Advanced Engineering Mathematics

Concepts and Solutions

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Abstract: This document covers key topics in advanced engineering mathematics, including differential equations, Laplace transforms, linear algebra, vector calculus, and optimization. It's designed as a compact reference with methods, formulas, and worked examples. The main reference textbook is *Advanced Engineering Mathematics* by Kreyszig et al. (2008).

Useful Identities & Tricks

It is often the case that complex problems are transferred into a different domain (e.g., via Laplace transforms) or reformulated as systems of ODEs, and complex algebra is the price we pay to solve such problems. This section provides tips and tricks to help solve some of the common problems encountered.

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1. Partial Fraction Decomposition

Being able to split the equation into fractions is a useful tool. To achieve this, set up the expected form, multiply both sides by the common denominator, then either: (1) Substitute convenient values (roots of the denominator) to find the coefficients, or (2) Expand and match coefficients of like powers. Common patterns:

$$\frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$
$$\frac{1}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+a}$$
$$\frac{1}{(s+a)^n} = \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n}$$
$$\frac{1}{(s+a)^2(s+b)} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$$
$$\frac{1}{(s+a)(s^2+b^2)} = \frac{A}{s+a} + \frac{Bs+C}{s^2+b^2}$$

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2. Polynomial Long Division

Once one root of a cubic is found, polynomial division reduces it to a quadratic, whose remaining roots follow easily. If the constant term is not large, try integer roots in $[-2, 2]$ first. Then,

Divide leading terms → multiply divisor by result → subtract → bring down → repeat.

Ex: $(x^3 - x^2 - x - 2) \div (x - 2)$

$$\begin{array}{r|l}
x^2 + x + 1 & \\
\hline
x - 2\,) & x^3 - x^2 - x - 2 \\
 & x^3 - 2x^2 \downarrow \\
\cline{2-2}
 & x^2 - x \\
 & x^2 - 2x \downarrow \\
\cline{2-2}
 & x - 2 \\
 & x - 2 \\
\cline{2-2}
 & 0
\end{array}$$

Result: $x^2 + x + 1$

3. Equalizing indices in sum operators

Equalizing indices (or "shifting indices") in sum operators ($\sum$) involves adjusting the lower/upper bounds and the general term formula simultaneously to make summation limits match, often for combining terms.

Shifting the Index: To shift the starting index from m to $m + r$, replace n with $n + r$ in the expression and subtract r from the upper bound to keep the number of terms consistent:

$$\sum_{n=m}^{p} a_n = \sum_{n=m+r}^{p+r} a_{n-r}$$

The idea is simple: if a backward shift is needed (starting the sum from $n-1$ instead of n), replace the lower limit $n \to n-1$, substitute $n \to n+1$ in the internal expression to compensate.

Example: To shift $\sum_{n=1}^{4} n^2$ to start at $n = 0$, it becomes $\sum_{n=0}^{3}(n+1)^2$.

4. Useful Trigonometric Equalities

$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh^2 x - \sinh^2 x = 1, \quad \cosh^2 x + \sinh^2 x = \cosh(2x)$

$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$

$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$

$\frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x$

5. Reduction Formulas

$$\int \sin^n \theta \, d\theta = -\frac{\sin^{n-1}\theta \cos\theta}{n} + \frac{n-1}{n} \int \sin^{n-2}\theta \, d\theta$$
$$\int \cos^n \theta \, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n} \int \cos^{n-2}\theta \, d\theta$$
$$\int \tan^n \theta \, d\theta = \frac{\tan^{n-1}\theta}{n-1} - \int \tan^{n-2}\theta \, d\theta$$
$$\int \sec^n \theta \, d\theta = \frac{\sec^{n-2}\theta \tan\theta}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2}\theta \, d\theta$$

6. Euler's Formula & Identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{i\pi} + 1 = 0$$

7. Chain Rule This rule applies to composite functions, where one function is nested inside another. The idea is similar to connected gears: to find the total rate of change, you multiply the rate of change of the outer function by the rate of change of the inner function.

To differentiate $y = f(g(x))$, let $u = g(x)$. Then $y = f(u)$ and:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For multivariable functions, the chain rule involves summing contributions from each intermediate variable:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}, \quad \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

8. Polar Coordinates A point is defined by distance r from the origin and angle θ from the positive x-axis. Useful for circular/rotational problems and simplifies integrals with radial symmetry.

Conversions:

$x = r\cos\theta, \quad y = r\sin\theta$

$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}$

Line and Area elements:

$ds = \sqrt{dr^2 + r^2 d\theta^2}$

$dA = r \, dr \, d\theta$

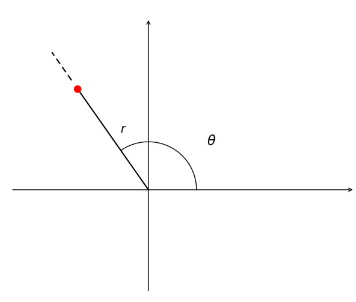

Figure 1: Polar coordinate system

Matrix Algebra

Characteristics of a Matrix

Commutativity: $AM = MA \Rightarrow$ matrices commute

Nilpotent (class p): $A^p = 0$ but $A^{p-1} \neq 0$

Involutory: $A^2 = I$, hence $A^{-1} = A$

Orthogonal: $A^T A = I$ (columns are orthonormal)

Orthonormal means: (1) unit vectors: $a^2 + c^2 = 1$, (2) perpendicular: $ab + cd = 0$

Unitary: $A^* A = I$ where $A^* = \overline{A}^T$ (conjugate transpose)

Symmetric: $A^T = A$ **Skew-Symmetric:** $A^T = -A$

Normal: $AA^T = A^T A$ (real), or $A^* A = AA^*$ (complex)

Orthogonal Vectors: Set is orthogonal if every dot product pair $= 0$

Inverse of a Matrix

$AA^{-1} = I$. Finding inverse: $A^{-1} = \dfrac{\text{adj}(A)}{\det(A)} = \dfrac{[C_{ij}]^T}{\det(A)}$

Cofactor: $C_{ij} = (-1)^{i+j} \det(A_{ij})$ where A_{ij} is A with row i, col j removed.

2×2 Inverse Formula: $A^{-1} = \dfrac{1}{\det A}\big((\text{tr}\, A)I - A\big)$

where trace = sum of diagonal elements.

Row Reduction Method: Augment $[A \mid I]$, row reduce to $[I \mid A^{-1}]$.

Example: $\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right]$

Determinants

If row/column is multiple of another: $\det = 0$

Properties: $\det(AB) = \det A \cdot \det B$, $\quad \det(A^T) = \det A$

$\det(kA) = k^n \det A$ for $n \times n$, $\quad \det(A^{-1}) = 1/\det A$

3×3 Determinant (Sarrus): $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$

4×4 (Cofactor Expansion): Select row with most zeros.

$\det A = a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} - a_{14}C_{14}$

Rank

$\text{rank}(A)$ = number of non-zero rows in echelon form

Full rank (rank $= n$) $\Leftrightarrow \det(A) \neq 0$

Rank $< n \Leftrightarrow \det(A) = 0$

Linear Systems ($\mathbf{Ax = b}$)

Unique solution: $\det(A) \neq 0$ (pivot in every variable column)

No solution: row $[0\ 0\ \cdots\ 0 \mid b]$ with $b \neq 0$

Infinite solutions: free variables exist (columns without pivots)

Linear Combination

Express $\mathbf{v}$ as $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$

Set up augmented matrix with basis vectors as columns, target as last column:

$$\left[\begin{array}{ccc|c} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{v} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{array}\right]$$

Solution: $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$

Cramer's Rule

For $A\mathbf{x} = \mathbf{b}$: $\quad x_i = \dfrac{\det(A_i)}{\det(A)}$

where A_i = matrix A with column i replaced by $\mathbf{b}$.

Characteristic Polynomial: $p(\lambda) = \det(A - \lambda I)$

Finding Eigenvalues: Solve $\det(A - \lambda I) = 0$

Finding Eigenvectors: Solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$

Cayley-Hamilton Theorem

Matrix satisfies its char. eqn: $p(A) = 0$ where $p(\lambda) = \det(A - \lambda I)$

Use: Express A^n in terms of I and A using the characteristic equation.

Finding A^{-1}: From $A^2 - A - 2I = 0$, isolate I: $2I = A^2 - A$, then multiply by A^{-1}: $2A^{-1} = A - I$

Diagonalization

Matrix Powers: $A^n = PD^nP^{-1}$ where $P = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots]$ (eigenvectors)

Matrix Exponential: $e^{At} = Pe^{Dt}P^{-1}$ where $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Steps: (1) Find eigenvalues/eigenvectors, (2) Form P, find P^{-1}, (3) Compute $Pe^{Dt}P^{-1}$

Diagonal Inverse: $D^{-1} = \begin{bmatrix} 1/d_1 & 0 & 0 \\ 0 & 1/d_2 & 0 \\ 0 & 0 & 1/d_3 \end{bmatrix}$

Gram-Schmidt Orthogonalization

Given vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots\}$, produce orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots\}$:

$\mathbf{u}_1 = \mathbf{v}_1 \quad \mathbf{u}_2 = \mathbf{v}_2 - \dfrac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$

$\mathbf{u}_3 = \mathbf{v}_3 - \dfrac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \dfrac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2$

Projection formula: $\text{proj}_{\mathbf{u}}\mathbf{v} = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$

Note: $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$

Matrix Exponential e^{At}

Fulmer's Method:

1. Find char. eqn: $|\lambda I - A| = 0$
2. Convert to DE: $\lambda^2 + 2\lambda + 6 \Rightarrow (D^2 + 2D + 6)y = 0$
3. Solve DE: $y = A_1e^{m_1t} + A_2e^{m_2t}$
4. Replace constants with ξ's: $e^{At} = \xi_1e^{m_1t} + \xi_2e^{m_2t}$
5. Differentiate both sides up to $(n-1)$th derivative
6. Set $t = 0$ for each equation to create system
7. Solve for ξ's, substitute back

Laplace Transforms

Definition of a Laplace Transform

The **Laplace transform** of a function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

In simple terms, the Laplace transform maps a function from the time domain t to the complex frequency domain s. Due to its linearity and its ability to convert derivatives into algebraic expressions in s, the resulting equation involves only algebraic terms, which are easier to manipulate. After simplification in the s-domain (see general shortcuts), the inverse Laplace transform is applied to obtain the solution in the time domain.

$$\begin{tikzcd}
\boxed{\text{Differential Equation}} \arrow[d, "\mathcal{L}"] & \\
\boxed{\text{Algebraic Equation in } s} \arrow[r, "\mathcal{L}^{-1}"] & \boxed{\text{Solution } f(t)}
\end{tikzcd}$$

Derivatives in s-Domain

Transform: $y(t) \to Y(s)$

First derivative: $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$

Second derivative: $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

Third derivative: $\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

Multiplication by t^n: $\mathcal{L}\{t^n g(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

Heaviside and Step Functions

Not all functions encountered in engineering are continuous. Step functions are piecewise-defined functions that change their value or functional form at specified points along the x-axis. They are commonly used to model systems with sudden inputs or switching behaviors.

Turn off at a: $\begin{cases} g(t), & t < a \\ 0, & t \geq a \end{cases} = g(t) - g(t)\, u(t-a)$

Turn on at a: $\begin{cases} 0, & t < a \\ g(t), & t \geq a \end{cases} = g(t)\, u(t-a)$

Switch at a: $\begin{cases} g(t), & t < a \\ h(t), & t \geq a \end{cases} = g(t) + [h(t) - g(t)]\, u(t-a)$

Properties of Laplace functions

Convolution: $(f * g)(t) = \displaystyle\int_0^t f(\tau)\, g(t-\tau)\, d\tau$

Laplace of convolution: $\mathcal{L}\{f * g\} = F(s) \cdot G(s)$

Integration property: $\mathcal{L}\left\{\displaystyle\int_0^t g(\tau)\, d\tau\right\} = \dfrac{1}{s} G(s)$

First Shifting Theorem (s-shifting)

If $\mathcal{L}\{f(t)\} = F(s)$, then: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Inverse: $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$

Common forms: $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$, $\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$, $\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$

Second Shifting Theorem

Forward: $\mathcal{L}\{u(t-a)\, g(t-a)\} = e^{-as} F(s)$

Inverse: $\mathcal{L}^{-1}\{e^{-as} F(s)\} = u(t-a)\, g(t-a)$

Rewriting $g(t)\, u(t-a)$: Can't apply shift theorem directly! Rewrite $g(t)$ in terms of $(t-a)$:

$$t\, u(t-1) = [(t-1)+1]\, u(t-1) = (t-1)u(t-1) + u(t-1)$$

Now each term fits the shift theorem: $\mathcal{L}\{(t-1)u(t-1)\} = e^{-s}/s^2$, $\quad \mathcal{L}\{u(t-1)\} = e^{-s}/s$

Periodic (Repeating) Functions

For $g(t)$ with period T: $\mathcal{L}\{g(t)\} = \frac{1}{1-e^{-sT}} \cdot \int_0^T e^{-st} g(t)\, dt$

If the function changes form, split the integral (multiplier stays): $\mathcal{L}\{g(t)\} = \frac{1}{1-e^{-sT}} \cdot \left[\int_0^a e^{-st} g_1(t)\, dt + \int_a^T e^{-st} g_2(t)\, dt\right]$

Tips for Inversion

1. **Completing the Square**
   For $s^2 + bs + c$, rewrite as
   $$\left(s + \tfrac{b}{2}\right)^2 + \left(c - \tfrac{b^2}{4}\right).$$
   Example: $s^2 + 6s + 34 = (s+3)^2 + 25$
   then apply the s-shifting theorem with $a = -3$ and $k = 5$.
2. **Partial Fraction Decomposition**
   Decompose rational functions into simpler terms:
   $$\frac{1}{(s+a)^n} = \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n}.$$
   See Useful Equalities for more examples.
3. **Standard Transform Matching**
   Before decomposing, check whether the expression matches a known Laplace pair (exponentials, sines, cosines, polynomials). This often avoids unnecessary algebra.

Solving Systems of ODEs

Method: (1) Take $\mathcal{L}$ of both equations (2) Substitute ICs: $\mathcal{L}\{x'\} = sX - x(0)$, etc. (3) Get algebraic system in $X(s), Y(s)$ (4) Solve for $X(s)$ and $Y(s)$ (elimination or Cramer's) (5) Inverse transform to get $x(t), y(t)$

Example setup: For $\begin{cases} 2x' + y' - 2x = 1 \\ x' + y' - 3x - 3y = 2 \end{cases}$

After $\mathcal{L}$: $\begin{cases} (2s-2)X + sY = \frac{1}{s} + \text{ICs} \\ (s-3)X + (s-3)Y = \frac{2}{s} + \text{ICs} \end{cases}$

Matrix exponentials: Laplace transforms can also be used for calculating exponentials of matrices as:

1. Compute $sI - A$ and $\det(sI - A)$
2. $(sI - A)^{-1} = \dfrac{\operatorname{adj}(sI - A)}{\det(sI - A)}$
3. Partial fractions on each entry
4. $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

Ordinary Differential Equations (ODEs)

0. Basics of Integration

Substitution: $\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$

By Parts: $\int u\,dv = uv - \int v\,du$

Order for choosing u (LIATE): Logs → Inverse trig → Algebraic → Trig → Exponential

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1. Integrating Factor – Converts linear to exact.

Form: $\dfrac{dy}{dx} + P(x)y = Q(x)$, $\quad \mu(x) = e^{\int P(x)\,dx}$

Multiply all sides by $\mu(x)$, and note the $\frac{d}{dx}$ on the left side.

2. Exact Equations

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

To solve: (1) Integrate M w.r.t. x, add $g(y)$. (2) Diff result w.r.t. y, set $= N(x,y)$. (3) Find $g'(y)$, integrate, write $\mu = C$.

3. Check for Homogeneous

$f(tx,ty) = t^n \cdot f(x,y) \Rightarrow$ degree n

4. Homogeneous to Separable

Sub $y = vx$: $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Or $x = uy$: $\dfrac{dx}{dy} = u + y\dfrac{du}{dy}$

5. Reduction of Order

First-Order System: For $x'' + ax' + bx = 0$: let $x_1 = x$, $x_2 = x'$. Then $X' = AX$, $A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$, $X(t) = e^{At}X(0)$

. . . .

For 2nd solution of $y'' + P(x)y' + Q(x)y = 0$: Given y_1, let $y_2 = v(x) \cdot y_1(x)$

$y_2' = v'y_1 + vy_1'$, $\quad y_2'' = v''y_1 + 2v'y_1' + vy_1''$. Plug in, simplify.
Tip: $w = v'$

6. Bernoulli Equations

$\dfrac{dy}{dx} + P(x)y = Q(x)y^n$

(1) Divide by y^n. (2) Sub $v = y^{1-n}$, so $v' = (1-n)y^{-n}y'$. (3) Plug in. (4) Use IF(μ) to solve, plug back.

7. Euler (Cauchy-Euler)

$x^2y'' + axy' + by = 0$

Let $x = e^t$, $t = \ln x$:

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Or trial $y = x^m$: $xy' = mx^m$, $x^2y'' = m(m-1)x^m$

8. Characteristic Roots For $y'' + ay' + by = 0$, assume $y = e^{mx}$ and solve the characteristic equation $m^2 + am + b = 0$.
Case I (distinct real): $y_h = C_1e^{m_1x} + C_2e^{m_2x}$
Case II (repeated): $y_h = (C_1 + C_2x)e^{mx}$
Case III (complex $\alpha \pm i\beta$):

$$y_h = e^{\alpha x}\left[C_1\cos(\beta x) + C_2\sin(\beta x)\right]$$

Differential Operator: $D^2y = \dfrac{d^2y}{dx^2}$, $\quad Dy = \dfrac{dy}{dx}$

9. Undetermined Coefficients

For RHS: e^{ax}, $\sin x$, $\cos x$, x^n

Steps: (1) Find y_h from char. roots. (2) Guess y_p from table. (3) Plug y_p in ODE, solve for coeffs. (4) $y = y_h + y_p$

RHS	Guess y_p
ke^{ax}	Ae^{ax}
kx^n	$A_nx^n + \cdots + A_1x + A_0$
$\sin(bx)$ or $\cos(bx)$	$A\cos(bx) + B\sin(bx)$
$e^{ax}\sin(bx)$	$e^{ax}[A\cos(bx) + B\sin(bx)]$
x^ne^{ax}	$(A_nx^n + \cdots + A_0)e^{ax}$
$x\sin(bx)$ or $x\cos(bx)$	$(Ax+B)\cos(bx) + (Cx+D)\sin(bx)$

Modification Rule: If $y_p \in y_h$, multiply by x^s (s = multiplicity of root)

Operator Shift: $P(D)[e^{ax}v(x)] = e^{ax}P(D+a)v(x)$

Ex: $P(D) = D^2(D-1)^3$, RHS $= e^xEx^3$ $(a = 1)$. Replace $D \to (D+1)$: $P(D+1) = (D+1)^2D^3$. Apply: $D^3(Ex^3) = 6E$, then $(D+1)^2(6E) = 6E$

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10. Variation of Parameters

$y'' + p(x)y' + q(x)y = g(x)$

Steps: (1) Find y_h roots $\Rightarrow y_1, y_2$. (2) Compute Wronskian $W \neq 0$. (3) Solve for u_1', u_2'. (4) Integrate. (5) $y_p = u_1y_1 + u_2y_2$. (6) $y = C_1y_1 + C_2y_2 + y_p$

2nd Order:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$u_1' = \frac{-y_2 \cdot g(x)}{W}, \quad u_2' = \frac{y_1 \cdot g(x)}{W}$$

$W_k = W$ with kth column replaced by $(0, 0, \ldots, 0, g(x))^T$

3rd Order:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$u_1' = \frac{1}{W}\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g & y_2'' & y_3'' \end{vmatrix}, \quad u_2' = \frac{1}{W}\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g & y_3'' \end{vmatrix}, \quad u_3' = \ldots$$

General nth Order:

$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0y = g(x)$

Given n homogeneous solutions $y_1, y_2, \ldots, y_n$:

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$y_p = \sum_{k=1}^{n} u_k(x)y_k(x)$ where $u_k' = \dfrac{W_k}{W}$

Series Solutions for ODEs

Power Series Method (Ordinary Points)

Recall that a power series is an expansion of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

For a linear differential equation

$$y'' + p(x)y' + q(x)y = r(x),$$

if $p(x), q(x), r(x)$ are analytic at x_0, then the solution is also analytic at x_0. This allows us to represent the solution as a power series and reduce the differential equation to an algebraic recurrence relation for the coefficients a_n. To solve such equations, we need to follow these steps:

1. Assume: $y = \sum_{n=0}^{\infty} a_n x^n$
$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

2. Shift indices to equalize powers of x^n

3. Write out and equalize with RHS

4. Develop recurrence for a_n, a_{n+1}, or a_{n+2}

5. Solution: $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

Common Series:
$e^{ax} = \sum \frac{(ax)^n}{n!}, \quad \sin x = \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum (-1)^n \frac{x^{2n}}{(2n)!}$

Initial Conditions: If $y(0), y'(0)$ given, verify by taking derivatives and substituting.

Taylor form: $y = \sum \frac{y^{(n)}(0)}{n!} x^n$

Frobenius Method (Singular Points)

When $p(x)$ or $q(x)$ contains singular terms such as $\frac{1}{x}$ or $\frac{1}{x^2}$, the standard power series method generally fails because the solution need not be analytic at the expansion point.

A point x_0 is called a **regular singular point** if $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ are analytic at x_0. Although the coefficients are singular, the differential equation retains enough structure to admit solutions of a generalized power series form.

In this case, the appropriate technique is the **Frobenius Method**, which extends the power series approach to handle regular singular points.

Assume: $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}$ where r is found from the **indicial equation**.
$y' = \sum_{n=0}^{\infty} (n + r) a_n (x - x_0)^{n+r-1}$
$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n (x - x_0)^{n+r-2}$
Substitute into ODE $\to$ indicial equation for r arises from the lowest power coefficient. The roots r_1, r_2 (with $r_1 \geq r_2$) determine the form of solutions.

Cases for Indicial Roots

Case I: Distinct roots, $r_1 - r_2 \notin \mathbb{Z}$
$y_1 = x^{r_1} \sum a_n x^n, \quad y_2 = x^{r_2} \sum b_n x^n$
Both solutions found directly by substitution.

Case II: Equal roots, $r_1 = r_2 = r$
$y_1 = x^r \sum a_n x^n$
$y_2 = y_1 \ln x + x^r \sum c_n x^n$

Abel's Formula: Given y_1, find y_2 via reduction of order:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{[y_1]^2}\,dx$$

where $P(x)$ is from the normalized form $y'' + P(x)y' + Q(x)y = 0$.

Case III: Roots differ by integer, $r_1 - r_2 = k \in \mathbb{Z}^+$
$y_1 = x^{r_1} \sum a_n x^n$ (use larger root)
For y_2: Try the second root r_2 directly as in Case I. This may work and give a valid second solution.
If the recurrence collapses (division by zero), then use:

$$y_2 = C y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} c_n x^n$$

where C is a constant (possibly zero) determined by substitution.

Constructing the Solution

After finding the series coefficients, write:

$$y(x) = x^r \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]$$

Factor out common terms and identify known functions (exponentials, trig functions, Bessel functions, etc.) when possible.

Bessel's Functions Bessel Functions are solutions to Bessel equations with the form:
$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$
They can be considered as a special case of the Frobenius solution when $r = n$ and they arise naturally when solving PDEs (heat, wave, Laplace) in cylindrical/spherical coordinates.
Bessel function of the first kind:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

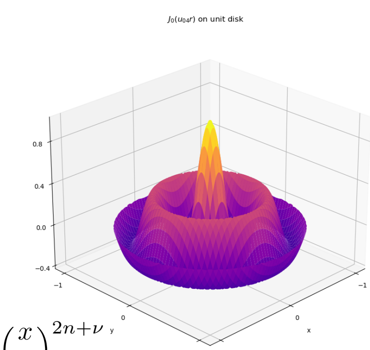

Figure 2: *
$J_0(u_{04} r)$ on unit disk

Case 1: $\nu \neq$ integer — J_ν and $J_{-\nu}$ are linearly independent:

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

Case 2: $\nu = n$ (integer) — $J_{-n} = (-1)^n J_n$, need Bessel function of the second kind:

$$y = c_1 J_n(x) + c_2 Y_n(x)$$

where $Y_n(x) = \lim_{\nu \to n} \dfrac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$

Note: $J_n(0)$ is bounded; $Y_n(0) \to -\infty$. For domains including origin, set $c_2 = 0$.

Partial Differential Equations

Partial Differential Equations (PDEs) arise in connection with various physical and geometrical problems when the functions involved depend on two or more independent variables. The order of the highest derivative is called the order of the equation. In general, the solution space to any given PDE is very large, requiring us to define boundary conditions (e.g., $f(0,t)$) and initial conditions (e.g., $f(x,0)$) to achieve a useful solution.

Method	When Useful
Separation of Variables	Simple geometries, homogenous BCs
Fourier Series/Transform	Periodic/infinite domains
Laplace Transform	Time-dependent, IVPs
Green's Functions	Inhomogeneous PDEs
Method of Characteristics	1st-order & hyperbolic
Finite Difference	Complex geom., any PDE
Finite Element	Irregular domains

Classification of 2nd-Order Linear PDE:

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

Type	Condition	Example
Hyperbolic	$B^2 - 4AC > 0$	Wave eq.
Parabolic	$B^2 - 4AC = 0$	Heat eq.
Elliptic	$B^2 - 4AC < 0$	Laplace eq.

Separation of Variables

Steps: 1. Assume $u(x,t) = X(x)T(t)$ and substitute into PDE. 2. Divide both sides to separate variables: $\frac{\text{terms in } t}{\text{coeff}} = \frac{\text{terms in } x}{\text{coeff}} = -\lambda$ 3. Check three cases for spatial ODE $X'' + \lambda X = 0$:

$\lambda < 0 \quad X = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ (or sinh/cosh)
$\lambda = 0 \quad X = c_1 + c_2 x$
$\lambda > 0 \quad X = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

4. Apply homogeneous BCs to find which λ gives nontrivial solutions $\Rightarrow$ eigenvalues λ_n, eigenfunctions $X_n(x)$. 5. Solve temporal ODE for each λ_n to get $T_n(t)$. 6. Form general solution: $u(x,t) = \sum_n c_n X_n(x) T_n(t)$ 7. Apply IC to find coefficients c_n using orthogonality.

BCs	$X_n(x)$, λ_n
$u(0,t) = u(L,t) = 0$	$\sin\frac{n\pi x}{L}$, $\lambda_n = \frac{n^2\pi^2}{L^2}$
$u_x(0,t) = u_x(L,t) = 0$	$\cos\frac{n\pi x}{L}$, $\lambda_n = \frac{n^2\pi^2}{L^2}$
$u(0,t) = u_x(L,t) = 0$	$\sin\frac{(2n-1)\pi x}{2L}$, $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$

2D Problems (e.g., $u_t = K(u_{xx} + u_{yy})$ on $[0,b] \times [0,c]$):

Assume $u = X(x)Y(y)T(t)$, separate $\Rightarrow$ eigenvalues in both x and y.

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{b} \sin\frac{m\pi y}{c} \times e^{-K[(n\pi/b)^2 + (m\pi/c)^2]t} \quad (1)$$

Finding A_{nm}: At $t = 0$: $u(x,y,0) = f(x,y)$. Multiply both sides by $\sin\frac{n\pi x}{b}\sin\frac{m\pi y}{c}$, integrate over domain, use orthogonality:

$$A_{nm} = \frac{4}{bc}\int_0^b\int_0^c f(x,y)\sin\frac{n\pi x}{b}\sin\frac{m\pi y}{c}\,dy\,dx$$

1D Case with Constant IC: If $u(x,0) = 1 = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$

Multiply by $\sin(m\pi x)$, integrate 0 to 1: $\int_0^1 \sin(m\pi x)\,dx = \frac{1-(-1)^m}{m\pi} = \frac{1}{2}B_m$

$$B_m = \frac{2(1-(-1)^m)}{m\pi} = \begin{cases} 0 & m \text{ even} \\ \frac{4}{m\pi} & m \text{ odd} \end{cases}$$

Solution: $u = \sum_{n=1,3,5,\ldots} \frac{4}{n\pi}\sin(n\pi x)e^{-n^2\pi^2 kt}$ or $u = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin((2k-1)\pi x)e^{-(2k-1)^2\pi^2 kt}$

Tip: $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

Systems of PDEs / Method of Characteristics

Given: $\mathbf{u}_t + A\mathbf{u}_x = 0$. Diagonalize: $A = PDP^{-1}$. Substitute $\mathbf{v} = P^{-1}\mathbf{u}$:

$\mathbf{v}_t + D\mathbf{v}_x = 0 \Rightarrow$ decoupled equations $v_{i,t} + \lambda_i v_{i,x} = 0$

Solve $v_t + cv_x = 0$: Characteristics: $\frac{dx}{dt} = c \Rightarrow x - ct = \text{const}$

Solution: $v = F(x - ct)$. If $v_t - cv_x = 0$: $v = F(x + ct)$

After solving, transform back: $\mathbf{u} = P\mathbf{v}$

Laplace Transform Method for PDEs

$\mathcal{L}\{u_t\} = sU(x,s) - u(x,0)$
$\mathcal{L}\{u_{tt}\} = s^2U(x,s) - su(x,0) - u_t(x,0)$
$\mathcal{L}\{u_x\} = \frac{\partial U}{\partial x} = U_x(x,s)$
$\mathcal{L}\{u_{xx}\} = \frac{\partial^2 U}{\partial x^2} = U_{xx}(x,s)$
$\mathcal{L}\{u_{xt}\} = sU_x(x,s) - u_x(x,0)$

Example: Heat Equation via Laplace Transform

PDE: $u_t = ku_{xx}$, $u(x,0) = f(x)$, $u(0,t) = 0$, $u(L,t) = 0$

Apply $\mathcal{L}$ in t: $sU - f(x) = kU_{xx}$

Rearrange: $U_{xx} - \frac{s}{k}U = -\frac{f(x)}{k}$ (ODE in x)

Solve ODE, apply BCs, then invert.

Duhamel's Principle For nonhomogeneous PDE with zero ICs: $u_t = ku_{xx} + F(x,t)$ or $u_{tt} = c^2u_{xx} + F(x,t)$. The idea is to treat the source term $F(x,\tau)$ as an initial condition at time τ and superpose solutions over all past times.

Steps: 1. Solve the corresponding homogeneous problem with IC $= F(x,\tau)$ at $t = \tau$. 2. Call this solution $v(x,t;\tau)$ (parameterized by τ). 3. Final solution: $u(x,t) = \int_0^t v(x,t;\tau)\,d\tau$

Sturm-Liouville Problems

Form: $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$

Properties: Eigenvalues λ_n are real; eigenfunctions y_n orthogonal with weight $r(x)$:

$$\int_a^b r(x)y_n(x)y_m(x)\,dx = 0 \quad (n \neq m)$$

Eigenfunction expansion: $f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$

$c_n = \frac{\int_a^b r(x)f(x)y_n(x)\,dx}{\int_a^b r(x)y_n^2(x)\,dx}$

Systems of ODEs and PDEs

Many physics and engineering problems include coupled differential equations. For example, n coupled springs will result in n coupled differential equations arising from the equalization of Newtonian and Hooke's Forces. First-order ODE systems can also be used as a reduction of order method with the introduction of new variables for successive derivatives:

$$y'' + 3y' + 2y = 0.$$

Define the state variables

$$x_1 = y, \qquad x_2 = y'.$$

Then the system can be written as the following first-order system:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In general, $\mathbf{x}' = A\mathbf{x}$ where A encodes the system coefficients.

Stability Classification (2D Systems)

Based on eigenvalues λ_1, λ_2 of matrix A:

Stable node:	both real, $\lambda_1, \lambda_2 < 0$
Unstable node:	both real, $\lambda_1, \lambda_2 > 0$
Saddle:	real, opposite signs
Stable spiral:	complex, $\mathrm{Re}(\lambda) < 0$
Unstable spiral:	complex, $\mathrm{Re}(\lambda) > 0$
Center:	purely imaginary $\lambda = \pm i\beta$

Quick check: $\tau = a + d$ (trace), $\Delta = ad - bc$ (det)

- $\Delta < 0$: saddle (unstable)
- $\Delta > 0$, $\tau < 0$: stable (node or spiral)
- $\Delta > 0$, $\tau > 0$: unstable (node or spiral)
- $\Delta > 0$, $\tau = 0$: center

Solution to homogenous system

1. **Real, distinct eigenvalues:** $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots$.
2. **Repeated real eigenvalue:** $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{u})$, where $(A - \lambda I)\mathbf{u} = \mathbf{v}$. If the multiplicity is three,

   $$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} + c_3\left(\tfrac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3\right)e^{\lambda t}.$$
3. **Complex eigenvalues** $\lambda = \alpha \pm i\beta$: $e^{\lambda t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$, $\mathbf{x}(t) = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$.

Non-Homogeneous Systems

For $\mathbf{Y}' = A\mathbf{Y} + \mathbf{F}(t)$ with A upper triangular, solve bottom-up:

Ex: $$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ 7e^{2t} \end{bmatrix}$$

Expand: $y_3' = 2y_3 + 7e^{2t}$, $y_2' = 2y_2 + y_3 + 3e^{2t}$, $y_1' = 2y_1 + y_2 + e^{2t}$

Step 1: Solve $y_3' - 2y_3 = 7e^{2t}$ with $\mu = e^{-2t}$: $\frac{d}{dt}(y_3 e^{-2t}) = 7 \Rightarrow y_3 = (7t + C_3)e^{2t}$

Step 2: Substitute y_3 into $y_2' - 2y_2 = y_3 + 3e^{2t}$, solve with $\mu = e^{-2t}$

Step 3: Substitute y_2 into $y_1' - 2y_1 = y_2 + e^{2t}$, solve with $\mu = e^{-2t}$

General formula: $\mathbf{x}(t) = e^{At}C + e^{At}\int e^{-As}\mathbf{F}(s)\,ds$

2nd Order Systems (Decoupling)

For coupled second-order systems $\mathbf{x}'' = A\mathbf{x} + \mathbf{f}(t)$, we use eigenvalue decomposition to decouple into independent ODEs.

Method: Let $\mathbf{x} = P\mathbf{u}$ where P is the matrix of eigenvectors of A.

Example: $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

Steps:

1. Find eigenvalues λ_1, λ_2 of A and construct eigenvector matrix P.
2. Substitute $\mathbf{x} = P\mathbf{u}$ into the system: $P\mathbf{u}'' = AP\mathbf{u} + \mathbf{f}$
3. Multiply both sides by P^{-1}: $\mathbf{u}'' = P^{-1}AP\mathbf{u} + P^{-1}\mathbf{f}$
4. Since $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2)$, the system decouples:

   $$u_1'' = \lambda_1 u_1 + (P^{-1}\mathbf{f})_1, \qquad u_2'' = \lambda_2 u_2 + (P^{-1}\mathbf{f})_2$$
5. Solve each independent ODE using standard methods (undetermined coefficients, variation of parameters).
6. Back-substitute to get original variables: $\mathbf{x} = P\mathbf{u}$

Matrix Exponentials e^{At}

Putzer's Method for (2×2)

$e^{At} = u_0 I + u_1 A$ for 2×2 matrices

For distinct eigenvalues λ_1, λ_2:

$$\begin{cases} e^{\lambda_1 t} = u_0 + u_1\lambda_1 \\ e^{\lambda_2 t} = u_0 + u_1\lambda_2 \end{cases}$$

Solving: $u_1 = \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$, $\quad u_0 = e^{\lambda_1 t} - u_1\lambda_1$

Series Expansion for e^{At}

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

For nilpotent matrices ($A^k = 0$ for some k), series terminates.

Diagonalization for e^{At}

If $A = PDP^{-1}$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$ and P has eigenvector columns:

$$e^{At} = Pe^{Dt}P^{-1}$$

where $e^{Dt} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots)$

Vector Calculus & Line Integrals

1. Vector Functions

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad \mathbf{r}'(t) = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$

$\frac{d}{dt}[f\mathbf{r}] = f'\mathbf{r} + f\mathbf{r}', \quad \frac{d}{dt}[\mathbf{r}_1 \cdot \mathbf{r}_2] = \mathbf{r}_1' \cdot \mathbf{r}_2 + \mathbf{r}_1 \cdot \mathbf{r}_2'$

$\frac{d}{dt}[\mathbf{r}_1 \times \mathbf{r}_2] = \mathbf{r}_1' \times \mathbf{r}_2 + \mathbf{r}_1 \times \mathbf{r}_2'$

---

2. Arc Length & TNB

$s = \int_a^b |\mathbf{r}'|\, dt, \quad \mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \quad \mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$

---

3. Curvature κ

$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \quad$ 2D: $\kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}$

Circle: $\kappa = 1/a \quad$ Helix: $\kappa = \frac{a}{a^2+c^2}$

---

4. Velocity & Acceleration

$\mathbf{v} = \mathbf{r}'$, Speed $= |\mathbf{v}|$, $\mathbf{a} = \mathbf{r}''$, $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$

$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}, \quad a_N = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$

---

5. Gradient & Directional Derivative

$\nabla f = \langle f_x, f_y, f_z \rangle, \quad \boxed{D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}}$

Note: "Derivative of f in direction of $\mathbf{v}$" means:

- Take partials of $f \to$ get ∇f, evaluate at given point
- Normalize $\mathbf{v} \to \mathbf{u} = \mathbf{v}/|\mathbf{v}|$
- Dot them: $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$

---

6. Tangent Plane & Normal Line

For $F(x, y, z) = c$ at point $P_0(x_0, y_0, z_0)$:

Step 1: Find $\nabla F = \langle F_x, F_y, F_z \rangle$

Step 2: Evaluate at P_0: $\mathbf{n} = \nabla F|_{P_0} = \langle a, b, c \rangle$

Step 3: Tangent: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Normal line: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$

Perpendicular tangent planes: Two surfaces $F = 0$, $G = 0$ have $\perp$ tangent planes where $\nabla F \cdot \nabla G = 0$

---

7. Divergence & Curl

For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$:

$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$= (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$

---

8. Vector Identities

$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + \mathbf{F} \cdot \nabla f$

$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \quad$ (BAC-CAB)

---

9. Line Integrals

Along C: $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$

$\int_C G\, dx = \int_a^b G\, x'(t)\, dt, \quad \int_C G\, dy = \int_a^b G\, y'(t)\, dt$

$\int_C G\, ds = \int_a^b G\sqrt{(x')^2 + (y')^2 + (z')^2}\, dt$

If $y = f(x)$: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

---

10. Work & $\int \mathbf{F} \cdot d\mathbf{r}$

For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and curve C:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

Parametric form ($\mathbf{r}(t)$, $a \le t \le b$):

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Work depends on the path unless $\mathbf{F}$ *is conservative.*

---

11. Conservative Vector Fields

$\mathbf{F}$ is **conservative** if there exists f such that $\mathbf{F} = \nabla f$.

On a simply connected domain, the following are equivalent:

$$\nabla \times \mathbf{F} = \mathbf{0} \iff \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \iff \text{path independence}$$

Fundamental Theorem for Line Integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

Shortcut: find f instead of integrating.

---

11. Green's Theorem

$\oint_C P\, dx + Q\, dy = \iint_D (Q_x - P_y)\, dA$

Transfers between line integrals and double integrals.

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12. Surface Area

For surface $z = f(x, y)$ over region R: $A(S) = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$

---

13. Polar Coordinates in Double Integrals

$x = r\cos\theta$, $y = r\sin\theta$, $dA = r\, dr\, d\theta$

$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Tip: $\int_0^\pi |\cos\theta|\, d\theta = 2\int_0^{\pi/2} \cos\theta\, d\theta$ (use symmetry for $|\cdot|$)

---

14. Stokes' Theorem

$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$

Transfers between line integrals and surface integrals.

---

15. Divergence Theorem

$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$

Transfers between surface integrals and volume integrals.

Optimization Problems

1. Discrete Least-Squares Regression

Goal: Find coefficients that minimize the sum of squared errors for data points (x_i, y_i).

Linear fit: $\hat{y} = a + bx$. Minimize $L(a,b) = \sum_{i=1}^{n}(y_i - a - bx_i)^2$

1.1 Take partial derivatives and set to zero:

$$\frac{\partial L}{\partial a} = -2\sum(y_i - a - bx_i) = 0$$

$$\frac{\partial L}{\partial b} = -2\sum x_i(y_i - a - bx_i) = 0$$

1.2 Simplify to normal equations and solve:

$$\sum y_i = na + b\sum x_i$$

$$\sum x_i y_i = a\sum x_i + b\sum x_i^2$$

1.3 Matrix style:$\mathbf{A}^T\mathbf{A}\vec{c} = \mathbf{A}^T\vec{y}$ where $\mathbf{A} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$,$\vec{c} = \begin{pmatrix} a \\ b \end{pmatrix}$

Closed-form solution: $b = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2}, \quad a = \bar{y} - b\bar{x}$

---

2. Continuous Least-Squares

Goal: Find constants a_i minimizing:

$$E(a_1, \ldots, a_n) = \int_{x_1}^{x_2}\left[f(x) - \sum_{i=1}^{n} a_i\phi_i(x)\right]^2 dx$$

2.1 Set $\frac{\partial E}{\partial a_i} = 0$ for each i.

2.2 Apply Leibniz rule (move derivative inside): $\int_{x_1}^{x_2} 2[f(x) - \hat{f}(x)](-\phi_i(x))\,dx = 0$

2.3 Normal equations:

$$\int_{x_1}^{x_2} f(x)\phi_i(x)\,dx = \sum_{j=1}^{n} a_j \int_{x_1}^{x_2} \phi_i(x)\phi_j(x)\,dx$$

Matrix form: $\mathbf{G}\vec{a} = \vec{b}$ where $G_{ij} = \int \phi_i\phi_j\,dx$ and $b_i = \int f\phi_i\,dx$

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3. Critical Points & Classification

Single variable: Set $f'(x) = 0$, solve for critical points.

Second derivative test (1D):

- $f''(c) > 0 \Rightarrow$ local minimum
- $f''(c) < 0 \Rightarrow$ local maximum
- $f''(c) = 0 \Rightarrow$ inconclusive

Two variables: Set $f_x = 0$ and $f_y = 0$, solve system.

Second derivative test (2D): Compute Hessian determinant at critical point:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

- $D > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum
- $D > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum
- $D < 0 \Rightarrow$ saddle point
- $D = 0 \Rightarrow$ inconclusive

Note: Always check boundary values for closed domains.

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4. Lagrange Multipliers

Goal: Optimize $f(x,y)$ subject to constraint $g(x,y) = c$.

Key idea: At constrained extrema, ∇f is parallel to ∇g (can't improve f while staying on constraint).

Method: Solve $\nabla f = \lambda\nabla g$ with constraint:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x,y) = c$$

Steps:

4.1 Write Lagrangian: $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - c)$

4.2 Set $\nabla\mathcal{L} = 0$: solve $\mathcal{L}_x = 0$, $\mathcal{L}_y = 0$, $\mathcal{L}_\lambda = 0$

4.3 Classify critical points by comparing f values or using bordered Hessian.

Multiple constraints: $\nabla f = \lambda_1\nabla g_1 + \lambda_2\nabla g_2 + \cdots$

Sensitivity: $\lambda = \frac{\partial f^*}{\partial c}$ (how optimal value changes if constraint relaxed)

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5. Common Optimization Problems

General approach:

5.1 Identify the objective function (minimize/maximize) and constraint.

5.2 Use constraints to reduce variables (substitution) or use Lagrange multipliers.

5.3 Differentiate, set to zero, solve for critical points.

5.4 Verify min/max using second derivative test or boundary analysis.

Tip: For distance problems, minimize D^2 instead of D (same critical points, easier algebra).

Example Types:

Distance/Proximity: Find point on $y = g(x)$ closest to (x_0, y_0). Minimize $f(x) = (x - x_0)^2 + (g(x) - y_0)^2$.

Beam problem: Rod over fence (height h, distance w). Use similar triangles: $y = \frac{h(w+x)}{x}$, minimize $L^2 = (w+x)^2 + y^2$.

Surface area/volume: For cylinder with fixed $V = \pi r^2 h$, minimize $S = 2\pi r^2 + 2\pi rh$. Substitute $h = V/(\pi r^2)$, differentiate. Optimal: $h = 2r$.

Path of least time (Snell's Law): Minimize $T(x) = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}$. Result: $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.