Mathematical Programming-1
Session-5

Farkas Lemma

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Farkas Lemma

- **Farkas' lemma** is a solvability theorem for a finite system of linear in equalities in mathematics. It was originally proven by the Hungarian mathematician Gyula Farkas.
- Farkas' lemma is the key result underpinning the linear programming duality and has played a central role in the development of mathematical optimization (alternatively, mathematical programming).
- It is used amongst other things in the proof of the Karush–Kuhn–Tucker theorem in nonlinear programming.

Farkas Lemma

Farkas' lemma — Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following two assertions is true:

1. There exists an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$.
2. There exists a $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{A}^\mathsf{T}\mathbf{y} \geq 0$ and $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$.

Farkas Lemma

- Here the notation $x>=0$ means that all components of the vector x are non negative.

Variants of Farkas Lemma

The Farkas Lemma has several variants with different sign constraints (the first one is the original version):[7]:92

- Either the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution with $\mathbf{x} \geq 0$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} \geq 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$.
- Either the system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ has a solution with $\mathbf{x} \geq 0$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} \geq 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$ and $\mathbf{y} \geq 0$.
- Either the system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ has a solution with $\mathbf{x} \in \mathbb{R}^n$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} = 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$ and $\mathbf{y} \geq 0$.
- Either the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution with $\mathbf{x} \in \mathbb{R}^n$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} = 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} \neq 0$.

Farkas Lemma : Certificate of Infeasibility

Let $A \in R^{m \times n}, b \in R^m$ and $x = [x_1, x_2, ..., x_n]^T$.

Then $Ax \geq b$ has no solution or is inconsistent iff

there exists $y \in R^m$ such that:

1. $y \geq 0$,
2. $A^T y = 0$ and
3. $b^T y < 0$.

The IDEA is if we could find a vector y, such that above conditions are satisfied, then we can conclude that the given system of linear equations have no solution.

Example on Farkas Lemma

Example:

Consider the following system of linear equations:

$$
\begin{aligned}
x_1 + x_2 + 2x_3 &\geq 1 \qquad (1)\\
-x_1 + x_2 + x_3 &\geq 2 \qquad (2)\\
x_1 - x_2 + x_3 &\geq 1 \qquad (3)\\
-x_2 - 3x_3 &\geq 0 \qquad (4)
\end{aligned}
$$

Here, $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

Solution…

Next we go for elimination of x_1:

$$\begin{aligned} (1)+(2) \implies \quad & 2x_2 + 3x_3 \geq 3 & (5)\\ (2)+(3) \implies \quad & 2x_3 \geq 3 & (6)\\ & -x_2 - 3x_3 \geq 0 & (4) \end{aligned}$$

Next we go for elimination of x_2:

$$\begin{aligned} \frac{1}{2} \times (5) \implies \quad & x_2 + \frac{3}{2}x_3 \geq \frac{3}{2} & (7)\\ & 2x_3 \geq 3 & (6)\\ & -x_2 - 3x_3 \geq 0 & (4) \end{aligned}$$

Solution…

Now,

$$(4) + (7) \implies \qquad \frac{-3}{2}x_3 \geq \frac{3}{2} \qquad (8)$$

$$2x_3 \geq 3 \qquad (6)$$

Next we go for elimination of x_3:

$$\frac{3}{4} \times (6) \implies \qquad \frac{3}{2}x_3 \geq \frac{9}{4} \qquad (9)$$

$$\frac{-3}{2}x_3 \geq \frac{3}{2} \qquad (8)$$

If we add (8) and (9), we have

$$0 \geq \frac{15}{4} \qquad \text{(this is a contradiction)}$$

Solution…

The contradiction is due to

$$
\begin{aligned}
(9) + (8) &= \frac{3}{4}(6) + (8) \\
&= \frac{3}{4}[(2) + (3)] + [(4) + (7)] \\
&= \frac{3}{4}[(2) + (3)] + (4) + \frac{1}{2}(5) \\
&= \frac{3}{4}[(2) + (3)] + (4) + \frac{1}{2}[(1) + (2)] \\
&= \frac{3}{4}(2) + \frac{3}{4}(3) + (4) + \frac{1}{2}(1) + \frac{1}{2}(2) \\
&= \frac{1}{2}(1) + \frac{5}{4}(2) + \frac{3}{4}(3) + (4)
\end{aligned}
$$

Therefore, the y vector is the set of the coefficients in the above equation, i.e., $y = \begin{bmatrix} 1/2 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$

Solution…

$$\text{Now, } A^T y = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 2 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)\cdot\left(\frac{1}{2}\right) + (-1)\cdot\left(\frac{5}{4}\right) + (1)\cdot\left(\frac{3}{4}\right) + (0)\cdot(1) \\ (1)\cdot\left(\frac{1}{2}\right) + (1)\cdot\left(\frac{5}{4}\right) + (-1)\cdot\left(\frac{3}{4}\right) + (-1)\cdot(1) \\ (2)\cdot\left(\frac{1}{2}\right) + (1)\cdot\left(\frac{5}{4}\right) + (1)\cdot\left(\frac{3}{4}\right) + (-3)\cdot(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution…

$$\text{Also, } b^T y = \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$$

$$= \left[(1) \cdot \left(\tfrac{1}{2}\right) + (-2) \cdot \left(\tfrac{5}{4}\right) + (1) \cdot \left(\tfrac{3}{4}\right) + (0) \cdot (1) \right]$$

$$= \left[-\tfrac{5}{4} \right] < 0$$

As the conditions are satisfied, we can conclude that the given system of equations is inconsistent.

End of Session

Certifying infeasibility

A well known result in linear algebra states that a system of linear equations $\mathbf{Ax} = \mathbf{b}$ (where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a tuple of variables) has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} \neq 0$.

It is easily seen that if such a $\mathbf{y}$ exists, then the system $\mathbf{Ax} = \mathbf{b}$ cannot have a solution. (Simply multiply both sides of $\mathbf{Ax} = \mathbf{b}$ on the left by $\mathbf{y}^\mathsf{T}$.) However, proving the converse requires a bit of work. A standard elementary proof involves using Gauss-Jordan elimination to reduce the original system to an equivalent system $\mathbf{Rx} = \mathbf{d}$ such that $\mathbf{R}$ has a row of zero, say in row i, with $d_i \neq 0$. The process can be captured by a square matrix $\mathbf{M}$ satisfying $\mathbf{MA} = \mathbf{R}$. We can then take $\mathbf{y}^\mathsf{T}$ to be the ith row of $\mathbf{M}$.

An analogous result holds for systems of linear inequalities. The following result is one of the many variants of **Farkas' Lemma**:

Theorem 2.1. With $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ as above, the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that

$$\mathbf{y} \geq \mathbf{0},\ \mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0},\ \mathbf{y}^\mathsf{T}\mathbf{b} > 0.$$

In other words, the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution if and only if one can infer the inequality $0 \geq \gamma$ for some $\gamma > 0$ by taking a nonnegative linear combination of the inequalities.

This result essentially says that there is always a certificate (the m-tuple $\mathbf{y}$ with the prescribed properties) for the infeasibility of the system $\mathbf{Ax} \geq \mathbf{b}$. This allows third parties to verify the claim of infeasibility without having to solve the system from scratch.

Example. For the system

$$\begin{aligned} 2x - y + z &\geq 2 \\ -x + y - z &\geq 0 \\ -y + z &\geq 0, \end{aligned}$$

adding two times the second inequality and the third inequality to the first inequality gives $0 \geq 2$. Hence, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a certificate of infeasibility.

We now give a proof of the lemma above.

Proof of Theorem 2.1. It is easy to see that if such a $\mathbf{y}$ exists, then the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution.

Conversely, suppose that the system $\mathbf{Ax} \geq \mathbf{b}$ has no solution. It suffices to show that we can infer the inequality $0 \geq \alpha$ for some postive α by taking nonnegative linear combination of the inequalities in the system $\mathbf{Ax} \geq \mathbf{b}$. If the system already contains an inequality $0 \geq \alpha$ for some positive α, then we are done. Otherwise, we show by induction on n that we can infer such an inequality.

Base case: The system $\mathbf{Ax} \geq \mathbf{b}$ has only one variable.

For the system to have no solution, there must exist two inequalites $ax_1 \geq t$ and $-a'x_1 \geq t'$ such that $a, a' > 0$ and $\frac{t}{a} > \frac{-t'}{a'}$. Adding $\frac{1}{a}$ times the inequality $ax_1 \geq t$ and $\frac{1}{a'}$ times the inequality $-a'x_1 \geq t'$ gives the inequality $0 \geq \frac{t}{a} + \frac{t'}{a'}$ with a positive right-hand side. This establishes the base case.

Induction hypothesis: Let $n \geq 2$ be an integer. Assume that given any system of linear inequalities $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$ in $n-1$ variables having no solution, one can infe the inequality $0 \geq \alpha'$ for some positive α' by taking a nonnegative linear combination of the inequalities in the system $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$.

Apply Fourier-Motzkin elimination to eliminate x_n from $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ to obtain the system $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$. As $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ has no solution, $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$ also has no solution.

By the induction hypothesis, one can infer the inequality $0 \geq \alpha$ for some positive α by taking a nonnegative linear combination of the inequalities in $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$. However, each inequality in $\mathbf{A}'\mathbf{x} \geq \mathbf{b}'$ can be obtained from a nonnegative linear combination of the inequalites in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$. Hence, one can infer the inequality $0 \geq \alpha$ by taking a nonnegative linear combination of nonnegative linear cominbations of the inequalities in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$. Since a nonnegative linear combination of nonnegative linear cominbations of the inequalities in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ is simply a nonnegative linear combination of the inequalities in $\mathbf{A}\mathbf{x} \geq \mathbf{b}$, the result follows.

Remark. Notice that in the proof above, if $\mathbf{A}$ and $\mathbf{b}$ have only rational entries, then we can take $\mathbf{y}$ to have only rational entries as well.

Corollary 2.2. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and let $\mathbf{b} \in \mathbb{R}^m$. Prove that

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} \geq \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} < 0$. Furthermore, if $\mathbf{A}$ and $\mathbf{b}$ are rational, $\mathbf{y}$ can be taken to be rational.

Worked examples

1. You are given that the following system has no solution.

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 1 \\ -x_1 + x_2 + x_3 &\geq 2 \\ x_1 - x_2 + x_3 &\geq 1 \\ -x_2 - 3x_3 &\geq 0. \end{aligned}$$

Obtain a certificate of infeasibility for the system.

Hide solution

The system can be written as $\mathbf{Ax} \geq \mathbf{b}$ with $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. So we need to find $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} > 0$. As the system of equations $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ is homogeneous, we could without loss of generality fix $\mathbf{y}^\mathsf{T}\mathbf{b} = 1$, thus leading to the system

$$\begin{aligned} \mathbf{y}^\mathsf{T}\mathbf{A} &= \mathbf{0} \\ \mathbf{y}^\mathsf{T}\mathbf{b} &= 1 \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

that we could attempt to solve directly. However, it is possible to obtain a $\mathbf{y}$ using the Fourier-Motzkin Elimination Method.

Let us first label the inequalities:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 1 \quad (1)\\ -x_1 + x_2 + x_3 &\geq 2 \quad (2)\\ x_1 - x_2 + x_3 &\geq 1 \quad (3)\\ -x_2 - 3x_3 &\geq 0. \quad (4) \end{aligned}$$

Eliminating x_1 gives:

$$\begin{aligned} -x_2 - 3x_3 &\geq 0 \quad (4)\\ 2x_2 + 3x_3 &\geq 3 \quad (5)\\ 2x_3 &\geq 3. \quad (6) \end{aligned}$$

Note that (5) is obtained from $(1) + (2)$ and (6) is obtained from $(2) + (3)$.

Multiplying (5) by $\frac{1}{2}$ gives

$$\begin{aligned} -x_2 - 3x_3 &\geq 0 \quad (4)\\ x_2 + \frac{3}{2}x_3 &\geq \frac{3}{2} \quad (7)\\ 2x_3 &\geq 3. \quad (6) \end{aligned}$$

Eliminating x_2 gives: \begin{align**} 2x_3 & \geq 3~~~~~~~~(6) \ - \frac{3}{2} x_3 & \geq \frac{3}{2}~~~~~~(8) \end{align*} where (8) is obtained from $(4) + (7)$.

Now $\frac{3}{4} \times (6) + (8)$ gives $0 \geq \frac{15}{4}$, a contradiction.

To obtain a certificate of infeasibility, we trace back the computations. Note that $\frac{3}{4}(6)+(8)$ is given by $\frac{3}{4}((2)+(3))+(4)+(7)$, which in turn is given by $\frac{3}{4}((2)+(3))+(4)+\frac{1}{2}(5)$, which in turn is given by $\frac{3}{4}((2)+(3))+(4)+\frac{1}{2}((1)+(2))$.

Thus, we can obtain $0 \geq \frac{15}{4}$ from the nonnegative linear combination of the original inequalities as follows: $\frac{1}{2}(1)+\frac{5}{4}(2)+\frac{3}{4}(3)+(4)$.

Therefore, $\mathbf{y} = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix}$ is a certificate of infeasibility.

(Check that $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} > 0$.

2. Prove Corollary 2.2.

Show solution

3. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and let $\mathbf{b} \in \mathbb{R}^m$. Prove that

$$\begin{aligned} \mathbf{A}\mathbf{x} &\geq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

has no solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y} \geq \mathbf{0},\ \mathbf{y}^\mathsf{T}\mathbf{A} \leq \mathbf{0}$ and $\mathbf{y}^\mathsf{T}\mathbf{b} > 0$.

Show solution

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex

$$A = \begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{(2/3 \times 3) - (0 \times 0)} \begin{bmatrix} 3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \times \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$