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**Models for Axiomatic Type Theory** 

Daniël Otten and Matteo Spadetto

### **Contents**

We explain and motivate Axiomatic Type Theory (ATT). (type theory without reductions)

We compare two semantics for a minimal version of ATT:

- comprehension categories: more traditional and well-studied closely follow the syntax and intricacies of type theory.
- path categories (Van den Berg, Moerdijk 2017): more concise take inspiration from homotopy theory.

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Path categories are equivalent to certain comprehension categories. This allows us to turn path categories into actual models as well.

We introduce a more fine-grained notion: display path categories, and show a similar equivalence.

We obtain the following diagram of 2-categories:

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Models for Axiomatic Type Theory

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Axiomatic Type Theory

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Axiomatic Type Theory

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#### Larger

If we define

$$0+n \equiv n,$$
  
 $(Sm)+n \equiv S(m+n),$   
 $m+0 = m,$   
 $m+(Sm)=S(m+n),$ 

then we can prove

But these proven eq are not definitional.

Agda allows you to make them definitional.

- Cubical Type Theory: only propositional  $\beta$ -rule for =-types.
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# Complexity and Conservativity

The complexity of type checking:

■ ETT: undecidable

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- ITT: nonelementary,
- ATT: quadratic

Does ETT prove more than ATT? Yes, namely

- binder extensionality (bindext).
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$$\begin{array}{c} \Gamma, x, x': A, p: x =_A x' \vdash C \text{ type} \\ \frac{\Gamma, x: A \vdash d: C[x/x', \operatorname{refl}_x/p]}{\Gamma, x, x': A, p: x =_A x' \vdash \operatorname{ind}_{C,d,p}^{\equiv}: C} (=\mathcal{E}), \end{array}$$

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# Minimal ATT

Lets start by considering the normal rules for =-types:

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## Without $\Pi$ -types, we have to strengthen the rules:

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# Minimal ATT

In ATT, we change the reduction to an axiom:

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Axiomatic Type Theory

#### How do we model this minimal ATT

- Follow the syntax and rules. (comprehension category)
  - We require:  $=_A$ , refl<sub>A</sub>, ind $^=_{C,c,v}$ , and  $\beta^=_{G,c,v}$ .
- Use intuition from homotopy theory. (path category)
  - $\,\,\,\,\,\,\,\,\,$  We require:  $\,=_A$ , refl $_A$ , and that refl $_A$  is an equivalence

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- lacksquare a category of contexts with terminal object  $\epsilon$
- a category of types
- for every type A a context map  $p_A : \Gamma.A \to \Gamma$ . (display map)
- for every type A in context  $\Gamma$  and context map  $\sigma:\Delta\to\Gamma$ , a type  $A[\sigma]$  in context  $\Delta$ . (substitution
- some other requirements

The terms of A are the maps  $a:\Gamma\to\Gamma.A$  such that  $p_A\circ a=\mathrm{id}_\Gamma$ 

- =-types: for A a type  $=_A$  and terms  $\operatorname{refl}_A$ ,  $\operatorname{ind}_{A,C,d}^=$ ,  $\beta_{A,C,d}^=$
- weak stability: for  $\sigma$  we have that  $=_A[\sigma]$  is also an =-type

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A path category is a category C with two classes of maps:

- fibrations: closed under pullbacks and compositions,
- (weak) equivalences: satisfying 2-out-of-6, so, if we have

$$A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \stackrel{h}{\longrightarrow} D$$

where  $g \circ f$  and  $h \circ g$  are equivalences, then f, g, h, and  $h \circ g \circ h$  are equivalences.

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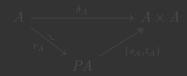
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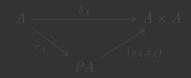
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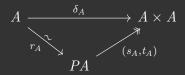
## Path Objects

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### **Homotopy Theory**

We call two maps  $f,g:A\to B$  homotopic, written  $f\simeq g$ , if there exists a map  $h:A\to PB$  such that  $s_B\circ h=f$  and  $t_B\circ h=g$ .

We call  $f:A\to B$  an homotopy equivalence, if there exists a map  $g:B\to A$  such that  $g\circ f\simeq \mathrm{id}_A$  and  $f\circ g\simeq \mathrm{id}_B$ .

The homotopy equivalences are precisely the weak equivalences

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### **Path Category** → **Comprehension Category**

We can view a path category  $\mathcal{C}$  as a comprehension category:

- the contexts are given by  $\mathcal{C}$
- the types are given by the full subcategory  $\mathcal{C}^{\mathsf{fib}} \subseteq \mathcal{C}^{\to}$ ,
- the display map for  $p \in \mathcal{C}^{\mathrm{fib}}$  is p itself,
- the substitution  $p[\sigma]$  is the pullback  $\sigma^*p$ .

- weakly stable =-types,
- weakly stable  $\Sigma$ -types with  $\beta$  and  $\eta$  reductions,
- contextuality (contexts are finite)

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We can turn a comprehension category C with weakly stable =  $\sum_{\beta,n}$  and contextuality into a path category by taking:

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#### References

- Benno van den Berg and leke Moerdijk (2017): Exact completion of path categories.
- Benno van den Berg (2018): Path categories and propositional identity types.
- Rafaël Bocquet (2020): Coherence of strict equalities in dependent type theories.
- Rafaël Bocquet (2021): Strictification of weakly stable type-theoretic structures using generic contexts.
- Martin Hofmann (1995): On the interpretation of type theory in locally Cartesian closed categories.
- Peter Lumsdaine and Michael Warren (2014): The local universes model, an overlooked coherence construction for dependent type theories.
- Nicolas Oury (2005): Extensionality in the calculus of constructions.
- Matteo Spadetto (2023): A conservativity result for homotopy elementary types in dependent type theory.
- Theo Winterhalter (2019): Formalisation and meta-theory of type theory.