Advanced Algorithms & Data Structures

Department of CSE

ADVANCED ALGORITHMS AND DATA STRUCTURES
23CS03HF

Topic:

All Shortest Path Problem

Session - 24

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AIM OF THE SESSION

To familiarize students with the concept of All pairs Shortest Path Problem

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1.Demonstrate :- All Pairs Shortest Path Problem.

2.Describe :- Solve All pairs shortest path problem using Floyd-Warshall Algorithm .

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define :- All Pairs Shortest path Problem.
2. Describe :- solve All Pairs shortest path problem using Floyd-Warshall Algorithm
3. Summarize:- Finding the shortest paths from all source vertices to all other vertices in a weighted graph.

- ***The problem:*** **find the shortest path between every pair of vertices of a graph**
- ***The graph*****: may contain negative edges but no negative cycles**
- ***A representation*****: a weight matrix where**
  W(i,j)=0 if i=j.
  W(i,j)=∞ if there is no edge between i and j.
  W(i,j)="weight of edge"

All Pairs Shortest Path

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Floyd-Warshall Algorithm

- Let $D^{(k)}[i,j]$=weight of a shortest path from v_i to v_j using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices in the path
  - $D^{(0)}=W$
  - $D^{(n)}=D$ which is the goal matrix
- How do we compute $D^{(k)}$ from $D^{(k-1)}$?

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Conditions

Case 1: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices does not use v_k.

Then $D^{(k)}[i,j]= D^{(k-1)}[i,j]$.

Case 2: A shortest path from v_i to v_j restricted to using only vertices from $\{v_1,v_2,\ldots,v_k\}$ as intermediate vertices does use v_k.

Then $D^{(k)}[i,j]= D^{(k-1)}[i,k]+ D^{(k-1)}[k,j]$.

The recursive definition

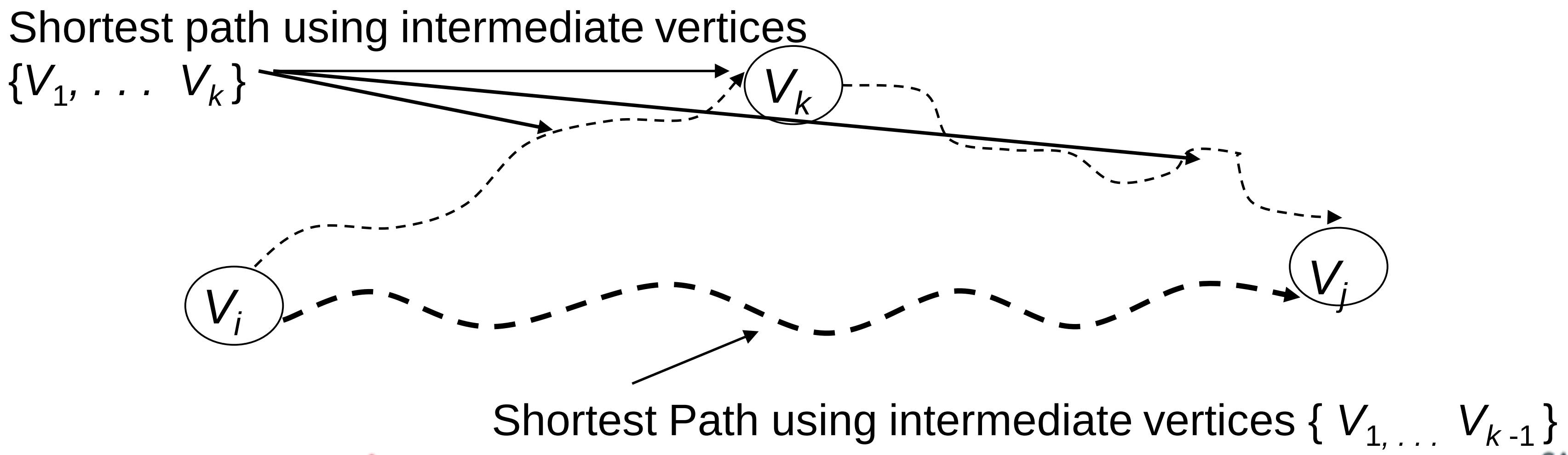
- Since

  $D^{(k)}[i,j]= D^{(k-1)}[i,j]$ or

  $D^{(k)}[i,j]= D^{(k-1)}[i,k]+ D^{(k-1)}[k,j]$.

- We conclude:

  $D^{(k)}[i,j]= \min\{ D^{(k-1)}[i,j], D^{(k-1)}[i,k]+ D^{(k-1)}[k,j] \}$.



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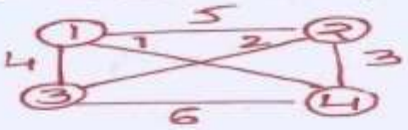
Implementation

Eg. Find the shortest distance from each vertex to other vertices in the below graph.



The All Pairs shortest-path problem is to determine a matrix A such that $A(i, j)$ is the length of a shortest path i, j.

From floyd's alg. we have the recurrence relation :

$$A(i,j) = \min\left\{ \min_{1 \le k \le n} \{ A^{k-1}(i,k) + A^{k-1}(k,j) \}, cost(i,j) \right\}$$

$$\Rightarrow A^{0}(i,j) = cost(i,j), \quad 1 \le i \le n, \quad 1 \le j \le n$$

$$\boxed{A^{k}(i,j) = \min\{ A^{k-1}(i,j), A^{k-1}(i,k) + A^{k-1}(k,j) \}, \quad k \ge 1}$$

The cost matrix for given graph is given by

$$A^{0} = cost(i,j) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 6 \\ 1 & 3 & 6 & 0 \end{bmatrix} \end{array}$$

where $cost(i,j) = \begin{cases} 0 & \text{if } i = j \\ \text{edge } cost(i,j) & \text{if } i \ne j \ \& \ (i,j) \in E \\ \infty & \text{if } i \ne j \ \& \ (i,j) \notin E \end{cases}$

and E is set of edges in graph 'G'.

When computing $A^{(1)}$,
- 1^{st} row & 1^{st} column can be copied from $A^{(0)}$ to $A^{(1)}$ as they remain constant.
- all diagonal elements will be 0's always.

$k \leftarrow 1$, $i \swarrow$, $j \swarrow$

$a^1_{23} = \min\{a^0_{23}, (a^0_{21} + a^0_{13})\}$
$= \min\{2, (5+4)\} = 2$

$a^1_{24} = \min\{a^0_{24}, (a^0_{21} + a^0_{14})\} = \min\{3, (5+1)\} = 3$

$a^1_{32} = \min\{a^0_{32}, (a^0_{31} + a^0_{12})\} = \min\{2, (4+5)\} = 2$

$a^1_{34} = \min\{a^0_{34}, (a^0_{31} + a^0_{14})\} = \min\{6, (4+1)\} = 5$

$a^1_{42} = \min\{a^0_{42}, (a^0_{41} + a^0_{12})\} = \min\{3, (1+5)\} = 3$

$a^1_{43} = \min\{a^0_{43}, (a^0_{41} + a^0_{13})\} = \min\{6, (1+4)\} = 5$

$$\therefore A^1 = \begin{bmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & \underline{5} \\ 1 & 3 & \underline{5} & 0 \end{bmatrix}$$

$$\text{lly } A^2 = \begin{bmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

$$\text{lly } A^3 = \begin{bmatrix} 0 & 5 & 4 & 1 \\ 5 & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & \underline{4} & 4 & 1 \\ \underline{4} & 0 & 2 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

SUMMARY

- **Objective**: Find the shortest paths from all source vertices to all other vertices in a weighted graph.

- **Approach**:
  - Initialize the solution matrix same as the input graph matrix as a first step.
  - Then update the solution matrix by considering all vertices as an intermediate vertex.
  - The idea is to pick all vertices one by one and updates all shortest paths which include the picked vertex as an intermediate vertex in the shortest path.
  - When we pick vertex number **k** as an intermediate vertex, we already have considered vertices **{0, 1, 2, .. k-1}** as intermediate vertices.

- **Algorithmic Solutions**: Algorithms like Floyd Warshall algorithm used to solve this problem efficiently.

SELF-ASSESSMENT QUESTIONS

The Floyd-Warshall algorithm is based on which programming paradigm?

(a) Divide and conquer
(b) Greedy method
(c) Dynamic programming
(d) Backtracking

What is the time complexity of the Floyd-Warshall algorithm for a graph with n vertices?

(a) $O(n^3)$
(b) $O(n^2)$
(c) $O(n^4)$
(d) $O(n\log n)$

TERMINAL QUESTIONS

1. Given the adjacency matrix of a graph:

$$\begin{bmatrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{bmatrix}$$

Use the Floyd-Warshall algorithm to compute the shortest path matrix. Show all intermediate steps.

2. Describe the differences between the Floyd-Warshall algorithm and Dijkstra's algorithm for solving shortest path problems.

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REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books :

1. Introduction to Algorithms, Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein., 3rd, 2009, The MIT Press.
2 Algorithm Design Manual, Steven S. Skiena., 2nd, 2008, Springer.
3 Data Structures and Algorithms in Python, Michael T. Goodrich, Roberto Tamassia, and Michael H. Goldwasser., 2nd, 2013, Wiley.
4 The Art of Computer Programming, Donald E. Knuth, 3rd, 1997, Addison-Wesley Professiona.

MOOCS :

1. https://www.coursera.org/specializations/algorithms?=
2.https://www.coursera.org/learn/dynamic-programming-greedy-algorithms#modules

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