1 Introduction

2 Rooted Trees

A *rooted tree* is a triple $t = (r, V, E)$, such that (V, E) is a finite, simple connected graph without cycles with vertex set V and edge set E. A vertex, $r \in V$, is selected from V and called the *root*. By convention each edge is directed away from r. We denote the order of a tree, t, by $|t|$ and the set of order n rooted trees is denoted $\mathcal{R}_n$. In addition we define $R := \bigcup R_n$.

A *labelled rooted tree* is a quadruple $t = (r, V, E, L)$ such that (r, V, E) is a rooted tree and

$$L : V \longrightarrow \{1, 2, \ldots, |t|\}$$

is a bijective map. We denote the set of order n labelled rooted trees by $\mathcal{L}_n$ and we define $\mathcal{L} = \bigcup \mathcal{L}_n$.

For any pair of vertices u and v of a rooted tree we write $u \leq v$ if u lies on the unique shortest path from r to v. A *random recursive tree* is a quadruple $t = (r, V, E, l)$ such that t is a labelled rooted tree and l is a labelling such that if $u \leq v$ then $l(u) < l(v)$. We denote the set of order n random recursive trees by $\mathcal{T}_n$ and we define $\mathcal{T} = \bigcup \mathcal{T}_n$.

Two rooted trees t_1 and t_2 with roots r_1 and r_2 and vertex sets $V(t_1)$ and $V(t_2)$ respectively are said to be isomorphic if there exists a bijection $f : V(t_1) \to V(t_2)$ such that vertices $u, v \in V(t_1)$ are adjacent if and only if $f(u), f(v) \in V(t_2)$ are adjacent and $f(r_1) = r_2$. If $V(t_1) = V(t_2)$ then f is called a rooted tree automorphism. The set of automorphisms of a tree, t, together with composition of maps forms a group denoted $\mathrm{Aut}(t)$. The order of the automorphsim group is

$$\sigma(t) := |\mathrm{Aut}(t)|$$

Consider a map $\phi : \mathcal{L} \to R$ that simply forgets the labels of a labelled by mapping

$$(r, V, E, L) \mapsto (r, V, E)$$

The map ϕ is clearly surjective but not injective hence we define $\beta(t) = |\phi^{-1}(t)|$ to be the number of possible isomorphism classes of labellings of a rooted tree $t \in \mathcal{R}$. There are $|t|!$ possible labellings of a tree $t \in \mathcal{R}$ hence there are

$$\beta(t) = \frac{|t|!}{\sigma(t)} \tag{1}$$

isomorphism classes of labellings of t.

Since $\mathcal{T} \leq \mathcal{L}$ we denote by ψ the restriction of ϕ to random recursive trees and define

$$\alpha(t) := |\psi^{-1}(t)|$$

however, to provide a result analagous to Equation 1 we require several additional definitions.

Let $t_1, t_2, \ldots, t_k$ be a forest of rooted trees: the rooted tree $B^+(t_1, t_2, \ldots, t_k)$ is built from this forest by introducing a new vertex, r (the root of $B^+(t_1, t_2, \ldots, t_k)$), and joining the roots of each tree $t_1, t_2, \ldots, t_k$ to r via an edge. Since every tree t can be written as $B^+(t_1, t_2, \ldots, t_k)$ if $|t| > 1$ we henceforth assume that each rooted tree $t = B^+(t_1, t_2, \ldots, t_k)$ if $|t| > 1$. For conveniance we will denote the rooted tree on 1 vertex by $\bullet$.

We have already seen one function $f : \mathcal{R} \to \mathbb{R}$ on rooted trees ($f(t) = |t|$), let us consider another: We define the tree factorial $t!$ recursively

$$\bullet! = 1 \tag{2}$$

$$t! = |t| \prod_{i=1}^{k} t_i! \tag{3}$$

Example 2.1.

Lemma 2.2. *For a rooted tree t,*

$$\alpha(t) = \frac{|t|!}{t!\sigma(t)} \tag{4}$$

Proof. There are $n!$ ways of labelling a tree $t \in \mathcal{R}_n$, however if l is a random recursive labelling every induced subtree t_v and a totally ordered set, S, of labels there is precisely one possible label $s \in S$ for vertex v (namely $s = \min(S)$). Therefore the factor we should divide out by is precisely $t!$. In addition, to calculate the number of isomorphism classes of random recursive trees we must again divide out by $\sigma(t)$. $\square$

Finally, let $\chi : \mathcal{F} \to \mathcal{R}$ be the map that "forgets" the embedding of a rooted plane tree. It is clear that χ is surjective but not injective hence we define

$$\gamma(t) = |\chi^{-1}|$$

the number of isomorphism classes of embeddings of a rooted tree t. In order to obtain a third relation analagous to Equations 1 and 4 we will describe a third function $w : \mathcal{R} \to \mathbb{R}$ recursively:

$$w(\bullet) = 1 \tag{5}$$

$$w(t) = k! \prod_{i=1}^{k} w(t_i) \tag{6}$$

Example 2.3. Do the same example set as before

Lemma 2.4.

$$\gamma(t) = \frac{w(t)}{\sigma(t)} \tag{7}$$

Proof. Let $B^+(t_1^{n_1}, t_2^{n_2}, \ldots, t_k^{n_k})$ denote a tree in which the root is incident to n_1 isomorphic copies of a tree t_1, n_2 isomorphic copies of t_2 and so forth. We remark that every rooted tree with at least 2 vertices can be written in this way. Let $t \in R$ and consider an induced subtree $t_v = B^+(t_1^{n_1}, t_2^{n_2}, \dots, t_k^{n_k})$. Vertex v contributes a factor of $\left(\sum_{i=1}^k n_i\right)!$ to $w(t)$ since it has a recursive definition. The order of automorphism group, $\sigma(t)$ can also be expressed recursively:

$$\sigma(\bullet) = 1 \tag{8}$$

$$\sigma(t) = n_1!n_2!\dots n_k! \prod_{i=1}^{k} \sigma(t_i)^{n_i} \tag{9}$$

Therefore vertex v contribute a factor of $\prod_{i=1}^k n_i!$ to $\sigma(t)$. Finally consider, $\gamma(t)$, the number of somorphism classes of embeddings of t. There are

$$\frac{\left(\sum_{i=1}^k n_i\right)!}{\prod_{i=1}^k n_i!}$$

non-isomorphic possibilities for the ordering of the children of v. □

3 Functions on Trees

3.1 Inductive Maps

Definition 3.1. Let $s = \{s_r\}_{r=0}^{\infty}$ be a sequence such that each $s_r \in \{0, 1\}$. The variety, $\mathcal{V}$, of trees is the collection of random recursive trees such that a vertex is permitted to have outdegree r only if $s_r = 1$ for each r.

Remark 1. *For a more general setting see [**?**].*

The *degree function* associated with a sequence s is the exponential generating function (EGF) defined as follows:

$$\phi(w) = \sum_{r \geq 0} s_r \frac{w^r}{r!}$$

Example 3.2. The collection, $\mathcal{B}$, of increasing binary trees are the subset of random recursive trees such that all vertices have outdegree either 0,1 or 2. The degree function for increasing binay trees is

$$\phi(w) = 1 + w + \frac{w^2}{2}$$

The degree function for random recursive trees is

$$\phi(w) = \exp(w).$$

Fix a variety of trees, $\mathcal{V}$ and let V_n be the number of trees of order n in the variety. The EGF of the variety of trees is

$$V_{\mathcal{V}}(z) = \sum_{n \geq 1} V_n \frac{z^n}{n!}$$

For example,

$$V_{\mathcal{T}}(z) = \sum_{n \geq 1} \frac{z^n}{n} \tag{10}$$

$$= \log\left(\frac{1}{1-z}\right) \tag{11}$$

since the number of random recursive trees of order n is $(n-1)!$.

Definition 3.3. Let $f = \{f_n\}_{n\geq 1}$ be a sequence of real numbers. A function $s : \mathcal{R} \to \mathbb{R}$ is called an *inductive map* if it is defineable by a relation,

$$s(t) = f_{|t|} + \sum_{i=1}^{k} s(t_i)$$

where $t = B^+(t_1, t_2, \ldots, t_k)$. Given a tree function s and a variety $\mathcal{V}$ the EGF of s over $\mathcal{V}$ is

$$S(z) = \sum_{t \in \mathcal{V}} s(t) \frac{z^{|t|}}{|t|!}$$

Example 3.4. Let $\delta_{i,j}$ be the usual kronecker delta. The following is a list of possible sequences f and the tree paramater measured by the corresponding function s:

(i) $f_n = 1$ for all n counts tree size.

(ii) $f_n = \delta_{n,i}$ counts the number of induced subtrees of order i.

(iii) $f_n = n$ counts path lengths.

In order to calculate $S(z)$ easily and effectively we appeal to the following Theorem of Bergeron [**?**].

Theorem 3.5.

$$S(z) = Y'(z) \int_0^z \frac{F'(t)}{Y'(t)} dt$$

where $F(z)$ is defined from $\mathcal{V}$ and the sequence f

$$F(z) = \sum_{n \geq 0} f_n V_n \frac{z^n}{n!}$$