Design and Analysis of Algorithms

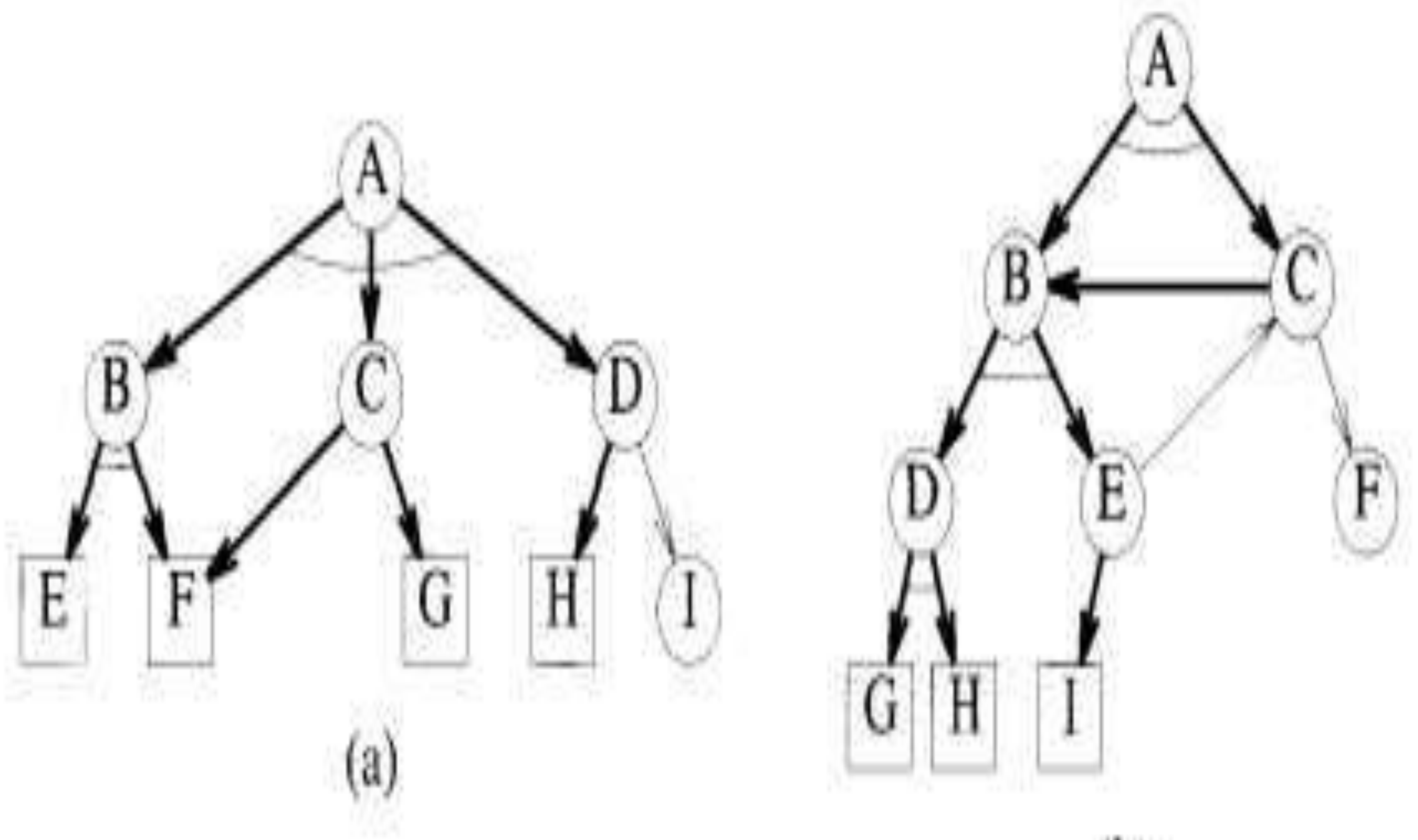
AND OR GRAPH DECISION PROBLEM

Session -36

AND/OR GRAPH DECISION PROBLEM(AOG)

- Many complex problems can be broken down into a series of subproblems such that the solution of all or some of these results in the solution of the original problem.
- These subproblems can be broken down further into sub subproblems, and so on, until the only problems remaining are sufficiently primitive as to be trivially solvable.
- This breaking down of a complex problem into several subproblems can be represented by a directed graph like structure in which nodes represent problems and descendents of nodes represent the subproblems associated with them

- To solve OR node requires either all descendents to be solved or only one descendent to be solved.
- To solve AND node requires all the descendents to be solved.
- The AND nodes are drawn with an arc across all edges leaving the node.
- Nodes with no descendents are called terminal. Terminal nodes represent primitive problems and are marked either solvable or not solvable.
- Solvable terminal nodes are represented by rectangles.
- An AND/OR graph need not always be a tree
- A solution graph is a subgraph of solvable nodes that shows that the problem is solved.



(a) (b)

Let us assume that there is a cost associated with each edge in the AND/OR graph.

The cost of a solution graph H of an AND/OR graph G is the sum of the costs of the edges in H.

The AND/OR graph decision problem (AOG) is to determine whether G has a solution graph of cost at most k, for k a given input.

Example 11.17 Consider the directed graph of Figure 11.15. The problem to be solved is P_1. To do this, one can solve node P_2, P_3, or P_7, as P_1 is an OR node. The cost incurred is then either 2, 2, or 8 (i.e., cost in addition to that of solving one of P_2, P_3, or P_7). To solve P_2, both P_4 and P_5 have to be solved, as P_2 is an AND node. The total cost to do this is 2. To solve P_3, we can solve either P_5 or P_6. The minimum cost to do this is 1. Node P_7 is free. In this example, then, the optimal way to solve P_1 is to solve P_6 first, then P_3, and finally P_1. The total cost for this solution is 3. □

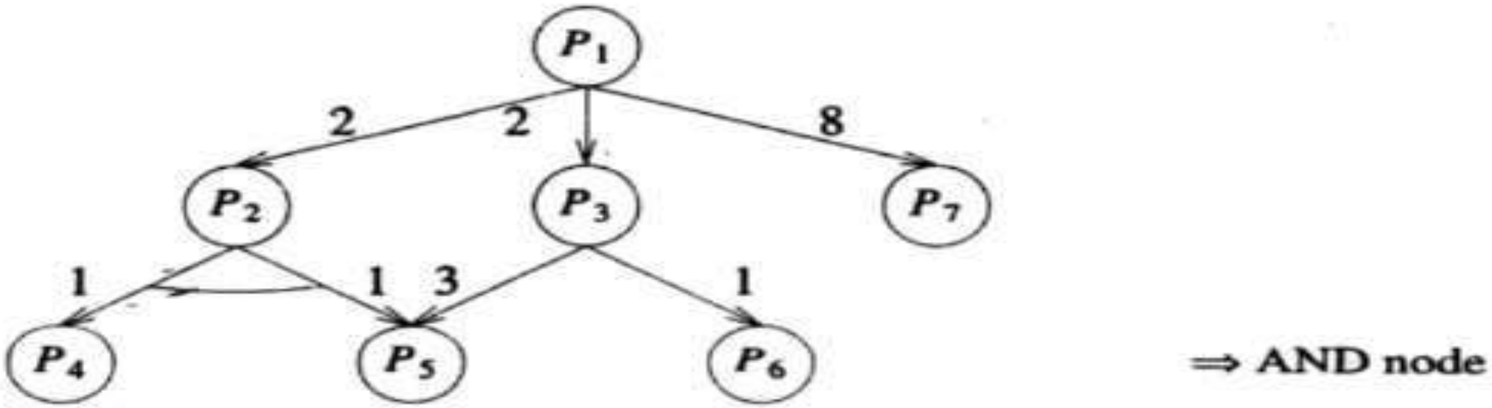


Figure 11.15 AND/OR graph

Theorem 11.7 CNF-satisfiability $\propto$ the AND/OR graph decision problem.

Proof: Let P be a propositional formula in CNF. We show how to transform a formula P in CNF into an AND/OR graph such that the AND/OR graph so obtained has a certain minimum cost solution if and only if P is satisfiable. Let

$$P = \bigwedge_{i=1}^{k} C_i, \quad C_i = \bigvee l_j$$

where the l_j's are literals. The variables of P, $V(P)$ are $x_1, x_2, \ldots, x_n$. The AND/OR graph will have nodes as follows:

1. There is a special node S with no incoming arcs. This node represents the problem to be solved.

2. The node S is an AND node with descendent nodes $P, x_1, x_2, \ldots, x_n$.

3. Each node x_i represents the corresponding variable x_i in the formula P. Each x_i is an OR node with two descendents denoted Tx_i and Fx_i respectively. If Tx_i is solved, then this will correspond to assigning a truth value of true to the variable x_i. Solving node Fx_i will correspond to assigning a truth value of false to x_i.

4. The node P represents the formula P and is an AND node. It has k descendents $C_1, C_2, \ldots, C_k$. Node C_i corresponds to the clause C_i in the formula P. The nodes C_i are OR nodes.

5. Each node of type Tx_i or Fx_i has exactly one descendent node that is terminal (i.e., has no edges leaving it). These terminal nodes are denoted $v_1, v_2, \ldots, v_{2n}$.

To complete the construction of the AND/OR graph, the following edges and costs are added:

1. From each node C_i an edge $\langle C_i, Tx_j \rangle$ is added if x_j occurs in clause C_i. An edge $\langle C_i, Fx_j \rangle$ is added if $\bar{x}_j$ occurs in clause C_i. This is done for all variables x_j appearing in the clause C_i. Clause C_i is designated an OR node.

2. Edges from nodes of type Tx_i or Fx_i to their respective terminal nodes are assigned a weight, or cost of 1.

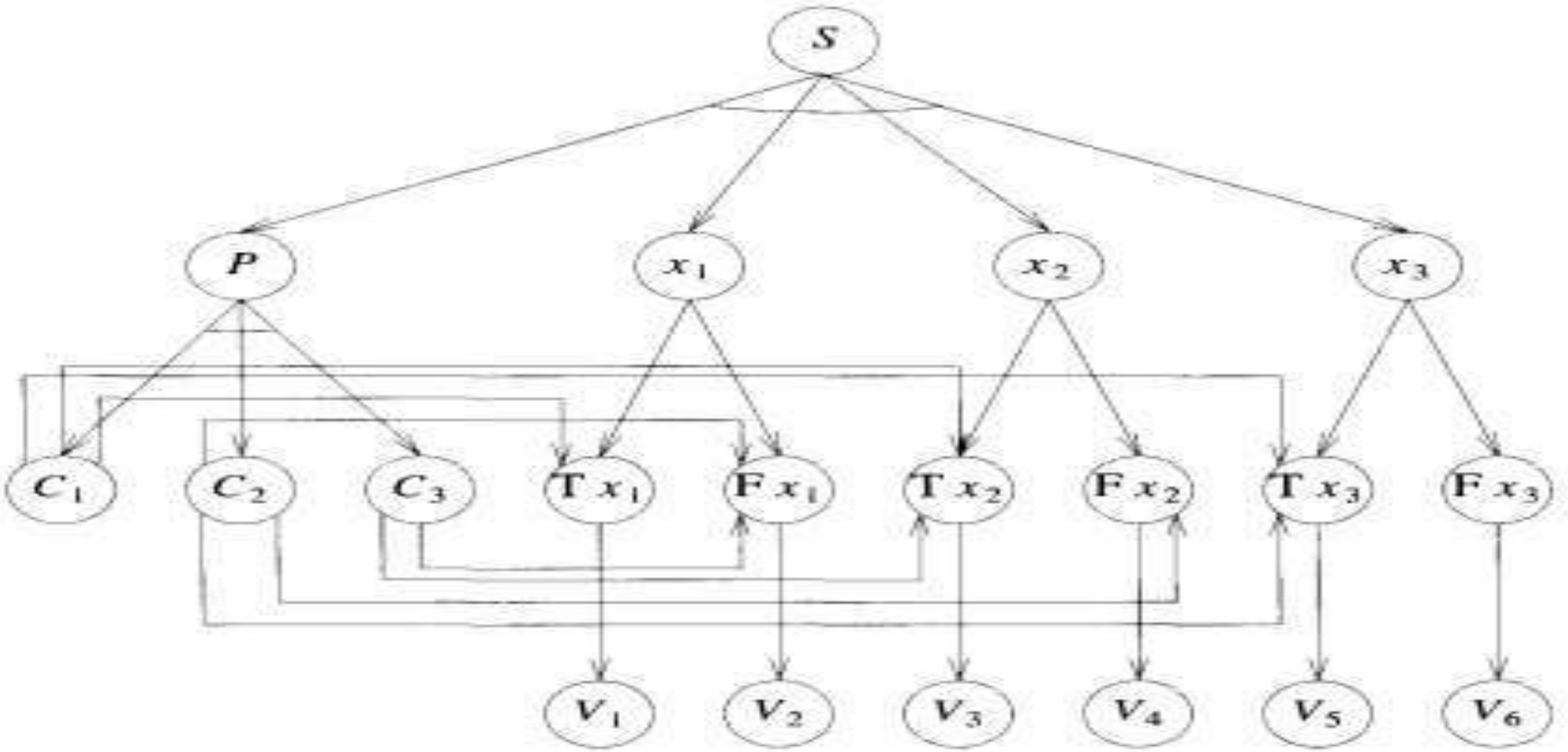
3. All other edges have a cost of 0.

Example 11.18 Consider the formula

$$P = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2); \quad V(P) = x_1, x_2, x_3; \quad n = 3$$

Figure 11.16 shows the AND/OR graph obtained by applying the construction of Theorem 11.7.

The nodes Tx_1, Tx_2, and Tx_3 can be solved at a total cost of 3. The node P costs nothing extra. The node S can then be solved by solving all its descendent nodes and the nodes Tx_1, Tx_2, and Tx_3. The total cost for this solution is 3 (which is n). Assigning the truth value of true to the variables of P results in P's being true. □



AND nodes joined by arc
All other nodes are OR

Questions:

1. Explain the terminology used in AOG?
2. Prove that CNF-Satisfiability can be reduced to AOG decision problem

THANK YOU