Department of Computer Science

THEORY OF COMPUTATION

23MT2014

Topic:

COMPLEXITY CLASSES: P & NP

Session - 24

AIM OF THE SESSION

To make students explain the fundamental concepts of complexity classes P and NP, their significance in computational theory, and the implications of problems classified within these classes.

INSTRUCTIONAL OBJECTIVES

By the end of this course, students will be able to:

1. Explain the definitions of the complexity classes P and NP.
2. Illustrate examples of problems in P and NP, highlighting the differences between them.
3. Discuss the importance of the P vs NP question and its implications for computer science and other fields.

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define and distinguish between the complexity classes P and NP.
2. Identify and categorize problems as belonging to P, NP, or NP-complete.
3. Evaluate the significance of the P vs NP problem and its potential impact on computational theory and practical applications.

CLASSIFYING PROBLEMS BY TIME COMPLEXITY

- **Efficiency Analysis:** **How does complexity change with input size?**
- **Resource Management:** **What resources are needed?**
- **Comparative Benchmarking:** **Which algorithm performs best?**
- **Scalability Prediction :** **How does performance change with size?**
- **Theoretical Foundation:** **What is the problem's complexity class?**
- **Optimization guidance:** **Where are the bottlenecks?**

CLASS OF LANGUAGES

- Let $t: \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Define the *time complexity class*, $\boldsymbol{TIME(t(n))}$, to be the collection of all languages that are decided by an $O(t(n))$ time Turing Machine.

POLYNOMIAL TIME

- The time taken by an algorithm (t) increases proportionally to the input size (n) raised to a power, k.
- Ex: n^2, n^3, n^4 etc.
- The time complexity is a polynomial function of the input size.

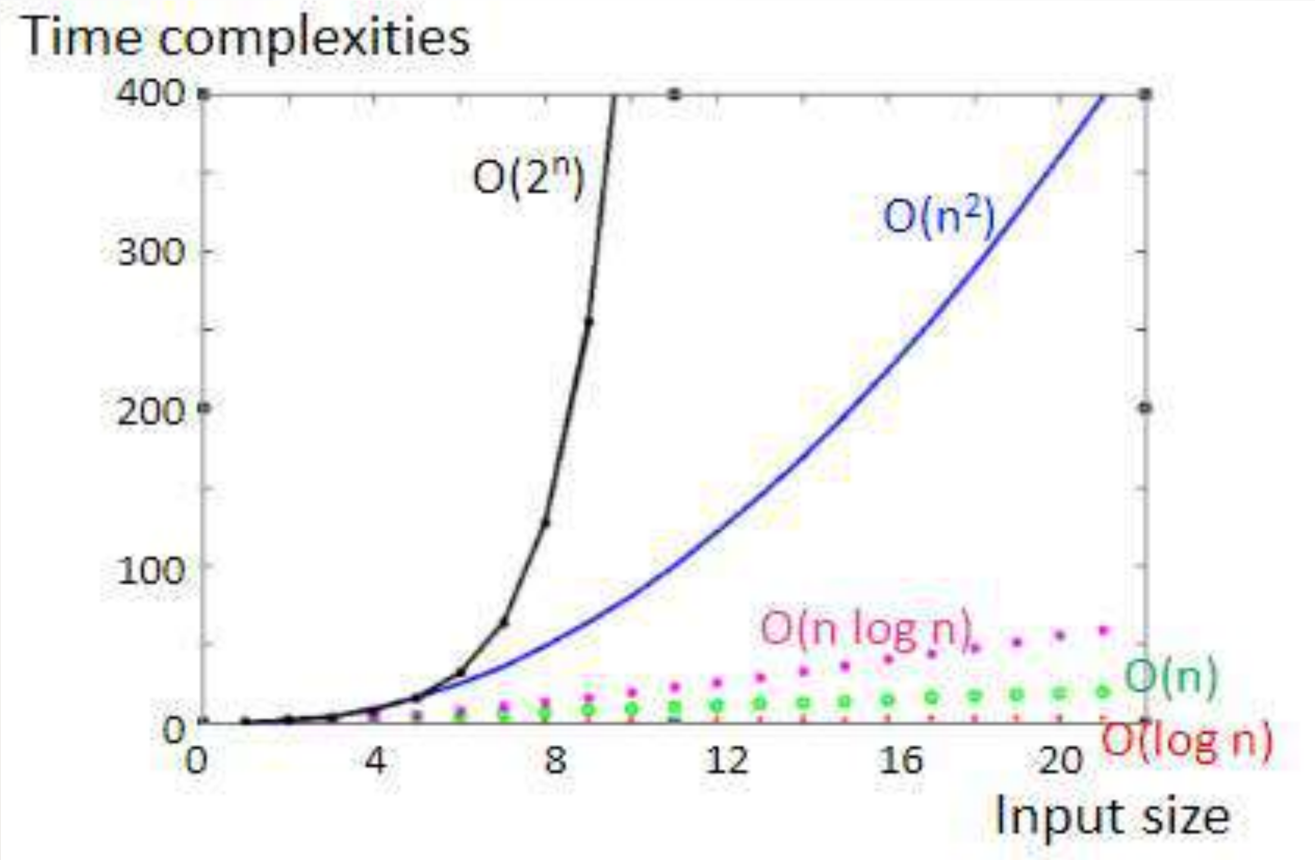
Ex: Quicksort, Merge sort

EXPONENTIAL TIME

- The time taken by the algorithm (t) increases exponentially with the input size (n).
- Ex: 2^n, 3^n, 5^n etc.
- The time complexity is an exponential function of the input size.
- Example:

1. Brute-force algorithm for solving Travelling salesperson problem
2. Solving the knapsack problem using dynamic programming has a time complexity of $O(2^n)$.

POLYNOMIAL VS. EXPONENTIAL TIME



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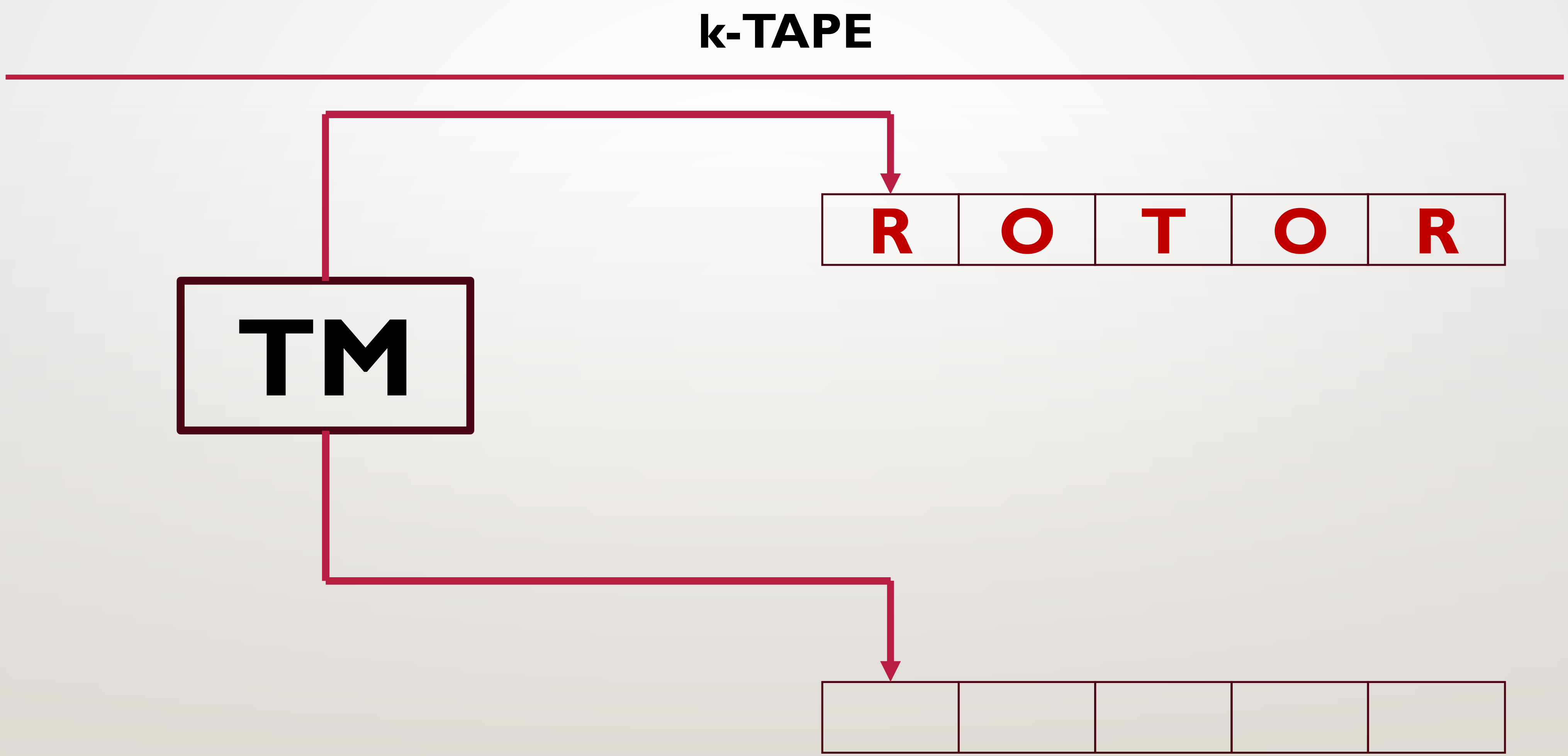

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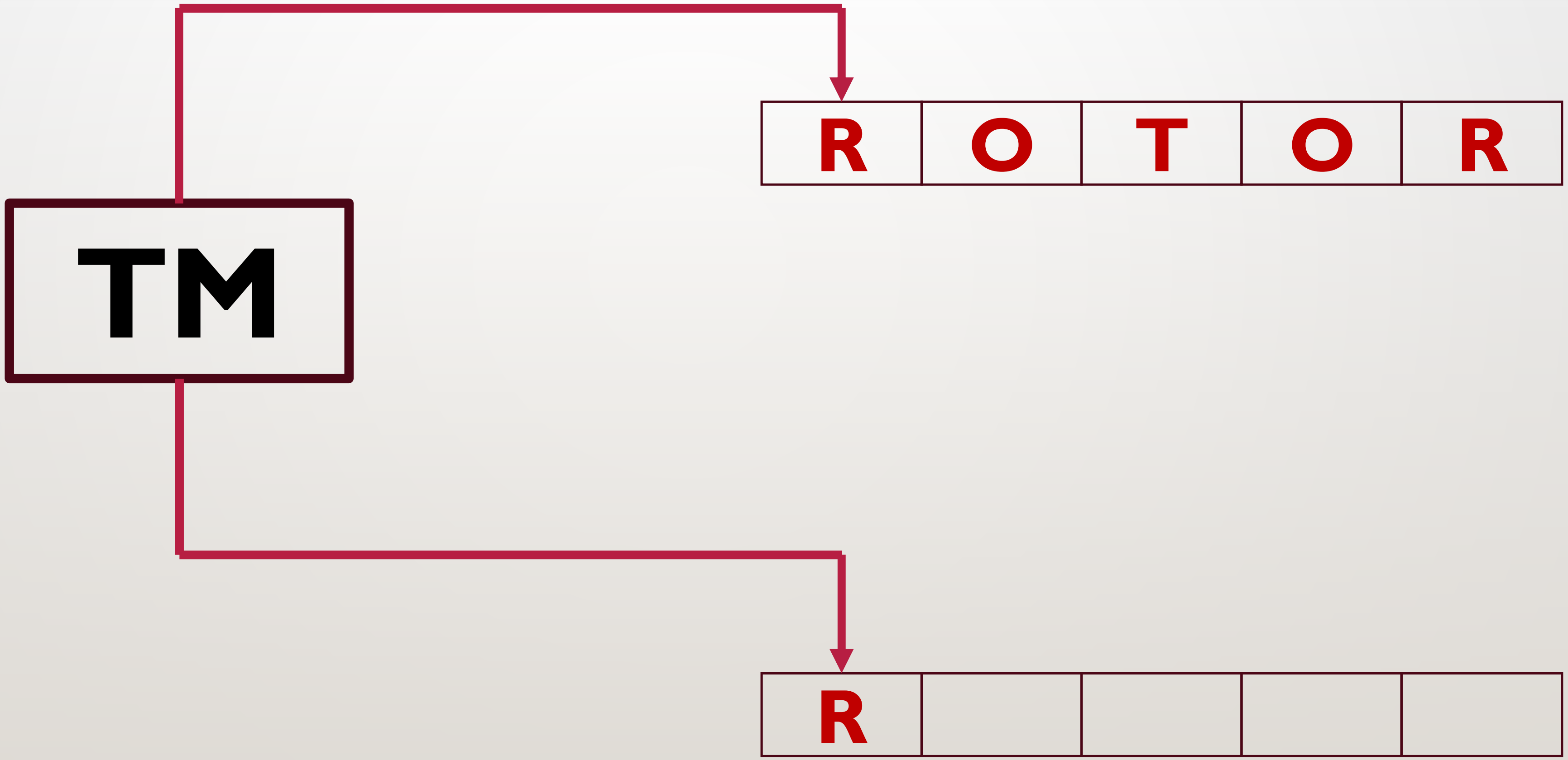

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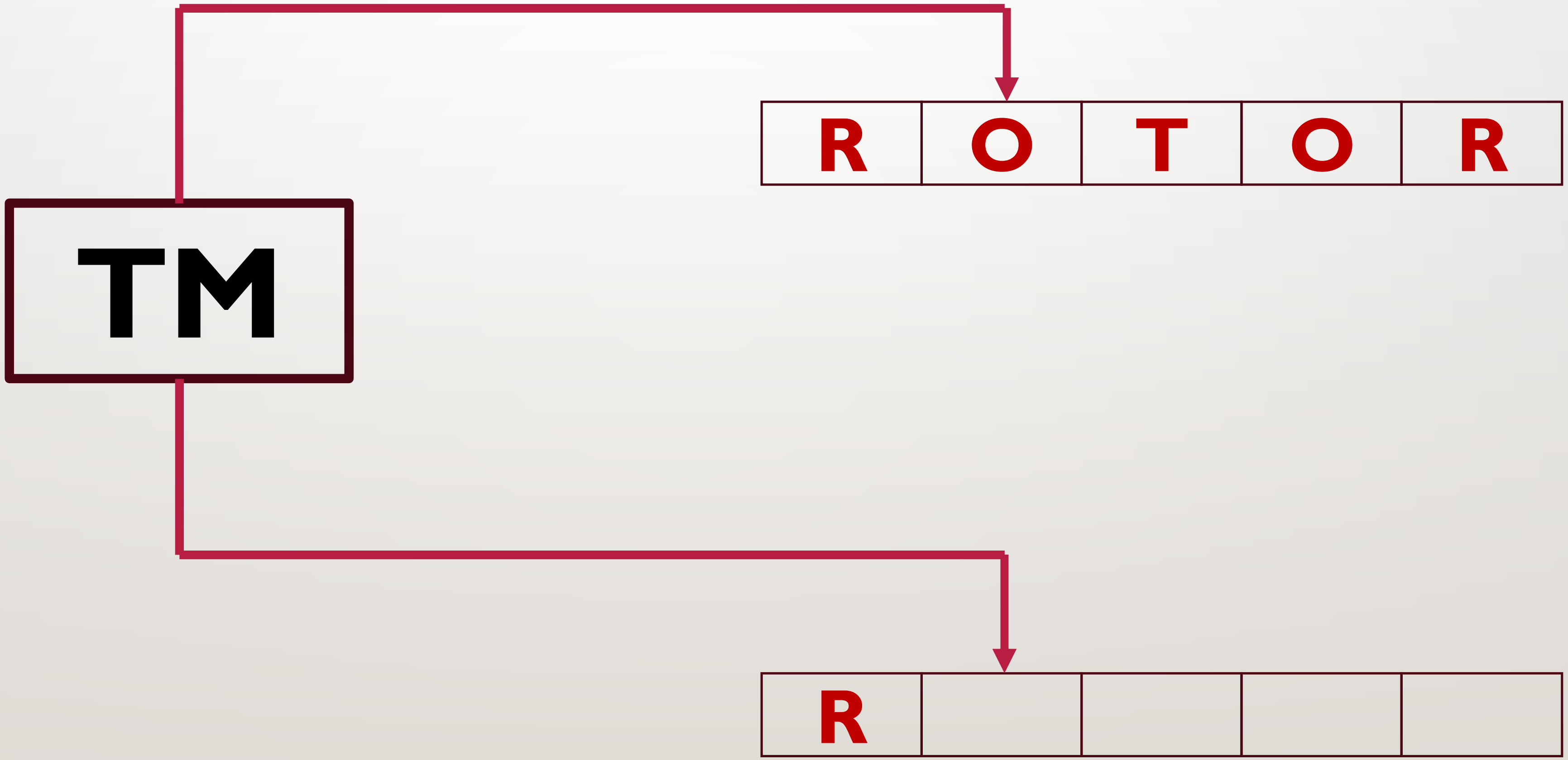

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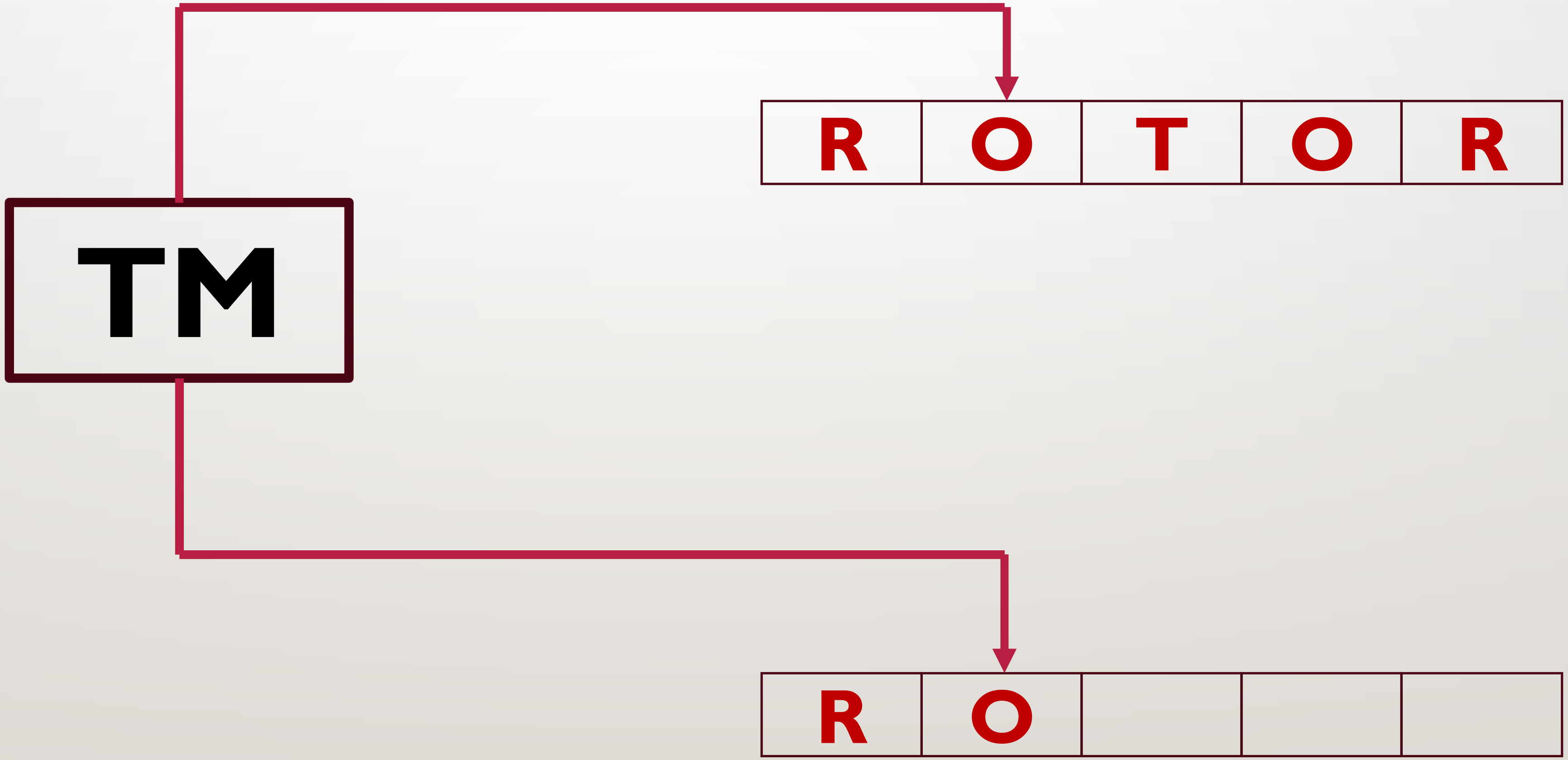

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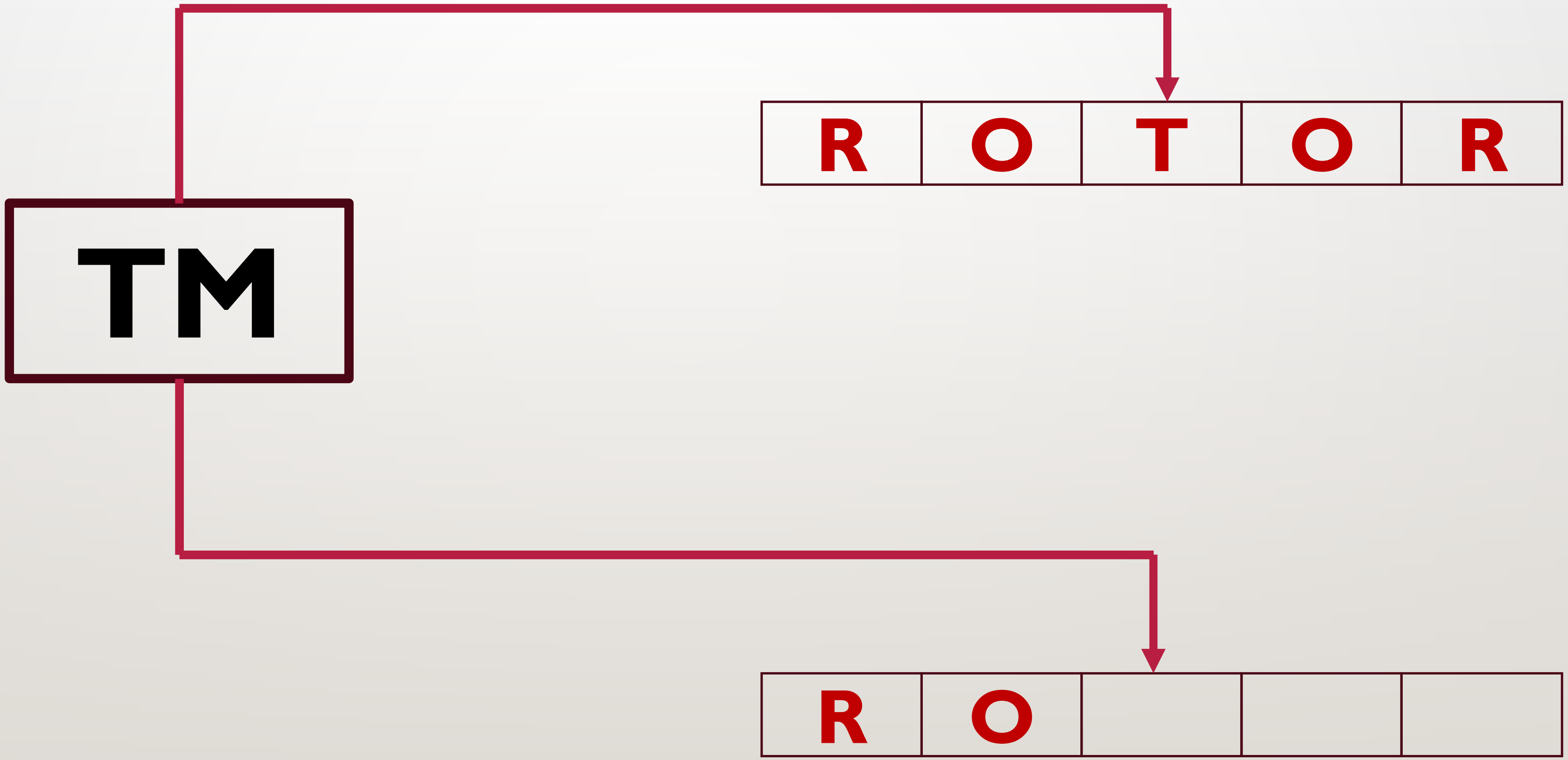

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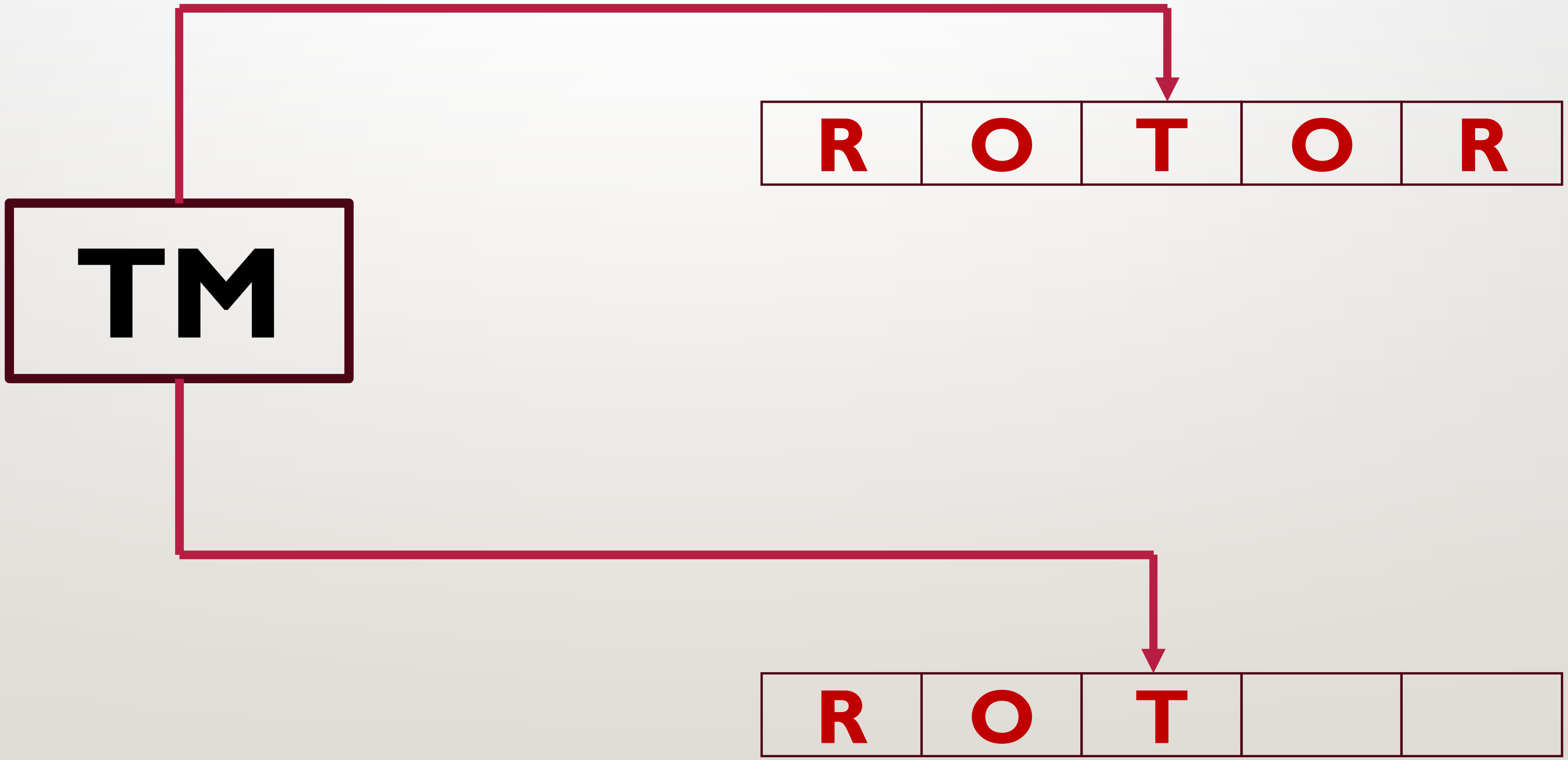

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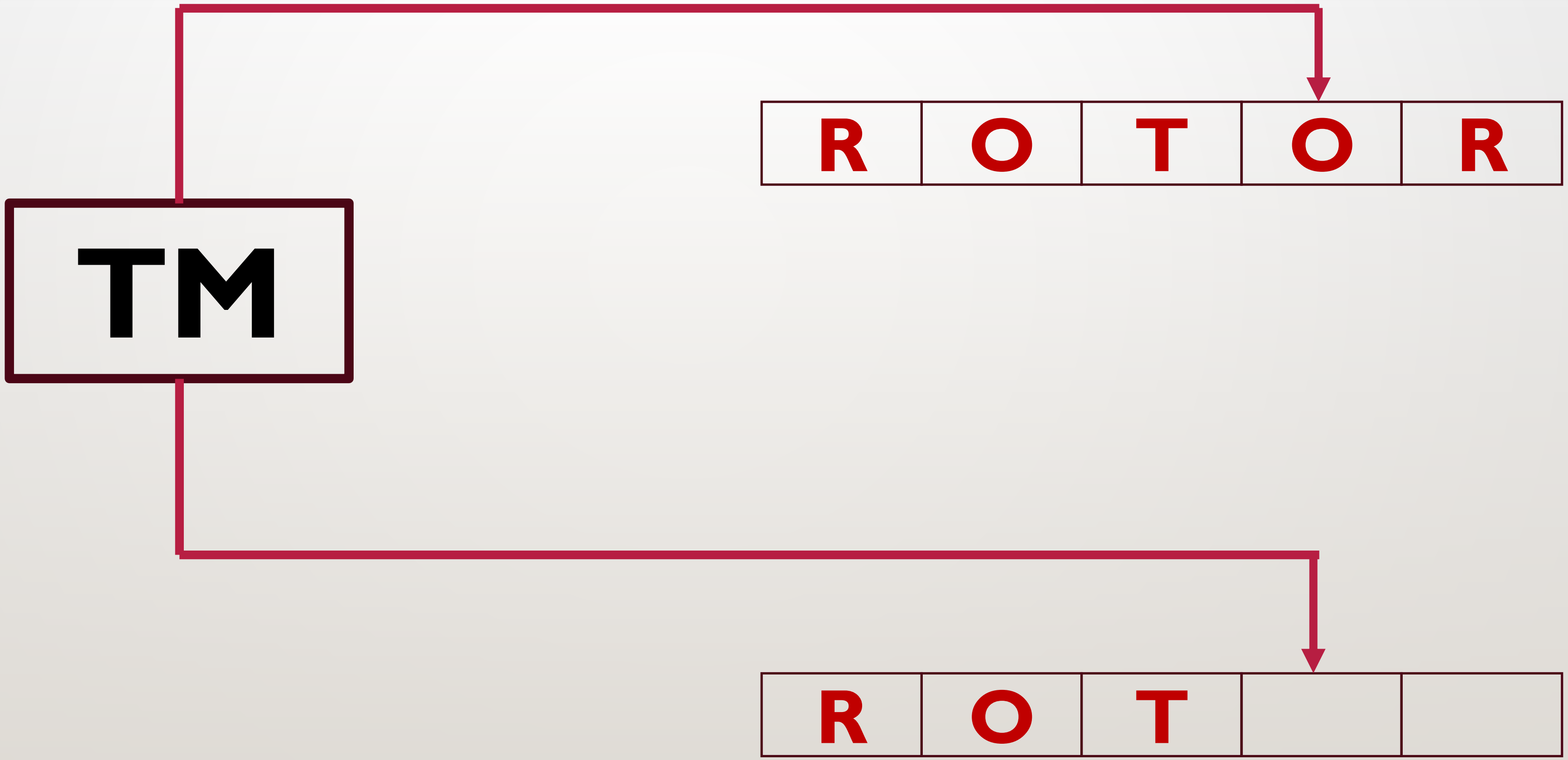

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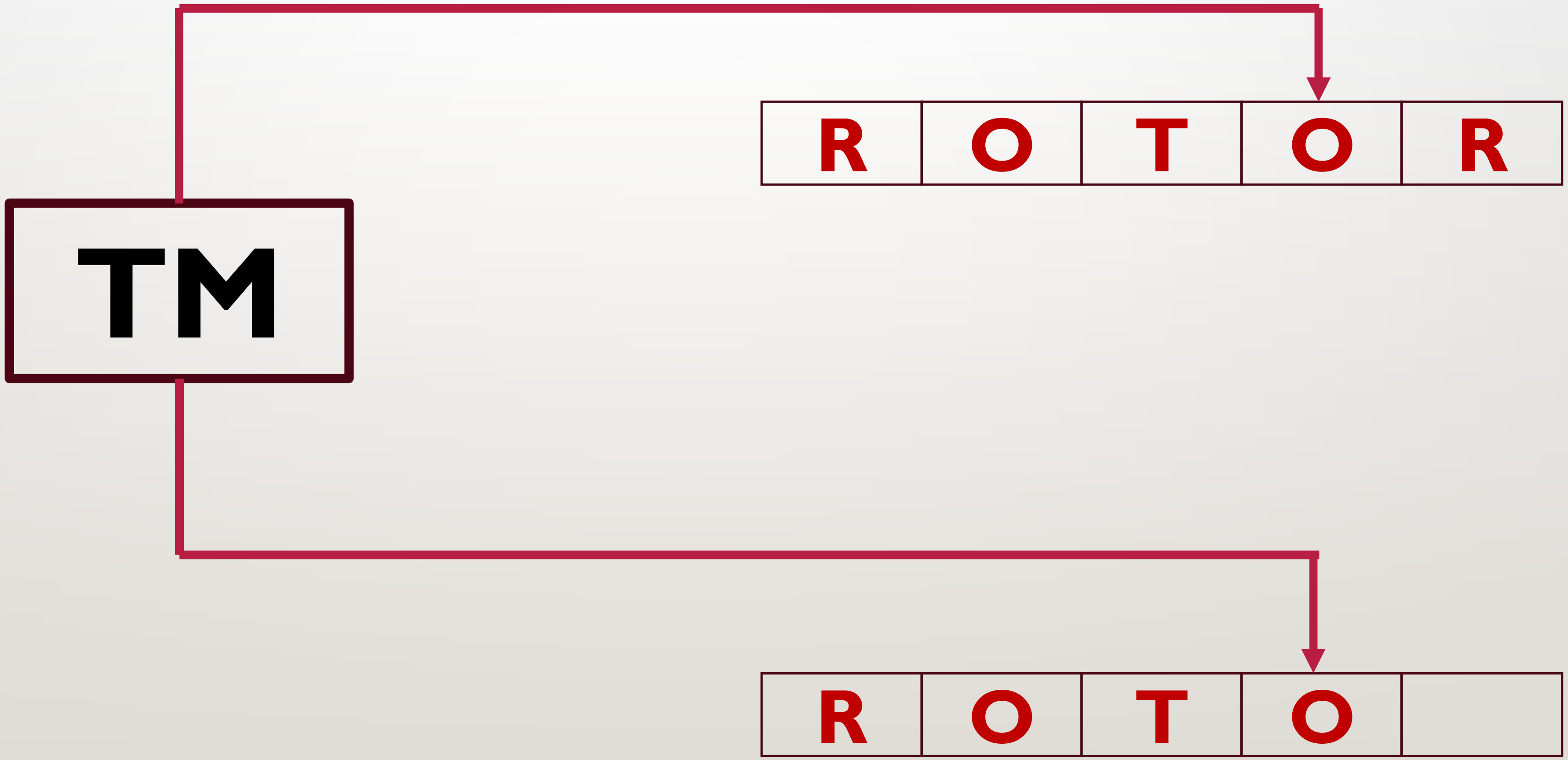

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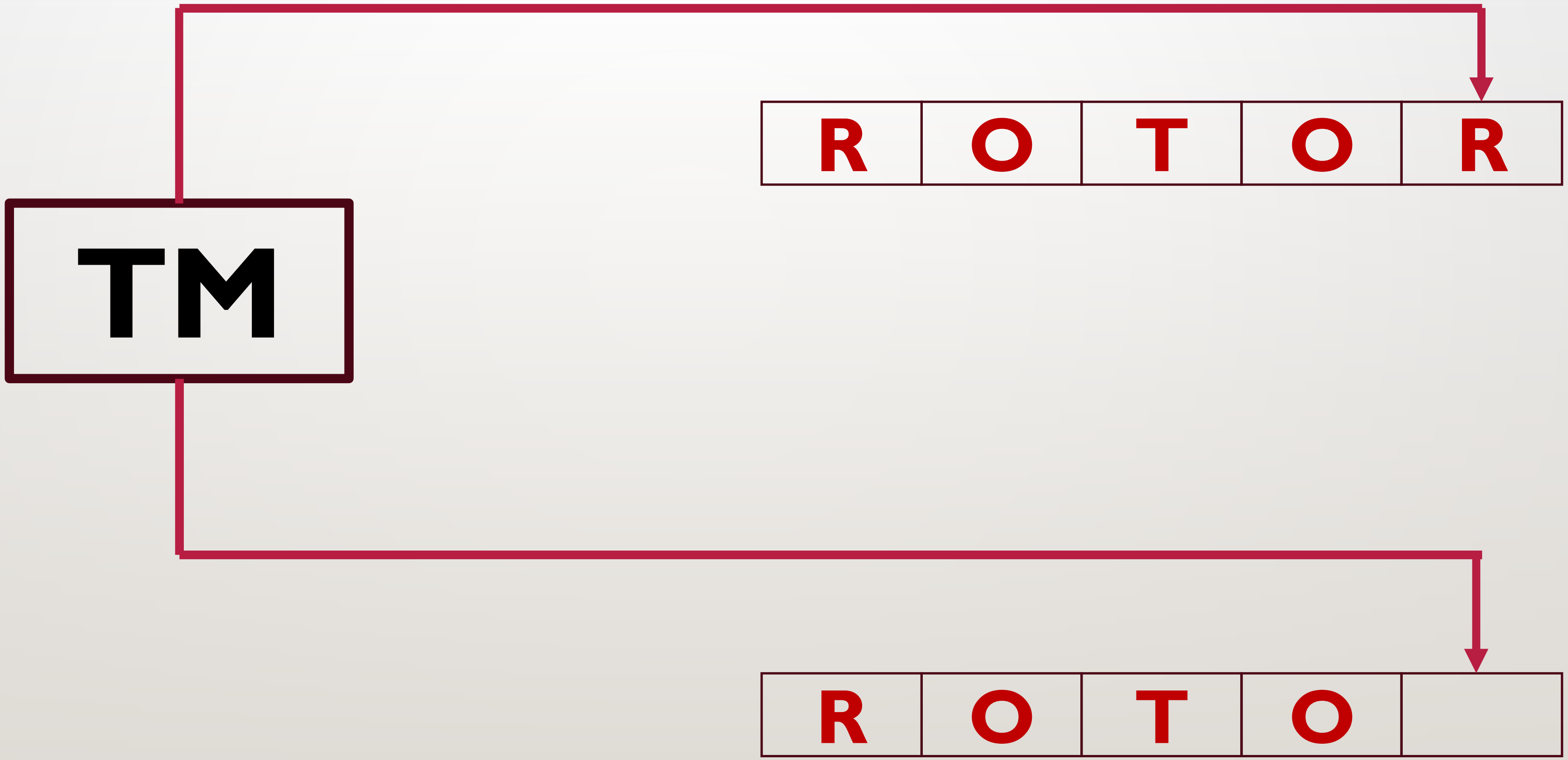

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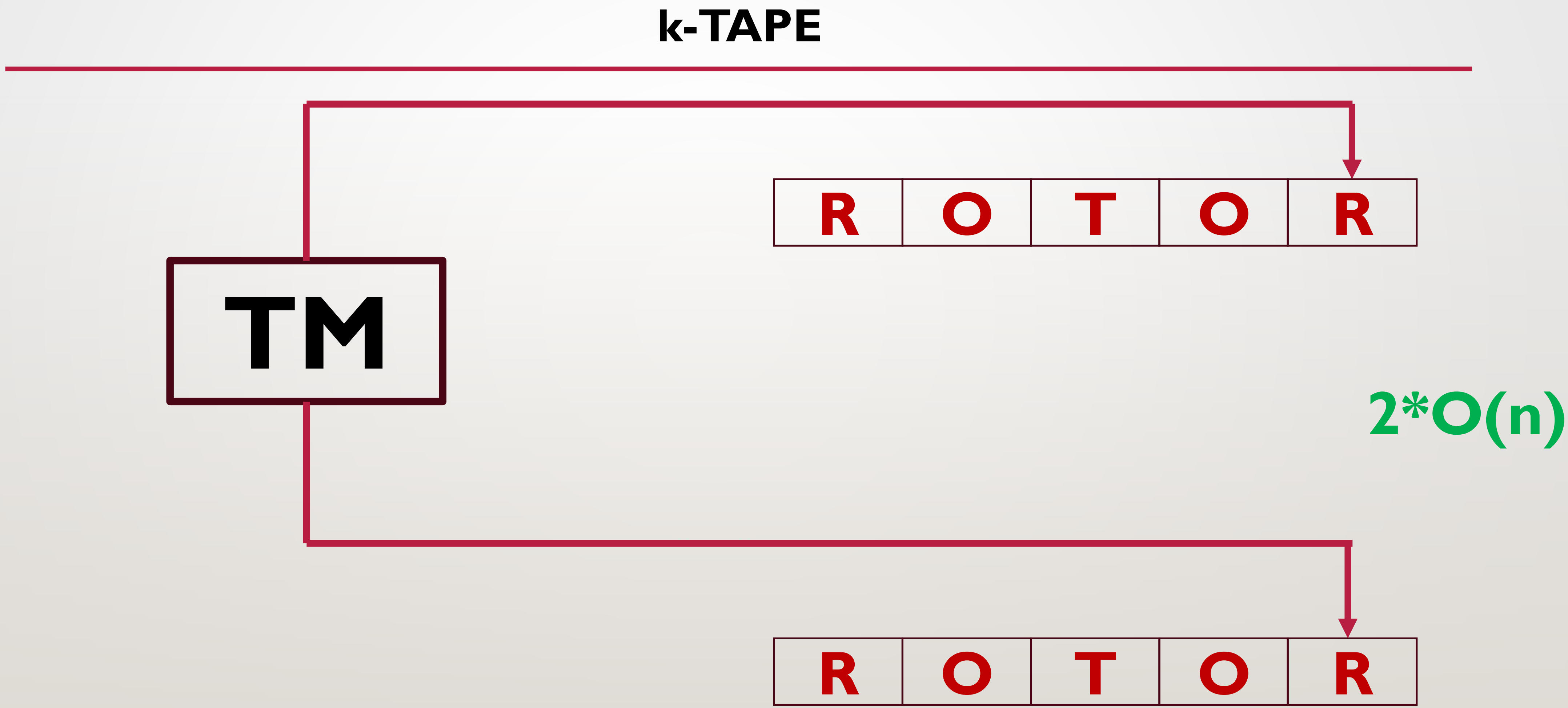

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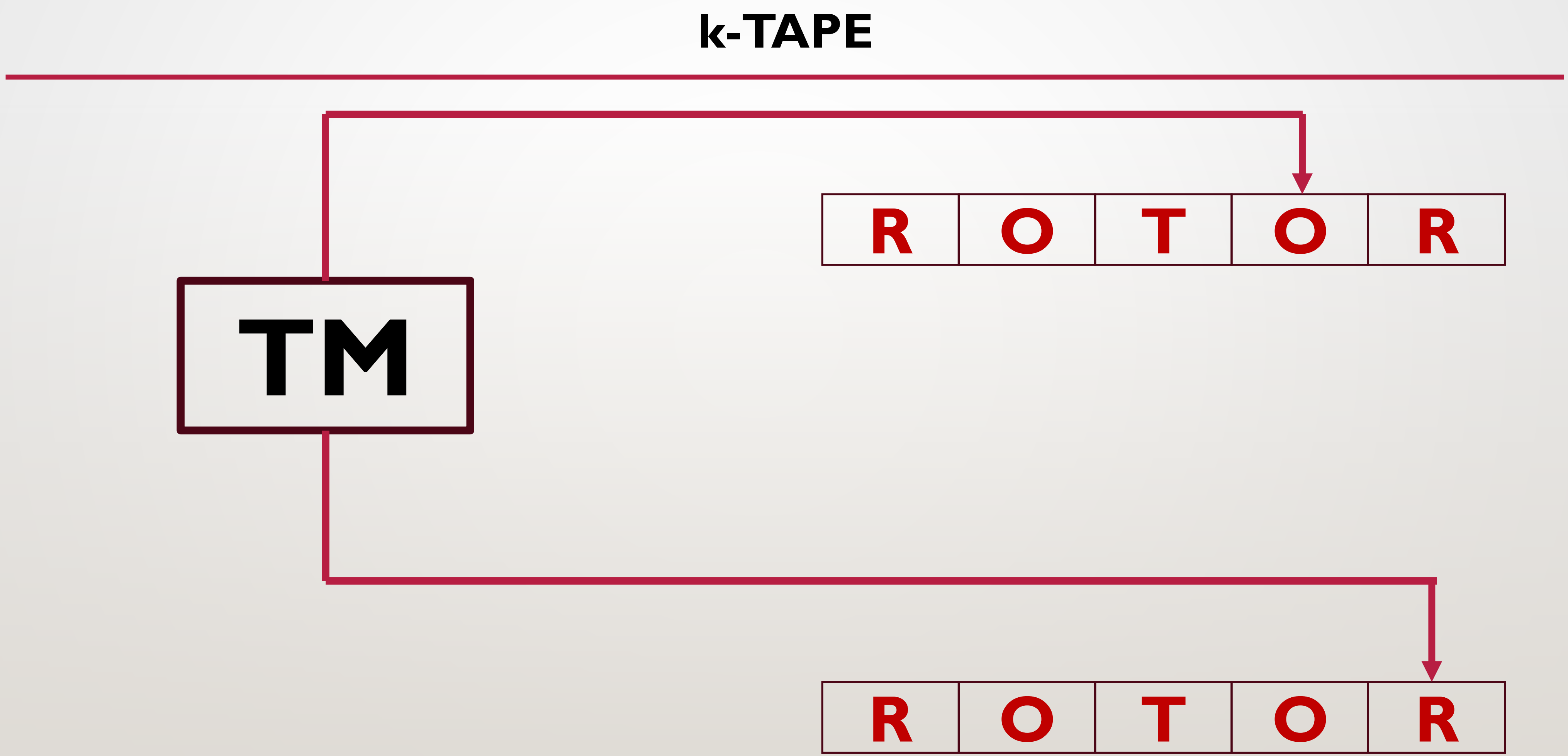

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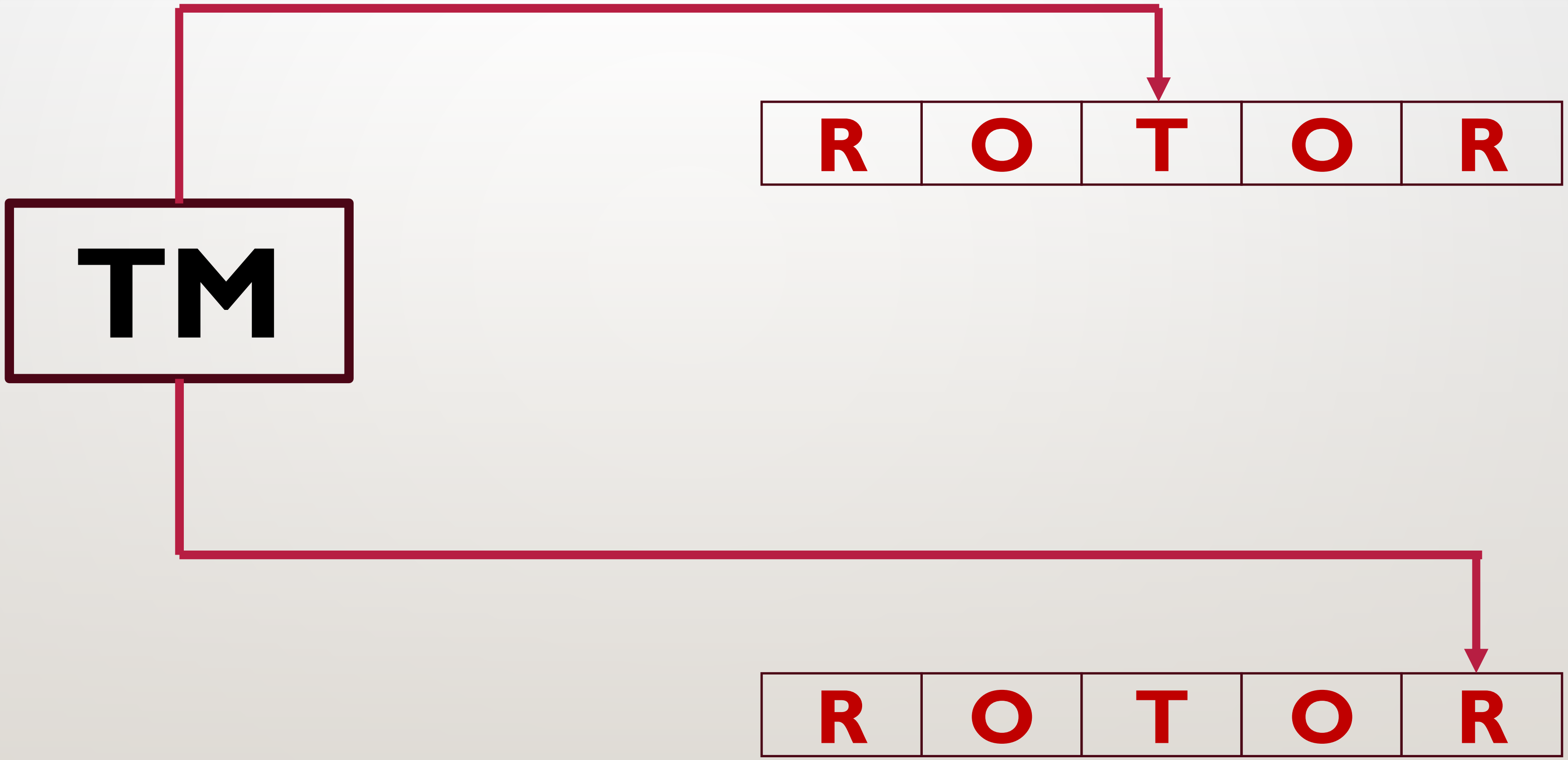

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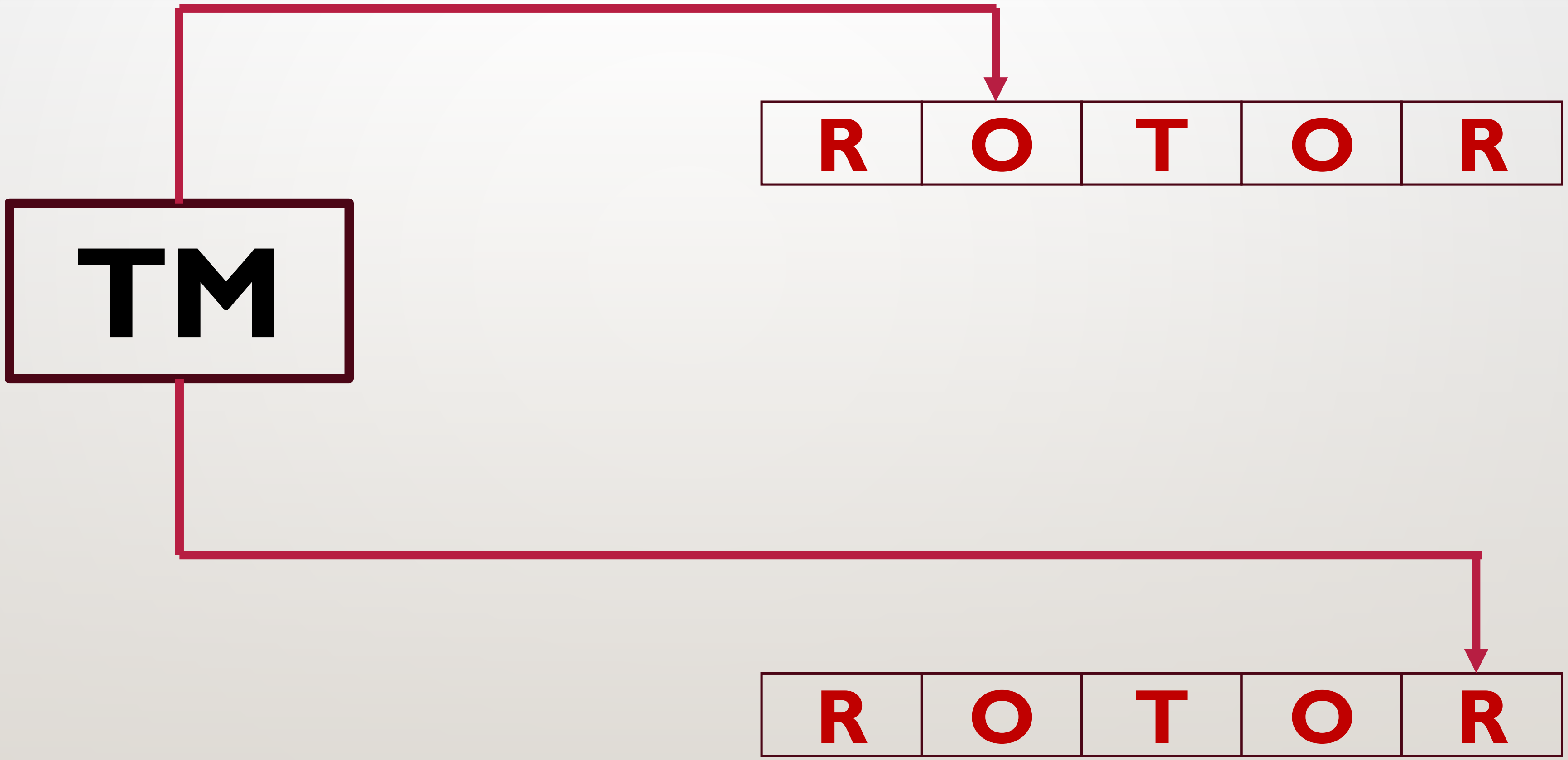

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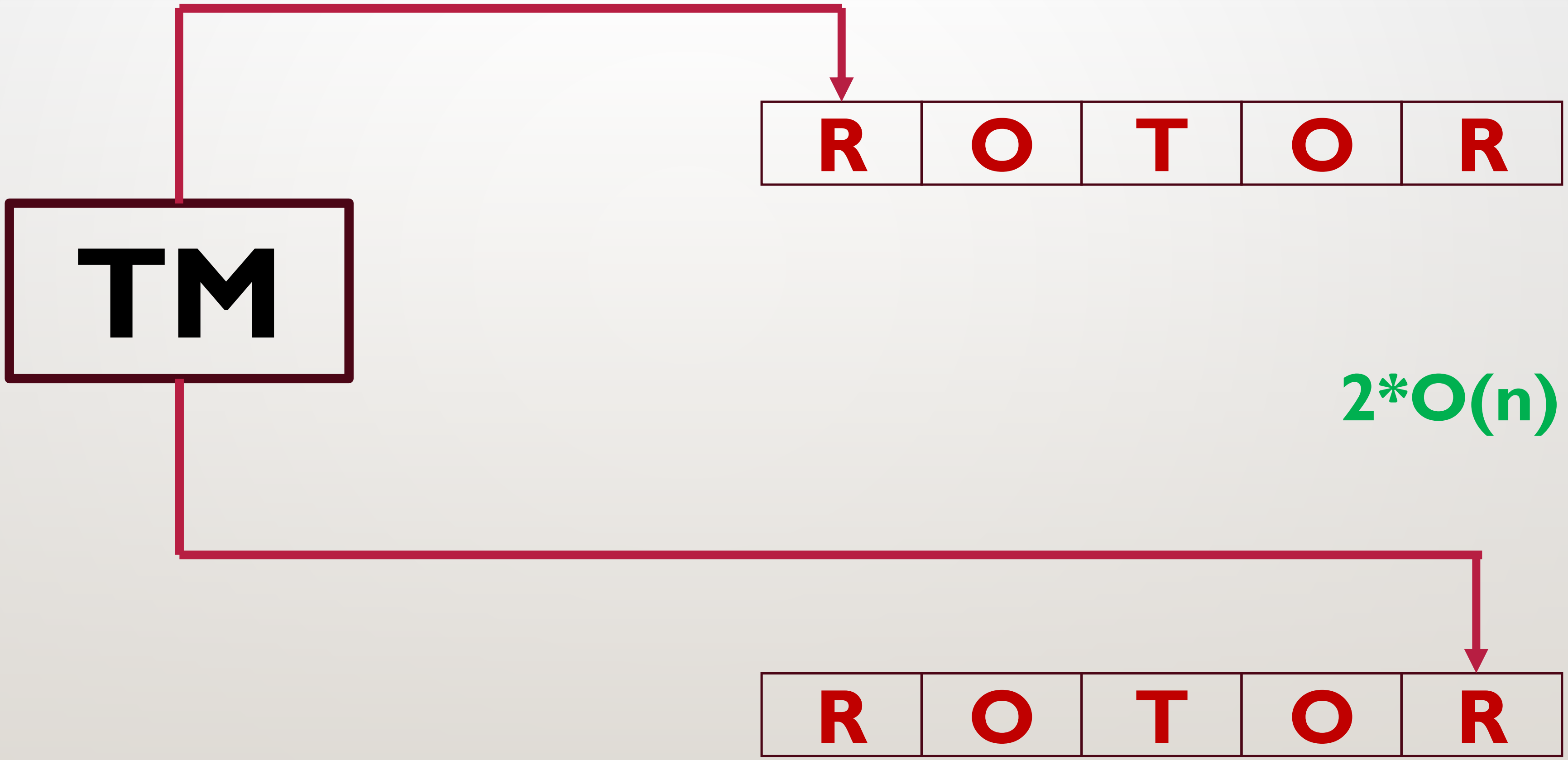

2*O(n) + O(n)

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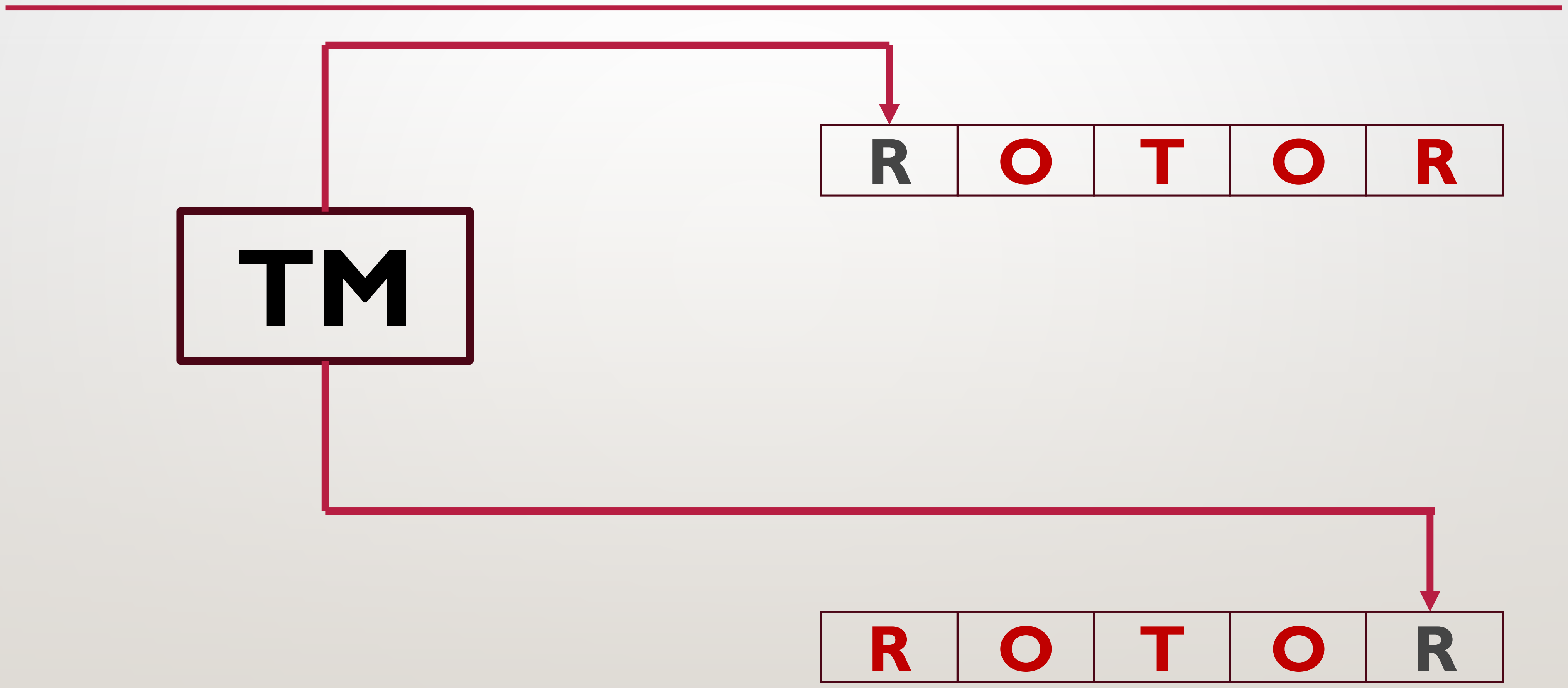

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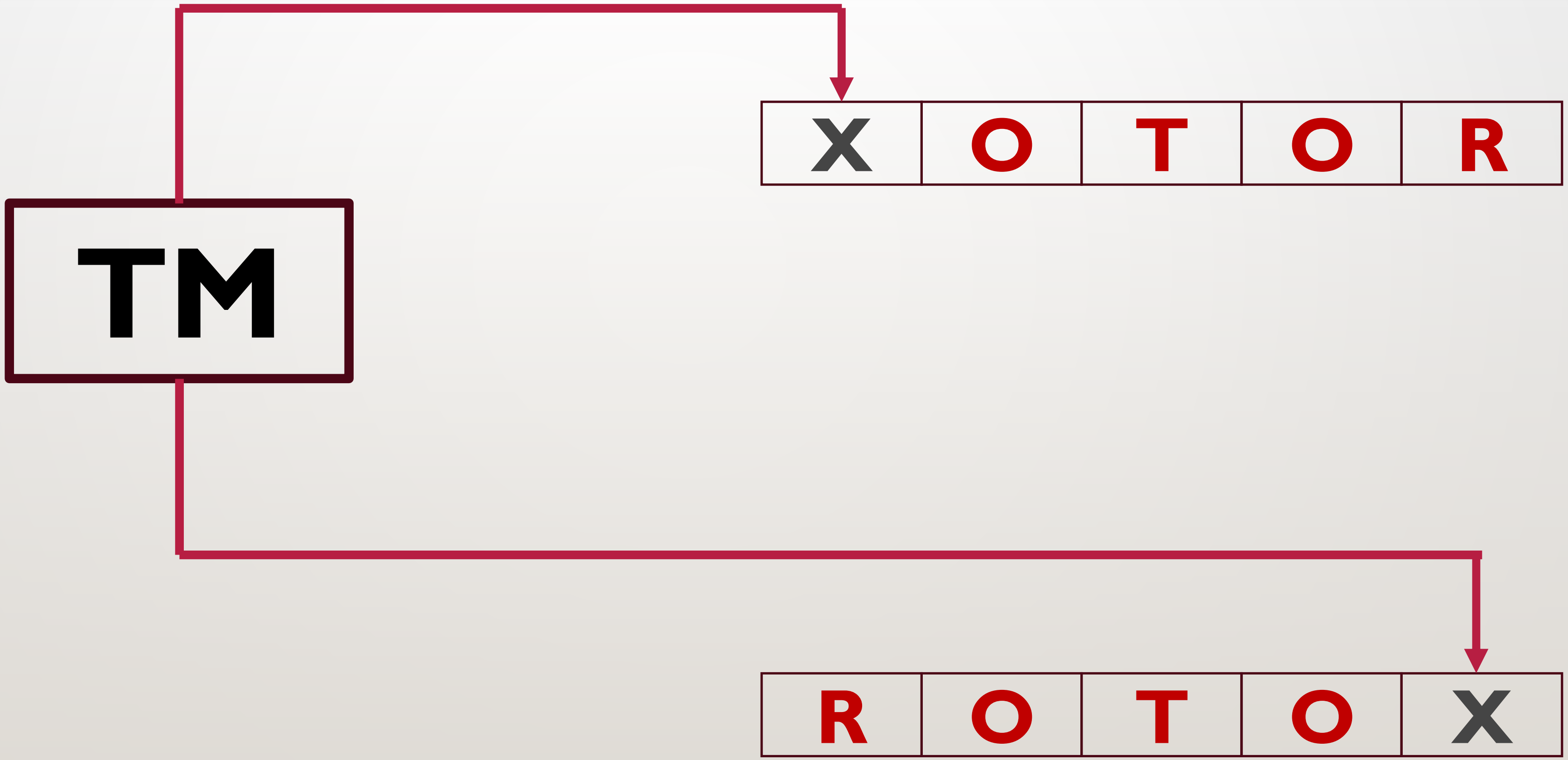

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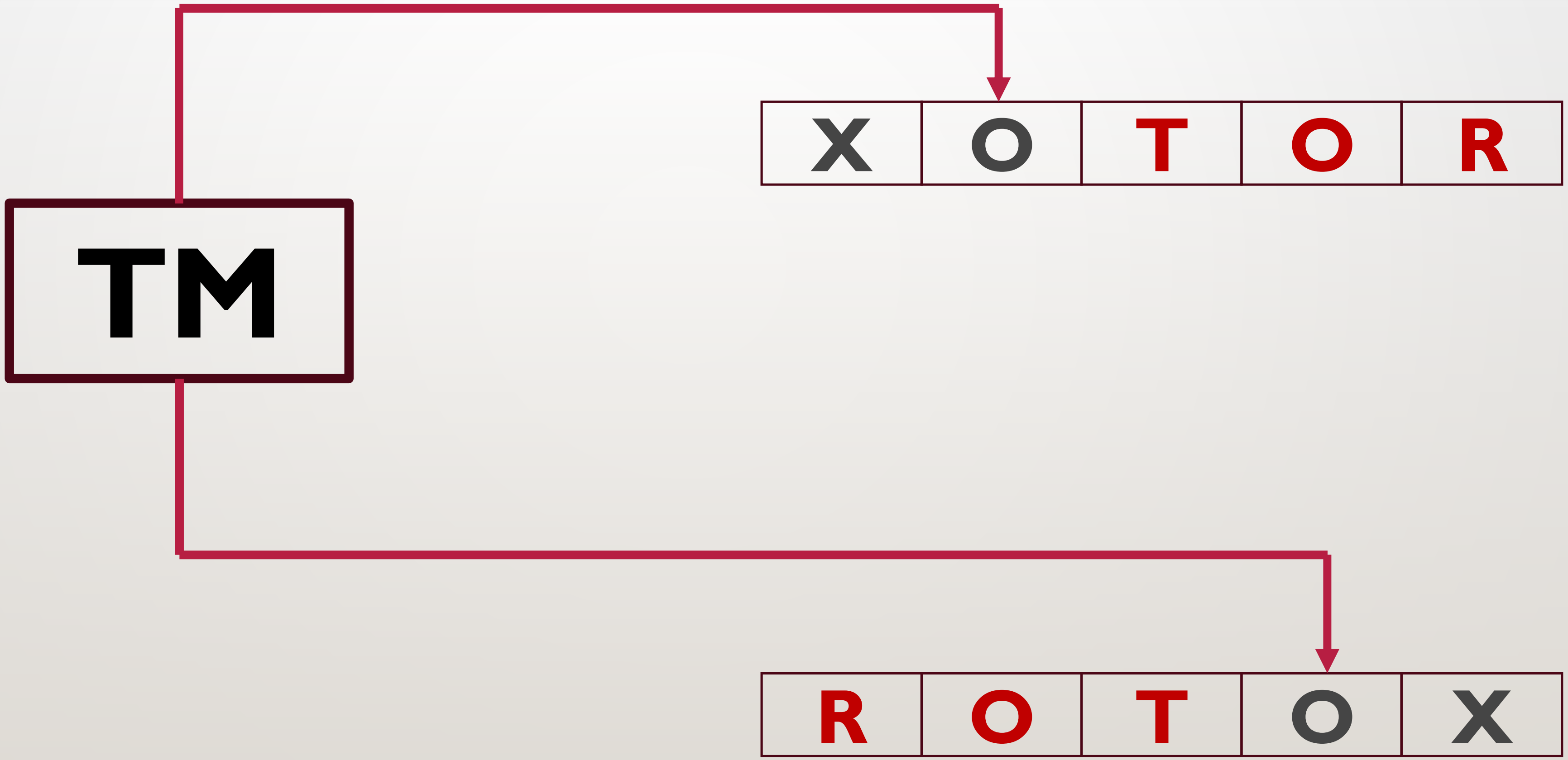

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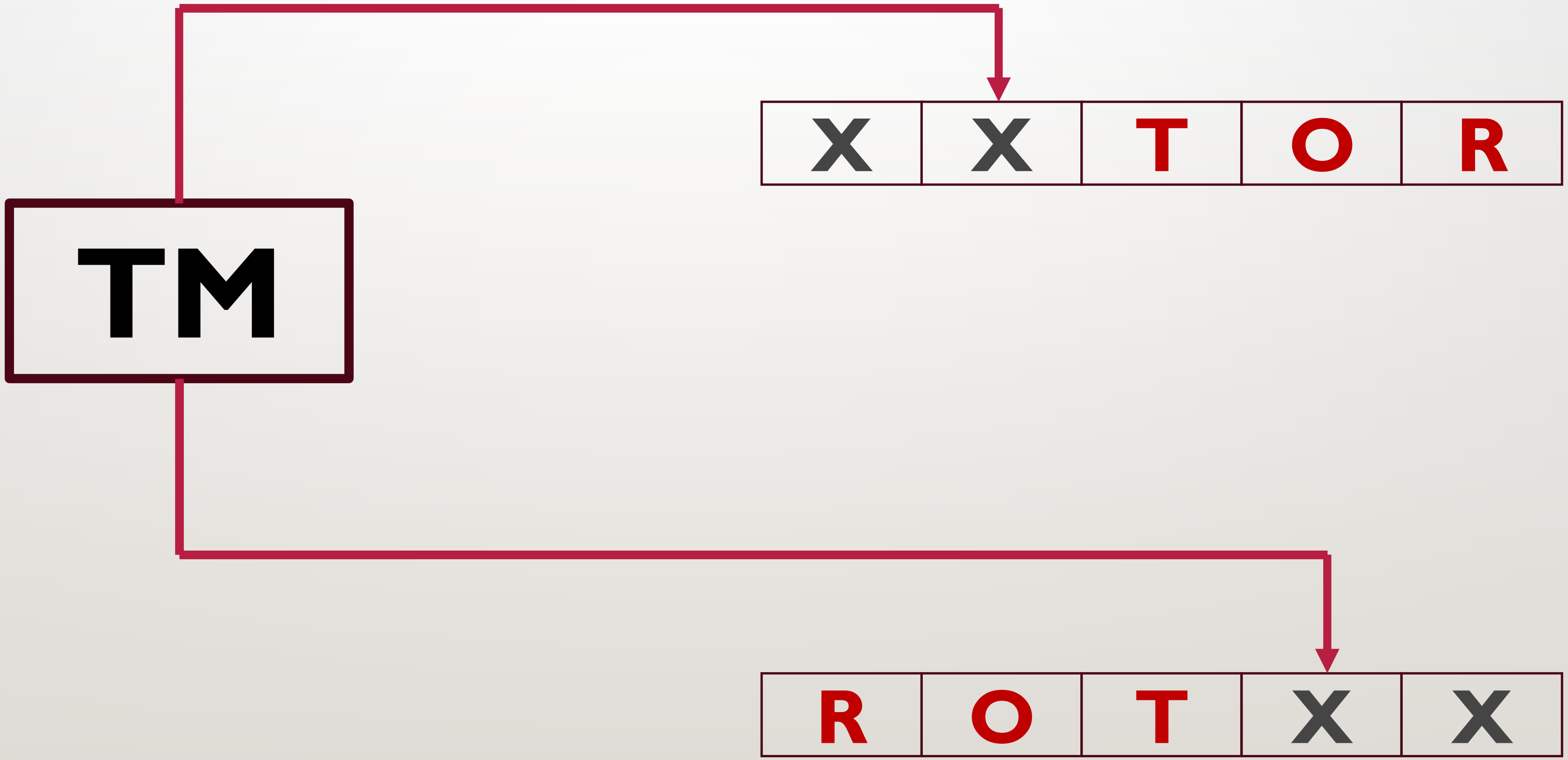

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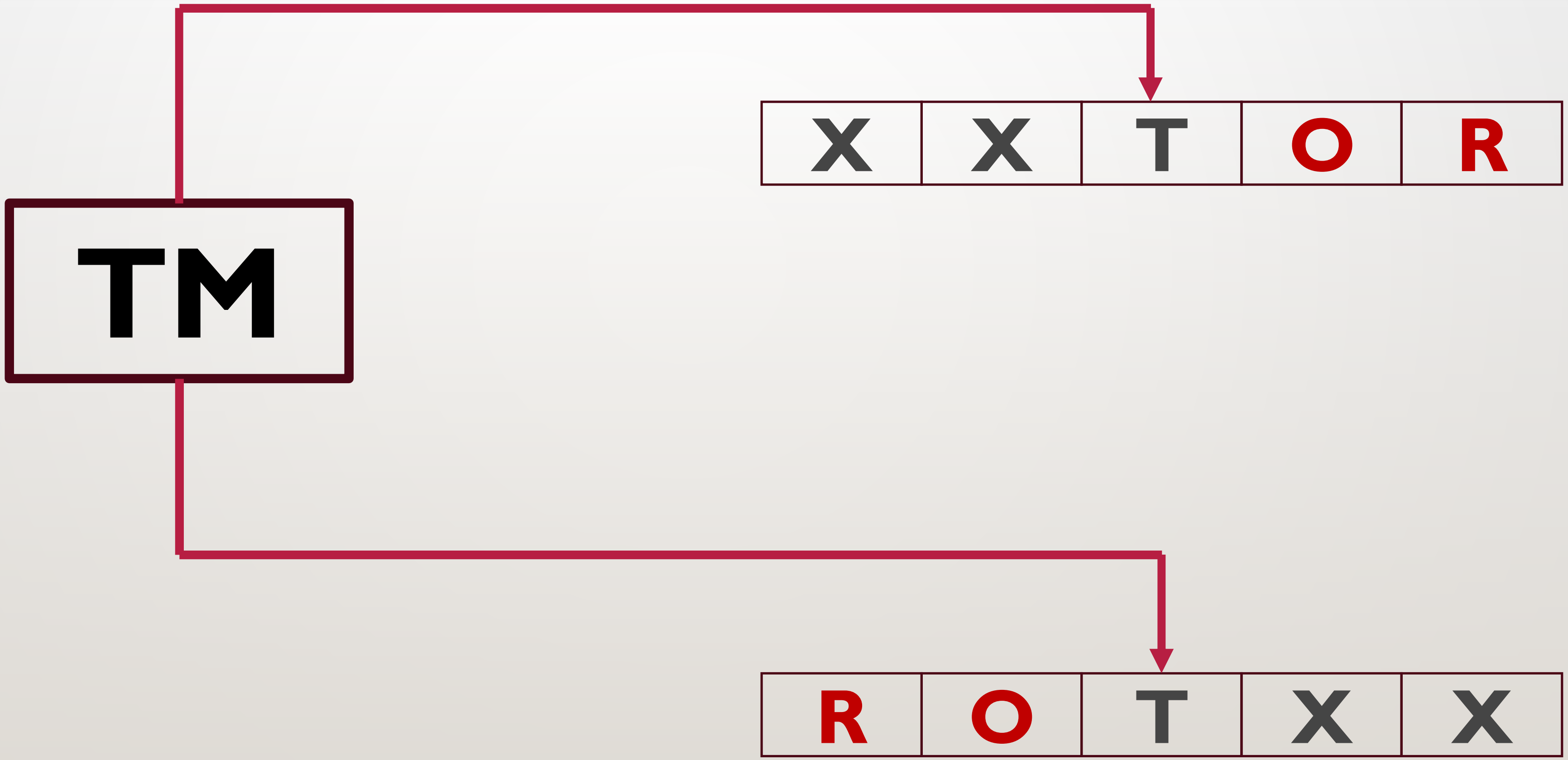

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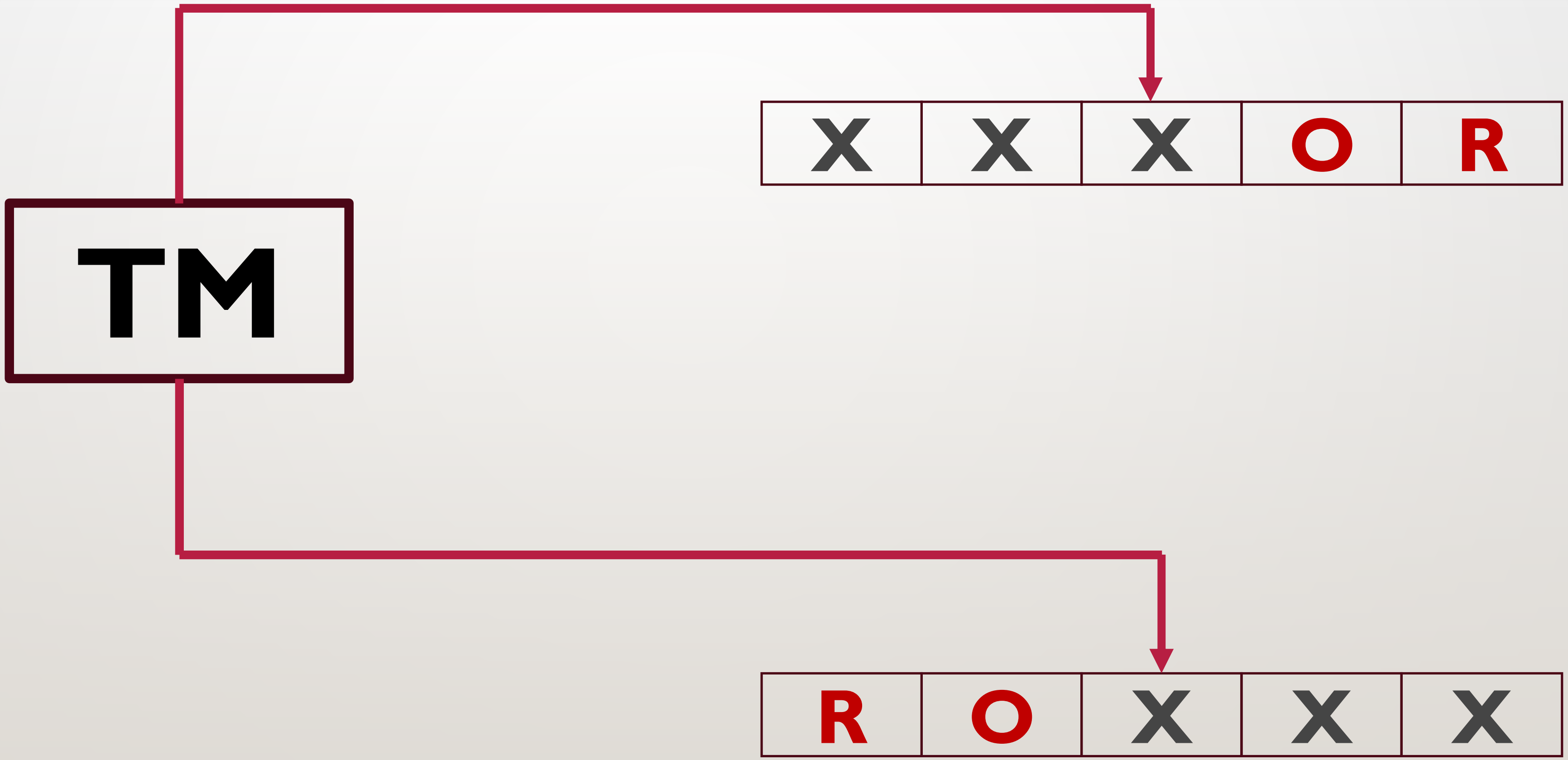

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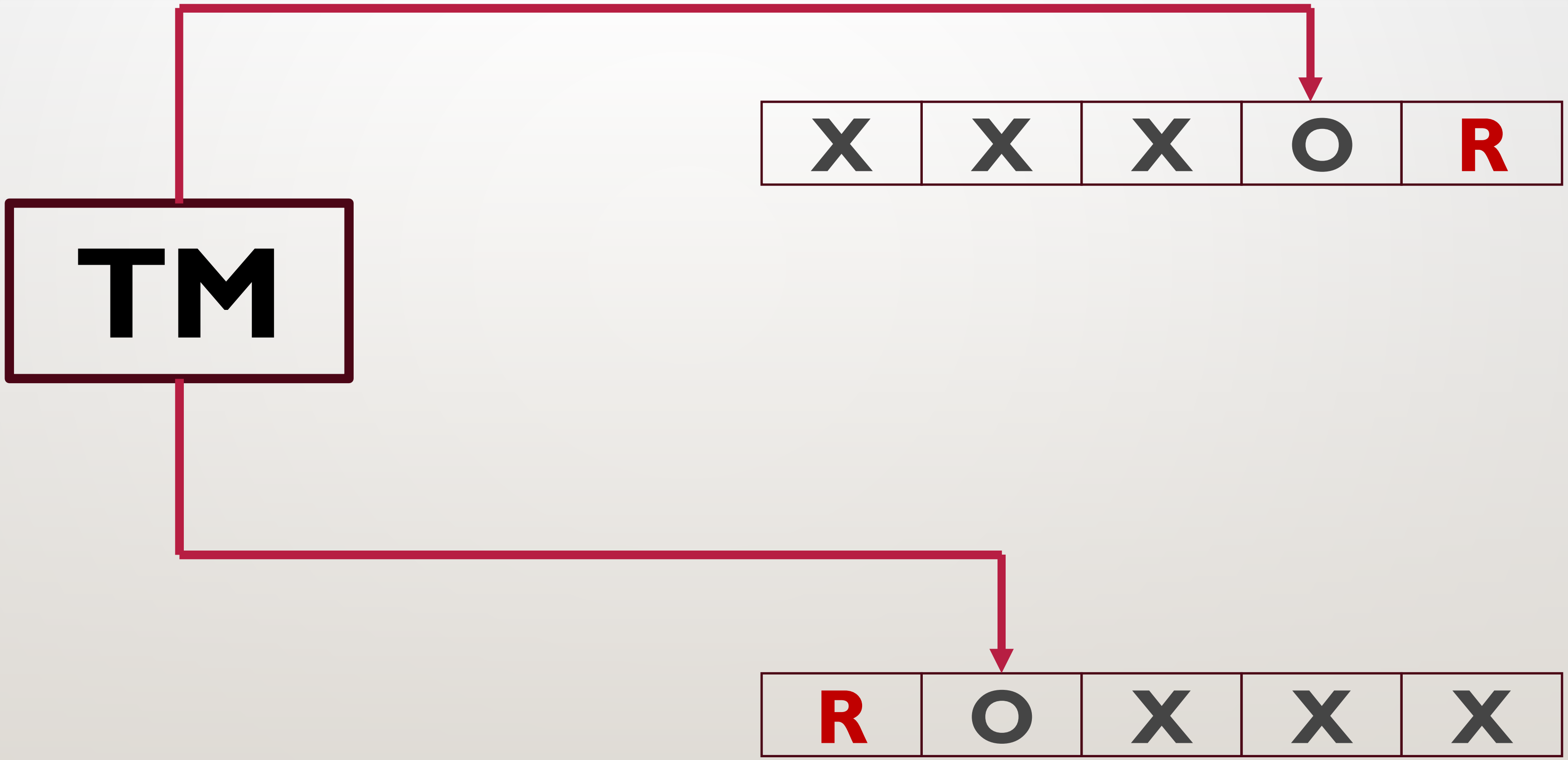

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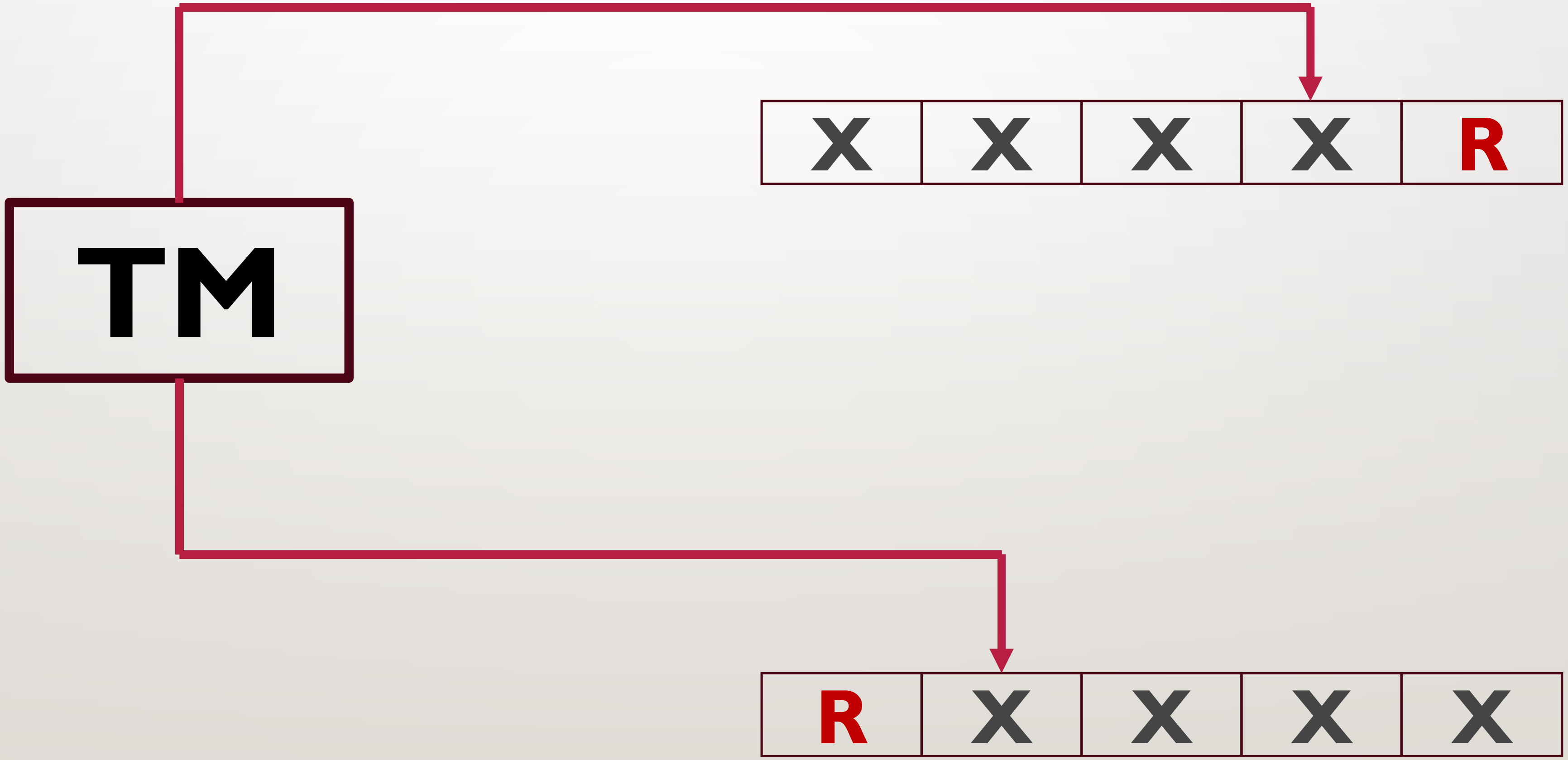

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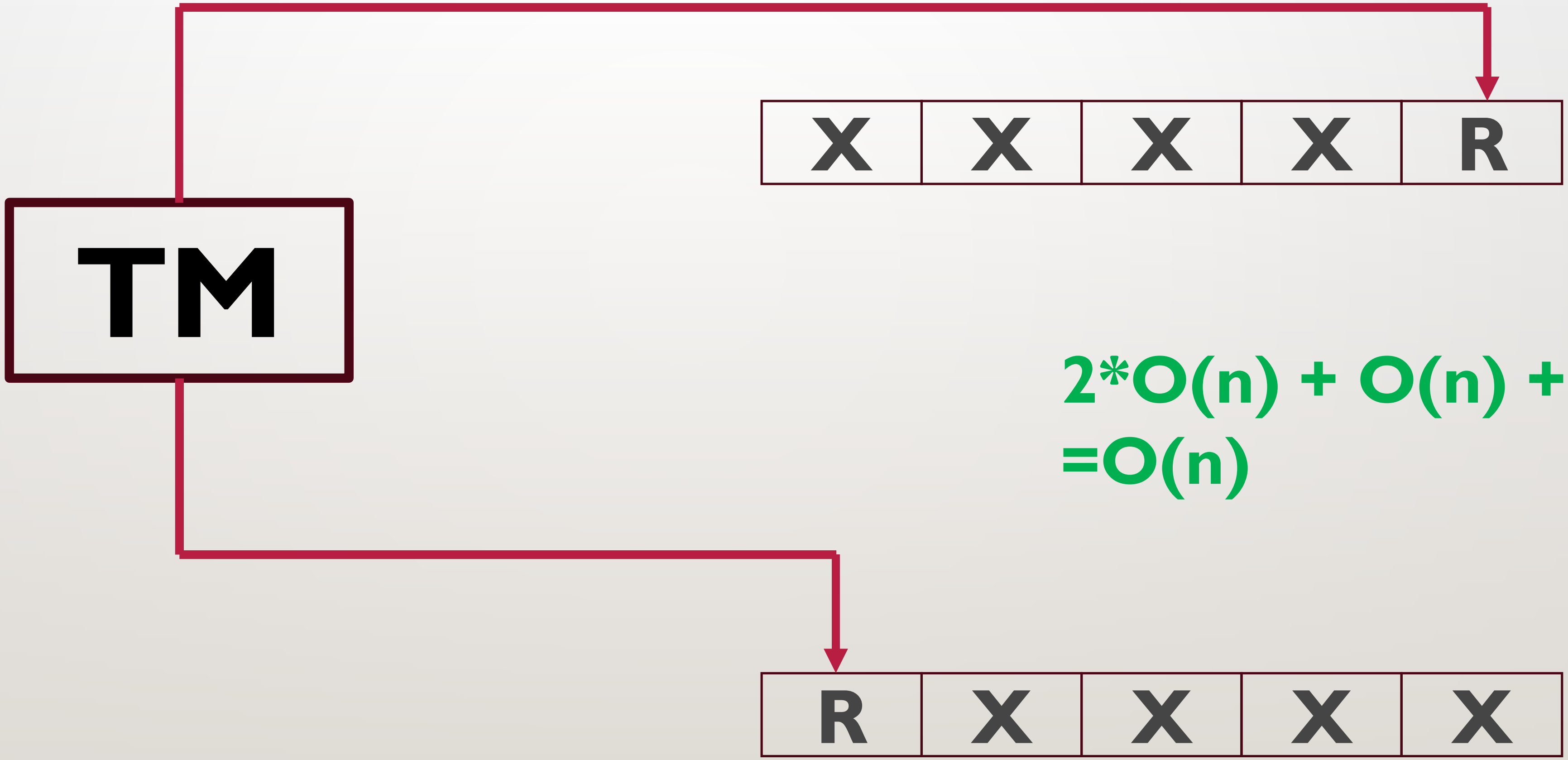

$$2*O(n) + O(n) + 2*O(n)$$
$$=O(n)$$

PALINDROME STRING VERIFICATION

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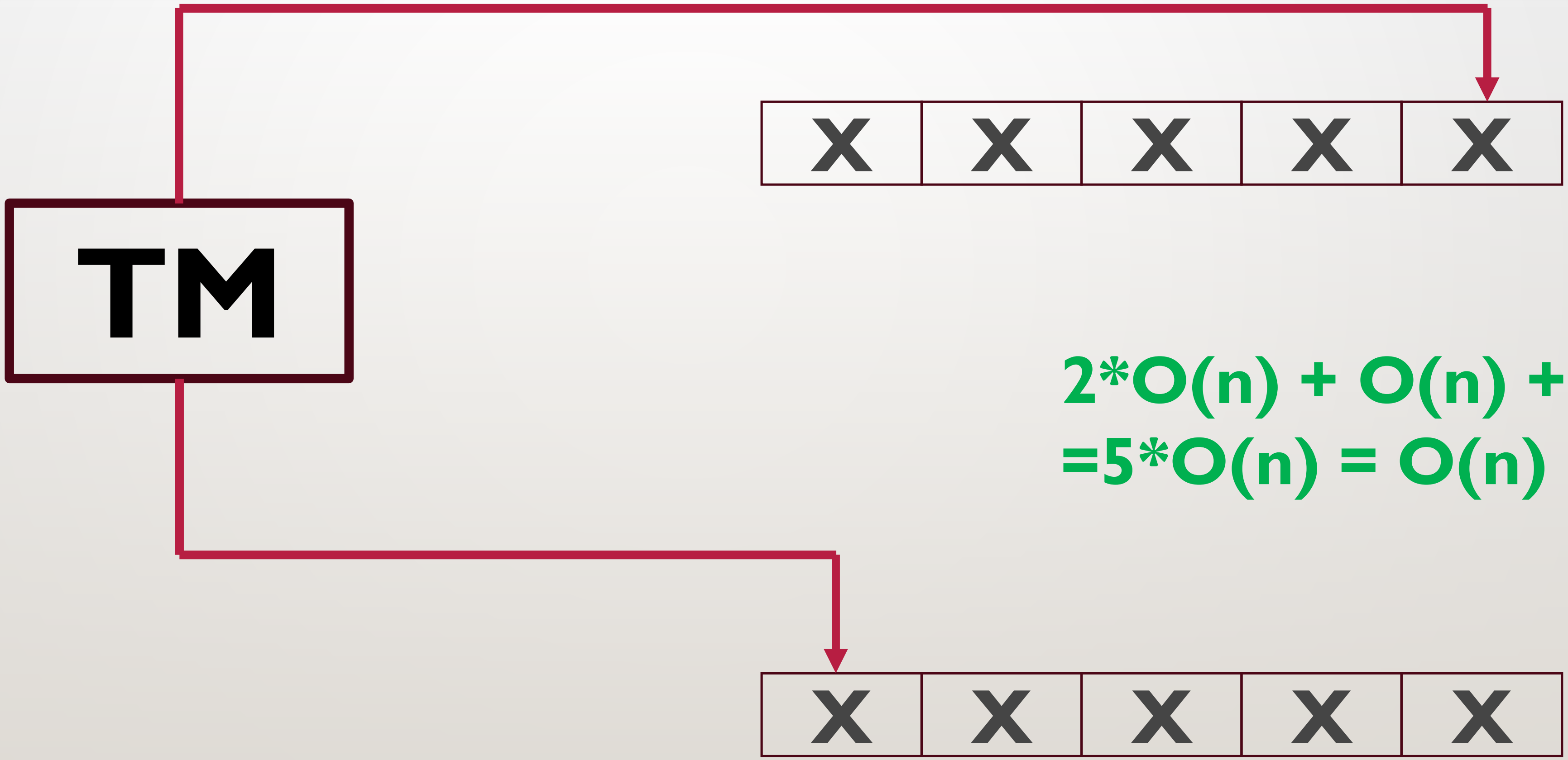

$$2*O(n) + O(n) + 2*O(n)$$
$$=5*O(n) = O(n)$$

PALINDROME STRING VERIFICATION

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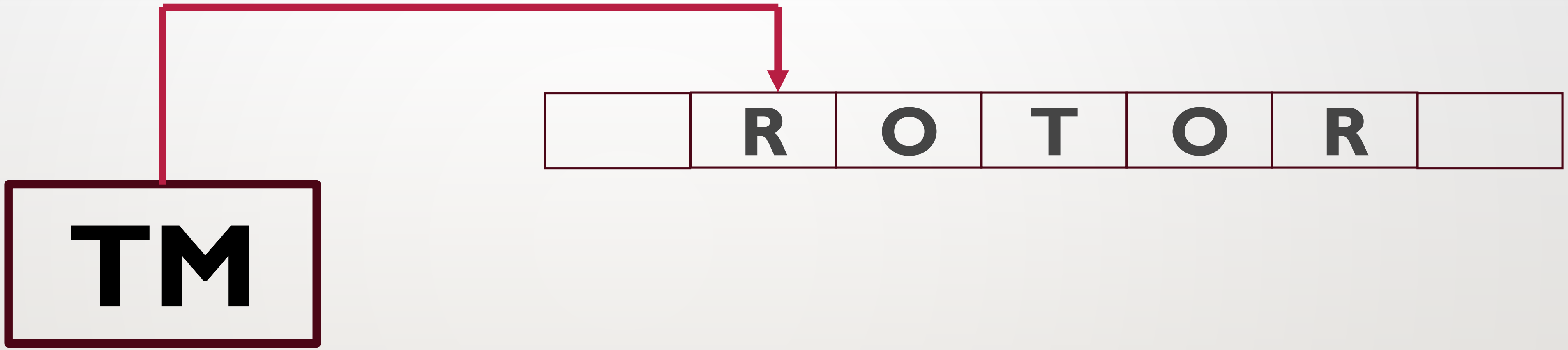

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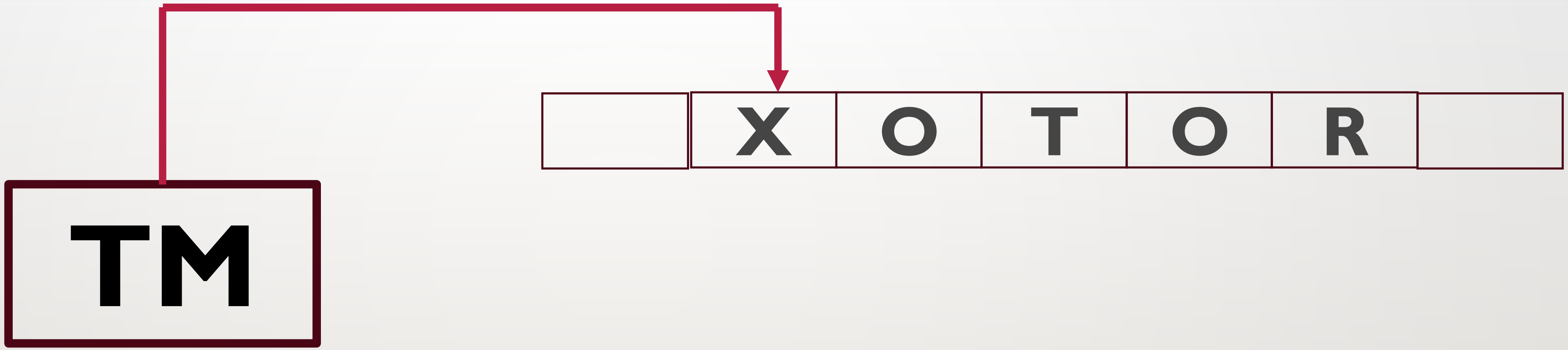

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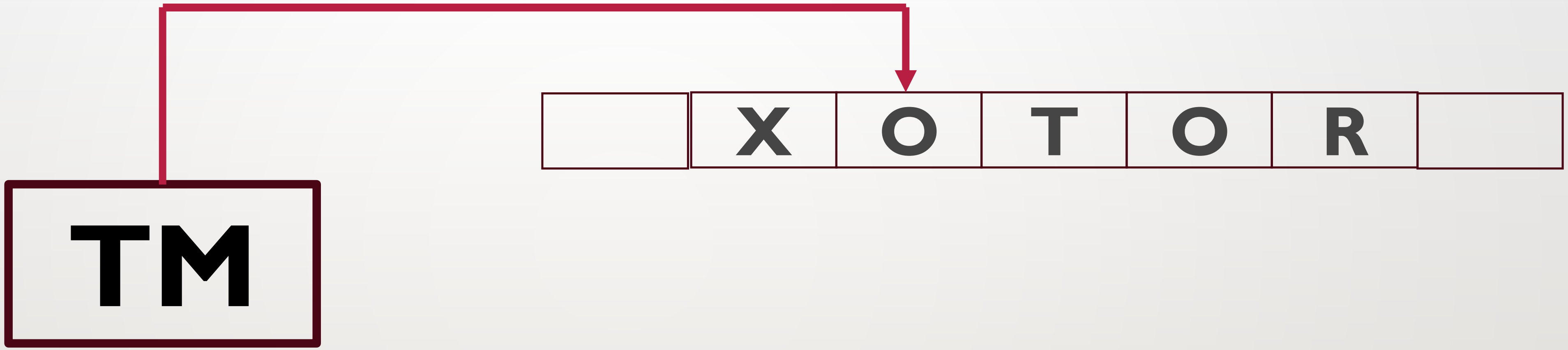

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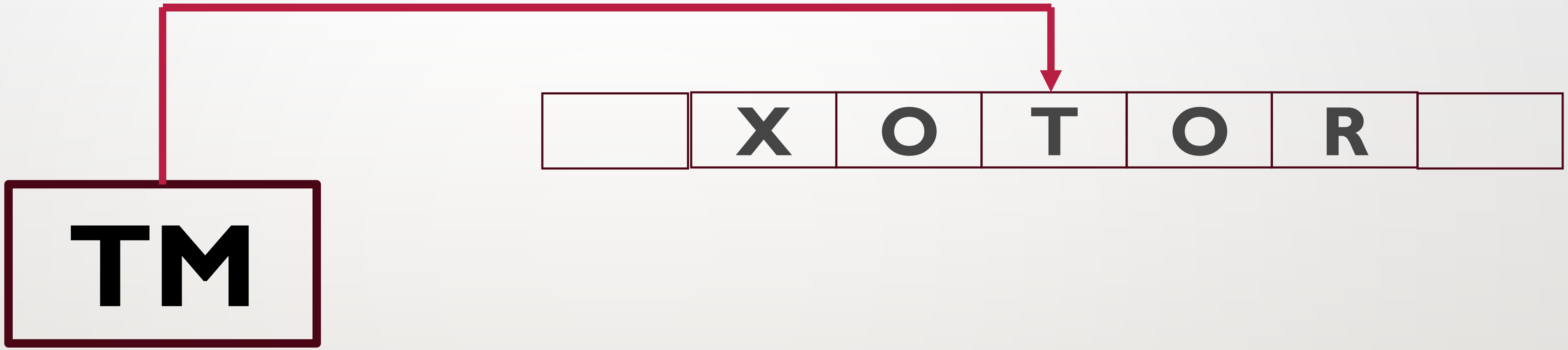

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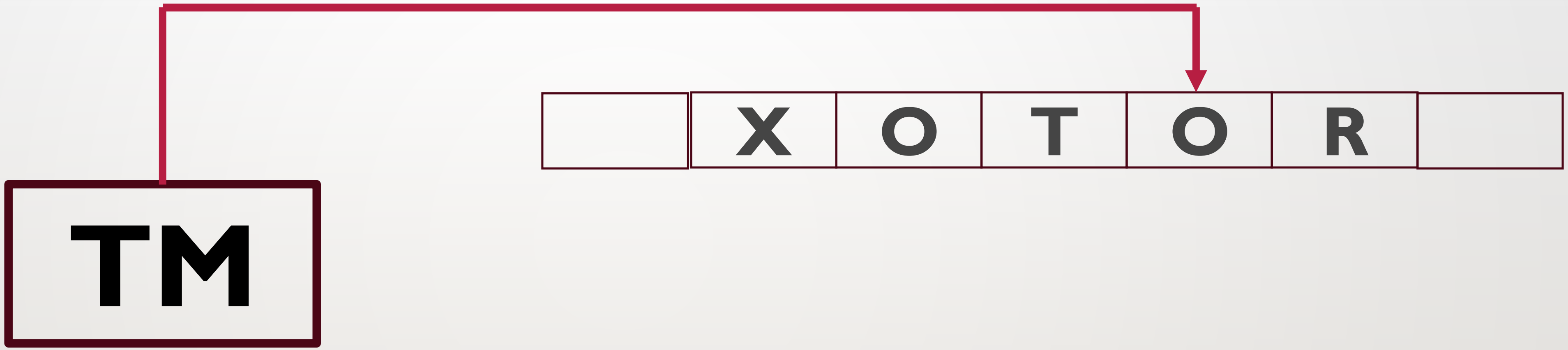

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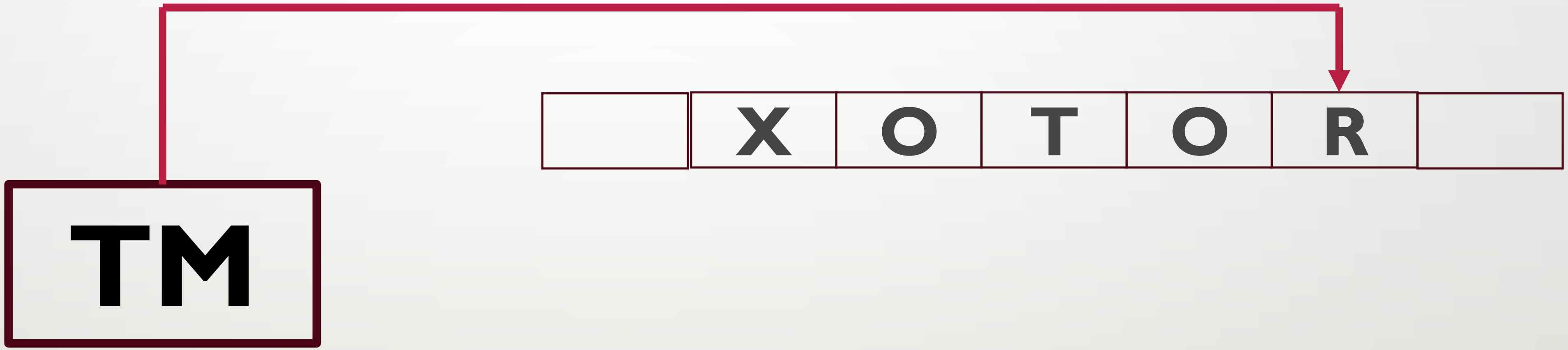

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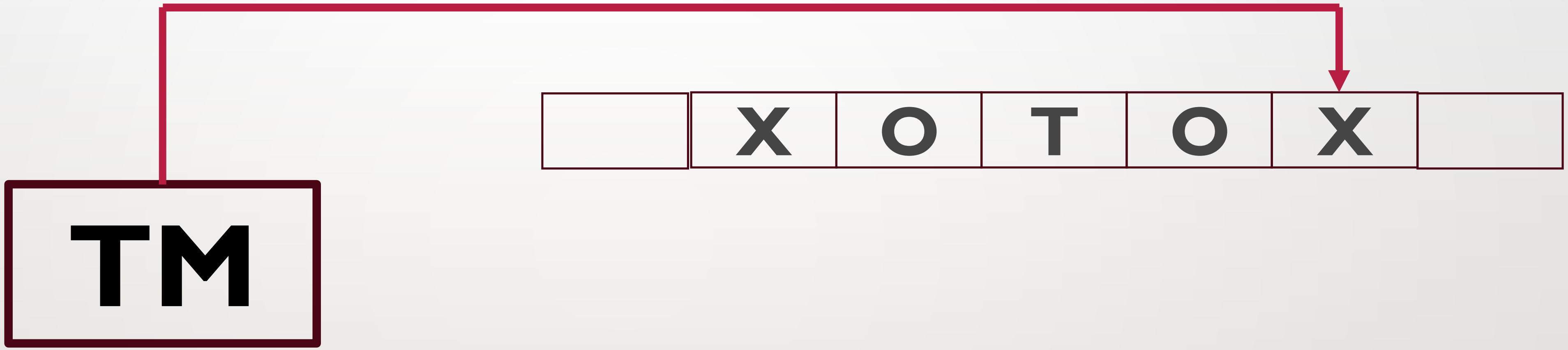

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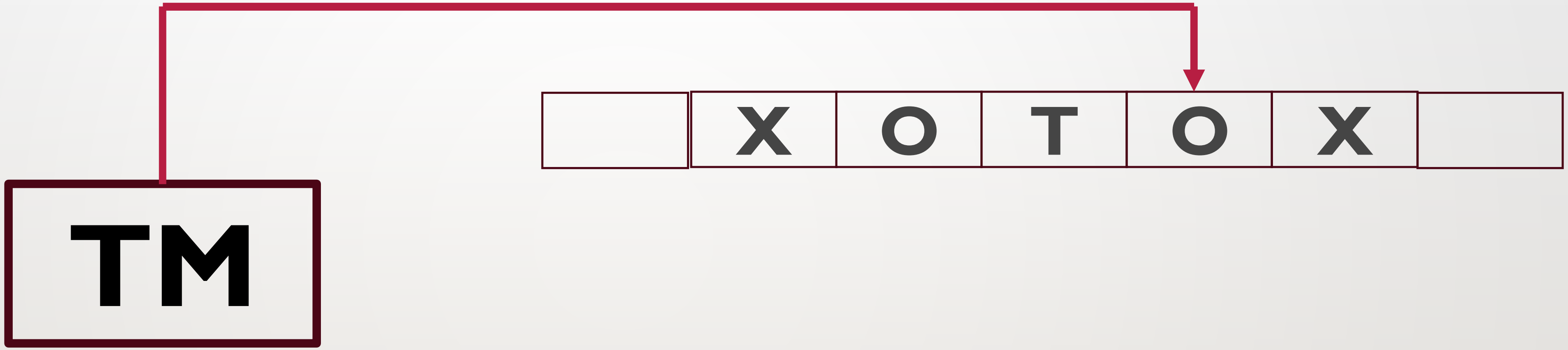

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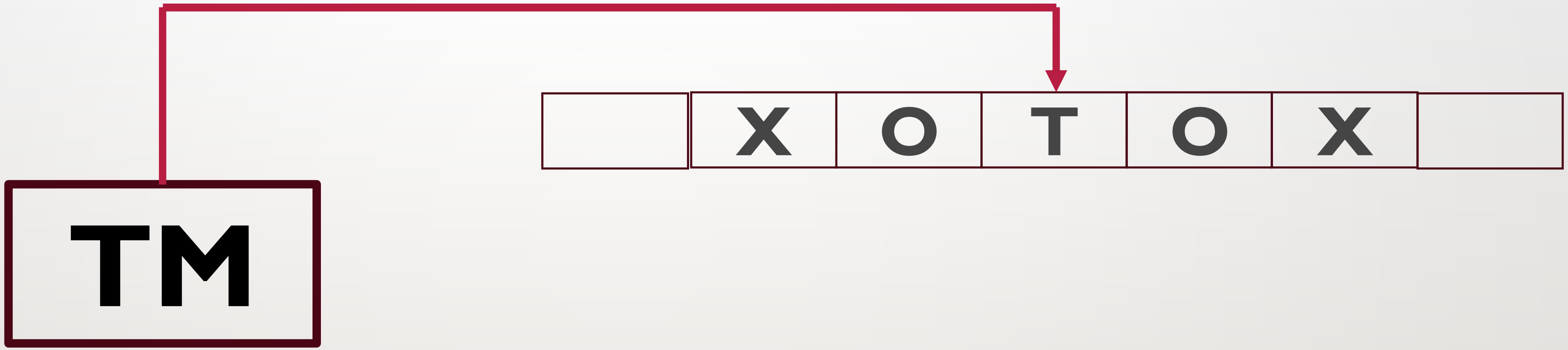

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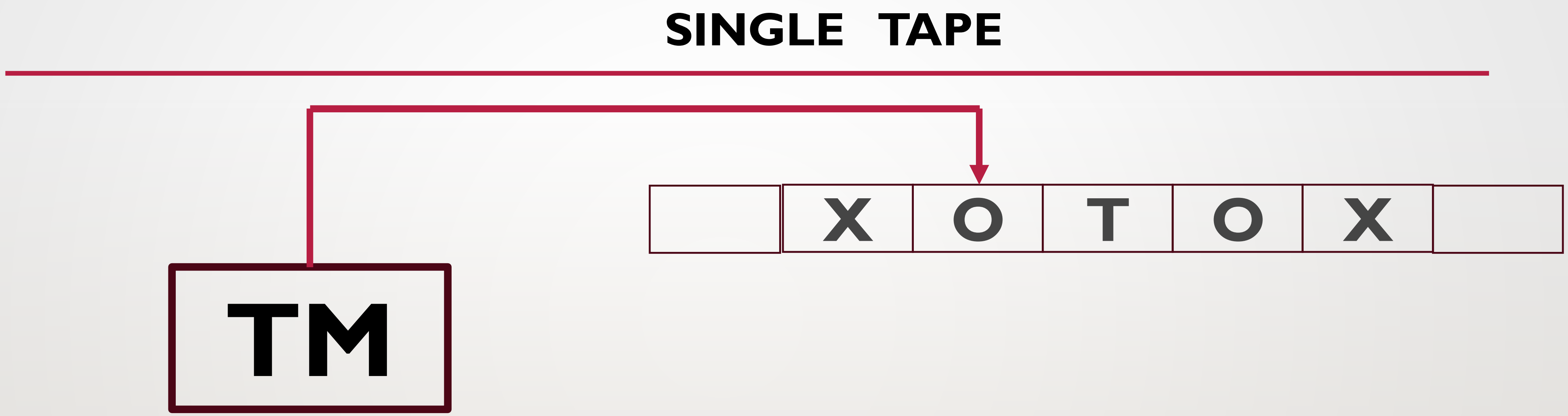

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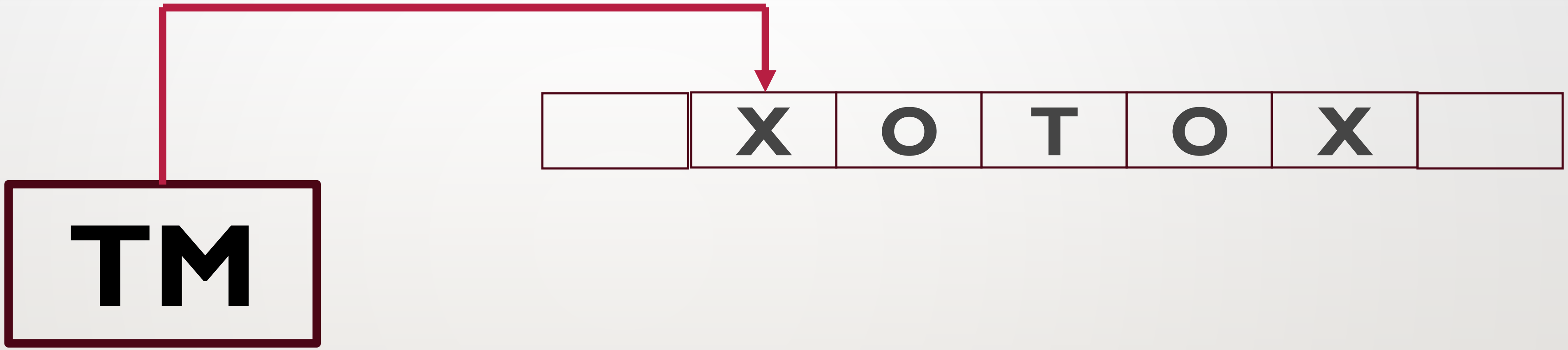

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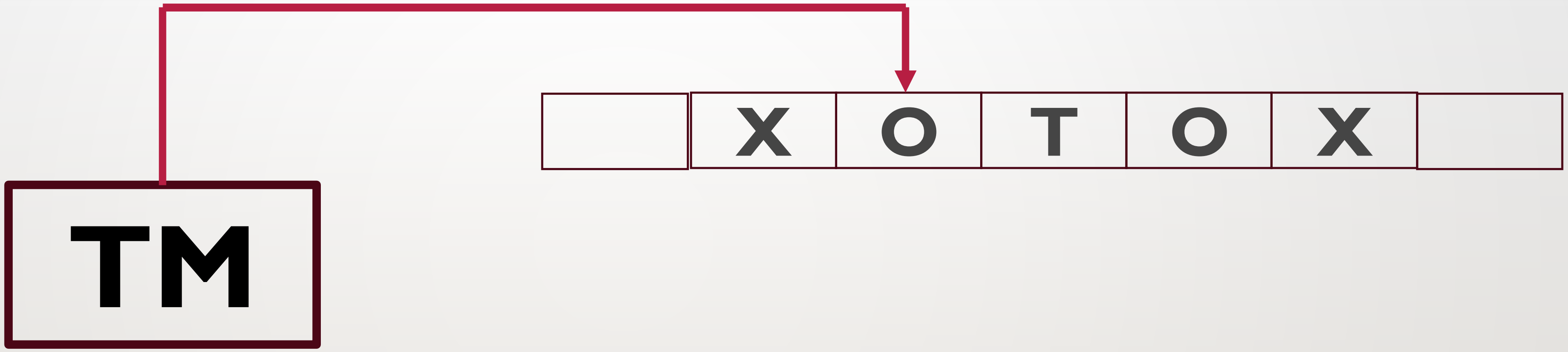

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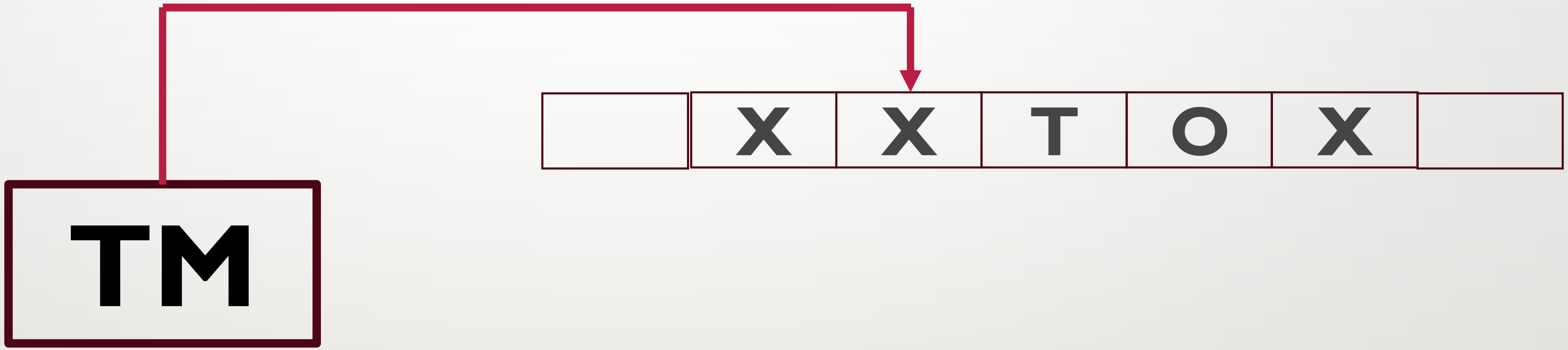

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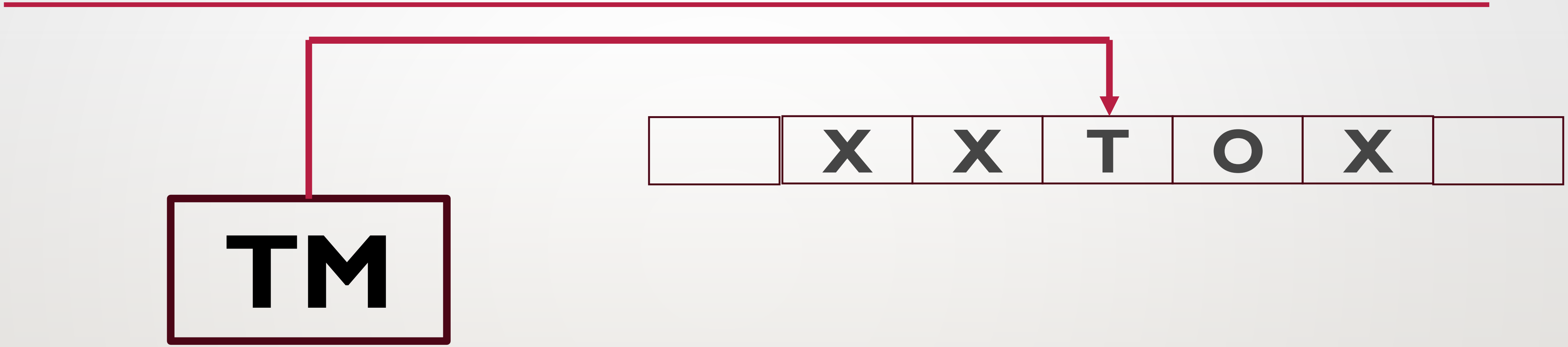

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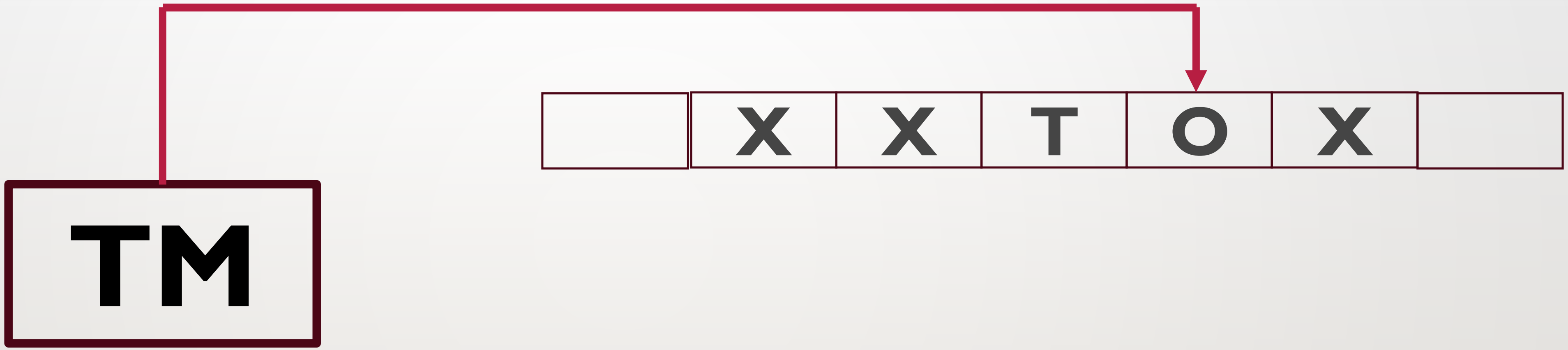

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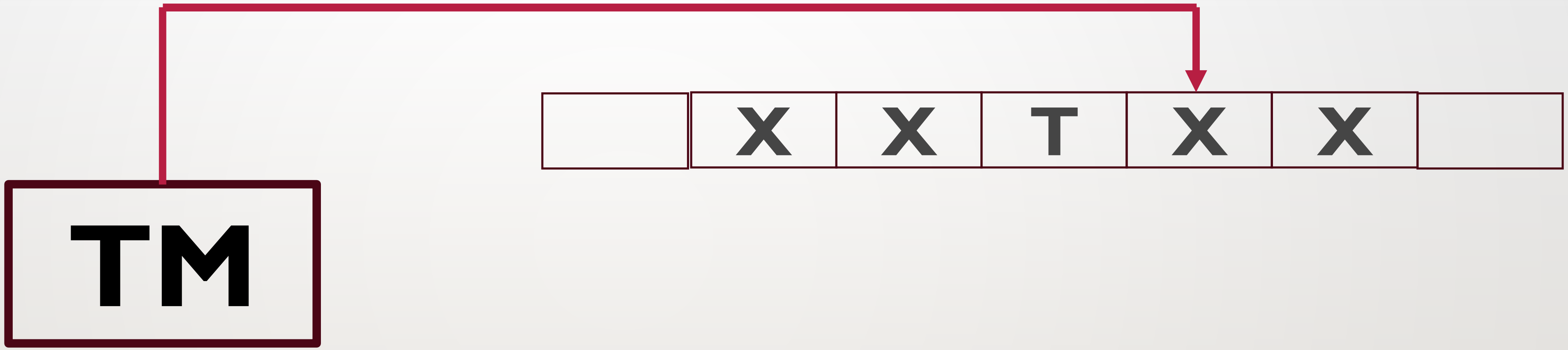

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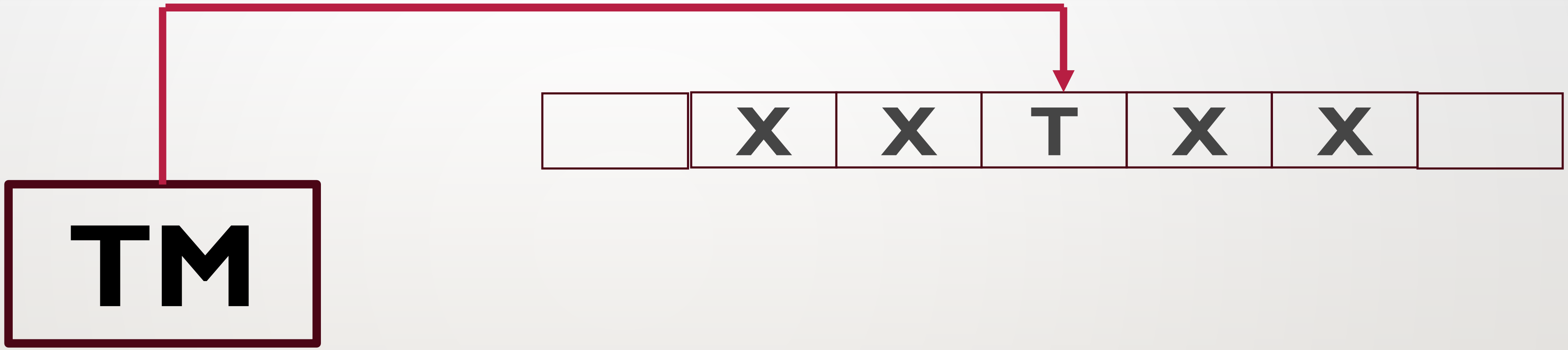

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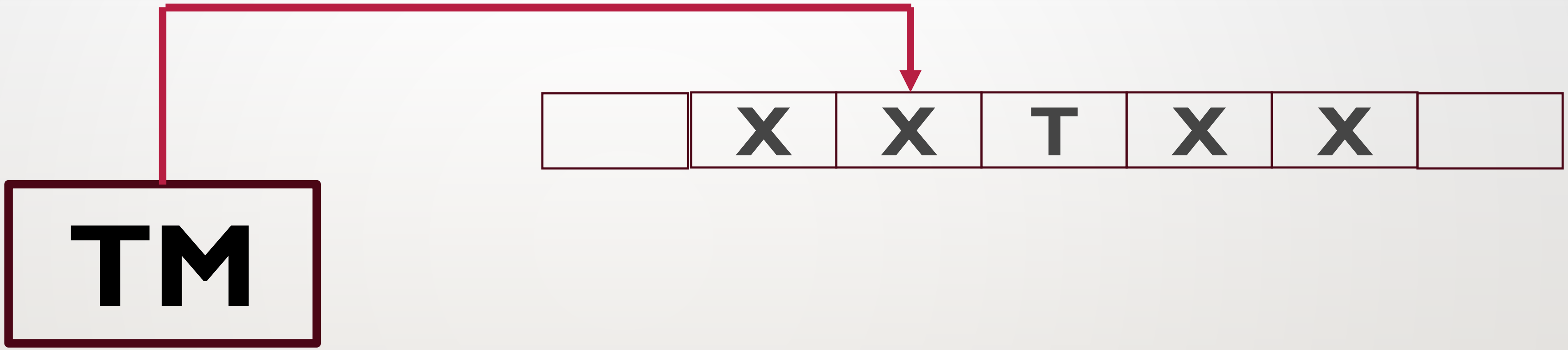

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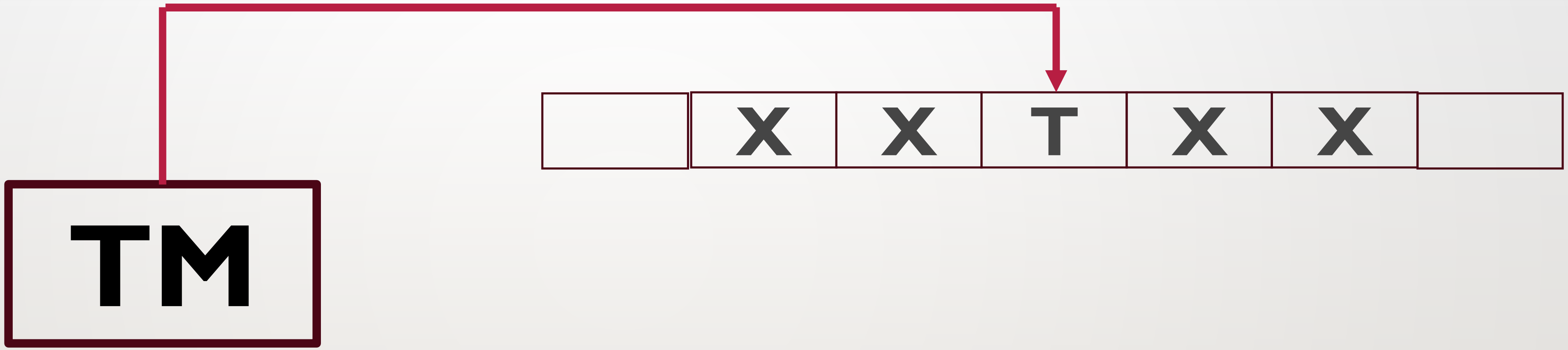

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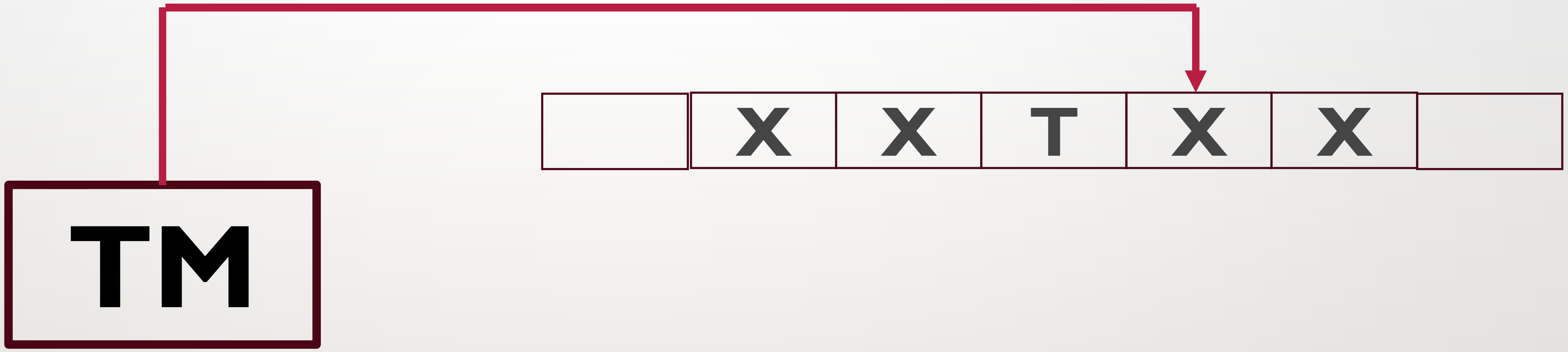

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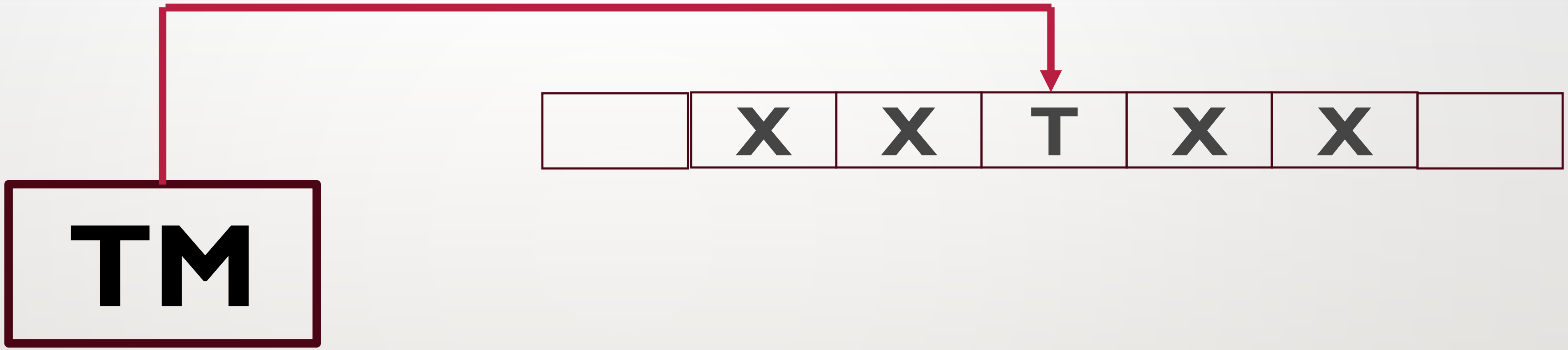

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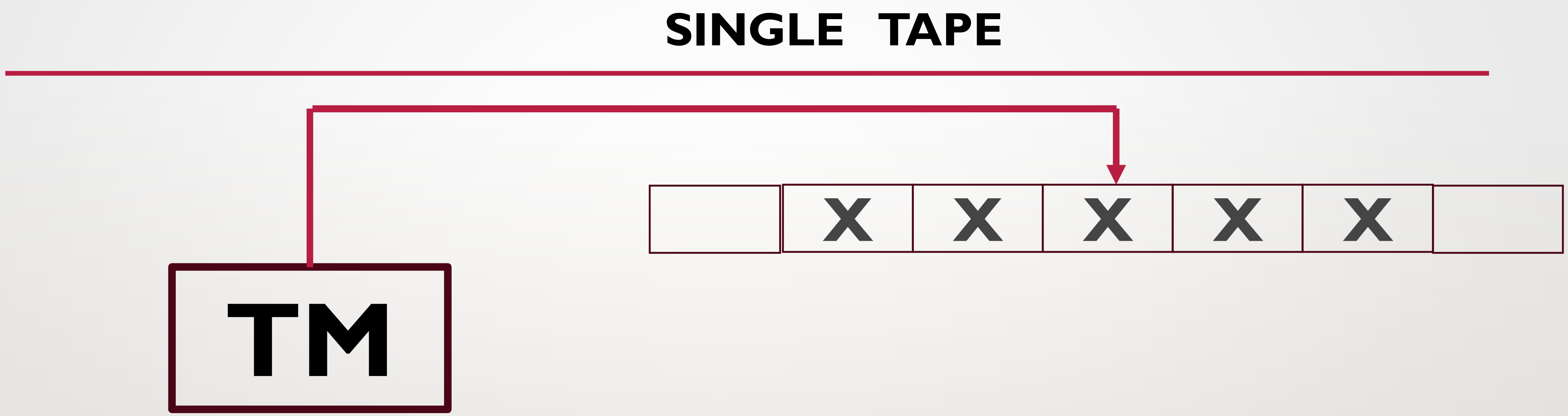

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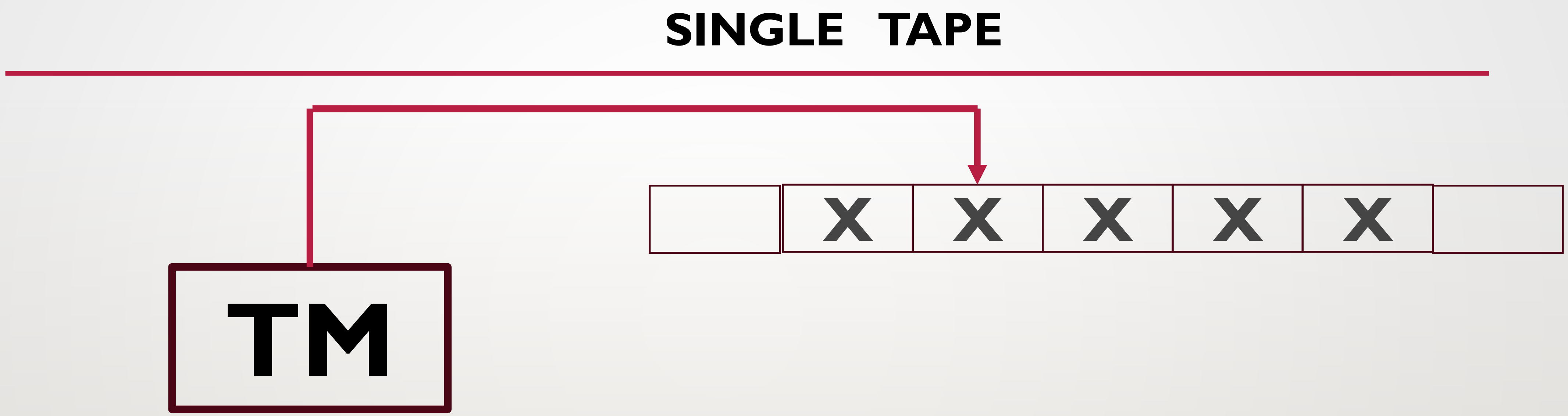

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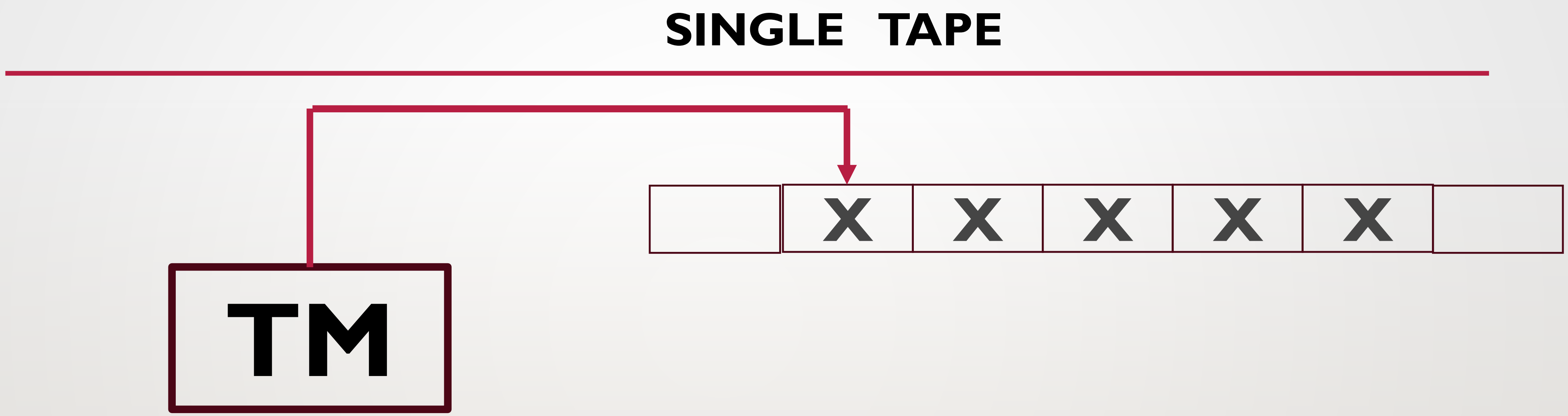

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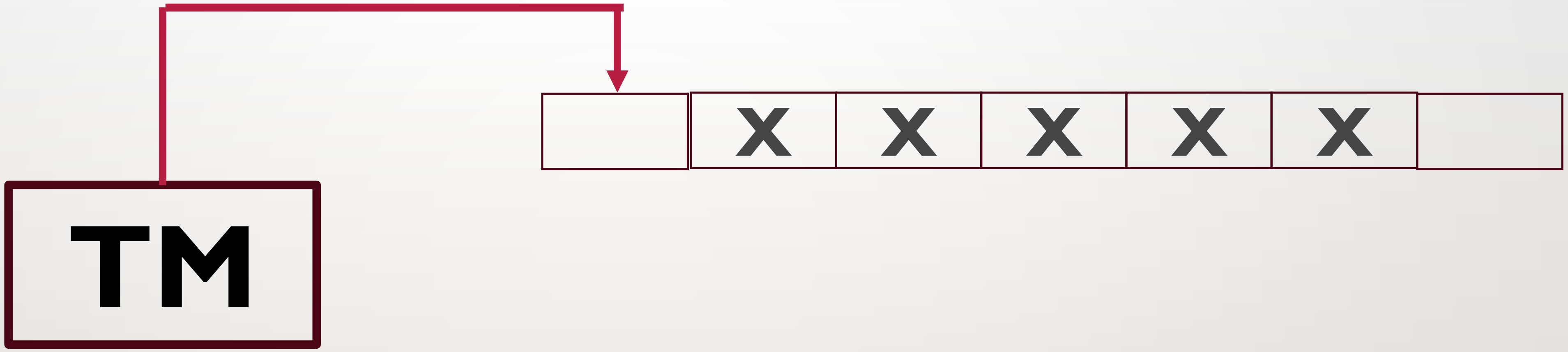

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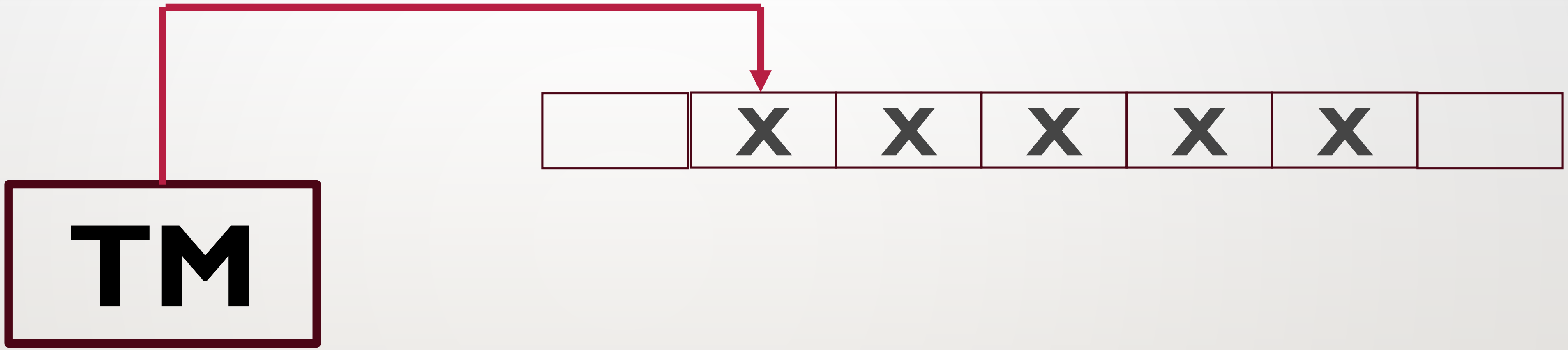

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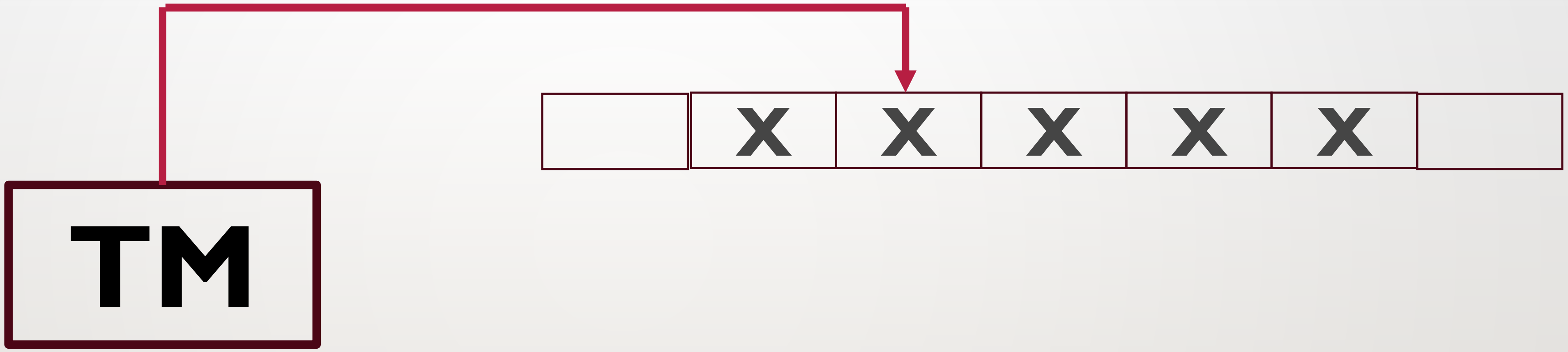

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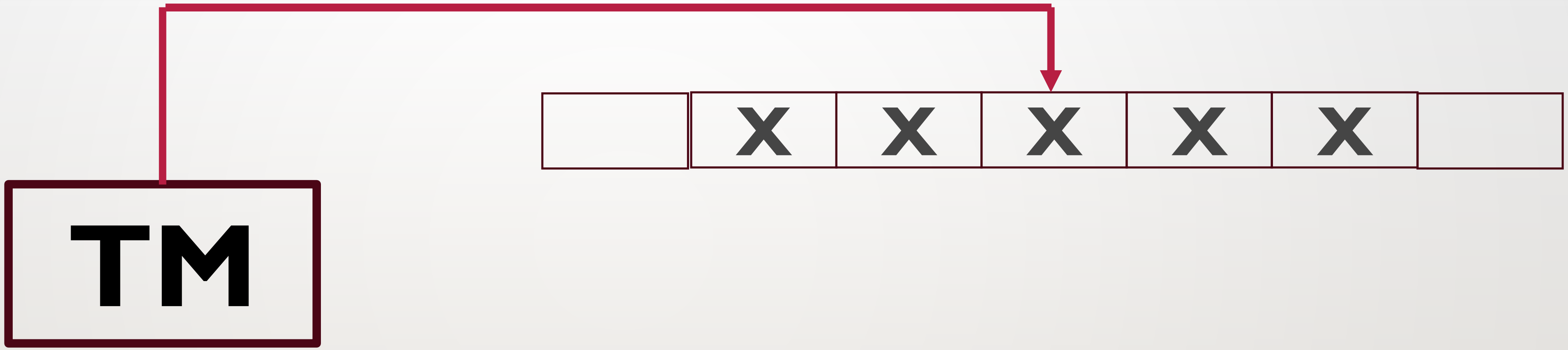

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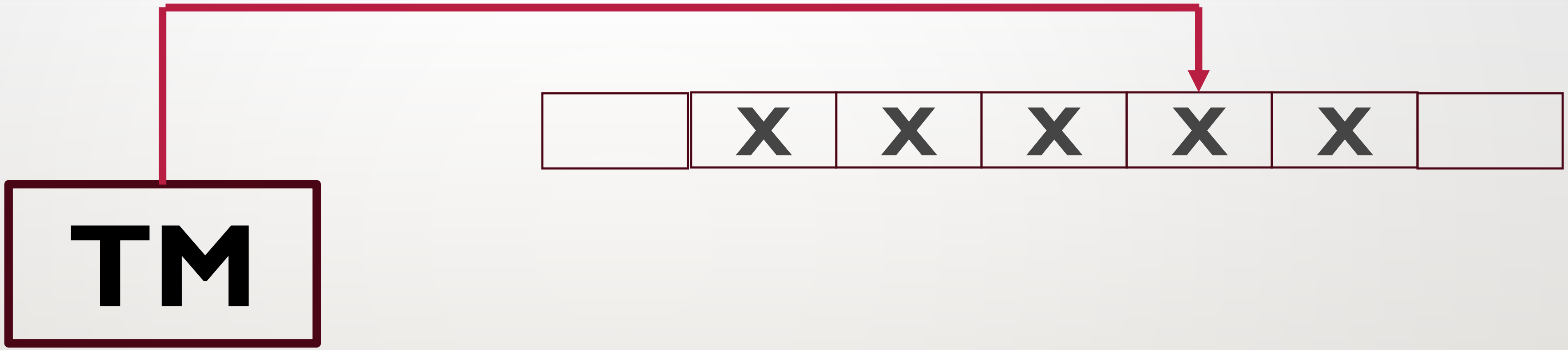

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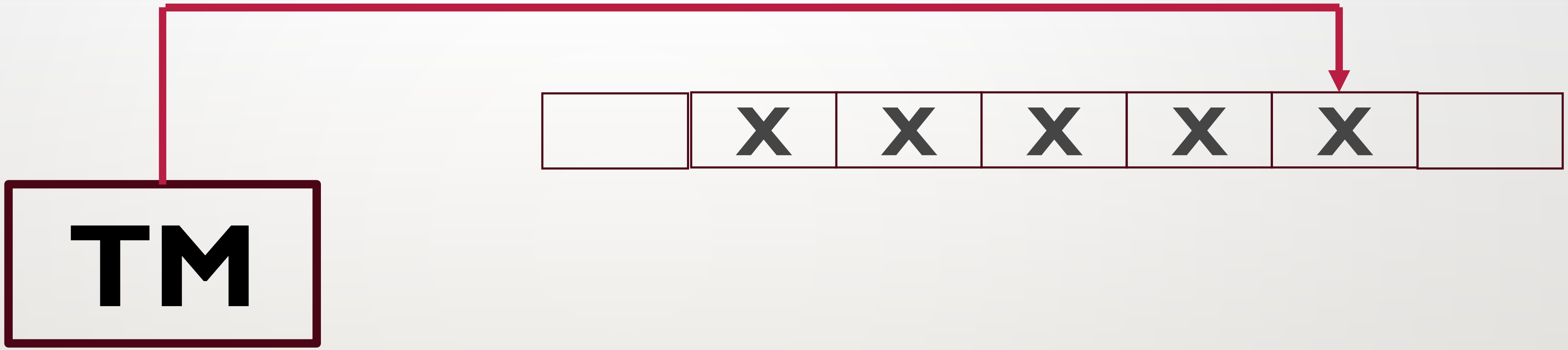

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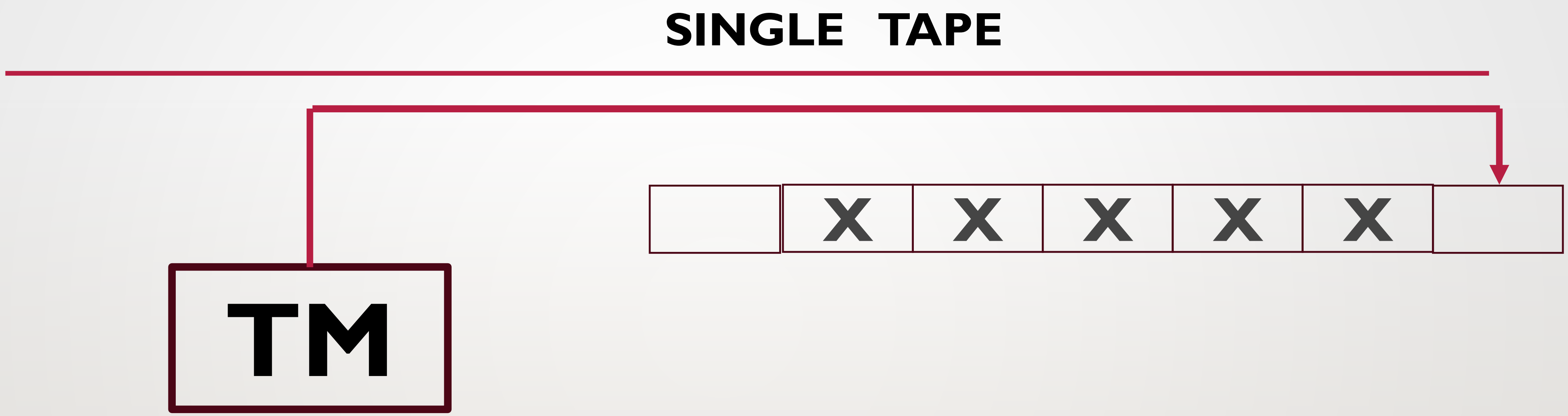

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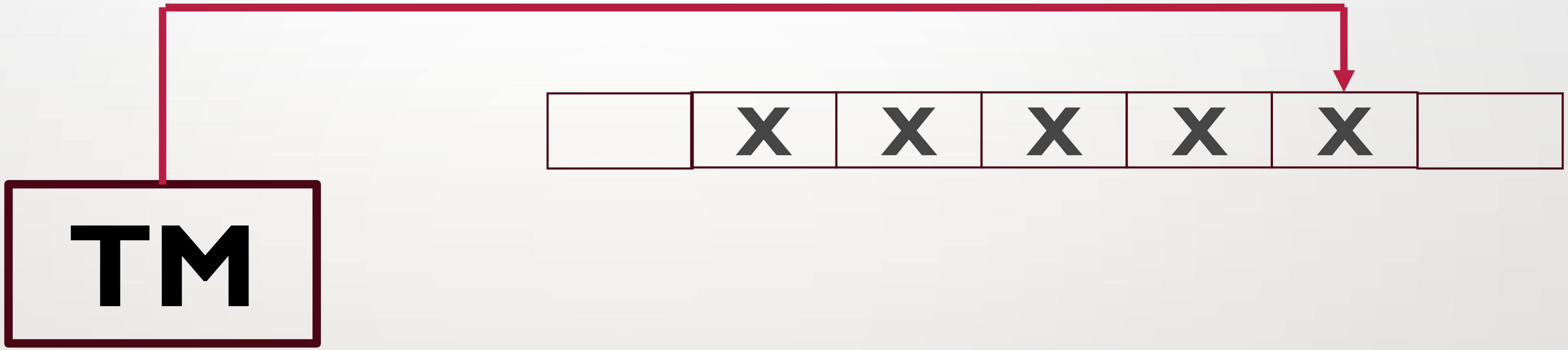


$O(n^2)$

CLASS OF LANGUAGES

- Let $t: \mathbb{N} \rightarrow \mathbb{R}^{+}$ be a function. Define the *time complexity class*, $\boldsymbol{TIME(t(n))}$, to be the collection of all languages that are decided by an $O(t(n))$ time Turing Machine.

CHALLENGES FACED

- Classifying problems by time complexity is challenging when grouping those requiring quadratic (n^2) or cubic (n^3) time into one set.
- This challenge stems from the dependency of classification on the underlying execution architecture.
- A classification system is needed to accommodate such problems within a polynomial time complexity class, regardless of execution architecture.

CLASS 'P'

- P is the class of languages decidable in polynomial time on a deterministic single-tape Turing Machine

$$P = \bigcup_{k=1}^{\infty} TIME(n^k)$$

P is class of problems realistically solvable on a computer.

Properties

- ***Robust*** i.e. independent of exact model of computation.
- ***Efficient*** i.e. It constitutes all efficient algorithms

CLASS 'P'

- *To show an algorithm belongs to P, we need to:*
  - **Provide polynomial upper bound on number of stages**
  - **Examine each stage to ensure it can run in polynomial time**

CLASS 'P'

Examples

Path problem

- Is there a path from s to t in a given graph G?
- PATH={<G, s, t> | G is directed graph with directed path from s to t}

Example: *Path problem*

- **Brute force approach**
- If G has m nodes, path cannot be more than m
- Upper bound on possible paths m^m
- Try each “path” one by one for legality and for linking s-to-t

Example: *Path problem*

- Breadth first search
- Place mark on node s
- Repeat until no new nodes marked
    - Scan all edges; If edge (a, b) found from marked node a to unmarked b, mark b
- If t is marked, accept; Otherwise, reject

CLASS 'P'

Breadth First Search Time Complexity

- Measure complexity based on number of nodes n
- Place mark on node s ← n steps (find node s in list of m nodes)
- Repeat until no new nodes marked ← n steps (if mark 1 new node/ loop)
- For each edge (a, b) with 'a' already marked, mark 'b' also ← $O(n^3)$

(n^2 max total edges, for each edge, search for 'a' to see if marked (n steps), then mark b in list if needed (n steps)

- In total: $n^2 \times (n + n) = 2n^3$
- If t marked, accept; else, reject ← n steps (find node t in list of n nodes)
- In total: $f(n)= n + n \times 2n^3 + n = 2m^4 + 2n = O(n^4)$ It's POLYNOMIAL!

CLASS 'P'

Example: RELPRIME

- Two numbers are relatively prime if **1** is the largest number that evenly divides them both
- **10** and **21** are relatively prime
- **10** and **22** are not relatively prime
- Solution: search all divisors from 2 until min(x, y) / 2
- min(x, y) / 2 numbers tried, min(x, y) /2 steps
- size of input n = length of binary encoding = $\log_2(\max(x, y))$
- 2^n steps – exponential complexity!

RELPRIME: Faster Solution

Euclidean algorithm

- E = On input <x,y>
- Repeat until y = 0
  - Assign x = x mod y
  - Exchange x and y
- Output x
- R = On input <x, y>, Run E on <x, y>
- If result is 1, accept; Otherwise, reject

RELPRIME: Simulating the Euclidean algorithm

x=10	y=21	Perform x % y
x=10	y = 21	x=y and y = (x MOD y)
x=21	y=10	Perform x % y
x=10	y=1	x=y, and y=(x MOD y)
x=0	y=1	

y=1 when x=0, so original numbers relatively prime

CLASS 'P'

RELPRIME: Euclidean Algorithm – complexity

x = x mod y ← new x always less than y

- **If old x is twice y or more, new x will be cut at least in half**
- **If old x between y and 2y, new x will be cut at least in half**
    - **new x = x – y**

Number of loops: $2*\log_2(\max(x, y))$

Length of input (in binary): $\log_2(x) + \log_2(y) = O(\log_2(\max(x, y)))$

Number of loops: $O(n)$

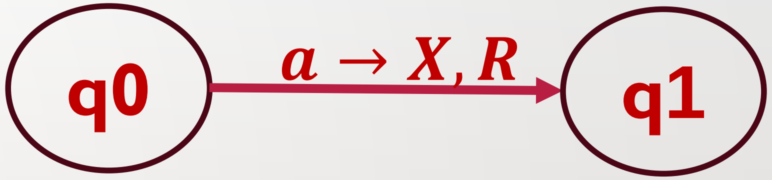
TURING MACHINE

A Decider for the language $L = \{a^n b^n c^n : n > 0\}$

Input Tape

a	a	b	b	c	c

q0aabbcc



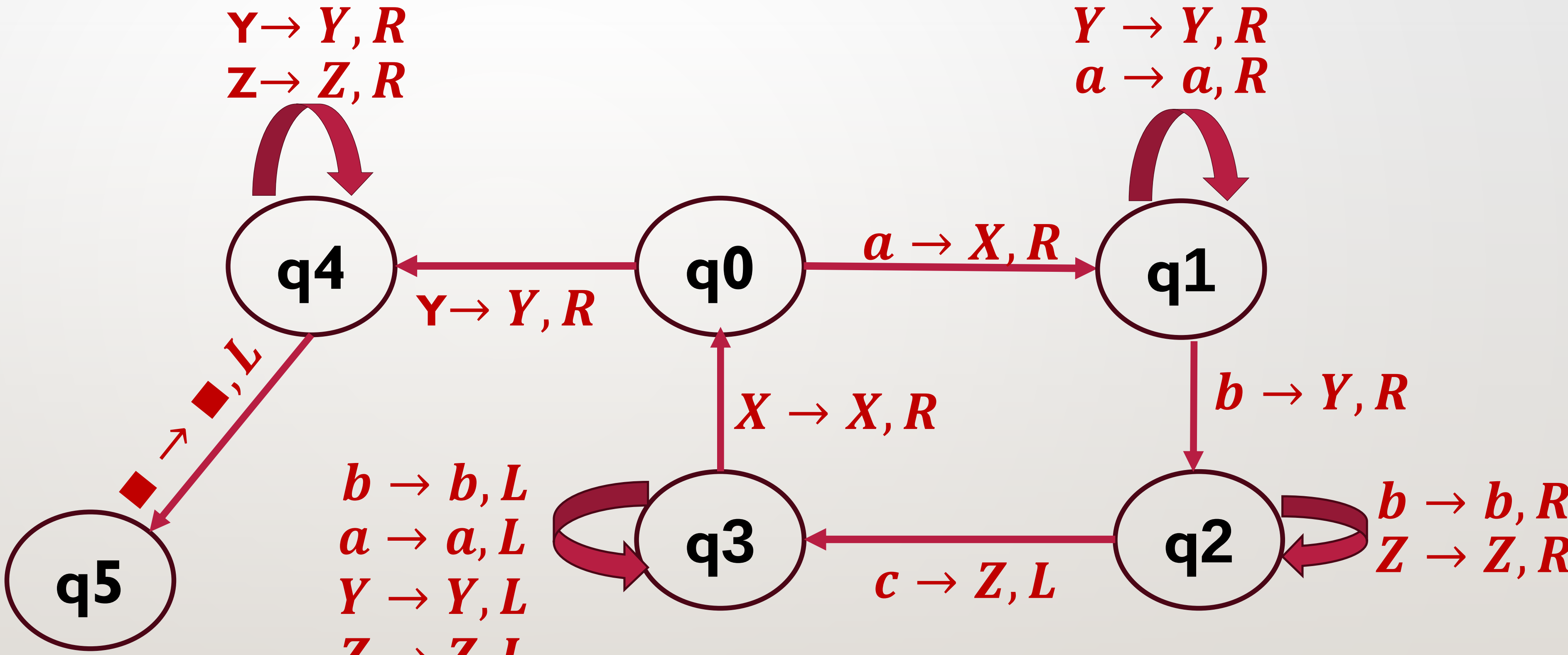
X	a	b	b	c	c

Configuration → xq1abbcc

A Decider for the language $L = \{a^n b^n c^n : n > 0\}$

A partial Computing Path

q0aabbcc
↓
xq1abbcc
↓
xaq1bbcc



TURING MACHINE

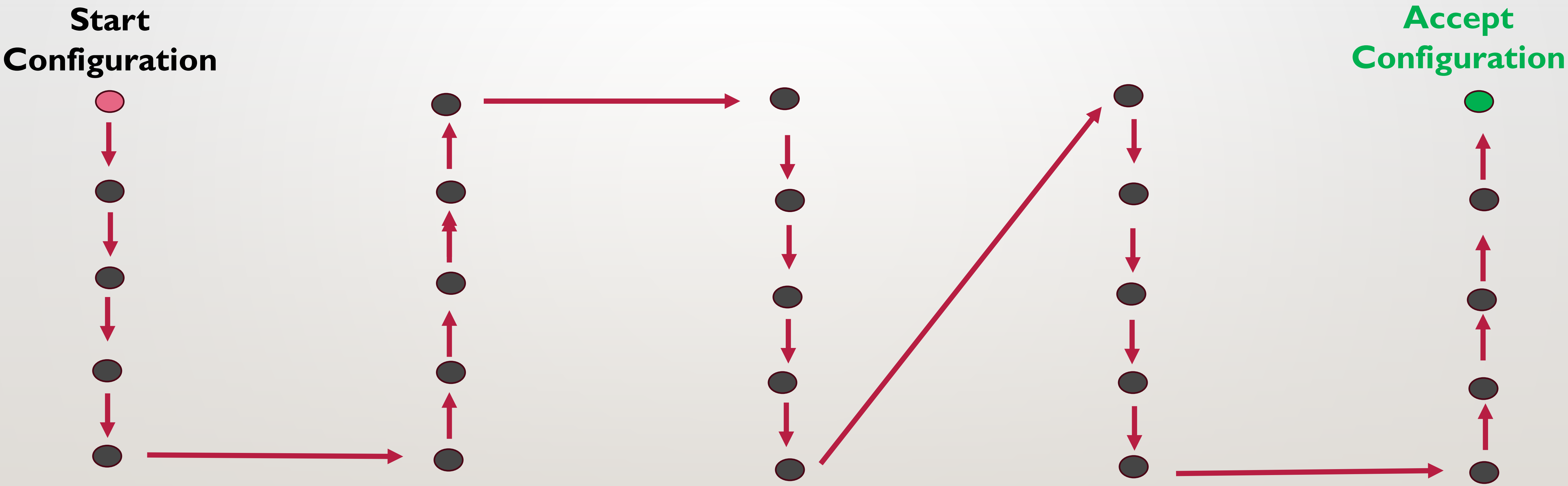
A Decider for the language $L = \{a^n b^n c^n : n > 0\}$

Start configuration | **Accept configuration**

q0aabbcc → xq1abbcc → xaq1bbcc → xayq2bcc → xaybq2cc → xaybq3zc → xayq3bzc → xaq3ybzc → xq3aybzc → q3xaybzc → xq0aybzc → xxq1ybzc → xxyq1bzc → xxyyq2zc → xxyyzq2c → xxyyq3zz → xxyq3yzz → xxq3yyzz → xq3xyyzz → xxq4yyzz → xxyq4yzz → xxyyq4zz → xxyyzq4z → xxyyzzq4■ → xxyyzq4z■

TURING MACHINE

A Decider for the language $L = \{a^n b^n c^n : n > 0\}$

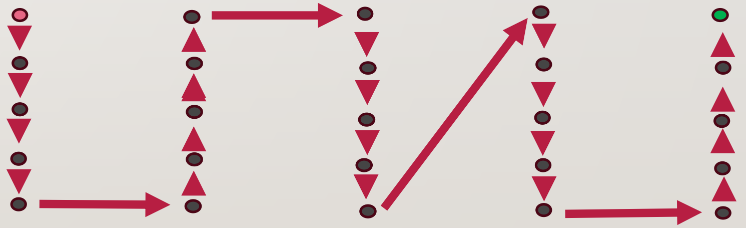


A Decider for the language $L = \{a^n b^n c^n : n > 0\}$

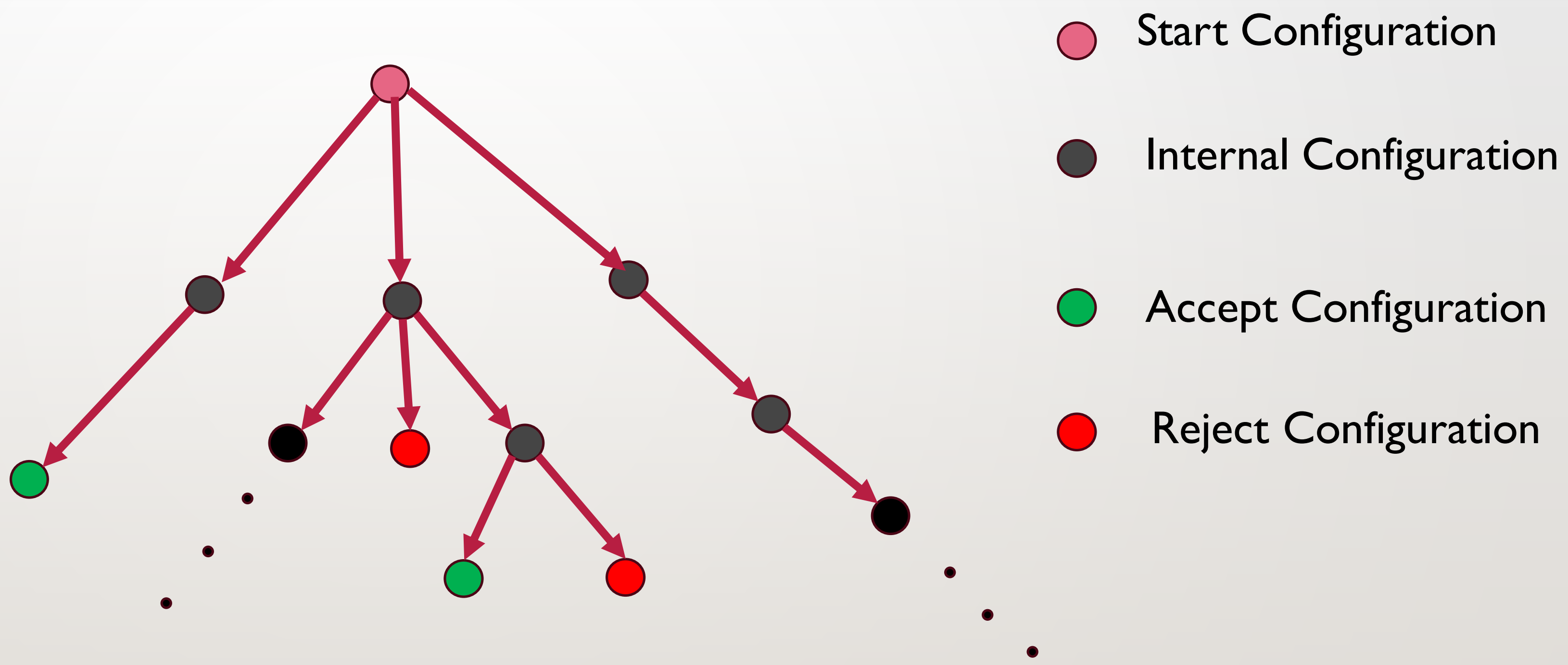
There is a Single Computing Path

TM halts on every input with either q_{accept} or q_{reject} Configuration.

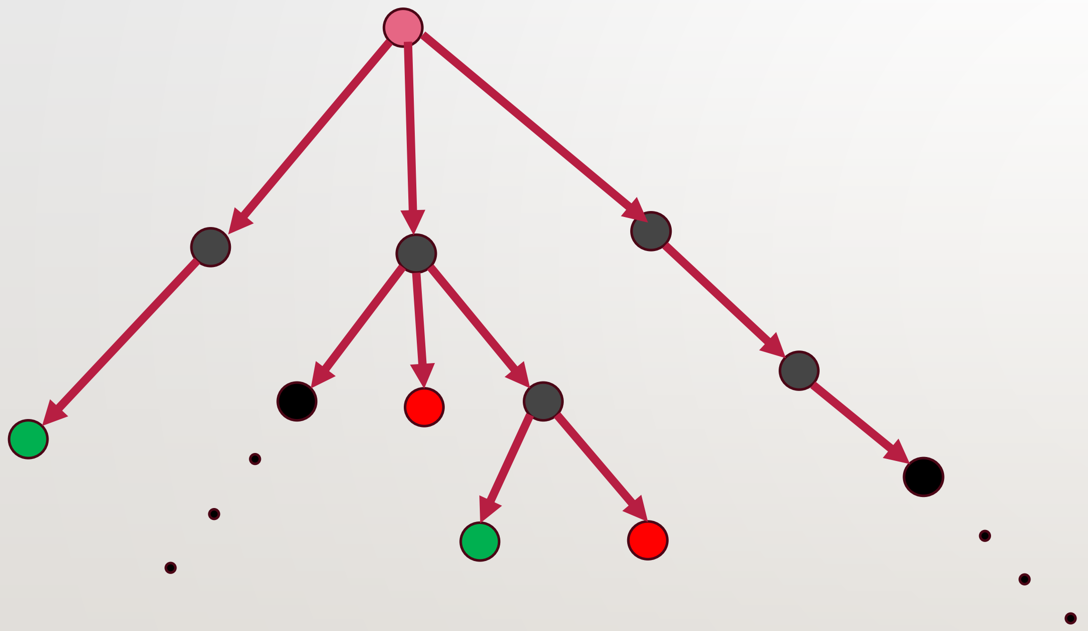
The TM we discussed is a **DETERMINISTIC** Turing Machine.



A Non-Deterministic TM



A Non-Deterministic TM

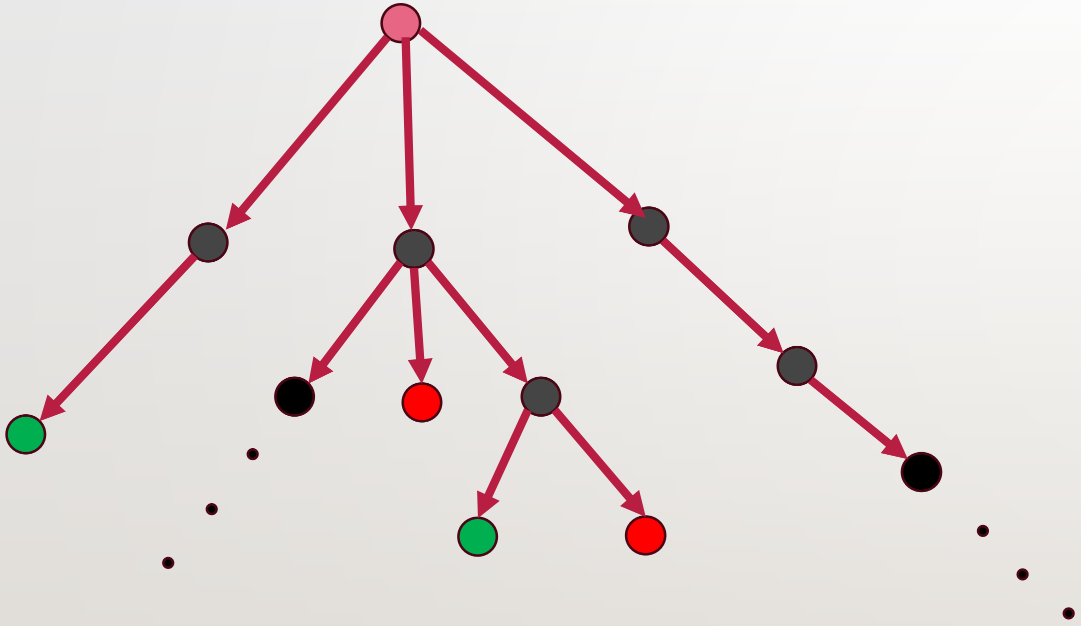


Multiple Computational Paths

Each path may lead to either q_{accept} or q_{reject} Configuration

Many of the computing paths may not be terminating

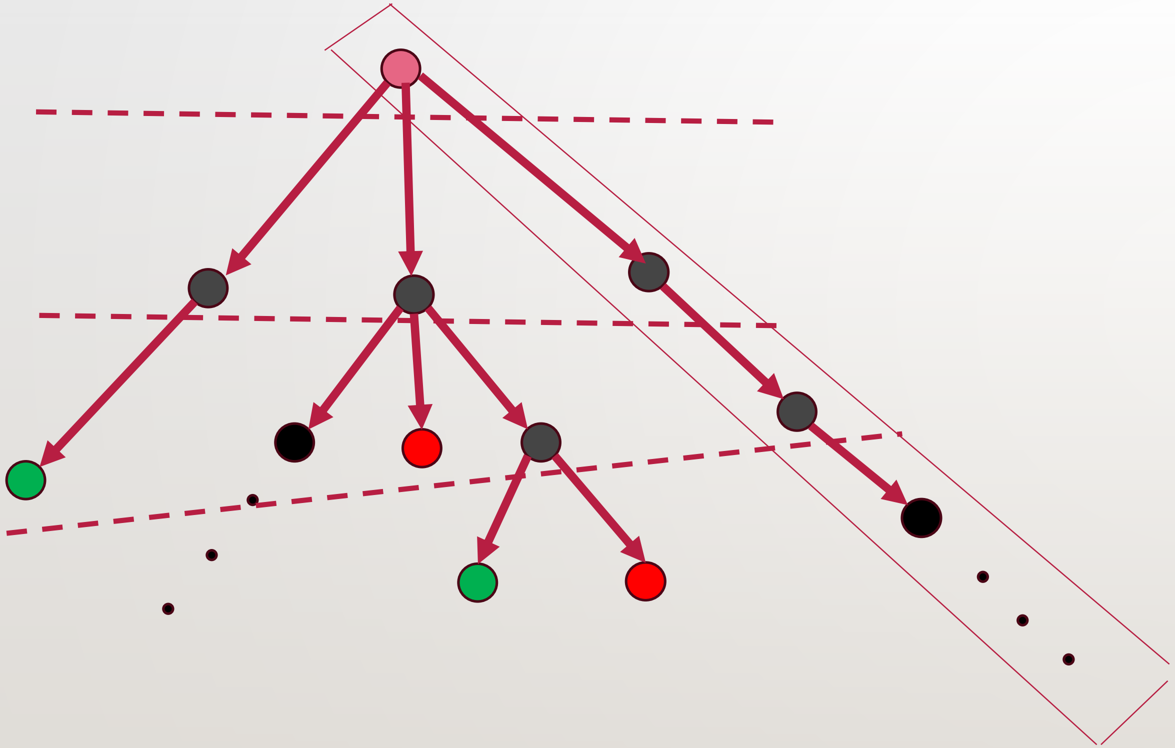
A Non-Deterministic TM



A string is accepted by a NDTM if any of the computing path reaches a q_{accept} configuration

we move from a uniquely determined sequence of computation steps to several possible sequences.

A Non-Deterministic TM



An NDTM has a magical parallelism

We can traverse the tree structure either using BFS or DFS method.

A Non-Deterministic TM

Given a language $L \subseteq \sum^*$, we determine the time $t(x)$ needed by a non-deterministic machine to process input $x \in \sum^*$ in the following way.

1. If $x \in L$ then $t(x)$ is defined as the length of the shortest path in the computation tree accepting x.
2. If $x \notin L$, the value of $t(x)$ is defined as the length of the shortest path in the computation tree.

A Non-Deterministic TM

Time $t(n)$ needed by T for processing input $x \in \sum^*$ with the length $|x| = n \in N$ is defined as the maximum of all finite $t(x)$ for $x \in \sum^*$ with $|x| = n$.

CLASS 'NP'

Non-Deterministic Polynomial Time

$$NTIME(t(n)) = \{L: L \text{ is a decided by a } NDTM \text{ in } O(t(n))\}$$

$$NP = \bigcup_{k=1}^{\infty} NTIME(n^k)$$

It is also known as given and verify model

Example: SUBSET-SUM

$$SUBSET - SUM =$$

$$\{< s, t >: s = \{x_1, x_2, \dots, x_n\} \; \exists \, t \; \subseteq \{1, 2, 3, \dots, k\} \; such \; that \sum_{i \in t} x_i = t$$

Example: SUBSET-SUM

$$SUBSET - SUM =$$

$$\{< s, t >: s = \{x_1, x_2, \ldots, x_n\} \exists\, t \subseteq \{1, 2, 3, \ldots, k\}\ such\ that \sum_{i \in t} x_i = t$$