

Department of AI & DS

CSE and CS&IT

COURSE NAME: PROBABILITY, STATISTICS AND QUEUING THEORY

COURSE CODE: 23MT2005

Topic

Joint random variables and probability functions

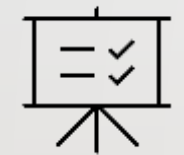
Session - 9

AIM OF THE SESSION



To familiarize students with the basic concept of Jointly distributed random variables

INSTRUCTIONAL OBJECTIVES



This Session is designed to:

1. Define Joint random variables and their probability functions
2. Describe Marginal and conditional functions
3. List out the properties of independence of jointly distributed random variables
4. Describe the applications of discrete and continuous for jointly distributed random variables

LEARNING OUTCOMES



At the end of this session, you should be able to:

1. Define joint random variables and their properties
2. Describe Probability functions of jointly random variables
3. Summarize the discrete and continuous case of jointly distributed random variables

CONTENTS

- ❖ Joint probability distributions
- ❖ Marginal distribution
- ❖ Conditional distribution

Joint Probability distributions

In real life, we are often interested in several random variables that are related to each other. For example, suppose that we choose a random family, and we would like to study

- the number of people in the family
- the household income
- the ages of the family members etc.

Each of these is a random variable, and we suspect that they are dependent.

In this session, we will learn tools to study joint distributions of two or more random variables.

The concepts are extensions of what we studied for one random variable in the previous classes.

Joint Probability distributions

Joint Probability distributions: If X and Y are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values

$$f(x,y) = P(X=x, Y=y);$$

the function $f(x, y)$ is a joint Probability distribution or Probability mass function of the random variables X and Y if

i) $f(x,y) \geq 0$ for all (x, y)

ii) $\sum_x \sum_y f(x, y) = 1$

iii) $P(X=x, Y=y) = f(x, y)$

Joint density function: The function $f(x, y)$ is a joint density function of the continuous random variables X and Y if

i) $f(x,y) \geq 0$ for all (x, y)

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$

iii) $P[(X,Y) \in A] = \int \int_A f(x, y) dx dy$

Marginal Distribution: The Marginal Distribution of X alone and of Y alone are

$$g(x) = \sum_y f(x, y) \quad \text{and}$$

$$h(y) = \sum_x f(x, y)$$

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{and}$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Conditional distribution: Let X and Y be two random variables, discrete or continuous. The conditional distribution of the random variable Y given that X =x is

$$f(y/x) = \frac{f(x, y)}{g(x)}, \quad g(x) > 0$$

Similarly the conditional distribution of the random variable X given that Y=y is

$$f(x/y) = \frac{f(x, y)}{h(y)}, \quad h(y) > 0$$

Statistical Independence: Two random variables X and Y, discrete or continuous with joint probability distribution of $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be statistically independent if and only if $f(x,y)=g(x)h(y)$ for all (x,y) within their range.

Mean of a joint probability Distribution:

Let X and Y be random variables with joint probability Distribution $f(x, y)$. The mean or expected value of the random variable $g(X, Y)$ is

$$\mu_{g(x)} = E(g(x, y)) = \sum_x \sum_y g(x, y) f(x, y) \text{ if } x \text{ and } y \text{ are discrete and}$$

$$\mu_{g(x)} = E(g(x, y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \text{ if } x \text{ and } y \text{ are continuous}$$

Variance of random variable:

Let X be a random variable with probability distribution $f(x)$ and mean μ . The variance of X is

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (X - \mu)^2 f(x), \quad \text{if } X \text{ is discrete and}$$

Note: $\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (X - \mu)^2 f(x), \text{ if } X \text{ is continuous}$

i) $\sigma^2 = E(X^2) - \mu^2$

ii) $E(aX+b) = aE(X)+b$

iii) $V(aX+b) = a^2 V(X)$

iv) If X and y are independent random variables, then $V(aX+bY) = a^2 V(X) + b^2 V(Y)$

EXAMPLE –Discrete Case

Two scanners are needed for an experiment. Of the five available, two have electronic defects, another one has a defect in the memory, and two are in good working order. Two units are selected at random.

- Find the joint probability distribution of X_1 =the number with electronic defects, and X_2 = the number with a defect in memory.
- Find the probability of 0 or 1 total defects among the two selected.
- Find the marginal probability distribution of X_1 .
- Find the conditional probability distribution of X_1 given $X_2=0$.

Solution:

a) Joint distribution of X_1 and X_2

		X_1		
		0	1	2
X_2	0	1/10	4/10	1/10
	1	2/10	2/10	0

EXAMPLE –Discrete Case

or symbolically the joint distribution of X_1 and X_2 is

$$\frac{{}^2c_{x_1} \times {}^1c_{x_2} \times {}^2c_{x_1-x_2}}{{}^5c_2} \quad \text{for } x_1 = 0,1,2, x_2 = 0,1 \text{ and } 0 \leq x_1 + x_2 \leq 2$$

b) Probability of total defects is 0 and 1

$$=P(x_1=0 \text{ and } x_2=0) + P(x_1=0 \text{ and } x_2=1) + P(x_1=1 \text{ and } x_2=0)$$

$$=1/10 + 2/10 + 4/10 = 0.7$$

c) Marginal probability distribution of X_1 and X_2

Let $f_1(x_1)$ and $f_2(x_2)$ be the marginal distribution of X_1 and X_2 respectively. They are shown in the following table.

		X_1			$f_2(x_2)$
		0	1	2	
X_2	0	1/10	4/10	1/10	
	1	2/10	2/10	0	
$f_1(x_1)$		3/10	6/10	1/10	

EXAMPLE –Discrete Case

d) The conditional distribution of X_1 given $X_2=x$ is defined as

$$f_1(x_1/x_2) = \frac{f(x_1, x_2)}{f_2(x_2)}$$

The conditional distribution of X_1 given $X_2=0$ is given by

$$f_1(0/0) = \frac{f(0,0)}{f_2(0)} = \frac{1/10}{6/10} = 1/6$$

$$f_1(1/0) = \frac{f(1,0)}{f_2(0)} = \frac{4/10}{6/10} = 4/6$$

$$f_1(2/0) = \frac{f(2,0)}{f_2(0)} = \frac{1/10}{6/10} = 1/6$$

Example-Continuous case

Given the joint density function

$$f(x, y) = \begin{cases} \frac{x(1 + 3y^2)}{4} & , \quad 0 < x < 2, 0 < y < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

find $g(x)$, $h(y)$, $f(x/y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2} / Y = \frac{1}{3})$

Solution: By definition

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{x(1 + 3y^2)}{4} dy$$

$$= \left(\frac{xy}{4} + \frac{xy^3}{4} \right) \Big|_{y=0}^{y=1} = \frac{x}{2}, \quad 0 < x < 2 \quad \text{and}$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^2 \frac{x(1 + 3y^2)}{4} dx$$

Example-Continuous case

$$= \left(\frac{x^2}{8} + \frac{3x^2y^2}{8} \right) \Big|_{x=0}^{x=2} = \frac{1+3y^2}{2}, \quad 0 < y < 1$$

Therefore,

$$f(x|y) = \frac{f(x, y)}{h(y)} = \frac{x(1 + 3y^2)/4}{(1 + 3y^2)/2} = \frac{x}{2}, \quad 0 < x < 2$$

and

$$f(x|y) = \frac{f(x, y)}{h(y)} = \frac{x(1 + 3y^2)/4}{(1 + 3y^2)/2} = \frac{x}{2}, \quad 0 < x < 2$$

$$P\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2} dx = \frac{3}{64}.$$

SUMMARY

In this session, the concepts of Jointly distributed random variables and its applications have discussed

1. Define Joint probability distribution function for discrete and continuous
2. Applications of Joint random variables in various applications of real life have observed.

SELF-ASSESSMENT QUESTIONS

If the density function of bivariate X and Y is given as:

$$f(x,y) = 3xy \text{ for } 0 \leq x \leq 1 \\ 0 \leq y \leq 1,$$

the marginal distribution of X is:

- a) $f_X(x) = 3x$
- b) $f_X(x) = (3/2)x$
- c) $f_X(x) = (3/4)x$
- d) none of the above...

2. Two random variables X and Y are said to be independent if:

- (a) $E(XY) = 1$
- (b) $E(XY) = 0$
- (c) $E(XY) = E(X) E(Y)$
- (d) $E(XY) = \text{any constant value}$

1. Describe the Marginal and conditional distributions.
2. List out the properties of jointly distributed random variables for discrete and continuous cases.
3. A privately owned cooperative store operates both a drive-in facility and walk-in facility. On a randomly selected day, let X and Y , respectively, be the proportion of the time that the drive-in and walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y), & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- a) Find the marginal density of X
- b) Find the marginal density of y
- c) Find the probability that the drive-in facility is busy less than one-half of the time.
- d) Find the probability that the drive in facility is fully less than one fourth of the time.

Reference Books:

1. Chapter 1 of TPI: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons, Inc.
2. Richard A Johnson, Miller & Freund's Probability and statistics for Engineers, PHI, New Delhi, 11th Edition (2011).

Sites and Web links:

- [Continuous Random Variables and their Distributions \(probabilitycourse.com\)](http://probabilitycourse.com)

Notes: sections 1 to 1.3 of <http://www.statslab.cam.ac.uk/~rrwl/prob/prob-weber.pdf>

3. Section 3.1.1 of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

Video: <https://www.youtube.com/watch?v=-5sOBWV0qH8&list=PLB45KifGiuHesi4PALNZSYZFhViVGQJK&index=19>

- https://www.probabilitycourse.com/chapter5/5_1_0_joint_distributions.php
- https://www.probabilitycourse.com/chapter5/5_2_0_continuous_vars.php

THANK YOU



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