CHAPTER

NINE

DISCRETE NUMERIC FUNCTIONS AND GENERATING FUNCTIONS

9.1 INTRODUCTION

We recall that a function is a binary relation that assigns to each element in the domain a unique value which is an element in the range. In this and the next chapter, we shall study the class of functions whose domain is the set of natural numbers and whose range is the set of real numbers. We shall refer to these functions as *discrete numeric functions* or, briefly, *numeric functions.* Numeric functions are of particular interest to us because we encounter them very often in digital computation.

We shall use boldface lower case letters to denote numeric functions. For a numeric function **a** we use $a_0, a_1, a_2, \ldots, a_r, \ldots$ to denote the values of the function at $0, 1, 2, \ldots, r, \ldots$.†

Although, in principle, we can specify a numeric function by exhaustively listing its values $(a_0, a_1, a_2, \ldots)$, in practice we need to use a representation that is not infinitely long. Thus,

$$a_r = 7r^3 + 1 \qquad r \geqq 0$$

$$b_r = \begin{cases} 2r & 0 \leqq r \leqq 11 \\ 3^r - 1 & r \geqq 11 \end{cases}$$

$$c_r = \begin{cases} -4 & r = 3, 5, 7 \\ 0 & \text{otherwise} \end{cases}$$

$$d_r = \begin{cases} 2 + r & 0 \leqq r \leqq 5 \\ 2 - r & r > 5 \quad r \text{ is odd} \\ 2/r & r > 5 \quad r \text{ is even} \end{cases}$$

† This is merely a simpler notation than what we introduced in Sec. 4.8, where we used **a**(0), **a**(1), **a**(2), ..., **a**(*r*), ...

are examples of concise specification of numeric functions. When there is one simple expression for the value of the numeric function for every r such as a_r above, we shall also use the notation

$$\mathbf{a} = 7r^3 + 1$$

Let us consider the following examples:

Example 9.1 Suppose we deposit \$100 in a savings account at an interest rate of 7 percent per year, compounded annually. At the end of the first year, the total amount in the account is \$107; at the end of the second year, the total amount in the account is \$114.49; at the end of the third year, the total amount in the account is \$122.50, and so on. The amount in the account at the end of each year can be described by a numeric function **a**, which can either be specified as (100, 107, 114.49, 122.50, ...) or as

$$a_r = 100(1.07)^r \qquad r \geqq 0$$

or as

$$\mathbf{a} = 100(1.07)^r$$

□

Example 9.2 Let a_r denote the altitude of an aircraft, in thousands of feet, at the rth minute. Suppose the aircraft takes off after spending 10 minutes on the ground, climbs up at a uniform speed to a cruising altitude of 30,000 feet in 10 minutes, starts to descend uniformly after 110 minutes of flying time, and lands 10 minutes later. We have

$$a_r = \begin{cases} 0 & 0 \leqq r \leqq 10 \\ 3(r-10) & 11 \leqq r \leqq 20 \\ 30 & 21 \leqq r \leqq 120 \\ 3(130-r) & 121 \leqq r \leqq 130 \\ 0 & r \geqq 131 \end{cases}$$

□

9.2 MANIPULATION OF NUMERIC FUNCTIONS

The *sum* of two numeric functions is a numeric function whose value at r is equal to the sum of the values of the two numeric functions at r. The *product* of two numeric functions is a numeric function whose value at r is equal to the product of the values of the two numeric functions at r. For example, consider the two functions **a** and **b** where

$$a_r = \begin{cases} 0 & 0 \leqq r \leqq 2 \\ 2^{-r} + 5 & r \geqq 3 \end{cases}$$

and

$$b_r = \begin{cases} 3 - 2^r & 0 \leqq r \leqq 1 \\ r + 2 & r \geqq 2 \end{cases}$$

Let **c** be the sum of the two functions **a** and **b**, which we shall denote **a** + **b**. Then

$$c_r = a_r + b_r = \begin{cases} 3 - 2^r & 0 \leqq r \leqq 1 \\ 4 & r = 2 \\ 2^{-r} + r + 7 & r \geqq 3 \end{cases}$$

Let **d** be the product of the two functions **a** and **b**, which we shall denote **ab**. Then

$$d_r = a_r b_r = \begin{cases} 0 & 0 \leqq r \leqq 2 \\ r \cdot 2^{-r} + 2^{-r+1} + 5r + 10 & r \geqq 3 \end{cases}$$

For example, if **a** denotes the monthly income of a husband and **b** denotes the monthly income of his wife, then **a** + **b** will be their joint monthly income. Let **a** denote the balance in a savings account in each month and **b** denote the monthly interest rate, which fluctuates from month to month. Then **ab** will be the interest earned in each month.

Let **a** be a numeric function and α be a real number. We use α**a** to denote a numeric function whose value at r is equal to α times a_r. The numeric function α**a** is called a *scaled version* of **a** with *scaling factor* α. For example, let **a** be a numeric function whose value at r is $(1.07)^r$. Then 100**a** is a numeric function whose value at r is $100(1.07)^r$. Clearly, if **a** describes the total amount in a savings account through the years for an initial deposit of \$1, then 100**a** describes the total amount in the account for an initial deposit of \$100.

Let **a** be a numeric function. We use $|\mathbf{a}|$ to denote a numeric function whose value at r is equal to a_r if a_r is nonnegative, and is equal to $-a_r$ if a_r is negative. For example, let

$$a_r = (-1)^r \left(\frac{2}{r^2}\right) \qquad r \geqq 0$$

Let **b** be $|\mathbf{a}|$, then

$$b_r = \frac{2}{r^2} \qquad r \geqq 0$$

For example, let **a** be a numeric function where a_r is the difference between the output voltage of the electrical power supply after r hours of operation and a nominal reference voltage. Thus, positive a_r means that the voltage after r hours of operation is above the reference voltage, while negative a_r means that the voltage after r hours of operation is below the reference voltage. Thus, $|\mathbf{a}|$ gives the deviation of the output voltage from the reference voltage.

Let **a** be a numeric function and i a positive integer. We use S^i**a** to denote a numeric function such that its value at r is 0 for $r = 0, 1, 2, \ldots, i - 1$ and is a_{r-i} for $r \geqq i$. For example, let

$$a_r = \begin{cases} 1 & 0 \leqq r \leqq 10 \\ 2 & r \geqq 11 \end{cases}$$

Let **b** be $S^5\mathbf{a}$. Then

$$b_r = \begin{cases} 0 & 0 \leqq r \leqq 4 \\ 1 & 5 \leqq r \leqq 15 \\ 2 & r \geqq 16 \end{cases}$$

As another example, let **a** describe the altitude of an aircraft as given in the foregoing. Then $S^{17}\mathbf{a}$ describes the altitude when takeoff is delayed by 17 minutes.

Let **a** be a numeric function and i a positive integer. We use $S^{-i}\mathbf{a}$ to denote a numeric function such that its value at r is a_{r+i} for $r \geqq 0$. For example, let

$$a_r = \begin{cases} 1 & 0 \leqq r \leqq 10 \\ 2 & r \geqq 11 \end{cases}$$

Let **b** be $S^{-7}\mathbf{a}$. Then

$$b_r = \begin{cases} 1 & 0 \leqq r \leqq 3 \\ 2 & r \geqq 4 \end{cases}$$

As another example, let **a** be a numeric function where a_r is the number of applicants whose height is r inches above the minimum height requirement to join the Army. It follows that the numeric function $S^{-2}\mathbf{a}$ describes the numbers of applicants in the height groups if two inches have been added to the minimum height requirement.

The *accumulated sum* of a numeric function **a** is a numeric function whose value at r is equal to $\sum_{i=0}^{r} a_i$. For example, let **a** describe the monthly earnings of an employee. Let **b** be the accumulated sum of **a**. Then **b** gives her accumulated earnings by month. As another example, let

$$a_r = 100(1.07)^r \qquad r \geqq 0$$

Let **b** be the accumulated sum of **a**. Then

$$b_r = \sum_{i=0}^{r} a_i = \sum_{i=0}^{r} 100(1.07)^i = \frac{10{,}000}{7}\,[(1.07)^{r+1} - 1] \qquad r \geqq 0$$

Note that if we deposit \$100 each year in a savings account at an annual compounded interest rate of 7 percent, then b_r is the total amount we have in the account after r years.

The *forward difference* of a numeric function **a** is a numeric function, denoted $\Delta\mathbf{a}$, whose value at r is equal to $a_{r+1} - a_r$. The *backward difference* of a numeric function **a** is a numeric function, denoted $\nabla\mathbf{a}$, whose value is equal to a_0 at 0 and is equal to $a_r - a_{r-1}$ at $r \geqq 1$. Thus, if **a** describes the total business income of a company in each month, $\Delta\mathbf{a}$ will describe the increase of income from the rth month to the $(r + 1)$st month and $\nabla\mathbf{a}$ will describe the increase of income for the rth month over the $(r - 1)$st month. For example, let **a** be a numeric function such that

$$a_r = \begin{cases} 0 & 0 \leqq r \leqq 2 \\ 2^{-r} + 5 & r \geqq 3 \end{cases}$$

Let **b** denote the forward difference of **a**, $\Delta\mathbf{a}$; then

$$b_r = \begin{cases} 0 & 0 \leqq r \leqq 1 \\ \frac{41}{8} & r = 2 \\ -2^{-(r+1)} & r \geqq 3 \end{cases}$$

Let **c** denote the backward difference of **a**, $\nabla\mathbf{a}$; then

$$c_r = \begin{cases} 0 & 0 \leqq r \leqq 2 \\ \frac{41}{8} & r = 3 \\ -2^{-r} & r \geqq 4 \end{cases}$$

We note that $S^{-1}(\nabla\mathbf{a}) = \Delta\mathbf{a}$.

Let **a** and **b** be two numeric functions. The *convolution* of **a** and **b**, denoted $\mathbf{a} * \mathbf{b}$, is a numeric function **c** such that

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_i b_{r-i} + \cdots + a_{r-1} b_1 + a_r b_0 = \sum_{i=0}^{r} a_i b_{r-i}$$

For example, let **a** and **b** be the two numeric functions such that

$$a_r = 3^r \qquad r \geqq 0$$

and

$$b_r = 2^r \qquad r \geqq 0$$

Then, for $\mathbf{c} = \mathbf{a} * \mathbf{b}$

$$c_r = \sum_{i=0}^{r} 3^i 2^{r-i} \qquad r \geqq 0$$

Example 9.3 Consider the problem of determining c_r, the number of sequences of length r that are made up of the letters $\{x, y, z, \alpha, \beta\}$, with the first portion of each sequence made up of English letters and the second portion made up of Greek letters. Since

$$a_r = 3^r$$

is the number of sequences of length r made up of the letters $\{x, y, z\}$ and

$$b_r = 2^r$$

is the number of sequences of length r made up of the letters $\{\alpha, \beta\}$,

$$c_r = \sum_{i=0}^{r} 3^i 2^{r-i}$$

is the answer to our question. □

Example 9.4 Consider again the example of a savings account that receives annually compounded interest at a rate of 7 percent per year. Suppose we

deposit \$100 in the account at the beginning, \$110 at the end of the first year, \$120 at the end of the second year, \$100(1 + 0.1r) at the end of the rth year, and so on. We want to know the total amount in the account at the end of the rth year. Let the numeric function **a** describe the yearly deposit, that is,

$$a_r = 100(1 + 0.1r) \qquad r \geqq 0$$

Let **b** be a numeric function such that

$$b_r = (1.07)^r \qquad r \geqq 0$$

Note that b_r is the amount in the savings account at the end of the rth year if \$1 was deposited into the account at the beginning (or at the end of the 0th year). Thus, if a_i dollars was deposited in the savings account at the end of the ith year, it will become $a_i b_{r-i}$ dollars $r - i$ years later (or at the end of the rth year). It follows that

$$c_r = \sum_{i=0}^{r} a_i b_{r-i}$$

is the total amount in the savings account at the end of the rth year. □

Example 9.5 We consider the financial operation of a company. For every \$100,000 of orders it receives, the company must borrow \$70,000 as working capital for materials, machinery, wages, and so on. In particular, the company will need \$30,000 immediately, \$20,000 in the next month, and \$5,000 in each of the subsequent four months. Thus, the needs for new working capital, in thousands of dollars, can be described by a numeric function **a** such that

$$a_r = \begin{cases} 30 & r = 0 \\ 20 & r = 1 \\ 5 & 2 \leqq r \leqq 5 \\ 0 & r \geqq 6 \end{cases}$$

For every \$1000 the company borrows, it must repay a total of \$1100 in 11 installments of \$100 each for 11 months starting with the month after the money was borrowed. Let **b** be a numeric function describing the repayment schedule, in thousands of dollars. We have

$$b_r = \begin{cases} 0 & r = 0 \\ 0.1 & r = 1, 2, \ldots, 11 \\ 0 & r \geqq 12 \end{cases}$$

Suppose the company has total orders of \$200,000 in each of the first 10 months and total orders of \$300,000 in subsequent months. We want to know the total repayment in the rth month. Let **c** be a numeric function such that

$$c_r = \begin{cases} 2 & 0 \leqq r \leqq 9 \\ 3 & r \geqq 10 \end{cases}$$

represents the orders received in the rth month, in hundreds of thousands of dollars. We note that $\mathbf{a} * \mathbf{b}$ will give the monthly repayment corresponding to a single order of \$100,000 at the beginning. We ask the reader to check that $\mathbf{a} * \mathbf{b} = (0, 3, 5, 5.5, 6, 6.5, 7, 7, 7, 7, 7, 7, 4, 2, 1.5, 1, .5, 0, 0, \ldots)$. Consequently, $\mathbf{c} * (\mathbf{a} * \mathbf{b})$ will give the repayment schedule of all debts in thousands of dollars.† Again, we ask the reader to check that $\mathbf{c} * (\mathbf{a} * \mathbf{b}) = (0, 6, 16, 27, \ldots)$. □

9.3 ASYMPTOTIC BEHAVIOR OF NUMERIC FUNCTIONS

Let **a** be a numeric function describing the market value of a piece of farm land we purchased. In particular, let a_r be the value of the land after r years. If we purchased the farm land for long-term investment, we would be most interested in how a_r increases or decreases for large r. Let **b** be a numeric function in which b_r is the time it takes for a computer to process the payroll of r employees. To assess the efficiency of the data processing operation, we are mostly interested in knowing the amount of time it takes to process payrolls with large numbers of employees, since when there is only a small number of employees, the total processing time will not be prohibitively long under any circumstance and, consequently, is not a serious concern. Let **c** be a numeric function in which c_r is the time it takes for a certain sorting algorithm to sort r numbers. We are interested in how c_r increases when we want to use the algorithm to sort tens of thousands of names in the preparation of a telephone directory. By the *asymptotic behavior* of a numeric function, we mean how the value of the function varies for large r. For example, for

$$a_r = 5 \qquad r \geqq 0$$

the value of the numeric function remains constant for increasing r. For

$$b_r = 5r^2 \qquad r \geqq 0$$

the value of the function increases for increasing r and is proportional to r^2, while for

$$c_r = 5 \log r \qquad r \geqq 0$$

the value of the function increases for increasing r and is proportional to $\log r$. Finally, for

$$d_r = \frac{5}{r} \qquad r \geqq 0$$

the value of the function decreases for increasing r, is proportional to $1/r$, and approaches 0 as a limit.

In many occasions, we are interested in comparing the asymptotic behavior of two numeric functions. To this end, we introduce the notion of *asymptotic dominance*. Let **a** and **b** be numeric functions. We say that **a** *asymptotically*

† We ask the reader to ascertain that expressing a_r and b_r in thousands of dollars and c_r in hundreds of thousands of dollars will yield the repayment schedule in thousands of dollars.

dominates **b**, or **b** is *asymptotically dominated* by **a**, if there exist positive constants k and m such that†

$$|b_r| \leqq ma_r \qquad \text{for } r \geqq k$$

For example, let **a** and **b** be two numeric functions such that

$$a_r = r + 1 \qquad r \geqq 0$$

$$b_r = \frac{1}{r} + 7 \qquad r \geqq 0$$

Then **a** asymptotically dominates **b** because, for $k = 7$ and $m = 1$,

$$|b_r| \leqq a_r \qquad \text{for } r \geqq 7$$

On the other hand, **b** does not dominate **a** asymptotically, since for any choice of k and m, there exists r_0 such that $r_0 \geqq k$ and $|a_{r_0}| > mb_{r_0}$. As another example, let **a** and **b** be two numeric functions such that

$$a_r = \frac{1}{100} r^2 - 1000$$

$$b_r = \begin{cases} -1{,}000{,}000 & 0 \leqq r \leqq 10 \\ -100{,}000r & r > 10 \end{cases}$$

Because for $k = 10^6 + 1$ and $m = 10$,

$$|b_r| \leqq 10a_r \qquad \text{for } r \geqq 10^6 + 1$$

we can conclude that **a** asymptotically dominates **b**.

Intuitively, that **a** asymptotically dominates **b** means that **a** grows faster than **b**. Thus, for sufficiently large r, the absolute value of b_r does not exceed a fixed proportion of a_r. For example, suppose we deposit \$1 in two separate bank accounts. After r years, our total deposit in account A becomes $1 + 0.2r$ dollars, and our total deposit in account B becomes $(1 + 0.03)^{4r}$ dollars.‡ Let

$$a_r = 1 + 0.2r \qquad r \geqq 0$$

$$b_r = (1 + 0.03)^{4r} \qquad r \geqq 0$$

We note that **b** asymptotically dominates **a** since, for $k = 9$ and $m = 1$,

$$|1 + 0.2r| \leqq (1 + 0.03)^{4r} \qquad \text{for } r \geqq 9$$

That is, the growth rate of our money in account B is higher than that in account A. Although in the first few years the total in account B is less than that in account A, the total in account B exceeds that in account A in the long run.

We leave the proofs of the following statements to the reader:

† Some authors use the condition $|b_r| \leqq m|a_r|$ for $r \geqq k$ instead. However, as the reader will see later on, it is meaningful to insist that a_r be positive.

‡ That is, account A pays 20 percent simple annual interest (a somewhat outdated banking practice), and account B pays 12 percent annual interest that is compounded quarterly.

1. For any numeric function **a**, $|\mathbf{a}|$ asymptotically dominates **a**.
2. If **b** is asymptotically dominated by **a**, then for any constant α, $\alpha\mathbf{b}$ is also asymptotically dominated by **a**.
3. If **b** is asymptotically dominated by **a**, then for any integer i, $S^i\mathbf{b}$ is asymptotically dominated by $S^i\mathbf{a}$.
4. If both **b** and **c** are asymptotically dominated by **a**, then for any constants α and β, $\alpha\mathbf{b} + \beta\mathbf{c}$ is also asymptotically dominated by **a**.
5. If **c** is asymptotically dominated by **b** and **b** is asymptotically dominated by **a**, then **c** is asymptotically dominated by **a**.
6. It is possible that **a** asymptotically dominates **b**, and **b** also asymptotically dominates **a**. For example, let

$$a_r = r^2 + r + 1 \qquad r \geqq 0$$

$$b_r = 0.05r^2 - r^{1/3} - 9 \qquad r \geqq 0$$

7. It is possible that **a** does not asymptotically dominate **b**, nor does **b** asymptotically dominate **a**. For example, let

$$a_r = \begin{cases} 1 & r \text{ is even} \\ 0 & r \text{ is odd} \end{cases}$$

$$b_r = \begin{cases} 0 & r \text{ is even} \\ 1 & r \text{ is odd} \end{cases}$$

8. It is possible that both **a** and **b** asymptotically dominate **c**, while **a** does not asymptotically dominate **b**, nor does **b** asymptotically dominate **a**. For example, let

$$a_r = \begin{cases} 1 & r = 3i \text{ or } 3i + 1 \\ 0 & r = 3i + 2 \end{cases}$$

$$b_r = \begin{cases} 1 & r = 3i \text{ or } 3i + 2 \\ 0 & r = 3i + 1 \end{cases}$$

$$c_r = \begin{cases} 1 & r = 3i \\ 0 & \text{otherwise} \end{cases}$$

For a given discrete numeric function **a**, let $O(\mathbf{a})$ denote the set of *all* numeric functions that are asymptotically dominated by **a**. $O(\mathbf{a})$ is read "order **a**" or "big-oh of **a**." Thus, if **b** is asymptotically dominated by **a**, then **b** is in the set $O(\mathbf{a})$. Instead of saying **b** is in the set $O(\mathbf{a})$, we often simply say **b** is $O(\mathbf{a})$. For example, let

$$\mathbf{a} = r^2$$

$$\mathbf{b} = 10r^2 + 25$$

$$\mathbf{c} = 5 - r$$

$$\mathbf{d} = \frac{1}{2} r^2 \log r - r^2$$

We note that **b** is $O(\mathbf{a})$, **c** is $O(\mathbf{a})$, but **d** is not $O(\mathbf{a})$. Also, **a** is $O(\mathbf{d})$, as are **b** and **c**.

When numeric function **a** can be expressed in a simple closed form, such as $\mathbf{a} = r^2$, we shall also write $O(\mathbf{a})$ as $O(r^2)$. Thus, when we say that a numeric function **b** is $O(r \log r)$, we mean that **b** is asymptotically dominated by the numeric function $\mathbf{a} = r \log r$. We want to remind the reader that **b** is $O(\mathbf{a})$ only means **b** does *not* grow faster than **a**; indeed, **b** could grow much slower than **a**. Thus, a statement such as "**b** grows too fast because **b** is $O(2^r)$" is not meaningful because we only know that **b** does not grow faster than 2^r. It is possible that **b** grows much slower than 2^r.

We note that the relations stated above in terms of asymptotic dominance can also be restated in terms of the "big-oh" notation as follows:

1. For any numeric function **a**, **a** is $O(|\mathbf{a}|)$.
2. If **b** is $O(\mathbf{a})$, then for any constant α, $\alpha\mathbf{b}$ is also $O(\mathbf{a})$.
3. If **b** is $O(\mathbf{a})$, then for any integer i, $S^i\mathbf{b}$ is $O(S^i\mathbf{a})$.
4. If both **b** and **c** are $O(\mathbf{a})$, then for any constants α and β, $\alpha\mathbf{b} + \beta\mathbf{c}$ is also $O(\mathbf{a})$.
5. If **c** is $O(\mathbf{b})$ and **b** is $O(\mathbf{a})$, then **c** is $O(\mathbf{a})$.
6. It is possible that **a** is $O(\mathbf{b})$, and **b** is also $O(\mathbf{a})$.
7. It is possible that **a** is not $O(\mathbf{b})$ and **b** is not $O(\mathbf{a})$.
8. It is possible that **c** is both $O(\mathbf{a})$ and $O(\mathbf{b})$, while **a** is not $O(\mathbf{b})$ and **b** is not $O(\mathbf{a})$.

Consider the following examples:

Example 9.6 Show that for $i \leqq j$, αr^i is $O(r^j)$. Note that for $k = 1$ and $m = 1$,

$$|r^i| \leqq r^j \qquad \text{for } r \geqq 1$$

It follows that r^i is $O(r^j)$. Consequently, αr^j is $O(r^j)$. □

Example 9.7 Let

$$\mathbf{a} = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_n r^n$$

We want to show that **a** is $O(r^n)$. Note that

$$\begin{aligned} |\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_n r^n| &\leqq |\alpha_0| + |\alpha_1| r + |\alpha_2| r^2 + \cdots + |\alpha_n| r^n \\ &\leqq (|\alpha_0| + |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|) r^n \end{aligned}$$

Since for $k = 1$ and $m = |\alpha_0| + |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$

$$|\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_n r^n| \leqq m r^n \qquad \text{for } r \geqq 1$$

a is asymptotically dominated by r^n. Consequently, **a** is $O(r^n)$.

In fact, the result can also be derived more directly. According to Example 9.6, for $i \leqq n$, $\alpha_i r^i$ is $O(r^n)$. Using the result in relation 4 above, we can conclude that **a** is $O(r^n)$. □

Example 9.8 We note that $O(1) \subset O(\log r) \subset O(r) \subset O(r^i) \subset O(\alpha^r) \subset O(r!)$. We ask the reader to confirm the results. □

Let **A** and **B** be two sets of numeric functions. We use the following notations to denote sets of numeric functions:

$$\mathbf{A} + \mathbf{B} = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$$

$$\alpha\mathbf{A} = \{\alpha\mathbf{a} \mid \mathbf{a} \in \mathbf{A}\}$$

$$\mathbf{A} \cdot \mathbf{B} = \{\mathbf{ab} \mid \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}\}$$

We state the following results and leave the proofs to the reader:

1. If **b** is $O(\mathbf{a})$, then $O(\mathbf{b})$ is a subset of $O(\mathbf{a})$. Consequently, if **b** is $O(\mathbf{a})$ and **a** is $O(\mathbf{b})$, then sets $O(\mathbf{a})$ and $O(\mathbf{b})$ are equal.
2. For any **a**, $O(\mathbf{a}) + O(\mathbf{a}) = O(\mathbf{a})$.
3. If $\mathbf{b} \in O(\mathbf{a})$, then $O(\mathbf{a}) + O(\mathbf{b}) = O(\mathbf{a})$.
4. For any constant α, $\alpha O(\mathbf{a}) = O(\alpha\mathbf{a}) = O(\mathbf{a})$.
5. For any **a** and **b**, $O(\mathbf{a})O(\mathbf{b}) = O(\mathbf{ab})$.

Let us consider the following examples:

Example 9.9 Let

$$\mathbf{a} = \tfrac{1}{3}r^3 + \tfrac{1}{2}r^2 + r$$

Clearly, **a** is $O(r^3)$. Also, **a** is in the set of numeric functions

$$\{\tfrac{1}{3}r^3\} + O(r^2)\dagger$$

as well as the set of numeric functions

$$\{\tfrac{1}{3}r^3 + \tfrac{1}{2}r^2\} + O(r)$$

In the literature, we frequently write

$$\mathbf{a} = O(r^3) \tag{9.1}$$

$$\mathbf{a} = \tfrac{1}{3}r^3 + O(r^2) \tag{9.2}$$

$$\mathbf{a} = \tfrac{1}{3}r^3 + \tfrac{1}{2}r^2 + O(r) \tag{9.3}$$

to mean‡

$$\mathbf{a} \in O(r^3)$$

$$\mathbf{a} \in \{\tfrac{1}{3}r^3\} + O(r^2)$$

$$\mathbf{a} \in \{\tfrac{1}{3}r^3 + \tfrac{1}{2}r^2\} + O(r)$$

† We remind the reader that this denotes the set of numeric functions **A** + **B**, where **A** is the set $\{\tfrac{1}{3}r^3\}$ and **B** is the set $O(r^2)$.

‡ Some authors prefer the interpretation:

$$\{\mathbf{a}\} \subseteq O(r^3)$$

$$\{\mathbf{a}\} \subseteq \{\tfrac{1}{3}r^3\} + O(r^2)$$

$$\{\mathbf{a}\} \subseteq \{\tfrac{1}{3}r^3 + \tfrac{1}{2}r^2\} + O(r)$$

Such slight difference in viewpoint is inconsequential.

Note that (9.1), (9.2), and (9.3) provide different information on the numeric function **a**. Because

$$\{\tfrac{1}{3}r^3 + \tfrac{1}{2}r^2\} + O(r) \subset \{\tfrac{1}{3}r^3\} + O(r^2) \subset O(r^3)$$

(9.3) is the strongest and (9.1) the weakest of the three relations concerning the behavior of numeric function **a**. In other words, (9.3) implies (9.2) which, in turn, implies (9.1), but not conversely. □

Example 9.10 Let

$$\mathbf{a} = r + O\left(\frac{1}{r}\right)$$

$$\mathbf{b} = \sqrt{r} + O\left(\frac{1}{\sqrt{r}}\right)$$

We have

$$\begin{aligned}
\mathbf{ab} &= \left(r + O\left(\frac{1}{r}\right)\right)\left(\sqrt{r} + O\left(\frac{1}{\sqrt{r}}\right)\right)^{\dagger} \\
&= r^{3/2} + rO\left(\frac{1}{\sqrt{r}}\right) + \sqrt{r}\,O\left(\frac{1}{r}\right) + O\left(\frac{1}{r}\right)O\left(\frac{1}{\sqrt{r}}\right) \\
&= r^{3/2} + O(\sqrt{r}) + O\left(\frac{1}{\sqrt{r}}\right) + O\left(\frac{1}{r^{3/2}}\right) \\
&= r^{3/2} + O(\sqrt{r})
\end{aligned}$$

□

Similar to the "big-oh" notation, we also want to introduce the "big-omega" and "big-theta" notations. For a given numeric function **a**, let $\Omega(\mathbf{a})$ denote the set of all numeric functions **b** such that there exist positive constants k and m with‡

$$|b_r| \geqq ma_r \qquad \text{for } r \geqq k$$

In other words, if **b** is in $\Omega(\mathbf{a})$, then **b** grows at least as fast as **a**. For example, let

$$\begin{aligned}
a_r &= 2r + 5 & r &\geqq 0 \\
b_r &= r - 2 & r &\geqq 0 \\
c_r &= r \log r & r &\geqq 0 \\
d_r &= r^{3/2} & r &\geqq 0
\end{aligned}$$

Both **b** and **c** are in $\Omega(\mathbf{a})$. However, neither **b** nor **c** is in $\Omega(\mathbf{d})$. In the literature, instead of saying **b** is in $\Omega(\mathbf{a})$, we often say **b** is $\Omega(\mathbf{a})$.

† Again, we remind the reader that this denotes the set of numeric functions $\mathbf{A} \cdot \mathbf{B}$, where **A** is the set $r + O(1/r)$ and **B** is the set $\sqrt{r} + O(1/\sqrt{r})$.

‡ The notation is meaningful only if a_r is positive.

For a given numeric function **a**, let $\Theta(\mathbf{a})$ denote the set of all numeric functions **b** such that there exist positive constants m, m', and k with†

$$ma_r \leqq |b_r| \leqq m'a_r \qquad \text{for } r \geqq k$$

In other words, if **b** is in $\Theta(\mathbf{a})$, then **b** grows at the same rate as **a**. Again, instead of saying **b** is in $\Theta(\mathbf{a})$, we often say **b** is $\Theta(\mathbf{a})$. Thus, for example, according to our discussion in Sec. 8.2, the time complexity of the bubble sort algorithm is $\Theta(r^2)$. Consequently, the time complexity of the problem of sorting is $O(r^2)$. On the other hand, the time complexity of the problem of selecting the largest of r numbers, and the time complexity of the problem of selecting the largest and smallest of r numbers are both $\Theta(r)$.

9.4 GENERATING FUNCTIONS

As we pointed out in Sec. 9.1, a numeric function can be specified by an exhaustive listing of its values. In this section, we introduce an alternative way of representing numeric functions.

Suppose we have an iron rod and wish to make a horseshoe out of it. Since hammering a cold iron rod into a horseshoe is quite tedious, we first place the iron rod in a furnace. We can then hammer the hot iron rod into a (hot) horseshoe. When we dip the hot horseshoe in a water tank to cool it, we obtain the (cold) horseshoe we want. The moral of our example is rather obvious: Our goal is to turn a (cold) iron rod into a (cold) horseshoe. However, instead of trying to achieve this goal directly, we change the cold iron rod into a hot iron rod first. Once we make a hot horseshoe out of the hot iron rod, we can cool the hot horseshoe and obtain the cold horseshoe we want. The process is shown in Fig. 9.1*a*.

Recall that given a positive number x we can compute its logarithm $\ln x$. Indeed, the logorithm of a number can be viewed as an alternative representation of the number, since from x we can compute $\ln x$, and from $\ln x$ we can compute x. Thus, when we want to compute xy, instead of multiplying the two numbers x and y directly, we can represent x as $\ln x$ and y as $\ln y$, compute $\ln x + \ln y$, which is equal to $\ln xy$, and then obtain the product xy from its alternative representation $\ln xy$. Similarly, when we want to compute x/y instead of dividing x by y directly, we can compute $\ln (x/y)$, which is equal to $\ln x - \ln y$, and then obtain x/y from its alternative representation $\ln (x/y)$. The process is depicted in Fig. 9.1*b*.

To a computer science student, the notion of alternative representation is a most familiar one. An alternative representation for a decimal number is its corresponding binary number. Thus, instead of adding, subtracting, multiplying,

† The notation is meaningful only if a_r is positive.

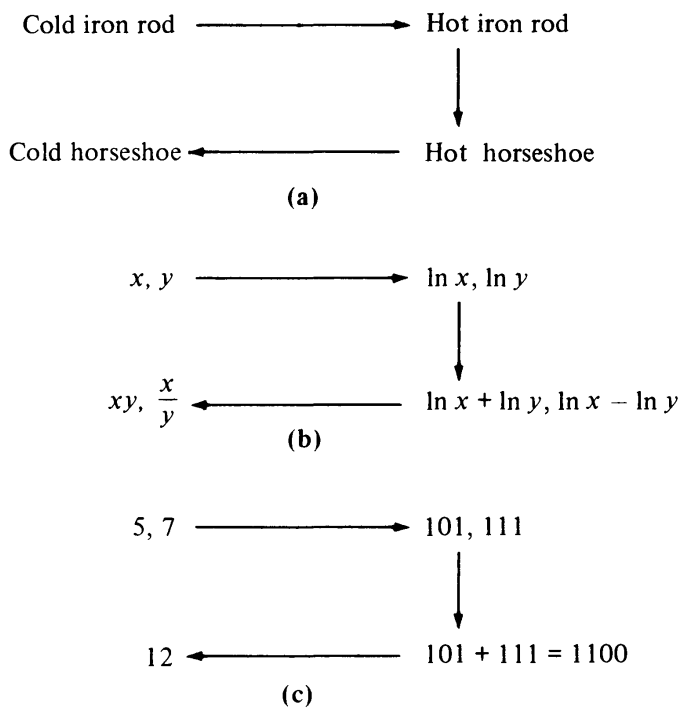

Figure 9.1

and dividing decimal numbers directly, we represent them as binary numbers, use a computer to carry out all arithmetic operations on binary numbers (which a computer can do effortlessly), and then obtain the results of our computation by converting the results in binary numbers into decimal numbers. Again, the process is shown in Fig. 9.1*c*.

These examples illustrate very well the notion of alternative representation for physical as well as mathematical entities. Furthermore, we saw that a *suitably chosen* alternative representation also leads to efficiency and easiness in some operations we wish to carry out.

We introduce now an alternative way to represent numeric functions. For a numeric function $(a_0, a_1, a_2, \ldots, a_r, \ldots)$, we define an infinite series

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

which is called the *generating function* of the numeric function **a**. Indeed, we can take the viewpoint that we have introduced nothing more than a new notation at this moment. Instead of writing down the values of a numeric function one by one and using commas as "separators" as in $(a_0, a_1, a_2, \ldots, a_r, \ldots)$, we choose a formal variable z and use the powers of z as "indicators" in an infinite series such that the coefficient of z^r is the value of the numeric function at r. For a numeric function **a**, we shall use the corresponding upper case letter and write $A(z)$ to denote the generating function of **a**. Clearly, given a numeric function, we can easily obtain its generating function, and conversely. For example, the generating function of the numeric function $(3^0, 3^1, 3^2, \ldots, 3^r, \ldots)$ is

$$3^0 + 3z + 3^2z^2 + 3^3z^3 + \cdots + 3^r z^r + \cdots \tag{9.4}$$

We note that the infinite series in (9.4) can be written in closed form as

$$\frac{1}{1 - 3z}$$

which is a rather compact way to represent the numeric function $(1, 3, 3^2, \ldots, 3^r, \ldots)$. Indeed, as we shall see, the reason we introduce the generating-function representation of numeric functions is the possibility of expressing infinite series in closed form so that they can be manipulated conveniently.

The reader can readily check that, corresponding to $\mathbf{b} = \alpha\mathbf{a}$, we have $B(z) = \alpha A(z)$. Thus, for example, the generating function of the numeric function

$$a_r = 7 \cdot 3^r \qquad r \geqq 0$$

is

$$A(z) = \frac{7}{1 - 3z}$$

and the generating function of the numeric function

$$a_r = 3^{r+2} \qquad r \geqq 0$$

is

$$A(z) = \frac{9}{1 - 3z}$$

It is also obvious that corresponding to $\mathbf{c} = \mathbf{a} + \mathbf{b}$, we have $C(z) = A(z) + B(z)$. Thus, the generating function of the numeric function

$$a_r = 2^r + 3^r \qquad r \geqq 0$$

is

$$A(z) = \frac{1}{1 - 2z} + \frac{1}{1 - 3z}$$

Also, corresponding to the generating function

$$A(z) = \frac{2 + 3z - 6z^2}{1 - 2z}$$

which can be written as

$$A(z) = 3z + \frac{2}{1 - 2z}$$

we have

$$a_r = \begin{cases} 2 & r = 0 \\ 7 & r = 1 \\ 2^{r+1} & r \geqq 2 \end{cases}$$

Let $\mathbf{a}$ be a numeric function and $A(z)$ its generating function. Let $\mathbf{b}$ be a numeric function such that

$$b_r = \alpha^r a_r$$

for some constant α. Since the generating function of **b** is

$$a_0 + \alpha a_1 z + \alpha^2 a_2 z^2 + \cdots + \alpha^r a_r z^r + \cdots$$
$$= a_0 + a_1(\alpha z) + a_2(\alpha z)^2 + \cdots + a_r(\alpha z)^r + \cdots$$

we have $B(z) = A(\alpha z)$. For example, since the generating function of the numeric function

$$a_r = 1 \qquad r \geqq 0$$

is

$$A(z) = \frac{1}{1 - z}$$

the generating function of the numeric function

$$a_r = \alpha^r \qquad r \geqq 0$$

is

$$A(z) = \frac{1}{1 - \alpha z}$$

As another example, corresponding to the generating function

$$A(z) = \frac{2}{1 - 4z^2}$$

which can be written as

$$A(z) = \frac{1}{1 - 2z} + \frac{1}{1 + 2z}$$

we have

$$a_r = 2^r + (-2)^r \qquad r \geqq 0$$

or

$$a_r = \begin{cases} 0 & r \text{ odd} \\ 2^{r+1} & r \text{ even} \end{cases}$$

Unfortunately, for the case $\mathbf{c} = \mathbf{ab}$, there is no simple form in which we can express $C(z)$ in terms of $A(z)$ and $B(z)$.

Let $A(z)$ be the generating function of **a**. It follows that $z^i A(z)$ is the generating function of $S^i\mathbf{a}$ for any positive integer i. For example, corresponding to

$$A(z) = \frac{z^4}{1 - 2z}$$

we have

$$a_r = \begin{cases} 0 & 0 \leqq r \leqq 3 \\ 2^{r-4} & r \geqq 4 \end{cases}$$

Also, $z^{-i}[A(z) - a_0 - a_1 z - a_2 z^2 - \cdots - a_{i-1} z^{i-1}]$ is the generating function of $S^{-i}\mathbf{a}$. Thus, the generating function of

$$a_r = 3^{r+2} \qquad r \geqq 0$$

can be computed as

$$\begin{aligned} A(z) &= z^{-2}\left(\frac{1}{1-3z} - 1 - 3z\right) \\ &= z^{-2}\,\frac{9z^2}{1-3z} \\ &= \frac{9}{1-3z} \end{aligned}$$

For another example, let

$$a_r = \begin{cases} 0 & 0 \leqq r \leqq 8 \\ 2 & r = 9 \\ 3 & r = 10 \\ 3 & r = 11 \\ 2 & r = 12 \\ 0 & r = 13 \\ 1 & r = 14 \\ 3 & r = 15 \\ 2 & r = 16 \\ 2 & r = 17 \\ 0 & r \geqq 18 \end{cases}$$

represent the hourly productivity of a machinist who comes to work at 8 a.m., where r indicates the hour of the day. Then

$$A(z) = z^9(2 + 3z + 3z^2 + 2z^3 + z^5 + 3z^6 + 2z^7 + 2z^8)$$

Suppose there are three machinists. One comes to work at 6 a.m., another at 8 a.m., and yet another at 9 a.m., and they simply shift their productivity according to their starting time. If we let $\mathbf{b}$ be a numeric function describing the total hourly productivity, we have

$$B(z) = (z^{-2} + 1 + z)A(z)$$

For $\mathbf{b} = \Delta\mathbf{a}$, we have

$$B(z) = \frac{1}{z}\,[A(z) - a_0] - A(z)$$

and for $\mathbf{b} = \nabla\mathbf{a}$, we have

$$B(z) = A(z) - zA(z)$$

The generating function representation of numeric functions is particularly useful in dealing with the convolution of numeric functions. Let $\mathbf{c} = \mathbf{a} * \mathbf{b}$. Since

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_{r-1} b_1 + a_r b_0$$

is the coefficient of z^r in the product

$$(a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots + b_r z^r + \cdots)$$

we conclude that the generating function $C(z)$ is equal to $A(z)B(z)$.

Example 9.11 Let $\mathbf{c} = \mathbf{a} * \mathbf{b}$ where

$$a_r = 3^r \qquad r \geqq 0$$

and

$$b_r = 2^r \qquad r \geqq 0$$

Since

$$A(z) = \frac{1}{1 - 3z}$$

and

$$B(z) = \frac{1}{1 - 2z}$$

we have

$$C(z) = A(z)B(z) = \frac{1}{1 - 3z}\,\frac{1}{1 - 2z} = \frac{3}{1 - 3z} - \frac{2}{1 - 2z}$$

or

$$c_r = 3(3)^r - 2(2)^r = 3^{r+1} - 2^{r+1}$$

□

Example 9.12 Let $\mathbf{a}$ be an arbitrary numeric function and $\mathbf{b}$ be the numeric function $(1, 1, 1, \ldots, 1, \ldots)$. Let

$$\mathbf{c} = \mathbf{a} * \mathbf{b}$$

so that

$$c_r = \sum_{i=0}^{r} a_i b_{r-i} = \sum_{i=0}^{r} a_i$$

Thus $\mathbf{c}$ is the accumulated sum of $\mathbf{a}$, and the generating function of $\mathbf{c}$ is

$$C(z) = \frac{1}{1 - z} A(z)$$

For example, by letting $A(z)$ be $1/(1 - z)$ we obtain the result that $1/(1 - z)^2$ is the generating function of the numeric function $(1, 2, 3, \ldots, r, \ldots)$.

□

Example 9.13 We wish to evaluate the sum

$$1^2 + 2^2 + 3^2 + \cdots + r^2$$

Let us first determine the generating function of the numeric function $(0^2, 1^2, 2^2, 3^2, \ldots, r^2, \ldots)$. Differentiating both sides of the identity

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots + z^r + \cdots$$

we obtain

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots + rz^{r-1} + \cdots$$

It follows that

$$\frac{d}{dz}\frac{z}{(1-z)^2} = 1^2 + 2^2 \cdot z + 3^2 \cdot z^2 + 4^2 \cdot z^3 + \cdots + r^2 \cdot z^{r-1} + \cdots$$

and that

$$z\frac{d}{dz}\frac{z}{(1-z)^2} = 0^2 + 1^2 \cdot z + 2^2 \cdot z^2 + 3^2 \cdot z^3 + 4^2 \cdot z^4 + \cdots + r^2 \cdot z^r + \cdots$$

Thus, $z(d/dz)[z/(1-z)^2]$, which is equal to $[z(1+z)]/(1-z)^3$, is the generating function of the numeric function $(0^2, 1^2, 2^2, 3^2, \ldots, r^2, \ldots)$, and it follows that $[z(1+z)]/(1-z)^4$ is the generating function of the numeric function $(0^2, 0^2 + 1^2, 0^2 + 1^2 + 2^2, 0^2 + 1^2 + 2^2 + 3^2, \ldots, 0^2 + 1^2 + 2^2 + 3^2 + \cdots + r^2, \ldots)$. According to the binomial theorem,† the coefficient of z^r in $1/(1-z)^4$ is

$$\frac{(-4)(-4-1)(-4-2)\cdots(-4-r+1)}{r!}(-1)^r = \frac{4 \times 5 \times 6 \times \cdots \times (r+3)}{r!}$$

$$= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$$

Therefore, the coefficient of z^r in the expansion of $[z(1+z)]/(1-z)^4$ is

$$\frac{r(r+1)(r+2)}{1 \cdot 2 \cdot 3} + \frac{(r-1)r(r+1)}{1 \cdot 2 \cdot 3} = \frac{r(r+1)(2r+1)}{6}$$

that is,

$$1^2 + 2^2 + 3^2 + \cdots + r^2 = \frac{r(r+1)(2r+1)}{6} \qquad \square$$

† The binomial theorem is

$$(1+z)^n = 1 + \sum_{r=1} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} z^r$$

where the upper limit of the summation is n if n is a positive integer, and is ∞ otherwise.

*9.5 COMBINATORIAL PROBLEMS

The generating-function representation of numeric functions is most useful in solving combinatorial problems. We shall present in this section some illustrative examples. Consider the numeric function **a** such that

$$a_r = C(n, r)$$

for a fixed n.† The generating function of **a** is

$$A(z) = C(n, 0) + C(n, 1)z + C(n, 2)z^2 + \cdots + C(n, r)z^r + \cdots + C(n, n)z^n$$

Let us consider a roundabout way to select r objects from n objects. We can first divide the n objects into two piles, one with k objects and the other with $n - k$ objects, for some fixed k less than n. We can then select i objects from the first pile, where there are $C(k, i)$ ways, and $r - i$ objects from the second pile, where there are $C(n - k, r - i)$ ways, for $i = 0, 1, 2, \ldots, r$. Thus, we have the equation

$$C(n, r) = \sum_{i=0}^{r} C(k, i)C(n - k, r - i)$$

It follows that we can write $\mathbf{a} = \mathbf{d} * \mathbf{e}$ where

$$D(z) = C(k, 0) + C(k, 1)z + C(k, 2)z^2 + \cdots + C(k, k)z^k$$

and

$$E(z) = C(n - k, 0) + C(n - k, 1)z + C(n - k, 2)z^2 + \cdots + C(n - k, n - k)z^{n-k}$$

Repeating the argument a sufficient number of times, we obtain

$$A(z) = (1 + z)^n \tag{9.5}$$

because the generating function of the numeric function $(C(1, 0), C(1, 1), 0, 0, \ldots)$ is $1 + z$. Let us also point out a very simple combinatorial argument that can also be used to derive (9.5). Consider the coefficient of the term z^r in the expansion of $(1 + z)^n$. In computing the product of the n factors $1 + z$, each factor will "contribute" either a 1 or a z. In particular, to make up the term z^r, r of the factors contribute a z each and $n - r$ of the factors contribute a 1 each. Consequently, the coefficient of z^r is the number of ways of selecting r of the n $1 + z$ factors to make up the term z^r, which, of course, is equal to $C(n, r)$. We show now some results that can be derived directly from (9.5).

Example 9.14 Setting z to 1 in (9.5), we obtain

$$C(n, 0) + C(n, 1) + \cdots + C(n, r) + \cdots + C(n, n) = 2^n$$

That is, the number of ways to select none, one, two, ..., or n objects from n objects is 2^n.

† We recall that $C(n, r) = 1$ for $r = 0$ and $C(n, r) = 0$ for $r > n$.

Setting z to -1 in (9.5), we obtain

$$C(n, 0) - C(n, 1) + C(n, 2) + \cdots + (-1)^r C(n, r) + \cdots$$
$$\cdots + (-1)^n C(n, n) = 0$$

or

$$C(n, 0) + C(n, 2) + C(n, 4) + \cdots = C(n, 1) + C(n, 3) + C(n, 5) + \cdots$$

That is, the number of ways of selecting an even number of objects from n objects is equal to the number of ways of selecting an odd number of objects. □

Example 9.15 The relation

$$C(n, r) = C(n-1, r) + C(n-1, r-1) \tag{9.6}$$

can be proved in several ways. By straightforward algebraic manipulation, we obtain

$$\begin{aligned} C(n, r) &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{n!}{r!\,(n-r)!}\left(\frac{n-r}{n} + \frac{r}{n}\right) \\ &= \frac{(n-1)!}{r!\,(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!} \\ &= C(n-1, r) + C(n-1, r-1) \end{aligned}$$

We can also prove (9.6) using a combinatorial argument: If we are to select r objects from n objects, there are $C(n-1, r)$ ways to select r objects so that a particular object is always excluded, and $C(n-1, r-1)$ ways so that this particular object is always included. Thus, (9.6) follows. The use of generating functions provides a third proof of (9.6). Note that $C(n, r)$ is the coefficient of z^r in $(1+z)^n$, which can be written as $(1+z)^{n-1} + z(1+z)^{n-1}$. Since the coefficient of z^r in $(1+z)^{n-1}$ is $C(n-1, r)$, and the coefficient of z^r in $z(1+z)^{n-1}$ is $C(n-1, r-1)$, (9.6) follows immediately. □

We present now some further examples:

Example 9.16 Let $(a_0, a_1, a_2, \ldots)$ be a numeric function such that a_r is equal to the number of ways to select r objects from 10 objects among which one, which will be denoted X, can be selected at most twice, one, which will be denoted Y, can be selected at most thrice, and the others can be selected only once. We claim that the generating function of **a** is

$$A(z) = (1 + z + z^2)(1 + z + z^2 + z^3)(1 + z)^8$$

Note that the coefficient of z^r in $A(z)$ is the number of ways to make up the term z^r from the factors $1 + z + z^2$, $1 + z + z^2 + z^3$, and the eight factors $1 + z$. The contribution from the factor $1 + z + z^2$ can be 1, z, or z^2, corresponding to selecting the object X zero times, once, or twice. The contribution from the factor $1 + z + z^2 + z^3$ can be 1 or z or z^2 or z^3, corresponding to selecting the object Y zero times, once, twice, or thrice. The contribution of each of the eight factors $1 + z$ can be 1 or z, corresponding to selecting each of the eight remaining objects zero times or once. □

Example 9.17 Let a_r denote the number of ways of selecting r objects from n objects with unlimited repetitions. Since each object can be selected as many times as we wish, a_r is equal to the coefficient of z^r in

$$(1 + z + z^2 + \cdots)^n$$

Note that the contribution from each of the factors $1 + z + z^2 + \cdots$ will correspond to the number of times one of the objects is selected. Since

$$(1 + z + z^2 + \cdots)^n = \left(\frac{1}{1 - z}\right)^n = (1 - z)^{-n}$$

we obtain

$$\begin{aligned} a_r &= (-1)^r \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} \\ &= \frac{(n + r - 1)!}{r!(n-1)!} \\ &= C(n + r - 1, r) \end{aligned}$$

This result is a result we derived earlier in Chap. 3. □

Example 9.18 Suppose we want to determine the number of ways in which $2t + 1$ marbles can be distributed among three distinct boxes so that no box will contain more than t marbles. We claim that the coefficient of z^{2t+1} in

$$A(z) = (1 + z + z^2 + \cdots + z^t)^3$$

will be our answer. Note that the coefficient of z^{2t+1} in $A(z)$ is the number of ways to make up the term z^{2t+1} from the three factors $1 + z + z^2 + \cdots + z^t$. The contribution from each factor $1 + z + z^2 + \cdots + z^t$ can be 1, z, z^2, ..., or z^t corresponding to having none, one, two, ..., or t marbles in a box. Since

$$\begin{aligned} (1 + z + z^2 + \cdots + z^t)^3 &= \left(\frac{1 - z^{t+1}}{1 - z}\right)^3 \\ &= (1 - 3z^{t+1} + 3z^{2t+2} - z^{3t+3})(1 - z)^{-3} \qquad (9.7) \end{aligned}$$

the coefficient of z^{2t+1} in (9.7) is the coefficient of z^{2t+1} in $(1-z)^{-3}$ minus three times the coefficient of z^t in $(1-z)^{-3}$. Thus, it is

$$C(3+2t+1-1, 2t+1) - 3C(3+t-1, t)$$

which is simplified as

$$C(2t+3, 2t+1) - 3C(t+2, t) \qquad \square$$

9.6 REMARKS AND REFERENCES

See chap. 3 of Beckenbach [1], chap. 2 of Liu [6], chaps. 1 and 5 of Liu and Liu [7], and chaps. 1 and 2 of Riordan [8] on numeric functions and their generating functions. For asymptotic behavior of numeric functions, see chap. 1 of Knuth [4] and chap. 5 of Stanat and McAllister [9]. We follow the notations introduced by Knuth [5]. See chap. 6 of Chung [2] and chap. 11 of Feller [3] on application of the technique of generating functions in probability theory.

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PROBLEMS

9.1 A ping-pong ball is dropped to the floor from a height of 20 m. Suppose that the ball always rebounds to reach half of the height from which it falls.

(*a*) Let a_r denote the height it reaches in the rth rebound. Sketch the numeric function **a**.

(*b*) Let b_r denote the loss in height during the rth rebound. Express b_r in terms of a_r. Sketch the numeric function **b**.

(*c*) A second ping-pong ball is dropped from a height of 6 m. to the same floor at the same time as the first ball reaches the highest point of its third rebound. Let c_r denote the height the second ball reaches in its rth rebound. Express c_r in terms of a_r.

9.2 In a process control system, a monitoring device measures the temperature inside a chemical reaction chamber once every 30 s. Let a_r denote the rth reading in degrees in centigrade. Determine an expression for a_r if it is known that the temperature rises from 100 to 120° at a constant rate in the first 300 s and stays at 120° from then on.

9.3 Let **a** be a numeric function such that a_r is equal to the remainder when the integer r is divided by 17. Let **b** be a numeric function such that b_r is equal to 0 if the integer r is divisible by 3, and is equal to 1 otherwise.

(*a*) Let $c_r = a_r + b_r$. For what values of r is $c_r = 0$? For what values of r is $c_r = 1$?

(*b*) Let $d_r = a_r b_r$. For what values of r is $d_r = 0$? For what values of r is $d_r = 1$?

9.4 Let **a** be a numeric function such that

$$a_r = \begin{cases} 2 & 0 \leqq r \leqq 3 \\ 2^{-r} + 5 & r \geqq 4 \end{cases}$$

(*a*) Determine $S^2\mathbf{a}$ and $S^{-2}\mathbf{a}$.

(*b*) Determine $\Delta\mathbf{a}$ and $\nabla\mathbf{a}$.

9.5 We introduce the notation

$$\Delta^2\mathbf{a} = \Delta(\Delta\mathbf{a})$$

$$\Delta^3\mathbf{a} = \Delta(\Delta^2\mathbf{a})$$

$$\cdots\cdots\cdots\cdots$$

$$\Delta^i\mathbf{a} = \Delta(\Delta^{i-1}\mathbf{a})$$

(*a*) Let **a** be a numeric function such that

$$a_r = r^3 - 2r^2 + 3r + 2$$

Determine $\Delta\mathbf{a}$, $\Delta^2\mathbf{a}$, $\Delta^3\mathbf{a}$, $\Delta^4\mathbf{a}$.

(*b*) Let **a** be a numeric function such that a_r is a polynomial of the form

$$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \cdots + \alpha_k r^k$$

Show that $\Delta^{k+1}\mathbf{a}$ is equal to 0.

9.6 (*a*) Let $\mathbf{c} = \mathbf{ab}$. Show that

$$\Delta c_r = a_{r+1}(\Delta b_r) + b_r(\Delta a_r)$$

(*b*) Let **a** and **b** be two numeric functions such that $a_r = r + 1$ and $b_r = \alpha^r$ for all $r \geqq 0$. Determine $\Delta(\mathbf{ab})$.

(*c*) Let **a** and **b** be two numeric functions. The quotient of **a** and **b** denoted $\mathbf{a}/\mathbf{b}$ is a numeric function whose value at r is equal to a_r/b_r. Let $\mathbf{d} = \mathbf{a}/\mathbf{b}$. Show that

$$\Delta d_r = \frac{b_r(\Delta a_r) - a_r(\Delta b_r)}{b_r b_{r+1}}$$

(*d*) Determine $\Delta(\mathbf{a}/\mathbf{b})$ for the numeric functions **a** and **b** in part (*b*).

9.7 Let **a** be a numeric function. Let $\Delta^{-1}\mathbf{a}$ denote the accumulated sum of **a**. Are the two numeric functions $\Delta(\Delta^{-1}\mathbf{a})$ and $\Delta^{-1}(\Delta\mathbf{a})$ the same?

9.8 Interest for money deposited in a savings account is paid at a rate of 0.5 percent per month, with interest compounded monthly. Suppose \$50 is deposited into a savings account each month for a period of five years. What is the total amount in the account four years after the first deposit? Twenty years after the first deposit?

9.9 Determine $\mathbf{a} * \mathbf{b}$ in the following. Give either a sketch or a closed-form expression of the resulting numeric function, whichever is more convenient.

(*a*)

$$a_r = \begin{cases} 1 & 0 \leqq r \leqq 2 \\ 0 & r \geqq 3 \end{cases}$$

$$b_r = \begin{cases} 1 & 0 \leqq r \leqq 2 \\ 0 & r \geqq 3 \end{cases}$$

(*b*)

$$a_r = 1 \qquad \text{for all } r$$

$$b_r = \begin{cases} 1 & r = 1 \\ 2 & r = 3 \\ 3 & r = 5 \\ -6 & r = 7 \\ 0 & \text{otherwise} \end{cases}$$

(*c*)

$$a_r = 2^r \qquad \text{for all } r$$

$$b_r = \begin{cases} 0 & 0 \leqq r \leqq 2 \\ 2^r & r \geqq 3 \end{cases}$$

9.10 Let **a**, **b**, and **c** be numeric functions such that $\mathbf{a} * \mathbf{b} = \mathbf{c}$. Given that

$$a_r = \begin{cases} 1 & r = 0 \\ 2 & r = 1 \\ 0 & r \geqq 2 \end{cases}$$

$$c_r = \begin{cases} 1 & r = 0 \\ 0 & r \geqq 1 \end{cases}$$

determine **b**.

9.11 Let

$$a_r = \begin{cases} 1 & r = 0 \\ 3 & r = 1 \\ 2 & r = 2 \\ 0 & r \geqq 3 \end{cases}$$

$$c_r = 5^r \qquad \text{for all } r$$

Determine **b** so that $\mathbf{a} * \mathbf{b} = \mathbf{c}$.

9.12 (*a*) Every particle inside a nuclear reactor splits into two particles in each second. Suppose one particle is injected into the reactor every second beginning at $t = 0$. How many particles are there in the reactor at the nth second?

(*b*) When a particle splits into two, one of them is actually the original particle and the other is a newly created particle. Suppose a particle has a lifetime of 10 seconds. That is, a particle created in the 0th second will vanish after the 10th second. Repeat part (*a*) and assume that all injected particles are newly created.

9.13 For a fee of \$1000, a radio station will provide a store with 10 min of commercial time each day for five consecutive days and then 2 min of commercial time each day for the next five consecutive days. Suppose that every minute of commercial time on a particular day will produce a total sale of \$500 on that day and a total sale of $\$500 \times 2^{-n}$ on the nth day from that day.

(*a*) Suppose a fee of \$1000 is paid to the radio station on the 0th day. Determine the numeric function **a**, where a_r is the total sale on the rth day.

(*b*) Suppose a fee of \$2000 is paid to the radio station daily for 10 consecutive days starting on the 0th day. Determine the numeric function **b** where b_r is the total sale on the rth day.

9.14 Let **a**, **b**, and **c** be three numeric functions:

$$\mathbf{a} = 3r - 2$$

$$\mathbf{b} = \frac{2}{r} + 7$$

$$\mathbf{c} = r \ln r$$

(*a*) Does **a** dominate **b** asymptotically? Does **a** dominate **c** asymptotically? Does **b** dominate **a** asymptotically? Does **b** dominate **c** asymptotically? Does **c** dominate **a** asymptotically? Does **c** dominate **b** asymptotically?

(*b*) Does **a** + **b** dominate **a** asymptotically? Does **a** + **b** dominate **c** asymptotically?

(*c*) Does **ab** dominate **a** asymptotically? Does **a** dominate **ab** asymptotically? Does **ab** dominate **c** asymptotically? Does **c** dominate **ab** asymptotically?

(*d*) Does Δ**a** dominate **b** asymptotically?

(*e*) Does the accumulated sum of **a** dominate **c** asymptotically?

9.15 Let

$$\mathbf{a} = 3^r$$

$$\mathbf{b} = 2^r$$

(*a*) Does **a** dominate **b** asymptotically?

(*b*) Does **b** dominate **a** asymptotically?

(*c*) Does **a*****b** dominate **a** asymptotically?

(*d*) Does **a*****b** dominate **b** asymptotically?

9.16 Let

$$a_r = \sum_{i=0}^{r} i^2$$

(*a*) Show that **a** is $O(r^3)$.

(*b*) Show that **a** is $(r^3/3) + O(r^2)$.

(*c*) Let **b** be a numeric function such that **b** is $O(r^3)$. It is necessary that **b** is $(r^3/3) + O(r^2)$?

(*d*) Let **b** be a numeric function such that **b** is $(r^3/3) + O(r^2)$. Is it necessary that **b** is $O(r^3)$?

9.17 Simplify the expression $[\ln r + O(1/n)][n + O(\sqrt{n})]$.

9.18 Let

$$\mathbf{a} = \ln r$$

Show that for $\varepsilon > 0$, **a** is $O(r^\varepsilon)$.

9.19 Given a numeric function **a**, let

$$b_r = \begin{cases} a_r + a_{r/2} + a_{r/4} + \cdots + a_{r/2^i} & r = 2^i \\ 0 & r \neq 2^i \end{cases}$$

If **a** is $O(\sqrt{r})$, show that **b** is $O(\sqrt{r} \log r)$.

9.20 Let

$$a_r = \sum_{i=0}^{\lceil \log_{3/2} r \rceil - 1} \log_2\left(\frac{2}{3}\right)^i r$$

$$= \log_2 r + \log_2\left(\frac{2}{3}\right) r + \log_2\left(\frac{4}{9}\right) r + \cdots$$

Show that a_r is $O(\log^2 r)$.

9.21 After careful analysis, a student concluded that a certain numeric function **a** is $O(r \log r)$. Upon hearing that, her roommate reminded her that she had forgotten to specify the base of the logarithm. Had she really forgotten something important?

9.22 (*a*) The time complexity of algorithm A is $O(r^2)$, and that of algorithm B is $O(r^2 \ln r)$. Can we conclude that algorithm A is superior to algorithm B?

(*b*) The time complexity of algorithm A is $O(r^2)$, and that of algorithm B is $\Omega(r^2 \ln r)$. Can we conclude that algorithm A is superior to algorithm B?

(*c*) The time complexity of algorithm A is $\Theta(r^2)$, and that of algorithm B is $\Theta(r^2 \ln r)$. Can we conclude that algorithm A is superior to algorithm B?

9.23 Two engineers were assigned the task of designing integrated circuit chips. Let a_r denote the total area of a chip that contains r transistors. For engineer A's design a_r is found to be $O(r^4)$. For engineer B's design, a_r is found to be $\Omega(\sqrt{r})$. Consequently, the manager concludes that when the number of transistors on a chip increases, A's design would occupy a large area and is therefore bad, while B's design would occupy a small area and is therefore good. Does he know what he is talking about?

9.24 A professor took his students out for beer on a Friday afternoon. After a couple of drinks, he declared: "The total amount of beer you are going to consume is big _____ of the size of my wallet, and is big _____ of your thirst." Granting the professor the privilege of abusing mathematical terminologies ever so slightly, can you fill in the blanks to make the statement meaningful?

9.25 Find a simple expression for the generating function of each of the following discrete numeric functions:

(*a*) $1, -2, 3, -4, 5, -6, \ldots$
(*b*) $1, 2/3, 3/9, 4/27, \ldots, (r+1)/3^r, \ldots$
(*c*) $1, 1, 2, 2, 3, 3, 4, 4, \ldots$
(*d*) $0 \times 1, 1 \times 2, 2 \times 3, 3 \times 4, \ldots$
(*e*) $0 \times 5^0, 1 \times 5^1, 2 \times 5^2, \ldots, r \times 5^r, \ldots$

9.26 Determine the generating function of the numeric function a_r, where

$$a_r = \begin{cases} 2^r & \text{if } r \text{ is even} \\ -2^r & \text{if } r \text{ is odd} \end{cases}$$

9.27 Determine the discrete numeric function corresponding to each of the following generating functions:

(*a*) $A(z) = \dfrac{1}{1 - z^3}$

(*b*) $A(z) = (1+z)^n + (1-z)^n$

(*c*) $A(z) = \dfrac{(1+z)^2}{(1-z)^4}$

(*d*) $A(z) = \dfrac{1}{5 - 6z + z^2}$

(*e*) $A(z) = \dfrac{z^5}{5 - 6z + z^2}$

(*f*) $A(z) = \dfrac{7z^2}{(1-2z)(1+3z)}$

(*g*) $A(z) = \dfrac{1 + z^2}{4 - 4z - z^2}$

(*h*) $A(z) = \dfrac{1}{(1-z)(1-z^2)(1-z^3)}$

9.28 (*a*) Let a_r denote the number of ways a sum of r can be obtained when two indistinguishable dice are rolled. Determine $A(z)$.

(*b*) Let b_r denote the number of ways a sum of r can be obtained when a single die is rolled once and then two indistinguishable dice are rolled once. Determine $B(z)$.

9.29 Let a_r denote the number of ways to seat 10 students in a row of r chairs so that no two students will occupy adjacent seats. Determine the generating function of the discrete numeric function $(a_0, a_1, a_2, \ldots, a_r, \ldots)$.

9.30 The lifespan of a rabbit is exactly 10 years. Suppose that there are two newborn rabbits at the beginning, and the number of newborn rabbits in each year is two times that in the previous year. Determine the number of rabbits there are in the rth year.

9.31 Let a_r denote the number of ways to select r balls from an infinite supply of red, blue, and white balls with the constraint that the number of red balls selected is always an even number. Determine the generating function of the numeric function **a**.

9.32 In how many ways can $3r$ balls be selected from $2r$ red balls, $2r$ blue balls, and $2r$ white balls?

9.33 Let a_r denote the number of ways to select r balls from 100 red balls, 50 blue balls, and 50 white balls with the constraints that the number of red balls selected is not equal to two, the number of blue balls selected is not equal to three, and the number of white balls selected is not equal to four. Determine the generating function of the numeric function **a**.

9.34 A set of r balls is selected from an infinite supply of red, blue, white, and yellow balls. A selection must satisfy the condition that either the number of red balls is even and the number of blue balls is odd, or the number of white balls is even and the number of yellow balls is odd. Let a_r denote the total number of such selections.

(*a*) Determine the generating function $A(z)$.

(*b*) Find a closed-form expression for a_r. Evaluate the expression for $r = 23$.

9.35 Let a_r denote the number of ways to divide r identical marbles into four distinct piles so that each pile has an odd number of marbles that is larger than or equal to three.

(*a*) Determine $A(z)$.

(*b*) Determine a closed-form expression for a_r.

9.36 (*a*) Determine the sum

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + i\binom{n}{i} + \cdots + n\binom{n}{n}$$

(*b*) If we write down all the combinations of n letters $\{1, 2, \ldots, n\}$, that is, all 1-combinations, all 2-combinations, how many times does each letter occur?

9.37 Determine the sum

$$C(n, 0) + 2C(n, 1) + 2^2C(n, 2) + \cdots + 2^nC(n, n)$$

9.38 Determine the sum

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0}$$

where $k \leqq n$ and $k \leqq m$.

9.39 (*a*) Let

$$A(z) = (1 + z)^{2n} + z(1 + z)^{2n-1} + \cdots + z^i(1 + z)^{2n-i} + \cdots + z^n(1 + z)^n$$

Show that

$$a_r = \begin{cases} \dbinom{2n+1}{r} & 0 \leqq r \leqq n \\ \dbinom{2n+1}{r} - \dbinom{n}{r-n-1} & n+1 \leqq r \leqq 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

(*b*) Evaluate the sum

$$\binom{2n}{n} + \binom{2n-1}{n-1} + \cdots + \binom{2n-i}{n-i} + \cdots + \binom{n}{0}$$

9.40 (*a*) Show that

$$\binom{r}{0}^2 + \binom{r}{1}^2 + \binom{r}{2}^2 + \cdots + \binom{r}{i}^2 + \cdots + \binom{r}{r}^2 = \binom{2r}{r}$$

(*b*) Show that the generating function for **a**, where

$$a_r = \binom{2r}{r}$$

is $(1 - 4z)^{-1/2}$.