SUBJECTCODE: 23MT2005
PROBABILITY STATISTICS AND QUEUING THEORY

Tutorial 3:
Applications of discrete probability distributions

Date of the Session: //_______________________________Time of the Session:______to________

Learning outcomes:

- Understanding the concept of Bernoulli trial.
- Apply Binomial and Poisson to the real-world problems

1. A basketball player can shoot a ball into the basket with a probability of 0.6. What is the probability that he misses the shot?

Solution:

Given:

- Probability of making the shot = 0.6

The probability of missing the shot is just the complement:

$$P(\text{miss}) = 1 - P(\text{make})$$

$$P(\text{miss}) = 1 - 0.6 = 0.4$$

So, the probability of missing the shot is **0.4**.

2. Determine the expected value and variance of Bernoulli distribution.

Solution:

1. Expected Value $E(X)$:

- Formula: $E(X) = \sum_x x \cdot P(X = x)$
- For Bernoulli, $X = 1$ with probability p, and $X = 0$ with probability $1 - p$.
- Calculation:

$$E(X) = (1 \cdot p) + (0 \cdot (1 - p)) = p$$

2. Variance $\mathrm{Var}(X)$:

- Formula: $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$
- For Bernoulli, since $X^2 = X$, $E(X^2) = E(X) = p$.
- Calculation:

$$\mathrm{Var}(X) = p - p^2 = p(1 - p)$$

3. In a manufacturing process, the probability that a component is defective is 0.1. A random sample of 20components is selected for quality inspection.

i) What is the probability that exactly 3 components are defective?
ii) What is the probability that at most 2 components are defective?
iii) What is the mean and standard deviation of the number of defective components?

Solution:

Binomial Distribution Formula:

The **probability mass function (PMF)** for a Binomial distribution is:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Where:

- $n = 20$ (sample size),
- k is the number of defective components,
- $p = 0.1$ (probability of defect),
- $1 - p = 0.9$ (probability of non-defect),
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ (binomial coefficient).

i) Probability that exactly 3 components are defective:

To calculate $P(X = 3)$:

$$P(X = 3) = \binom{20}{3}(0.1)^3(0.9)^{17}$$

Step-by-step:

1. **Binomial coefficient:**

$$\binom{20}{3} = \frac{20!}{3!(20-3)!} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$$

2. **Substitute into the formula:**

$$P(X = 3) = 1140 \cdot (0.1)^3 \cdot (0.9)^{17}$$

ii) Probability that at most 2 components are defective:

To calculate $P(X \leq 2)$, we sum the probabilities for $X = 0, 1, 2$:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

For each k, use the binomial formula:

1. $P(X = 0)$:

$$P(X = 0) = \binom{20}{0}(0.1)^0(0.9)^{20}$$

2. $P(X = 1)$:

$$P(X = 1) = \binom{20}{1}(0.1)^1(0.9)^{19}$$

3. $P(X = 2)$:

$$P(X = 2) = \binom{20}{2}(0.1)^2(0.9)^{18}$$

iii) Mean and Standard Deviation:

For a Binomial distribution, the **mean** and **standard deviation** are given by:

- **Mean μ:**

$$\mu = n \cdot p = 20 \cdot 0.1 = 2$$

- **Standard deviation σ:**

$$\sigma = \sqrt{n \cdot p \cdot (1 - p)} = \sqrt{20 \cdot 0.1 \cdot 0.9} \approx 1.342$$

4. Which conditions for the binomial distribution, if any, fail to hold in the following situations?

(a)The number of persons getting Corona in a room of 30 persons.

b) Consider 2 out of 20 PCs are defective. We randomly select 3 for testing.

Is this a binomial experiment?

Solution:

(a) The number of persons getting Corona in a room of 30 persons:

- **Condition failure:** The probability is not constant (depends on individuals) and the trials are not independent (shared exposure).
- **Conclusion: Not a Binomial distribution.**

(b) 2 out of 20 PCs are defective. We randomly select 3 for testing:

- **Condition failure:** The probability is not constant (sampling without replacement) and the trial are not independent.
- **Conclusion: Not a Binomial distribution.**

5. In a particular restaurant, an average of 3 out of every 5 customers ask for water with their meal. A random sample of 10 customers is selected. Find the probability that

(i) Exactly 6 ask for water with their meal,

(ii) Less than 9 ask for water with their meal.

(iii) No one asks for water with their meal.

(iv) At most 2 ask for water with their meal.

(v) At least 3 ask for water with their meal.

Solution:

Binomial Distribution Formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where:

- $n = 10$ (sample size),
- $p = 0.6$ (probability of asking for water),
- $q = 1 - p = 0.4$ (probability of not asking for water),
- k is the number of customers asking for water.

(i) Exactly 6 ask for water with their meal:

$$P(X = 6) = \binom{10}{6} (0.6)^6 (0.4)^4$$

1. **Binomial coefficient:**

$$\binom{10}{6} = \frac{10!}{6!(10-6)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

2. **Substitute into the formula:**

$$P(X = 6) = 210 \cdot (0.6)^6 \cdot (0.4)^4$$

$$P(X = 6) \approx 0.2508$$

(ii) Less than 9 ask for water with their meal:

$$P(X < 9) = P(X = 0) + P(X = 1) + \cdots + P(X = 8)$$

For each k, we calculate:

$$P(X = k) = \binom{10}{k}(0.6)^k(0.4)^{10-k}$$

Then sum up the probabilities for $k = 0$ to $k = 8$.

$$P(X < 9) \approx 0.9536$$

(iii) No one asks for water with their meal:

$$P(X = 0) = \binom{10}{0}(0.6)^0(0.4)^{10}$$

1. **Binomial coefficient:**

$$\binom{10}{0} = 1$$

2. **Substitute into the formula:**

$$P(X = 0) = 1 \cdot (0.6)^0 \cdot (0.4)^{10}$$

$$P(X = 0) \approx 0.0001$$

(iv) At most 2 ask for water with their meal:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

For each k from 0 to 2, we calculate:

$$P(X = k) = \binom{10}{k} (0.6)^k (0.4)^{10-k}$$

$$P(X \leq 2) \approx 0.0123$$

(v) At least 3 ask for water with their meal:

$$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

We calculate $P(X < 3)$ by summing the probabilities for $X = 0, 1, 2$, and then subtract from 1.

$$P(X \geq 3) \approx 0.9877$$

6. In a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.

Solution:

Step 1: Determine the probability $P(X \geq 3)$

We are looking for the probability that a sample contains **at least 3 defective parts**. This is calculated as:

$$P(X \geq 3) = 1 - P(X < 3)$$

where:

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

To calculate each term $P(X = k)$ for $k = 0, 1, 2$, we use the Binomial distribution formula:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Where:

- $n = 20$ (sample size),
- $p = 0.1$ (probability of a defective part),
- k is the number of defective parts in the sample.

Step 2: Calculate $P(X=0)$, $P(X=1)$, and $P(X=2)$

1. $P(X=0)$:

$$P(X=0)=\binom{20}{0}(0.1)^0(0.9)^{20}$$

$$P(X=0)=1\times(1)\times(0.9)^{20}\approx 0.1216$$

2. $P(X=1)$:

$$P(X=1)=\binom{20}{1}(0.1)^1(0.9)^{19}$$

$$P(X=1)=20\times(0.1)\times(0.9)^{19}\approx 0.2684$$

3. $P(X=2)$:

$$P(X=2)=\binom{20}{2}(0.1)^2(0.9)^{18}$$

$$P(X=2)=190\times(0.01)\times(0.9)^{18}\approx 0.3020$$

Step 3: Sum the probabilities for $X = 0, 1, 2$

$$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X < 3) = 0.1216 + 0.2684 + 0.3020 = 0.6920$$

Step 4: Calculate $P(X \geq 3)$

$$P(X \geq 3) = 1 - P(X < 3)$$

$$P(X \geq 3) = 1 - 0.6920 = 0.3080$$

Step 5: Calculate the expected number of samples with at least 3 defective parts

Now, we multiply the probability $P(X \geq 3)$ by the total number of samples (1000):

$$\text{Expected number of samples} = 1000 \times P(X \geq 3) = 1000 \times 0.3080 = 308$$

Conclusion:

Out of 1000 samples, approximately **308** samples are expected to contain at least 3 defective parts.

7. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state?

Solution:

1. **Binomial distribution formula:**

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where:

- $n = 9$,
- $p = 0.25$,
- $k = 0, 1, 2, 3$.

2. **Calculate for each k:**

- $P(X = 0) = \binom{9}{0}(0.25)^0(0.75)^9 \approx 0.0751$
- $P(X = 1) = \binom{9}{1}(0.25)^1(0.75)^8 \approx 0.2254$
- $P(X = 2) = \binom{9}{2}(0.25)^2(0.75)^7 \approx 0.2816$
- $P(X = 3) = \binom{9}{3}(0.25)^3(0.75)^6 \approx 0.2279$

3. **Add the probabilities:**

$$P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X < 4) \approx 0.0751 + 0.2254 + 0.2816 + 0.2279 = 0.8100$$

8. A computer crashes on average once every 4 months,

a) What is the probability that it will not crash in a period of 4 months?

b) What is the probability that it will crash once in a period of 4 months?

c) What is the probability that it will crash twice in a period of 4 months?

d) What is the probability that it will crash three times in a period of 4 months?

Solution:

1. **Poisson distribution formula:**

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

   Where $\lambda = 1$ (since the average crash rate is 1 per 4 months).

2. **a) Probability that it will not crash in a period of 4 months ($k = 0$):**

$$P(X = 0) = \frac{1^0 e^{-1}}{0!} = e^{-1} \approx 0.3679$$

3. **b) Probability that it will crash once in a period of 4 months ($k = 1$):**

$$P(X = 1) = \frac{1^1 e^{-1}}{1!} = e^{-1} \approx 0.3679$$

4. **c) Probability that it will crash twice in a period of 4 months ($k = 2$):**

$$P(X = 2) = \frac{1^2 e^{-1}}{2!} = \frac{1 \cdot e^{-1}}{2} \approx 0.1839$$

5. **d) Probability that it will crash three times in a period of 4 months ($k = 3$):**

$$P(X = 3) = \frac{1^3 e^{-1}}{3!} = \frac{1 \cdot e^{-1}}{6} \approx 0.0613$$

Final Answers:

- a) $P(X = 0) \approx 0.3679$
- b) $P(X = 1) \approx 0.3679$
- c) $P(X = 2) \approx 0.1839$
- d) $P(X = 3) \approx 0.0613$

9. A customer care center receives on average 2.5 calls every hour.

a) What is the probability that it will receive at most 4 calls every hour?

b) What is the probability that it will receive at least 5 calls every hour?

Solution:

Poisson Distribution Formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $\lambda = 2.5$ (average number of calls),
- k is the number of calls,
- $e \approx 2.71828$.

a) Probability that the center will receive at most 4 calls every hour ($P(X \leq 4)$):

This means calculating the sum of the probabilities from $P(X = 0)$ to $P(X = 4)$.

$P(X = 0)$:

$$P(X = 0) = \frac{2.5^0 e^{-2.5}}{0!} = e^{-2.5} \approx 0.0821$$

$P(X = 1)$:

$$P(X = 1) = \frac{2.5^1 e^{-2.5}}{1!} = 2.5e^{-2.5} \approx 0.2053$$

$P(X = 2)$:

$$P(X = 2) = \frac{2.5^2 e^{-2.5}}{2!} = \frac{6.25e^{-2.5}}{2} \approx 0.2566$$

$P(X = 3)$:

$$P(X = 3) = \frac{2.5^3 e^{-2.5}}{3!} = \frac{15.625e^{-2.5}}{6} \approx 0.2139$$

$P(X = 4)$:

$$P(X = 4) = \frac{2.5^4 e^{-2.5}}{4!} = \frac{39.0625e^{-2.5}}{24} \approx 0.1337$$

Sum for $P(X \leq 4)$:

$$P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$P(X \leq 4) \approx 0.0821 + 0.2053 + 0.2566 + 0.2139 + 0.1337 = 0.8916$$

b) Probability that the center will receive at least 5 calls every hour ($P(X \geq 5)$):

$$P(X \geq 5) = 1 - P(X \leq 4)$$

$$P(X \geq 5) = 1 - 0.8916 = 0.1084$$

Final Results:

- **a)** $P(X \leq 4) \approx 0.8916$
- **b)** $P(X \geq 5) \approx 0.1084$

10. The probability that a person will die from a certain respiratory infection is 0.002. Find the probability that fewer than 5 of the next 2000 so infected will die. Also find Mean and Variance.

Solution:

Given:

- The average number of deaths $\lambda = 4$ (since $\lambda = n \cdot p = 2000 \times 0.002$).

Poisson Distribution Formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:

- $\lambda = 4$ (mean number of deaths),
- k is the number of deaths.

a) Probability that fewer than 5 people will die ($P(X < 5)$):

We need to find the probability for $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, $P(X = 3)$, and $P(X = 4)$ and then sum them.

$P(X = 0)$:

$$P(X = 0) = \frac{4^0 e^{-4}}{0!} = e^{-4} \approx 0.0183$$

$P(X = 1)$:

$$P(X = 1) = \frac{4^1 e^{-4}}{1!} = 4e^{-4} \approx 0.0733$$

$P(X = 2)$:

$$P(X = 2) = \frac{4^2 e^{-4}}{2!} = \frac{16e^{-4}}{2} \approx 0.1465$$

$P(X = 3)$:

$$P(X = 3) = \frac{4^3 e^{-4}}{3!} = \frac{64e^{-4}}{6} \approx 0.1953$$

$P(X = 4)$:

$$P(X = 4) = \frac{4^4 e^{-4}}{4!} = \frac{256e^{-4}}{24} \approx 0.1953$$

Sum for $P(X < 5)$:

$$P(X < 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$P(X < 5) \approx 0.0183 + 0.0733 + 0.1465 + 0.1953 + 0.1953 = 0.6287$$

So, the probability that fewer than 5 of the 2000 infected people will die is approximately **0.6287.**

b) Mean and Variance of the Poisson Distribution:

For a Poisson distribution, the **mean** (μ) and **variance** (σ^2) are both equal to λ.

So, in this case:

- **Mean** = $\lambda = 4$
- **Variance** = $\lambda = 4$

Final Results:

- **a)** $P(X < 5) \approx 0.6287$
- **b)** Mean = 4, Variance = 4

Viva Questions

1. How to approximate the Binomial to Poisson distribution

The **Poisson distribution** can approximate the **Binomial distribution** when:

- n is large,
- p is small (i.e., p close to 0),
- $\lambda = n \cdot p$ is moderate.

The approximation formula is:

$$P(X = k) \approx \frac{\lambda^k e^{-\lambda}}{k!}$$

Where $\lambda = n \cdot p$.

2. Find the binomial distribution if the mean is 6 and variance is 4.

From the formulas:

- Mean: $\mu = n \cdot p$,
- Variance: $\sigma^2 = n \cdot p \cdot (1 - p)$.

Given:

- $\mu = 6$,
- $\sigma^2 = 4$.

Solve:

1. $n \cdot p = 6$,
2. $n \cdot p \cdot (1 - p) = 4$.

By solving, we get:

- $n = 18$,
- $p = \frac{1}{3}$.

Thus, the Binomial distribution has parameters $n = 18$ and $p = \frac{1}{3}$.

(For Evaluators use only)

Comment of the Evaluator (if Any)	Evaluator's Observation
	Marks Secured:___________out of__________
	Full Name of the Evaluator:
	Signature of the Evaluator:
	Date of Evaluation