

BEALE'S METHOD

• Beale's Method In Beale's method we solve Quadratic Programming problem and in this method we does not use the Kuhn-Tucker condition. At each iteration the objective function is expressed in terms of non basic variables only. Let the QPP be given in the form

$$Maximize \ f(X) = CX + \frac{1}{2}X^TQX$$

subject to AX = b, $X \ge 0$.

Where

$$X = (x_1, x_2, ..., x_{n+m})$$

c is
$$1 \times n$$

A is
$$m \times (n+m)$$

and Q is symmetric and every QPP with linear constraints.











Algorithm

Step 1

First express the given QPP with Linear constraints in the above form by introducing slack and surplus variable.

Step 2

Now select arbitrary m variables as basic and remaining as non-basic.

Now the constraints equation AX = b can be written as

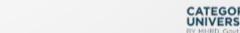
$$BX_B + RX_{NB} = b \Rightarrow X_B = B^{-1}b - B^{-1}RX_{NB}$$

where

 X_B -basic vector X_{NB} -non-basic vector

and the matrix A is partitioned to submatrices B and R corresponding to X_B and X_{NB} respectively.









Step 3

Express the basis X_B in terms of non-basic X_{NB} only, using the given additional constraint equations, if any.

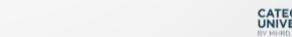
Step 4

Express the objective function f(x) in terms of X_{NB} only using the given and additional constraint, if any. Thus we observe that by increasing the value of any of the non-basic variables, the value of the objective function can be improved. Now the constraints on the new problem become

$$B^{-1}RX_NB \leqslant B^{-1}b \qquad (since X_B \geqslant 0)$$

Thus, any component of X_{NB} can increase only until $\frac{\partial f}{\partial x_{NB}}$ becomes zero or one or more components of X_B are reduced to zero.









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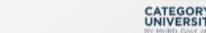
Step 5

Now we have m + 1 non-zero variables and m + 1 constraints which is a basic solution to the extended set of constraints.

Step 6

We go on repeating the above procedure until no further improvement in the objective function may be obtain by increasing one of the non-basic variables.









Quadratic Programming problem (9PP)

The general form of the QPP is

$$f(x) = cx + \frac{1}{2}x^TQ x$$

s.t.
$$Ax \leq b$$
 and $x \geq 0$

where Q is a symmetric matrix and b, c are the real vectors.









Beale's method

Beale developed a technique of solving the QPP that Does Not Use the Kuhn-Tucker conditions.

- His technique involves partitioning variables into basic (x_B) and non-basic variables (x_{NB}) .
- Express x_B & objective function f in terms of x_{NR} variables.





Example 1: Use Beale's method to solve QPP:

Maximize
$$f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$
 s.t. $x_1 + 2x_2 \le 4$;
$$x_1, x_2 \ge 0$$

Solution: Write in Standard form:

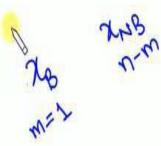
Maximize
$$f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

s.t. $x_1 + 2x_2 + \hat{s_1} = 4$;
 $x_1, x_2, s_1 \ge 0$

Here,
$$m = 1$$
; $n = 3$.







Max
$$f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

s.t. $x_1 + 2x_2 + s_1 = 4$;
 $x_1, x_2, s_1 \ge 0$

Let us choose arbitrary S_1 as initial basic variable,

i.e.,
$$x_B = (s_1)$$
; $x_{NB} = (x_1, x_2)$.









Working Rule

Express $x_B & f$ in terms of x_{NB} as

To find the entering variable:

To find the leaving variable:

$$\begin{aligned} \text{Max}\,f(x) &= 4x_1 + 6x_2 - x_1^2 - 3x_2^2 \\ \text{s.t.}\,x_1 + 2x_2 + s_1 &= 4 \ ; \\ x_1, x_2, s_1 &\geq 0 \end{aligned}$$

$$x_B = (s_1)$$
; $x_{NB} = (x_1, x_2)$

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Iteration I:

Express $x_B & f$ in terms of x_{NB} as

$$s_1 = 4 - x_1 - 2x_2$$

and
$$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = -2x_1 + 4; \quad \frac{\partial f}{\partial x_2} = -6x_2 + 6$$

At the point
$$x_{NB} = (x_1, x_2) = 0$$
,

$$\frac{\partial f}{\partial x_1}=4$$
 & $\frac{\partial f}{\partial x_2}=6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

$$x_B = (s_1); x_{NB} = (x_1, x_2)$$

$$\chi_{B} = (S_1, \chi_2); \gamma_{NB} = (\chi_1, \chi_1)$$

To find the leaving variable:

$$\begin{aligned} \min\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} &= \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\} \\ &= 1 \text{ corresponding to } \frac{\gamma_{20}}{|\gamma_{22}|}. \end{aligned}$$

Thus, introduce a free non-basic variable

$$u_1 = \frac{1}{2} \frac{\partial f}{\partial x_2} \qquad i.e., u_1 = 3 - 3x$$



Iteration 2:

$$\chi_2 = \underbrace{\frac{3-u_1}{3}}$$

$$s_1 = 4 - x_1 - \frac{2}{3}(3 - u_1)$$

$$\sqrt{2-x_1+\frac{2}{3}u_1}$$

And
$$f = 4x_1 + 6\left(\frac{3-u_1}{3}\right) - x_1^2 - 3\left(\frac{3-u_1}{3}\right)^2$$

$$=3-x_1^2-\frac{u_1^2}{3}+4x_1$$

$$s_1 = 4 - x_1 - 2x_2$$

$$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$$y_1 = 3 - 3x_2$$

$$x_B = (s_1, x_2); x_{\underline{NB}} = (x_1, u_1)$$









Iteration 2:

$$x_2 = \frac{3-u_1}{3} \; ; \; \; s_1 \; = 2-x_1 + \frac{2}{3}u_1$$
 And $f = 3-x_1^2 - \frac{u_1^2}{3} + 4x_1$

To find the entering variable:

Partial derivative of f w.r.t. x_1, u_1 :

$$\frac{\partial f}{\partial x_1} = -2 x_1 + 4 ; \frac{\partial f}{\partial u_1} = -\frac{2}{3} u_1$$

At the point $x_{NB} = (x_1, u_1) = 0$, we get

$$\frac{\partial f}{\partial x_1} = 4$$
; $\frac{\partial f}{\partial u_1} = 0$

Thus, ENTERING variables is x_1 .

$$x_B = (s_1, x_2); x_{NB} = (x_1, u_1)$$

To find the leaving variable:

$$\begin{split} \mathit{Min} \left\{ &\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{11}|} \right\} &= \mathit{Min} \left\{ \frac{2}{|-1|}, \frac{1}{|0|}, \frac{4}{|-2|} \right\} \\ &= 2 \quad \mathsf{corresponding to} \, \frac{\gamma_{10}}{|\gamma_{11}|}. \end{split}$$

Thus, define a free non-basic variable

$$u_2 = \frac{1}{2} \frac{\partial f}{\partial x_1}$$

i.e.,
$$u_2 = -x_1 + 2$$











Iteration 3:

Express x_B & f in terms of x_{NB} as

$$x_1 = 2 - u_2; \quad x_2 = 1 - \frac{1}{3}u_1;$$

$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

$x_B = (s_1, x_2, \underline{x_1}); \quad x_{NB} = (u_1, \underline{u_2})$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1 \; ; \; \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0 \quad ; \quad \frac{\partial f}{\partial u_2} = 0$$

Thus, current x_B is the optimal.

Answer:

$$x_1 = 2$$
; $x_2 = 1$; $s_1 = 0$
& Max $f = 7$



Iteration 3.

Express x_B & f in terms of x_{NB} as

$$x_1 = 2 - u_2; \quad x_2 = 1 - \frac{1}{3}u_1;$$

$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

$x_B = (s_1, x_2, x_1); \quad x_{NB} = (u_1, u_2)$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1 \; ; \; \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0 \quad ; \quad \frac{\partial f}{\partial u_2} = 0 \qquad \leq \bigcirc$$

Thus, current x_B is the optimal.

Answer:

$$x_1 = 2$$
; $x_2 = 1$; $s_1 = 0$
8 Max $f = 7$



Solution:

Step:1

$$Max. Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$
(1)

subject to

$$x_1 + 2x_2 + x_3 = 2 \tag{2}$$

and $x_1, x_2, x_3 \ge 0$

taking
$$X_B = (x_1); X_{NB} = {x_2 \choose x_3}$$

and
$$x_1 = 2 - 2x_2 - x_3$$







Step:2

put (3) in (1), we get

$$\begin{aligned}
Max. & f(x_2, x_3) = 4(2 - 2x_2 - x_3) + 6x_2 - 2(2 - 2x_2 - x_3)^2 - 2(2 - 2x_2 - x_3)x_2 - 2x_2^2 \\
\frac{\partial f}{\partial x_2} &= -2 + 8(2 - 2x_2 - x_3) + 8x_2 - 4x_2 - 2(2 - x_3) \\
\frac{\partial f}{\partial x_3} &= -4 + 4(2 - 2x_2 - x_3) + 2x_2
\end{aligned}$$

Now
$$\frac{\partial f}{\partial x_2}_{(0,0)} = 10$$

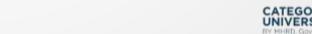
$$\frac{\partial f}{\partial x_3}_{(0,0)} = 4$$

Here '+ve' value of $\frac{\partial f}{\partial x_i}$ indicates that the objective function will increase if x_i increased

. Similarly '-ve' value of $\frac{\partial f}{\partial x_i}$ indicates that the objective function will decrease if x_i is decrease. Thus, increase in x_2 will give better improvement in the objective function.

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Step:3

f(x) will increase if x_2 increased.

If x_2 is increased to a value greater then 1, x_1 will be negative.

Since
$$x_1 = 2 - 2x_2 - x_3$$

$$x_3 = 0; \frac{\partial f}{\partial x_2} = 0$$

$$\Rightarrow 10 - 12x_2 = 0$$

$$\Rightarrow x_2 = \frac{5}{6}$$

Min.
$$(1, \frac{5}{6}) = \frac{5}{6}$$

The new basic variable is x_2 .











Second Iteration:

Step:1

$$let X_B = (x_2), X_{NB} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$x_2 = 1 - \frac{1}{2}(x_1 + x_3)$$

Step:2

Substitute (4) in (1)

$$\begin{aligned} Max. \ f(x_1, x_3) &= 4x_1 + 6(1 - \frac{1}{2}(x_1 + x_3)) - 2x_1^2 - 2x_1(1 - \frac{1}{2}(x_1 + x_3)) - 2(1 - \frac{1}{2}(x_1 + x_3))^2 \\ \frac{\partial f}{\partial x_1} &= 1 - 3x_1, \qquad \frac{\partial f}{\partial x_2} &= -1 - x_3 \\ \frac{\partial f}{\partial x_1} &= 1 \end{aligned}$$

$$\frac{\partial f}{\partial x_1} = 1$$

$$\frac{\partial f}{\partial x_3}_{(0,0)} = -1$$

This indicates that x_1 can be introduce to increased objective function.











Step:3

$$x_2 = 1 - \frac{1}{2}(x_1 + x_3)$$
 and $x_3 = 0$

If x_1 is increased to a value greater then 2, x_2 will become negative.

$$\frac{\partial f}{\partial x_1} = 0$$

$$\Rightarrow 1 - 3x_1 = 0$$

$$\Rightarrow x_1 = \frac{1}{3}$$

$$Min. (2, \frac{1}{3}) = \frac{1}{3}$$
Therefore $x_1 = \frac{1}{3}$
Hence $x_1 = \frac{1}{3}$, $x_2 = \frac{5}{6}$, $x_3 = 0$
and $Max. f(x) = \frac{25}{6}$







