23MT2004 - Mathematical Programming

Topic: Geometric Programming with equality constraints

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Geometric Programming with equality constraints

AIM OF THE SESSION

- To familiarize students with the method of Geometric Programming to optimize an objective function, subject to an equality constraint.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Learn about optimization of posynomial objective function, subject to an equality constraint.
2. Learn about the degree of difficulty in the case of GP with constraints.

Learning Outcomes

1. Optimize a posynomial objective function, subject to an equality constraint using Geometric Programming.
2. Determine when a GP problem with constraints is not feasible, has a unique solution or has multiple solutions.

GP to solve a constrained optimization problem

1. A constrained optimization problem:

$$\text{Minimize } f(x), x = (x_1, x_2, \ldots, x_n),$$
$$\text{subject to constraints } g(x) = \alpha,$$
$$\text{where } \alpha \text{ is a positive constant.}$$

In the above case, both f(x) and g(x) are posynomials.
For example:

$$\text{Minimize } f(x) = 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3$$
$$\text{subject to constraints}$$
$$4x_1x_2 + 2x_2x_3 = 8$$
$$\text{where } x_1, x_2.x_3 \geq 0$$

RECALL: Arithmetic-Geometric mean inequality

$$\frac{u_1 + u_2 + \ldots + u_n}{n} \geq (u_1 * u_2 * \ldots * u_n)^{\frac{1}{n}}$$

$$u_i \geq 0 \ \forall i \ \ n \in N$$

$$\sum_{i=1}^{n} u_i = 1 \ \forall i \ \ n \in N$$

$$\sum_{i=1}^{n} \frac{u_i}{n} \geq \prod_{i=1}^{n} (u_i)^{\frac{1}{n}}$$

$$\text{Let } \delta_i = \frac{1}{n} \ \forall i$$

$$\sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} (u_i)^{\delta_i}$$

Arithmetic-Geometric mean inequality (contd.)

Let $\delta_1 + \delta_2 + \ldots + \delta_n = \lambda$
$\implies \frac{\delta_1}{\lambda} + \frac{\delta_2}{\lambda} + \ldots + \frac{\delta_n}{\lambda} = 1$

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} (\frac{u_i}{\delta_i/\lambda})^{\delta_i/\lambda} \tag{1}$$

$$(\sum_{i=1}^{n} u_i)^{\lambda} = \prod_{i=1}^{n} (\frac{\lambda u_i}{\delta_i})^{\delta_i} \tag{2}$$

$$= (\lambda)^{\delta_1+\delta_2+\ldots+\delta_n} \prod_{i=1}^{n} (\frac{u_i}{\delta_i})^{\delta_i} \tag{3}$$

$$= \lambda^{\lambda} \prod_{i=1}^{n} (\frac{u_i}{\delta_i})^{\delta_i} \tag{4}$$

where $u_i, \delta_i \geq 0$

Geometric Programming: Optimization with equality constraints

Let us solve the minimization problem

$$\text{Minimize } f(x) = 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3$$
$$\text{subject to constraints}$$
$$4x_1x_2 + 2x_2x_3 = 8$$
$$\text{where } x_1, x_2.x_3 \geq 0$$

Solution:

$$\begin{aligned} f(x) &= 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3 \\ &= U_1 + U_2 \\ &\geq (\frac{u_1}{\delta_1})^{\delta_1}(\frac{u_2}{\delta_2})^{\delta_2} \end{aligned}$$

where $u_1, u_2, \delta_1, \delta_2 \geq 0$ and $\delta_1 + \delta_2 = 1$

Geometric Programming: Optimization with equality constraints (contd.)

Let us consider the constraint equation

$$\frac{1}{2}x_1x_2 + \frac{1}{4}x_2x_3 = 1$$

we have

$$U_3 + U_4 = 1$$
$$(U_3 + U_4)^\lambda = 1^\lambda$$

From Arithmetic mean - Geometric mean in equality, we get

$$(U_3 + U_4)^\lambda \geq (\lambda)^\lambda (\frac{u3}{\delta_3})^{\delta_3} (\frac{u4}{\delta_4})^{\delta_4}$$

Geometric Programming: Optimization with equality constraints (contd.)

Now, combine the objective function and constraint equation,

$$f(x) \geq (\frac{u_1}{\delta_1})^{\delta_1}(\frac{u_2}{\delta_2})^{\delta_2}(\lambda)^{\lambda}(\frac{u3}{\delta_3})^{\delta_3}(\frac{u4}{\delta_4})^{\delta_4}$$

$$\geq (\lambda)^{\lambda}(\frac{40x_1^{-1}x_2^{-1}x_3^{-1}}{\delta_1})^{\delta_1}(\frac{40x_1x_3}{\delta_2})^{\delta_2}(\frac{x_1x_2}{2\delta_3})^{\delta_3}(\frac{x_2x_3}{4\delta_4})^{\delta_4}$$

comparing the powers of x_1, x_2, x_3 on LHS and RHS, we get

$$-\delta_1 + \delta_2 + \delta_3 + 0\delta_4 = 0 \tag{5}$$

$$-\delta_1 + \delta_2 + 0\delta_3 + \delta_4 = 0 \tag{6}$$

$$-\delta_1 + 0\delta_2 + \delta_3 + \delta_4 = 0 \tag{7}$$

$$\delta_1 + \delta_2 + 0\delta_3 + 0\delta_4 = 1 \tag{8}$$

solving the above equations by elimination, we get
$\delta_1 = \frac{2}{3}, \delta_2 = \delta_3 = \delta_4 = \frac{1}{3}, \lambda = \delta_3 + \delta_4 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

Geometric Programming: Optimization with equality constraints (contd.)

Objective Function	Constraint Equation
$f(x) = 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3$	$g(x) = \frac{1}{2}x_1x_2 + \frac{1}{4}x_2x_3 = 1$
$U_1 + U_2 = 1$	$U_3 + U_4 = 1$
$f(x) \geq (\frac{u_1}{\delta_1})^{\delta_1}(\frac{u_2}{\delta_2})^{\delta_2}$	$(\frac{u_3}{\delta_3})^{\delta_3}(\frac{u_4}{\delta_4})^{\delta_4}$
$\frac{u_1}{\delta_1} = \frac{u_2}{\delta_2}$	$\frac{u_3}{\delta_3} = \frac{u_4}{\delta_4}$
	$\frac{u_3}{\delta_3} = \frac{u_4}{\delta_4} = \frac{u_3+u_4}{\delta_3+\delta_4} = \frac{1}{2/3}$

We have 3 equations and 3 variables (x_1, x_2, x_3).

The values of the variables and the minimum value of f can be computed.

Self-Assessment Questions

1. In a GP problem, an equality constraint function could be of the following form:
   - A. All the 3 below options are OK.
   - B. LHS is a posynomial and RHS is a positive number
   - C. LHS is a posynomial and RHS is a negative number
   - D. LHS is a polynomial and RHS is a positive number
2. If there are 3 terms in the objective function of 4 variables, and 2 terms in the constraint function, the degree of difficulty is
   - A. Negative
   - B. Zero
   - C. One
   - D. Uncountable

Terminal Questions

1. Write an example of a GP problem with equality constraints whose degree of difficulty is 1.
2. Compute the degree of difficulty and solve the following problem using Geometric Programming:

$$\begin{aligned} \text{Minimize } f(x) = 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3 \\ \text{subject to constraints} \\ x_1 \times x_2 = 100 \\ \text{where } x_1, x_2.x_3 \geq 0 \end{aligned}$$