Department of AI & DS
CSE and CS&IT

COURSE NAME: PROBABILITY, STATISTICS AND QUEUING THEORY

COURSE CODE: 23MT2005

Topic

Normal distribution

Session - 8

AIM OF THE SESSION

To familiarize students with Normal distribution and its applications

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Define Normal distribution
2. Demonstrate the properties of Normal distribution
3. Describe characteristics of Normal distribution
4. Solving problems of Normal distribution with its applications

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define Normal distribution
2. Describe the properties and Characteristics of Normal distribution
3. Summarize the concept with their applications

CONTENTS

- Normal distribution
- Properties of Normal curve
- Standard Normal distribution
- Applications

NORMAL DISTRIBUTION

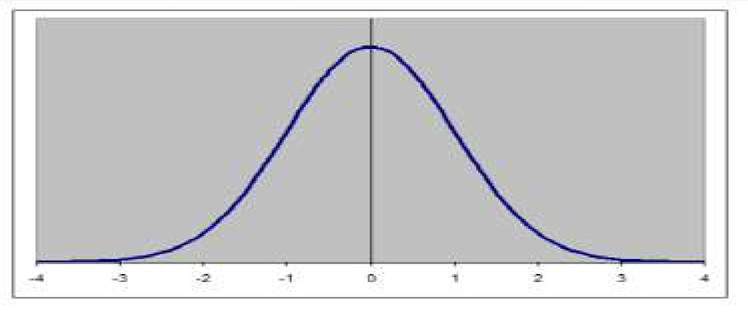
The most important continuous probability distribution in the entire field of statistics is the **Normal distribution**.

It is graph, called the normal curve, is the bell shaped curve, which describes approximately many phenomena that occur in nature, industry, and research.

Physical measurements in areas such as meteorological experiments, rainfall studies, and measurements of manufactured parts are often more than adequately explained with a normal distribution.

The Normal Distribution (N.D.) was first discovered by De-Moivre as the limiting form of the binomial model in 1733. The normal distribution is often referred to as the Gaussian distribution, in honour of Karl Friedrich Gauss (1777-1855), who also derived its equation from a study of errors in repeated measurements of the same quantity.

Normal distribution



The basic from of normal distribution is that of a bell, it has single mode and is symmetric about its central values. The flexibility of using normal distribution is due to the fact that the curve may be centered over any number on the real line and it may be flat or peaked to correspond to the amount of dispersion in the values of random variable.

Definition

A random variable X is said to follow a Normal Distribution with parameter mean (μ) and variance (σ^2) if its density function is given by the probability law

$$f(x) = n(x; u; \sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty$$

Symbolically we can represent the distribution of normal variate as $X \sim N(\mu, \sigma^2)$

Standard Normal distribution

Standard normal distribution:

The distribution of a random variable with mean '0' and variance '1' is called a standard normal distribution. If Z is a standard normal variate then Z~N(0,1).

The probability density function of the standard normal variate Z is given by the probability law

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < +\infty$$

The standard normal distribution, N (0, 1), is very important because probabilities of any normal distribution can be calculated from the probabilities of the standard normal distribution.

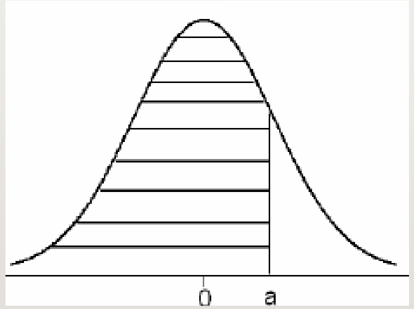
Standard Normal distribution

Note:

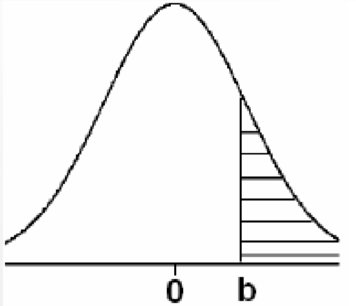
1. If X is a normal random variable with mean μ and standard deviation σ, then $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable and hence $P(x_1 < X < x_2) = P(\frac{x_1-\mu}{\sigma} < Z < \frac{x_2-\mu}{\sigma})$

2. Suppose Z ~ N(0, 1) is standard normal variate then by using the standard normal distribution area tables, we can calculate the various probabilities as explained below:

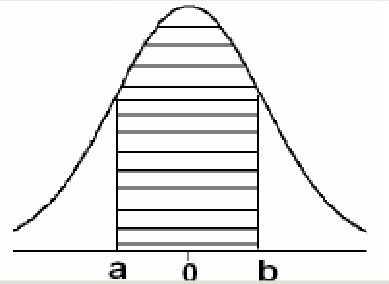
i) $P(Z<a) \cong P(Z\le a)$. This probability can be read from the table and is described in the following figure



ii) $P(Z>b)$. This probability can be represented by using the following graph of standard normal distribution and it cannot be read directly from the standard normal tables



iii) $P(a \leq Z \leq b)$. This probability can be represented by using the following graph of normal distribution



$\therefore$ $P(a \leq Z \leq b) = P(Z \leq b) – P(Z \leq a)$, where $P(Z \leq b)$ and $P(Z \leq a)$ are available directly from standard normal tables.

The Properties of normal probability curve

1. The mode which is point on the horizontal axis where the curve is a maximum, occurs at $x=\mu$. Hence the mean, median and mode of normal distribution are equal.

2. The curve is symmetric about a vertical axis through the mean μ.

3. The curve has its points of inflexion at $x=\mu\pm\sigma$, it concave downward if $\mu-\sigma<X<\mu+\sigma$, and is concave upward otherwise.

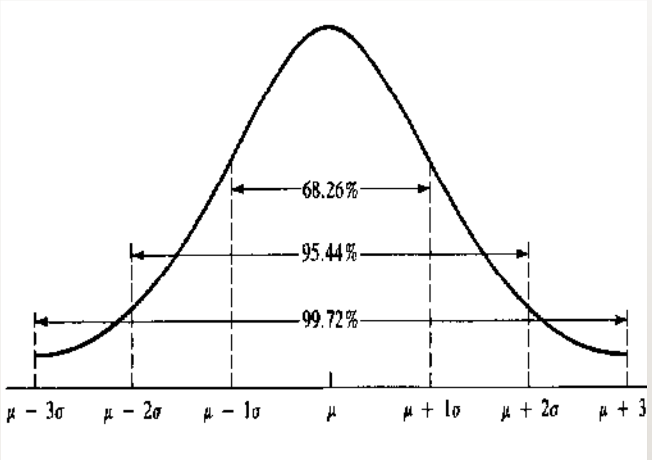
4. The curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.

5. The total area under the curve and above the horizontal axis $=1$.

IMPORTANT FACTS RELATED TO THE SESSION

The total area under the normal curve ($\int_{-\infty}^{+\infty} f(x)dx = 1$) is distributed as follows

- $(\mu - \sigma) < x < (\mu + \sigma)$ covers 68.26% of the area
- $(\mu - 2\sigma) < x < (\mu + 2\sigma)$ covers 95.44% of the area
- $(\mu - 3\sigma) < x < (\mu + 3\sigma)$ covers 99.74% of the area, and it can be represented as follows



Z Table

The table shows cumulative probabilities for the standard normal curve.

Cumulative probabilities for NEGATIVE z-values are shown first. SCROLL DOWN to the 2nd page for POSITIVE z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Z table

Cumulative probabilities for POSITIVE z-values are shown below.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

EXAMPLE 1

1. With an eye toward improving performance, industrial engineers studied the ability of scanners to read the bar codes of various food and household products. The maximum reduction in power, just before the scanner cannot read the bar code at a fixed dictionary is called the maximum attenuation. This quantity, measured in decibles, varies from product to product: After collecting the data, the engineers decided to model the variation in maximum attenuation as a normal distribution with mean 10.1 dB and standard deviation 2.7 dB.

a) For the next food product, what is the probability that its maximum attenuation is between 8.5 dB and 13.0 dB?

b) According to the normal model, what proportion of the products has maximum attenuation between 8.5 dB and 13.0 dB?

c) What proportion of the products has maximum attenuation greater than 15.1 dB?

Solution: Let X be the maximum attenuation of the next product, Then X is a normal variable with μ=10.1 and σ=2.7.

EXAMPLE 1

$$z = \frac{X - 10.1}{2.7}$$

a) Probability that the maximum attenuation of the next product is between 8.5 dB and 13.0 dB.

$=P(8.5 \le X \le 13.0)=P(\frac{8.5-10.1}{2.7} \le X \le \frac{13.0-10.1}{2.7})=P(-0.59 \le Z \le 1.07)=P(Z \le 1.07)-P(Z \le -0.59)$

$=0.8577-0.2776=0.5801.$

b) 0.5801 is the proportion of the product having maximum attenuation between 8.5 and 13.0 dB

c) Proportion of the products having maximum attenuation greater than 15.1 dB

$=P(X>15.1)=P(Z>(15.1-10.1)/(2.7))=P(Z>1.85)=1-0.9678=0.0322$

EXAMPLE 2

The actual amount of instant coffee that a filling machine puts into "4-ounce" jars may be looked upon as a random variable having a normal distribution with σ=0.04 Ounce. If only 2% of the jars are to contain less than 4 ounce, what should be the mean fill of these jars?

Solution: Let X be the amount in ounces of instant coffee that is put into jars. Then the distribution of X is normal with mean μ and standard deviation 0.04 ounce.

We are given that $P(X<\mu)=0.02$

Then $P(\frac{X-\mu}{0.04}<\frac{4-\mu}{0.04}) = 0.02$

$P(Z<\frac{4-\mu}{0.04}) = 0.02 \Rightarrow \frac{4-\mu}{0.04} = -2.05 \Rightarrow \mu=4.082$ ounce.

SUMMARY

In this session, the concepts of Normal distribution and its have described

1. Define Normal distribution and its properties
2. Applications of Normal distribution

SELF-ASSESSMENT QUESTIONS

1. If $X \sim N(\mu,\sigma^2)$, the points of inflexion of normal distribution curve are:

(a) $\pm\mu$

(b) $\mu\pm\sigma$

(c) $\sigma+\mu$

(d) $\pm\sigma$

2. If X is a normal variate with mean 20 and variance 64, the probability that X lies between 12 and 32 is

(a) 0.4332

(b) 0.1189

(c) 0.7475

(d) 0.5

TERMINAL QUESTIONS

1. Describe the importance of Normal distribution

2. List out properties of Normal distribution

3. Given a Standard Normal distribution, find the area under the curve which lies

a) To the left of z=1.43;

b) to the right of z=-0.89

c) between z=-2.16 and z=-0.65

d) to the left of z=-1.39

e) to the right of z=1.96

TERMINAL QUESTIONS

4. Given a standard normal distribution find the value of k such that

a) $P(z < k) = 0.0427;$

b) $P(z > k) = 0.2946;$

c) P(-0.93<Z<k)=0.7235.

5. In a test on 2000 electric bulbs it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for

(a) more than 2150 hours (b) less than 1950 hours

(c) more than 1920 hours and but less than 2160 hours.

(d) exactly 1960 hours.

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. Chapter 1 of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons, Inc.

2. Richard A Johnson, Miller& Freund's Probability and statistics for Engineers, PHI, New Delhi, 11th Edition (2011).

Sites and Web links:

- Continuous Random Variables and their Distributions (probabilitycourse.com)

Notes: sections 1 to 1.3 of http://www.statslab.cam.ac.uk/~rrw1/prob/prob-weber.pdf

3. Section 3.1.1 of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

Video: https://www.youtube.com/watch?v=- 5sOBWV0qH8&list=PLeB45KifGiuHesi4PALNZSYZFhViVGQJK&index=19

THANK YOU

Team – PSQT EVEN SEMESTER 2024-25