Department of AI & DS
CSE and CS&IT

COURSE NAME: PROBABILITY, STATISTICS AND QUEUING THEORY

COURSE CODE: 23MT2005

Topic

Test of significance for two Means using t and Z

Session - 17

AIM OF THE SESSION

To familiarize students with the basic concept of test of significance for difference of two means using t and Z

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Demonstrate the null and alternative hypothesis for t-test
2. Describe the procedure of t-test for two means
3. List out the test statistic for t and Z
4. Describe the procedure of Z-test for two means

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define Null and alternative hypothesis of test of significance for two means
2. Describe the procedure for t-test
3. Summarize the importance of t and Z tests in making decision about the sample

Test of significance for difference of two means (Large samples)

Let $\bar{x}_1$ be the mean of a sample size n_1 from a population with mean μ_1 and variance σ_1^2 and let

$\bar{x}_2$ be the mean of an independent sample size n_2 from another population with mean μ_2 and variance σ_2^2. Then, since sample sizes are large,

$\bar{x}_1 \sim N(\mu_1, \sigma_1^2/n_1)$, $\bar{x}_2 \sim N(\mu_2, \sigma_2^2/n_2)$

also $\bar{x}_1$-$\bar{x}_2$, being the difference of two independent normal variate is also a normal variate. The value of Z (Standard normal variate) corresponding to $\bar{x}_1$-$\bar{x}_2$ is given by

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - E(\bar{x}_1 - \bar{x}_2)}{S.E.(\bar{x}_1 - \bar{x}_2)} \sim N(0,1)$$

Under the null hypothesis, H_0: $\mu_1 = \mu_2$, i.e., there is no significant difference between the sample means, we get

$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu_1 - \mu_2 = 0$;

Test of significance for difference of two means (Large samples)

$$V(\bar{x}_1-\bar{x}_2)= \sigma_1^2/n_1+ \sigma_2^2/n_2,$$

The covariance term vanishes, since the sample mean $\bar{x}_1$ and $\bar{x}_2$ are independent.

Thus under H_0: $\mu_1=\mu_2$, the test statistic becomes (for large samples)

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}} \sim N(0,1)$$

Case: 1: If $\sigma_1^2 = \sigma_2^2 = \sigma^2$, i.e, if the samples have been drawn from the population with common standard deviation σ, then under H_0: $\mu_1=\mu_2$,

$$Z = \frac{\bar{x}_1-\bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1}\right)+\left(\frac{1}{n_2}\right)}} \sim N(0,1)$$

Test of significance for difference of two means (Large samples)

Case 2: If σ is not known, then its estimate based on the sample variances is used. If the sample sizes are not sufficiently large, then an unbiased estimated of σ^2 is given by

$$\hat{\sigma}^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 + n_2 - 2)}, since$$

But since sample sizes are large, $S_1^2 \cong s_1^2, S_2^2 \cong s_2^2, n_1 - 1 \cong n_1, n_2 - 1 \cong n_2$.

Therefore, in practice for large samples, the following estimate of σ^2 without any serious error is used:

$$\hat{\sigma}^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2},$$

Case: 3: If $\sigma_1^2 \neq \sigma_2^2$ are not known, then they are estimated from sample values.

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}} \sim N(0,1)$$

The means of two single large samples of 1,000 and 2,000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn from the same population of standard deviation 2.5 inches? Test at 5% level of significance

Solution: In usual notations, we are given

$n_1=1,000$, $n_2=2,000$, $\bar{x}_1=67.5$ inches, $\bar{x}_2=68.0$ inches.

Null hypothesis: H_0: $\mu_1=\mu_2$ and $\sigma=2.5$ inches, ie., the samples have been drawn from the same population of standard deviation 2.5 inches.

Alternative hypothesis: H_1: $\mu_1\neq\mu_2$ (Two-tailed)

Choose the level of significance $\alpha=5\%$

Test statistics: Under H_0, the test statistic is:

Example

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\left(\frac{1}{n_1}\right) + \left(\frac{1}{n_2}\right)}} \sim N(0,1)(\textit{since samples are large})$$

$$Z = \frac{67.5 - 68.0}{2.5\sqrt{\left(\frac{1}{1000}\right) + \left(\frac{1}{2000}\right)}} = -5.1$$

Conclusion: Since $|z| = 5.1 > 3,$ the value is highly significant and we reject the null hypothesis and conclude that the samples are certainly not from the same population and standard deviation 2.5.

Example

In a certain factory there are two independent processes manufacturing the same item. The average weight in a sample of 250 items produced from one process is found to be 120 ozs. With a standard deviation of 12 ozs. While the corresponding figures in a sample of 400 items from the other process are 124 and 14. Obtain the standard error of difference between the two sample means. Is this difference significant? Also find the 99%, 95% confidence limits for the difference in the average weights of items produced by the two processes respectively.

Solution:

In the usual notations, we are given

$n_1=250$, $n_2=400$, $\bar{x}_1=120$ ozs, $\bar{x}_2=124$ ozs.

$s_1=12$ oz$=\hat{\sigma}_1$, $s_2=14$ oz$=\hat{\sigma}_2$

Null hypothesis: H_0: $\mu_1=\mu_2$, ie., the samples do not differ significantly.

Alternative hypothesis: H_1: $\mu_1\neq\mu_2$ (Two-tailed)

Choose the level of significance $\alpha=5\%$ and 1%

Example

Test statistic: Under H_0, the test statistics is :

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})}} \cong \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})}} \sim N(0,1) \quad \text{(since samples are large).}$$

Where Standard error(S.E.)= $S.E.|\bar{x}_1 - \bar{x}_2| = \sqrt{(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})} \cong \sqrt{(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})}$

$$Z = \frac{120 - 124}{\sqrt{(\frac{144}{250} + \frac{196}{400})}} \sim N(0,1)$$

$$Z = 3.87$$

Conclusion: $|z| = 3.87 > 3$, the null hypothesis is rejected and we conclude that there is significant difference between the sample means.

Example

- **99% confidence limits for** $|\mu_1 - \mu_2|$, ie., for the difference in the average weights of items produced by two processes, are:
  $|\bar{x}_1 - \bar{x}_2| \pm 2.58 S.E. |\bar{x}_1 - \bar{x}_2|$ =4±2.58*1.034=6.67 and 1.33
  1.33<$|\mu_1 - \mu_2|$<6.67.

- **95% confidence limits for** $|\mu_1 - \mu_2|$, ie., for the difference in the average weights of items produced by two processes, are:
  $|\bar{x}_1 - \bar{x}_2| \pm 1.96 S.E. |\bar{x}_1 - \bar{x}_2|$ =4±1.96*1.034=6.03 and 1.98<$|\mu_1 - \mu_2|$<6.03.

t- TEST FOR DIFFERENCE OF TWO MEANS

Suppose we want to test if two independent samples x_i (i=1,2,…,n_1) and y_j(j=1,2,..,n_2) of sizes n_1 and n_2 have been drawn from two normal populations with means μ_x and μ_y respectively.

Under the null hypothesis H_0 that the samples have been drawn from the normal populations with means μ_x and μ_y and under the assumption that the population variance are equal,

ie., $\sigma_x^2=\sigma_y^2=\sigma^2$ (say), the statistic

$$t = \frac{(\bar{x}-\bar{y})-(\mu_x-\mu_y)}{S\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}} \cong \frac{(\bar{x}-\bar{y})-(\mu_x-\mu_y)}{\sigma\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}}$$

Under null hypothesis,

$$t = \frac{(\bar{x}-\bar{y})}{S\sqrt{(\frac{1}{n_1}+\frac{1}{n_2})}} \sim t_{n_1+n_2-2}$$

t- TEST FOR DIFFERENCE OF TWO MEANS

Where $\bar{x} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i,\ \ \bar{y} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i,$

$$S^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_i (x_i - \bar{x})^2 + \sum_i (y_i - \bar{y})^2\right]$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]$$

$$\text{(OR)}\quad S^2 = \frac{1}{n_1 + n_2 - 2}[n_1 s_1^2 + n_2 s_2^2]$$

and it follows t-distribution with n_1+n_2-2 degrees of freedom.

Example

The following random samples are measurements of the heat-producing capacity (in millions of calories per ton) of specimens of coal from two mines:

Mine 1: 8260   8130   8350   8070   8340

Mine 2: 7950   7890   7900   8140   7920   7840

Use the 0.01 level of significance to test whether the difference between the means of these two samples is significant.

Solution:

Step:1 : Set up Null hypothesis: $\mu_1-\mu_2=0$

Step: 2: Set up Alternative hypothesis: $\mu_1-\mu_2\neq0$

Step: 3: Choose the level of significance $\alpha=0.01$

Example

Step: 4 Calculations

$$\overline{\mathrm{X}} = \frac{1}{5}(8260 + 8130 + 8350 + 8070 + 8340) = 8230$$

$$\overline{\mathrm{Y}} = \frac{1}{6}(7950 + 7890 + 7900 + 8140 + 7920 + 7840) = 7940$$

$$\mathrm{S}^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_i (\mathrm{x_i} - \bar{\mathrm{x}})^2 + \sum_i (\mathrm{y_i} - \bar{\mathrm{y}})^2\right]$$

$$\mathrm{S}^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]$$

$$S_1^2 = \frac{\sum_i (\mathrm{x_i} - \bar{\mathrm{x}})^2}{n_1 - 1}, \qquad S_2^2 = \frac{\sum_i (\mathrm{x_i} - \bar{\mathrm{x}})^2}{n_2 - 1}$$

$$S_1^2 = \frac{63000}{4} = 15750, \qquad S_2^2 = \frac{54600}{5} = 10920$$

Example

$$S^2 = 13066.7$$

$$S = 114.31$$

$$t = \frac{8230 - 7940}{114.31\sqrt{\left(\frac{1}{5} + \frac{1}{6}\right)}} = 4.19 \sim t_{n_1+n_2-2}$$

Conclusion: t_{cal} value =4.19 exceeds 3.250, the null hypothesis must be rejected at level α=0.01. We conclude that the average heat producing capacity of the coal from the two mines is not the same.

i) 95% confidence interval for the population mean μ are:

$$(\bar{x}-\bar{y}) \pm t_{0.05} * S\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

ii) 99% confidence interval for the population mean μ are:

For Small samples (n<=30)

$$(\bar{x}-\bar{y}) \pm t_{0.01} * S\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

iii) 95% confidence interval for the population mean μ are:

$$(\bar{x}_1-\bar{x}_2) \pm 1.96\left(\sigma\sqrt{\left(\frac{1}{n_1}\right)+\left(\frac{1}{n_2}\right)}\right)$$

iv) 99% confidence interval for the population mean μ are:

For Large samples (n>30)

$$(\bar{x}_1-\bar{x}_2) \pm 2.58\left(\sigma\sqrt{\left(\frac{1}{n_1}\right)+\left(\frac{1}{n_2}\right)}\right)$$

ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION

t Table

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
two-tails	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Z table

Level of significance	Two tailed test	One tailed test	
		Right tailed test	Left tailed test
0.10 (90% confidence)	1.645	1.28	-1.28
0.05(95%)	1.96	1.645	-1.645
0.01 (99%)	2.58	2.33	-2.33

ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION

P-Value approach: The P –value is the probability of obtaining a value for the test statistics that is as extreme as or more extreme than the value actually observed. Probability is calculated under the null hypothesis.

If the alternative hypothesis is right sided i.e., H_1: $\mu>\mu_0$ then only values greater than the observed value are more extreme.

If the alternative hypothesis is left sided i.e., H_1: $\mu<\mu_0$ then only values less than the observed value are more extreme.

For two sided alternatives, values in both tails need to be considered.

Z cal compare with z table value, Z cal$\leq$z table we accept null hypothesis, Z cal >z table we reject H_0.

P cal compare with P table, PCal<Ptable we reject null hypothesis, P cal $\geq$Ptable we accept H_0

SUMMARY

In this session, the concept of test of significance for single mean have described

1. Define Null and Alternative hypothesis of significance for difference oof two means
2. Discuss in detail about the level of t-test and z-test for difference of two means.

SELF-ASSESSMENT QUESTIONS

When "between groups variance' is substantially greater than the "within-groups variance", the difference between the means may be ascribed only to

A) Sampling error
B) Measurement error
C) Chance error
D) Constant error

A group of 10 students was randomly drawn from class 12 and was given yoga training for three weeks. Their wellness life style was compared to with another similarly selected group which did not undergo such training. Which type of statistical test will be appropriate for testing the tenability of Null hypothesis?

A. Independent t-test
B: Dependent t-test
C: Wilcoxon-t-test
D: Sign test

TERMINAL QUESTIONS

1. Describe the difference between t and Z test for difference of two means.

2 The dynamic modulus of concrete is obtained for two different concrete mixes. For the first mix, n_1=33, $\bar{x}$=115.1, and s_1=0.47 psi. For the second mix, n_2=31, $\bar{y}$=114.6, and s_2=0.38. Test, with α=0.05, the null hypothesis of equality of mean dynamic modulus versus the two-sided alternative. And also obtain a 95% confidence interval for the difference in mean dynamic modulus.

3. To determine whether the car ownership affects a student's academic achievement, two random samples of 100 male students were each drawn from the student body. The grade point average for the n_1=100 non-owners of cars had an average and variance equal to $\bar{x}_1$ =2.70 and s_1^2=0.36., while $\bar{x}_2$ =2.54 and s_2^2=0.40 for the n_2=100 car owners. Do the data present sufficient evidence to indicate a difference in the mean achievements between car owners and non owners of cars? Test using α=0.05.

ii) Also obtain 95% confidence interval for the difference in average academic achievements of the car owners and non-car owners. Using the confidence interval, can you conclude that there is a difference in the population means for the two groups of students?

TERMINAL QUESTIONS

4. Measuring specimens of nylon yarn taken from two spinning machines, it was fwith a sound that 8 specimens from the first machine had a mean denier of 9.67 with a standard deviation of 1.81, while 10 specimens from the second machine had a mean denier of 7.43 with a standard deviation of 1.48. Assuming that the populations sampled are normal and have the same variance, test the null hypothesis $\mu_1-\mu_2=1.5$ against the alternative hypothesis $\mu_1-\mu_2>1.5$ at the 0.05 level of significance.

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. William Feller, An Introduction to Probability Theory and Its Applications: Volime 1, Third Edition, 1968 by John Wiley & Sons, Inc.
2. Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

3. Richard A Johnson, Miller& Freund's Probability and statistics for Engineers, PHI, New Delhi, 11th Edition (2011).

Sites and Web links:

1. https://www.khanacademy.org/math/statistics-probability/significance-tests-one-sample/more-significance-testing-videos/v/small-sample-hypothesis-test

THANK YOU

Team – PSQT EVEN SEMESTER 2024-25