MATHEMATICAL PROGRAMMING

CO2- INTEGER PROGRAMMING – GOMORY CUT PLANE METHOD

SESSION 9

AIM OF THE SESSION

To familiarize students with the method of solving ILP using Gomory cut plane method.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Introduce Gomory Cut plane method
2. Discuss methods to solve ILP using Gomory Cut plane method

LEARNING OUTCOMES

At the end of this session, the students should be able to:

1. Understand the solution for an ILP using Gomory Cut plane method

THE CUTTING–PLANE ALGORITHM

An Algorithm for solving Pure integer and mixed integer programming problems was developed by R. E. Gomory.

1. Relax the integer requirements.
2. Solve the resulting LP problem using Simplex Method.
3. If all the basic variables have integer values, Optimality of the Integer programming problem is reached. So go step 7; otherwise go to step 4.
4. Examine the constraints corresponding to the current optimal solution. For each Basic Variable with non-integer solution in the current optimal table, find the fractional part, f_i , Therefore, $b_i = [b_i] + f_i$, where $[b_i]$ is the integer part of b_i, and f_i is the positive fractional part of b_i.
5. Choose the largest fraction among various f_i ; i.e., Max (f_i). Treat the constraint corresponding to the maximum fraction as the source row (equation). Based on the source equation, develop an additional constraint (Gomory's constraint / fractional cut) as shown:
   - $-f_i = S_i$ – Summation ((f_i)(Non–Basic Variable))
6. Add the fractional cut as the last row in the latest optimal table and proceed further using dual simplex method, and find the new optimum solution. If the new optimum solution is integer then go to step 7; otherwise go to step 4.
7. Print the integer solution [X's and Z – Values]

THE CUTTING–PLANE ALGORITHM

EXAMPLE:

Max. $Z = 5X_1 + 8X_2$

Subject to:

$X_1 + 2X_2 \leq 8$

$4X_1 + X_2 \leq 10$

$X_1, X_2 \geq 0$ and integers

Standard Form:

Max. $Z = 5X_1 + 8X_2 + 0S_1 + 0S_2$

Subject to:

$X_1 + 2X_2 + S_1 = 8$

$4X_1 + X_2 + S_2 = 10$

$X_1, X_2, S_1,$ and $S_2 \geq 0$ and integers

THE CUTTING–PLANE ALGORITHM (CONT…)

Max. $Z = 5X_1 + 8X_2 + 0S_1 + 0S_2$

Subject to:

$X_1 + 2X_2 + S_1 + 0S_2 = 8$

$4X_1 + X_2 + 0S_1 + S_2 = 10$

X_1, X_2, S_1, and $S_2 \geq 0$ and integers

Initial Simplex Table:

Contribution Per Unit	C_j	5	8	0	0		
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	bj (solution)	Ratio
0	S_1	1	2	1	0	8	8/2 = 4*
0	S_2	4	1	0	1	10	10/1 = 10
Total Profit	(Z_j)	0	0	0	0	0	
Net Contribution	($C_j - Z_j$)	5	8*	0	0		

THE CUTTING–PLANE ALGORITHM (CONT…)

Initial Table:

Contribution Per Unit	C_j	5	8	0	0		
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	SOLUTION	Ratio
0	S_1	1	2	1	0	8	8/2 = 4*
0	S_2	4	1	0	1	10	10/1 = 10
Total Profit	(Z_j)	0	0	0	0	0	
Net Contribution	($C_j - Z_j$)	5	8*	0	0		

Iteration # 1:

Contribution Per Unit	C_j	5	8	0	0		
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	SOLUTION	Ratio
8	X_2	1/2	1	1/2	0	4	8
0	S_2	7/2	0	-1/2	1	6	12/7*
Total Profit	(Z_j)	4	8	4	0	32	
Net Contribution	($C_j - Z_j$)	1*	0	-4	0		

THE CUTTING–PLANE ALGORITHM (Cont…)

Iteration # 2:

CB_i	C_j	5	8	0	0	Solution
	Basic variable	X_1	X_2	S_1	S_2	
8	X_2	0	1	4/7	–1/7	22/7
5	X_1	1	0	–1/7	2/7	12/7
	Z_j	5	8	27/7	2/7	236/7
	$C_j - Z_j$	0	0	–27/7	–2/7	

All the Values of $(C_j - Z_j) \leq 0$; So, the current solution is optimal for LP.

$$X_1 = 12/7, X_2 = 22/7 \text{ and } Z = 236/7$$

However, the values of the decision variables X_1 & X_2 are not integers, so, the solution is **not optimum for Integer Programming.**

THE CUTTING–PLANE ALGORITHM (Cont...)

CB_i	C_j	5	8	0	0	
	Basic variable	X_1	X_2	S_1	S_2	Solution
8	X_2	0	1	4/7	−1/7	22/7
5	X_1	1	0	−1/7	2/7	12/7
	Z_j	5	8	27/7	2/7	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7	

All the Values of $(C_j - Z_j) \le 0$; So, the current solution is optimal for linear programming.

$X_1 = 12/7$, $X_2 = 22/7$ and $Z = 236/7$

STEP #4: **Summary of Integer & Fractional Parts**

Basic Variable in the above Optimal table	b_i	$[b_i] + f_i$
X_1	12/7	1 + (5/7)
X_2	22/7	3 + (1/7)

Maximum fraction value

THE CUTTING–PLANE ALGORITHM (CONT…)

CB_i	C_j / Basic variable	5 / X_1	8 / X_2	0 / S_1	0 / S_2	Solution
8	X_2	0	1	4/7	−1/7	22/7
5	X_1	1	0	−1/7	2/7	12/7
	Z_j	5	8	27/7	2/7	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7	

STEP #5: The fractional part, f_1, is the maximum. So, Select the Row "X_1" as the Source row for developing first cut.

$X_1 - 1/7S_1 + 2/7S_2 = 12/7$

$(1+0)X_1 + (-1+6/7)S_1 + (0+2/7)S_2 = 1+5/7$

$0 \quad + \quad 6/7 \quad + \quad 2/7 \quad >= \; 5/7$

$0X_1 \; + \quad 6/7S_1 \quad + \quad 2/7S_2 \quad = \quad 5/7 + S_3$

$$-5/7 = S_3 - 6/7S_1 - 2/7S_2$$

This is the fractional cut proposed by Gomory :

$$-f_i = S_i - \text{Summation } ((f_i)(\text{Non–Basic Variable}))$$

THE CUTTING–PLANE ALGORITHM (CONT…)

STEP # 5: The fractional cut is:

$-f_i = S_i -$ Summation $((f_i)$(Non–Basic Variable))

$\mathbf{-5/7 = S_3 - 6/7S_1 - 2/7S_2}$ ==> 0X1 +0X2 –(6/7)S1 -(2/7)S2 +1S3 = -5/7

STEP # 6: This cut is added to the table which we get in Iteration # 2 (Optimal Table Solution for Linear Programming), and further solved using dual simplex method.

CB_i	C_j	5	8	0	0	0	
	Basic variable	X_1	X_2	S_1	S_2	S_3	Solution
8	X_2	0	1	4/7	−1/7	0	22/7
5	X_1	1	0	−1/7	2/7	0	12/7
0	S_3	0	0	-6/7	$\boxed{-2/7}$	1	-5/7*
	Z_j	5	8	27/7	2/7	0	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7*	0	

THE CUTTING–PLANE ALGORITHM (CONT…)

Only the third row (Containing S_3) has a negative solution value. Therefore, S_3 (LEAVING Variable) leaves the basis.

	C_j	5	8	0	0	0	
CB_i	Basic variable	X_1	X_2	S_1	S_2	S_3	Solution
8	X_2	0	1	4/7	−1/7	0	22/7
5	X_1	1	0	−1/7	2/7	0	12/7
0	S_3	0	0	−6/7	$\boxed{-2/7}$	1	−5/7*
	Z_j	5	8	27/7	2/7	0	236/7
	$C_j - Z_j$	0	0	−27/7	−2/7*	0	

For ENTERING Variable;

Ratio = $(C_j - Z_j)$ / (Pivot Row <0)

--	--	**9/2**	**1**	--

The smallest ratio is "1" and the corresponding variable is "S_2". So, the variable "S_2" enters the basis.

THE CUTTING–PLANE ALGORITHM (CONT…)

Contribution Per Unit C_j		5	8	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	SOLUTION
8	X_2	0	1	1	0	- 1/2	7/2
5	X_1	1	0	-1	0	1	1
0	S_2	0	0	3	1	-7/2	5/2
Total Profit (Z_j)		5	8	3	0	1	33
Net Contribution ($C_j - Z_j$)		0	0	-3	0	-1	

The Solution is still non-integer. So, develop a fractional cut. The Basic variables X_2 and S_2 are not integers.

THE CUTTING–PLANE ALGORITHM (CONT…)

Contribution Per Unit	C_j	5	8	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	SOLUTION
8	X_2	0	1	1	0	- 1/2	7/2
5	X_1	1	0	-1	0	1	1
0	S_2	0	0	3	1	-7/2	5/2
Total Profit (Z_j)		5	8	3	0	1	33
Net Contribution ($C_j - Z_j$)		0	0	-3	0	-1	

STEP #4:

Summary of Integer & Fractional Parts

Basic Variable in the above Optimal table	**b_i**	**$[b_i] + f_i$**
X_2	7/2	3 + 1/2
S_2	5/2	2 + 1/2

STEP # 5: Here, the fractional parts are the same for X_2 & S_2. But, we preferred the fractional part of the X_2. So, Select the Row "X_2" as the Source row for developing cut

THE CUTTING–PLANE ALGORITHM (CONT…)

Contribution Per Unit C_j		5	8	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	SOLUTION
8	X_2	0	1	1	0	- 1/2	7/2
5	X_1	1	0	-1	0	1	1
0	S_2	0	0	3	1	-7/2	5/2
Total Profit (Z_j)		5	8	3	0	1	33
Net Contribution ($C_j - Z_j$)		0	0	-3	0	-1	

STEP # 5: Here, the fractional parts are the same for X_2 & S_2. But, we preferred the fractional part of the X_2. So, Select the Row "X_2" as the Source row for developing cut

$7/2 = X_2 + S_1 - 1/2S_3 \rightarrow (3+ 1/2) = (1+0)X_1 + (1+0)S_1 + (-1+1/2)S_3$

The Corresponding fractional cut is:

$-f_i = S_i -$ Summation $((f_i)$(Non–Basic Variable))

$\mathbf{-1/2 = S_4 - 1/2S_3}$

THE CUTTING–PLANE ALGORITHM (CONT…)

$-1/2 = \quad S_4 - 1/2S_3$

STEP # 6: This cut is added to the simplex table, and further solved using dual simplex method.

CB_i	C_j	5	8	0	0	0	0	
	Basic variable	X_1	X_2	S_1	S_2	S_3	S_4	Solution
8	X_2	0	1	1	0	−1/2	0	7/2
5	X_1	1	0	−1	0	1	0	1
0	S_2	0	0	3	1	−7/2	0	5/2
0	S_4	0	0	0	0	$\boxed{-1/2}$	1	−1/2*
	Z_j	5	8	3	0	1	0	33
	$C_j - Z_j$	0	0	−3	0	−1*	0	

For ENTERING Variable;
Ratio = $(C_j - Z_j)$ / (Pivot Row <0)

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The smallest positive ratio is "2" and the corresponding variable is "S_3". So, the variable "S_3" enters the basis.

THE CUTTING–PLANE ALGORITHM (CONT…)

Contribution Per Unit	C_j	5	8	0	0	0	0	
C_{Bi}	Basic Variables (B)	X_1	X_2	S_1	S_2	S_3	S_4	SOLUTION
8	X_2	0	1	1	0	0	-1	4
5	X_1	1	0	-1	0	0	2	0
0	S_2	0	0	3	1	0	-7	6
0	S_3	0	0	0	0	1	-2	1
Total Profit (Z_j)		5	8	3	0	0	2	32
Net Contribution ($C_j - Z_j$)		0	0	-3	0	0	-2	

So, The values of all the basic variables are integers. So, the optimality is reached and the corresponding results are summarized as follows:

$X_1 = 0$, $X_2 = 4$ and Z (Optimum) = 32

THE CUTTING–PLANE ALGORITHM

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1. Relax the integer requirements.
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5. Choose the largest fraction among various f_i ; i.e., Max (f_i). Treat the constraint corresponding to the maximum fraction as the source row (equation). Based on the source equation, develop an additional constraint (Gomory's constraint / fractional cut) as shown:
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7. Print the integer solution [X's and Z – Values]