings of a rooted tree  $t \in \mathcal{R}$ . There are  $|t|!$  possible labellings of a tree  $t \in \mathcal{R}$  hence there are

$$\beta(t) = \frac{|t|!}{\sigma(t)} \tag{1}$$

isomorphism classes of labellings of  $t$ .

Since  $\mathcal{T} \leq \mathcal{L}$  we denote by  $\psi$  the restriction of  $\phi$  to random recursive trees and define

$$\alpha(t) := |\psi^{-1}(t)|$$

however, to provide a result analogous to Equation 1 we require several additional definitions.

Let  $t_1, t_2, \dots, t_k$  be a forest of rooted trees: the rooted tree  $B^+(t_1, t_2, \dots, t_k)$  is built from this forest by introducing a new vertex,  $r$  (the root of  $B^+(t_1, t_2, \dots, t_k)$ ), and joining the roots of each tree  $t_1, t_2, \dots, t_k$  to  $r$  via an edge. Since every tree  $t$  can be written as  $B^+(t_1, t_2, \dots, t_k)$  if  $|t| > 1$  we henceforth assume that each rooted tree  $t = B^+(t_1, t_2, \dots, t_k)$  if  $|t| > 1$ . For convenience we will denote the rooted tree on 1 vertex by  $\bullet$ .

We have already seen one function  $f : \mathcal{R} \rightarrow \mathbb{R}$  on rooted trees ( $f(t) = |t|$ ), let us consider another: We define the tree factorial  $t!$  recursively

$$\bullet! = 1 \tag{2}$$

$$t! = |t| \prod_{i=1}^k t_i! \tag{3}$$

**Example 2.1.**

**Lemma 2.2.** For a rooted tree  $t$ ,

$$\alpha(t) = \frac{|t|!}{t! \sigma(t)} \tag{4}$$

*Proof.* There are  $n!$  ways of labelling a tree  $t \in \mathcal{R}_n$ , however if  $l$  is a random recursive labelling every induced subtree  $t_v$  and a totally ordered set,  $S$ , of labels there is precisely one possible label  $s \in S$  for vertex  $v$  (namely  $s = \min(S)$ ). Therefore the factor we should divide out by is precisely  $t!$ . In addition, to calculate the number of isomorphism classes of random recursive trees we must again divide out by  $\sigma(t)$ .  $\square$

Finally, let  $\chi : \mathcal{F} \rightarrow \mathcal{R}$  be the map that “forgets” the embedding of a rooted plane tree. It is clear that  $\chi$  is surjective but not injective hence we define

$$\gamma(t) = |\chi^{-1}|$$

the number of isomorphism classes of embeddings of a rooted tree  $t$ . In order to obtain a third relation analagous to Equations 1 and 4 we will describe a third function  $w : \mathcal{R} \rightarrow \mathbb{R}$  recursively:

$$w(\bullet) = 1 \tag{5}$$

$$w(t) = k! \prod_{i=1}^k w(t_i) \tag{6}$$

**Example 2.3.** Do the same example set as before

**Lemma 2.4.**

$$\gamma(t) = \frac{w(t)}{\sigma(t)} \tag{7}$$

*Proof.* Let  $B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$  denote a tree in which the root is incident to  $n_1$  isomorphic copies of a tree  $t_1$ ,  $n_2$  isomorphic copies of  $t_2$  and so forth. We remark that every rooted tree with at least 2 vertices can be written in this way. Let  $t \in \mathcal{R}$  and

consider an induced subtree  $t_v = B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$ . Vertex  $v$  contributes a factor of  $\left(\sum_{i=1}^k n_i\right)!$  to  $w(t)$  since it has a recursive definition. The order of automorphism group,  $\sigma(t)$  can also be expressed recursively:

$$\sigma(\bullet) = 1 \tag{8}$$

$$\sigma(t) = n_1!n_2!\dots n_k! \prod_{i=1}^k \sigma(t_i)^{n_i} \tag{9}$$

Therefore vertex  $v$  contribute a factor of  $\prod_{i=1}^k n_i!$  to  $\sigma(t)$ . Finally consider,  $\gamma(t)$ , the number of somorphism classes of embeddings of  $t$ . There are

$$\frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!}$$

non-isomorphic possibilities for the ordering of the children of  $v$ . □

### 3 Functions on Trees

In this section we will build up a 3-part tool kit of functions  $f : \mathcal{R} \rightarrow \mathbb{R}$  which will be used to calculate properties of rooted trees such as path length and order.

#### 3.1 Inductive Maps

**Definition 3.1.** Let  $s = \{s_r\}_{r=0}^\infty$  be a sequence such that each  $s_r \in \{0, 1\}$ . A *variety*,  $\mathcal{V}$ , of trees is a collection of random recursive trees such that each vertex is permitted to have outdegree  $r$  only if  $s_r = 1$ .

**Remark 1.** For a more general setting see [?].

The *degree function* associated with a sequence  $s = \{s_r\}_{r=0}^\infty$  is the exponential generating function (EGF) defined as follows:

$$\phi(w) = \sum_{r \geq 0} s_r \frac{w^r}{r!}$$

**Example 3.2.** The collection,  $\mathcal{B}$ , of increasing binary trees are the set of all random recursive trees such that each vertex has outdegree either 0,1 or 2. The degree function for increasing binay trees is

$$\phi(w) = 1 + w + \frac{w^2}{2}$$

The degree function for random recursive trees is

$$\phi(w) = \exp(w).$$

Fix a variety,  $\mathcal{V}$ , and define  $V_n$  to be the number of trees of order  $n$  in  $\mathcal{V}$ . The EGF of the variety of trees is

$$V_{\mathcal{V}}(z) = \sum_{n \geq 1} V_n \frac{z^n}{n!}$$

For example,

$$V_{\mathcal{T}}(z) = \sum_{n \geq 1} \frac{z^n}{n} \tag{10}$$

$$= \log \left( \frac{1}{1-z} \right) \tag{11}$$

since the number of random recursive trees of order  $n$  is  $(n-1)!$ .

**Definition 3.3.** Let  $f = \{f_n\}_{n \geq 1}$  be a sequence of real numbers. A function  $s : \mathcal{R} \rightarrow \mathbb{R}$  is called an *inductive map* if it is definable by a relation,

$$s(t) = f_{|t|} + \sum_{i=1}^k s(t_i)$$

where  $t = B^+(t_1, t_2, \dots, t_k)$ .

Given a tree function  $s$  and a variety  $\mathcal{V}$  the EGF of  $s$  over  $\mathcal{V}$  is

$$S(z) = \sum_{t \in \mathcal{V}} s(t) \frac{z^{|t|}}{|t|!}$$

**Example 3.4.** Let  $\delta_{i,j}$  be the usual kronecker delta. The following is a list of possible sequences  $f$  and the tree parameter measured by the corresponding function  $s$

- (i)  $f_n = 1$  for all  $n$  counts tree size.
- (ii)  $f_n = \delta_{n,i}$  counts the number of induced subtrees of order  $i$ .
- (iii)  $f_n = n$  counts path lengths.

In order to calculate  $S(z)$  easily and effectively we appeal to the following Theorem of Bergeron [?].

**Theorem 3.5.**

$$S(z) = Y'(z) \int_0^z \frac{F'(t)}{Y'(t)} dt$$

where  $F(z)$  is defined from  $\mathcal{V}$  and the sequence  $f$

$$F(z) = \sum_{n \geq 0} f_n V_n \frac{z^n}{n!}$$

### 3.2 Butcher Series

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a suitably differentiable function and consider differential equations of the form

$$\dot{y} = f(y)$$

and  $y(s_0) = y_0$ . Note that  $f'(y)$  is a linear map (the Jacobian),  $f''(y)$  is a bilinear map and so on.

**Definition 3.6.** Let  $t = B^+(t_1, t_2, \dots, t_k)$  be a rooted tree. An *elementary differential* is a map  $F(t) : \mathbb{R} \rightarrow \mathbb{R}$  defined recursively:

$$F(\bullet)(s) = f(s) \tag{12}$$

$$F(t)(s) = f^m(y)(F(t_1)(y), \dots, F(t_k)(y)) \tag{13}$$

Let

$$G(s) = \sum_{n=0}^{\infty} f^{(n)}(s_0) \frac{s^n}{n!}$$

**Theorem 1.** *The solution of*

$$x(s) = x_0 + \int_{s_0}^s G(x(s')) ds'$$

is

$$x(s) = x_0 + \sum \frac{(s - s_0)^{|t|}}{|t|!} \alpha(t) F(t)(s_0)$$

**Example 3.7.** Let  $x_0 = s_0 = 0$  and consider the case  $f^k(t)(0) = k!$  for all trees  $t \in R$ , then

$$f^{(k)}(t)(0) = k!(F(t_1)(0), \dots, F(t_k)(0))$$

thus  $G(s) = \frac{1}{(1-s)}$  and  $F(t)(s) = w(t)$ .

### 3.3 Bare Green Functions

## 4 Rooted Tree Automorphisms

## 5 Applications of Tree Functions