Course Title: **Mathematical programming (2-2-0-0)**

Topic : **Ellipsoid method**

CO-1

Session - 7

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AIM

To familiarize students with the concept of ellipsoid method in linear programming

INSTRUCTIONAL OBJECTIVES

This session is designed to:

1. Illustrate in detail of ellipsoid method in linear programming

LEARNING OUTCOMES

At the end of this unit, you should be able to:

1. Solve the formulation of ellipsoid method in linear programming

ELLIPSOID METHOD FOR OPTIMIZATION

The ellipsoid method is an iterative optimization algorithm used for solving convex optimization problems. It was introduced by Arkadi Nemirovski and David B. Yudin in 1979. The ellipsoid method is particularly useful for problems where the objective function and constraints are described by linear or affine functions.

The basic idea of the ellipsoid method is to iteratively update an ellipsoid that contains the feasible region of the convex set defined by the constraints. The algorithm narrows down the feasible region in each iteration until a solution with the desired accuracy is found.

The main steps of the ellipsoid method:

- **Initialization**
- **Iteration**
- **Feasibility Test**
- **Optimality Test**

OVERVIEW OF THE ELLIPSOID METHOD

- **Initialization:**
  - Start with an initial ellipsoid that contains the feasible region.
  - Choose an initial feasible point within the ellipsoid.
- **Iteration:**
  - At each iteration, compute the center and volume of the current ellipsoid.
  - Update the ellipsoid to a smaller one that is closer to the optimal solution. This is typically done by scaling and translating the ellipsoid.
- **Feasibility Test:**
  - Check if the current ellipsoid contains a feasible solution. If yes, proceed to the next step. If no, update the ellipsoid and continue the iteration.
- **Optimality Test:**
  - Check if the current ellipsoid is sufficiently small. If yes, the algorithm terminates, and the current solution is considered optimal. If no, go back to the iteration step.

ELLIPSOID METHOD FOR OPTIMIZATION

The ellipsoid method utilizes ellipsoids to represent the feasible region of the optimization problem. At each iteration, the ellipsoid is updated based on the information obtained from the evaluations of the objective function and constraints. The key idea is to iteratively refine the search space, narrowing down the region where the optimal solution is likely to be found.

While the ellipsoid method is a theoretically powerful algorithm for convex optimization, it may not be the most practical choice for large-scale problems due to its computational complexity. More efficient algorithms, such as interior-point methods, are often preferred in practice for solving convex optimization problems.

ELLIPSOID METHOD FOR OPTIMIZATION

- We next consider the following optimization problem and its dual:

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subject to} \quad & Ax \geq b \end{aligned} \qquad\qquad \begin{aligned} \text{maximize} \quad & b'p \\ \text{subject to} \quad & A'p = c \\ & p \geq 0. \end{aligned}$$

- By strong duality, both the primal and dual optimization problems have optimal solutions if and only if the following system of linear inequalities is feasible:

$$b'p = c'x, \qquad Ax \geq b, \qquad A'p = c, \qquad p \geq 0.$$

COMPLEXCITY OF ELLIPSOID METHOD

- Let Q be the feasible set of this system of inequalities.
- We can apply the ellipsoid method to decide whether Q is nonempty.
- If it is indeed nonempty and a feasible solution (x, p) is obtained, then x is an optimal solution to the original optimization problem and p is an optimal solution to its dual.

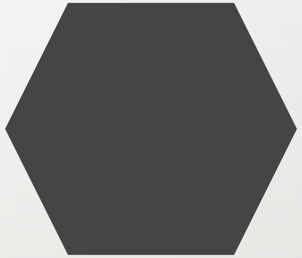
Ellipsoid Method

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$

Feasibility asks if there exists an x such that

$$\mathbf{c}^T\mathbf{x} \geq K$$
$$A\,\mathbf{x} \leq \mathbf{b}$$

Feasibility via Ellipsoid Method



Feasible region of LP

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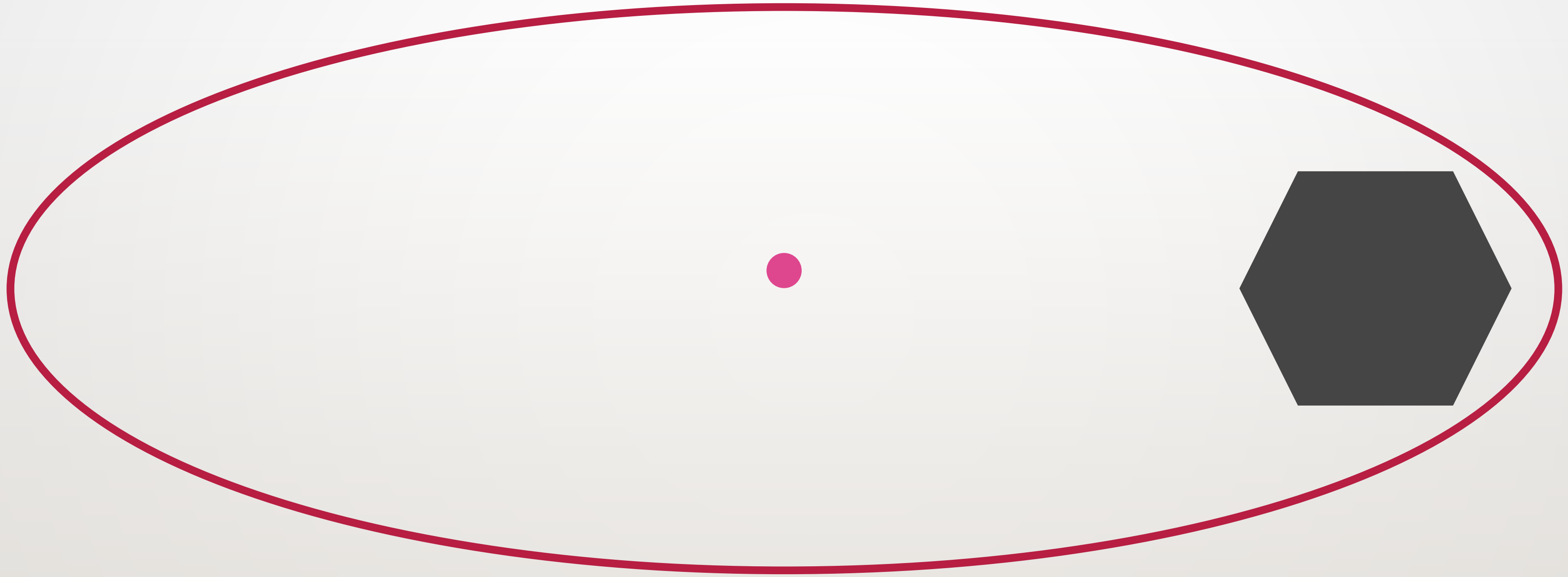
Feasibility via Ellipsoid Method

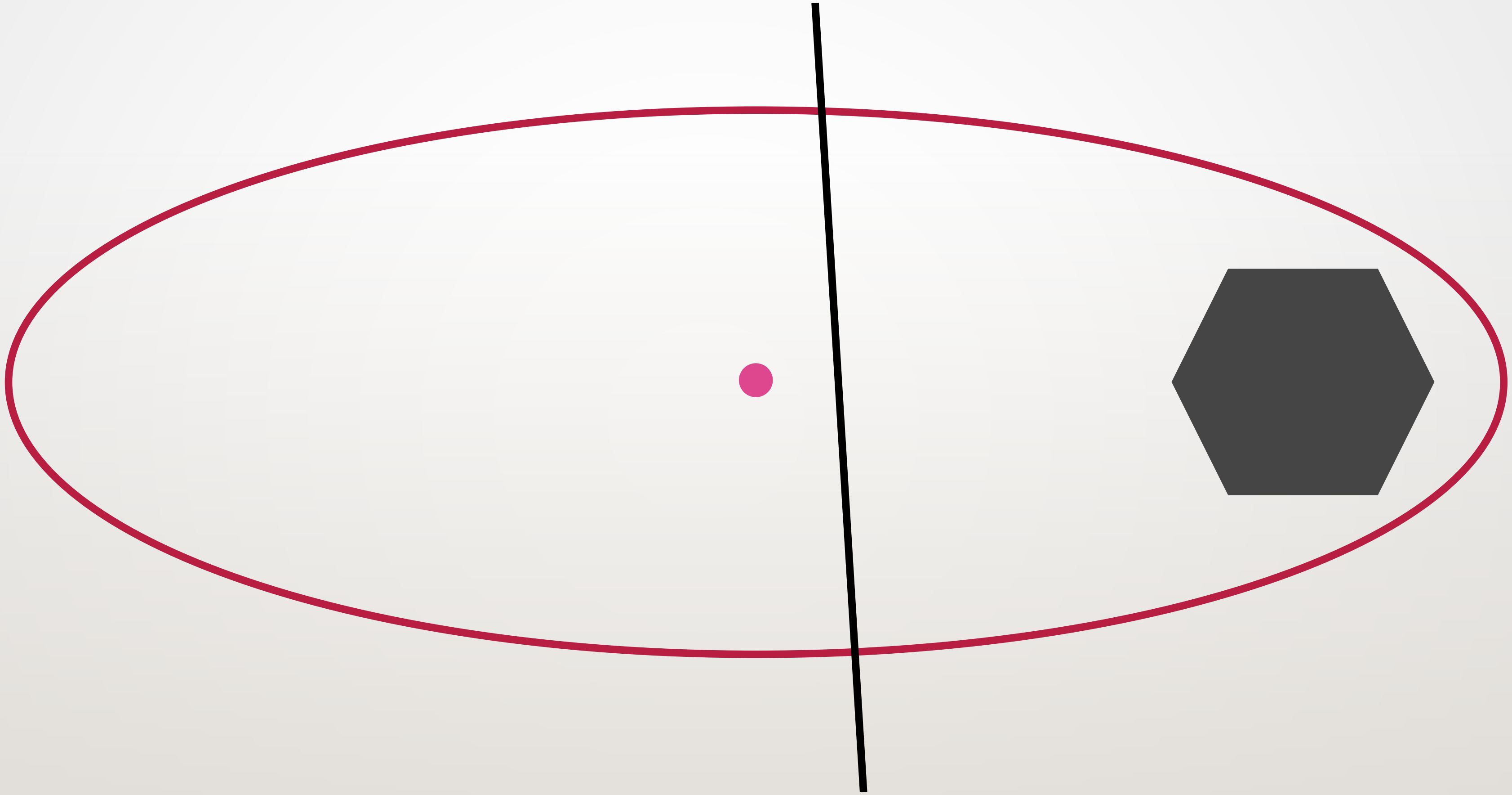
Ellipsoid containing feasible region of LP

Feasibility via Ellipsoid Method



Centroid of ellipsoid

Feasibility via Ellipsoid Method



Separating hyperplane for centroid

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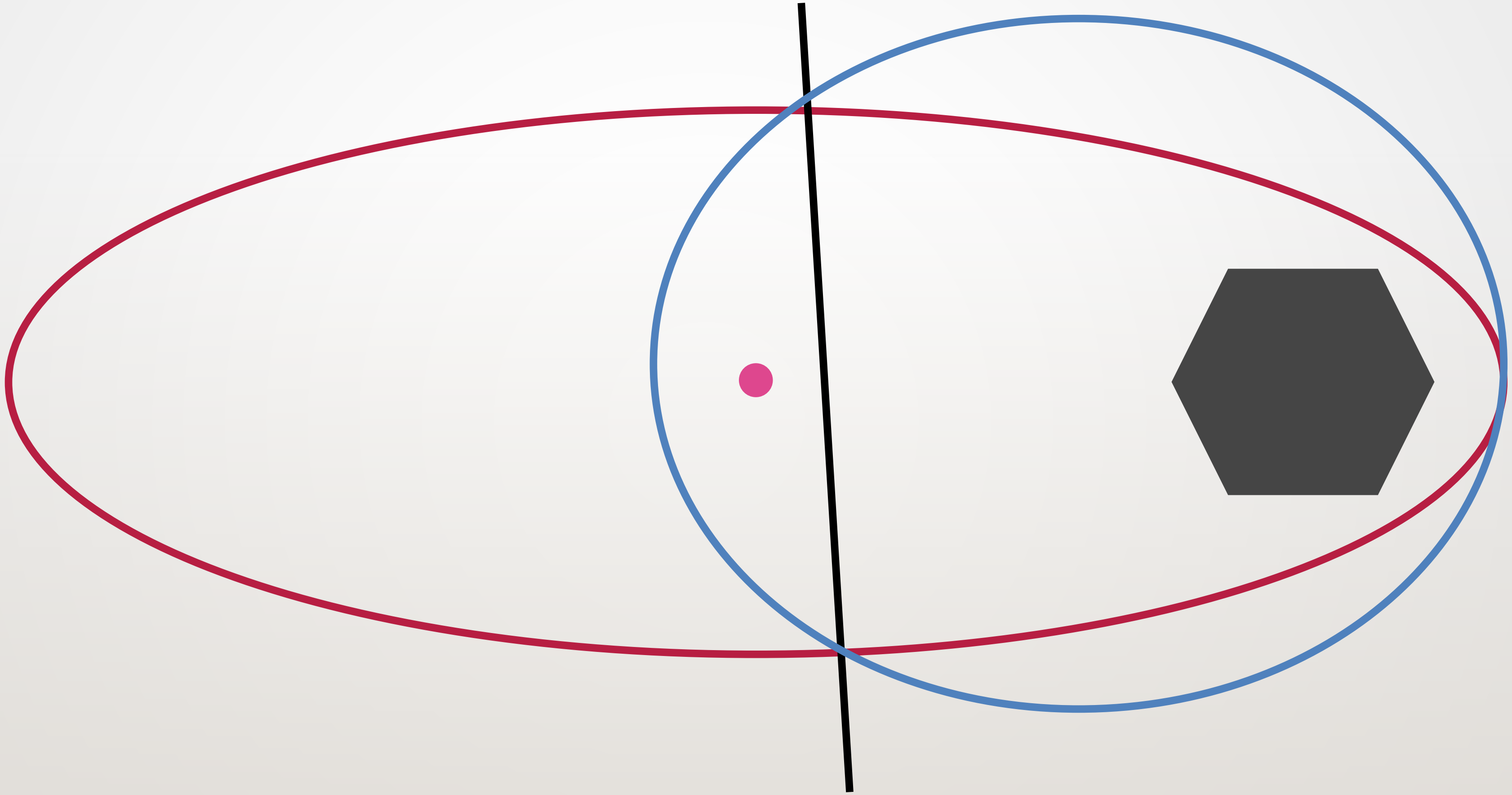
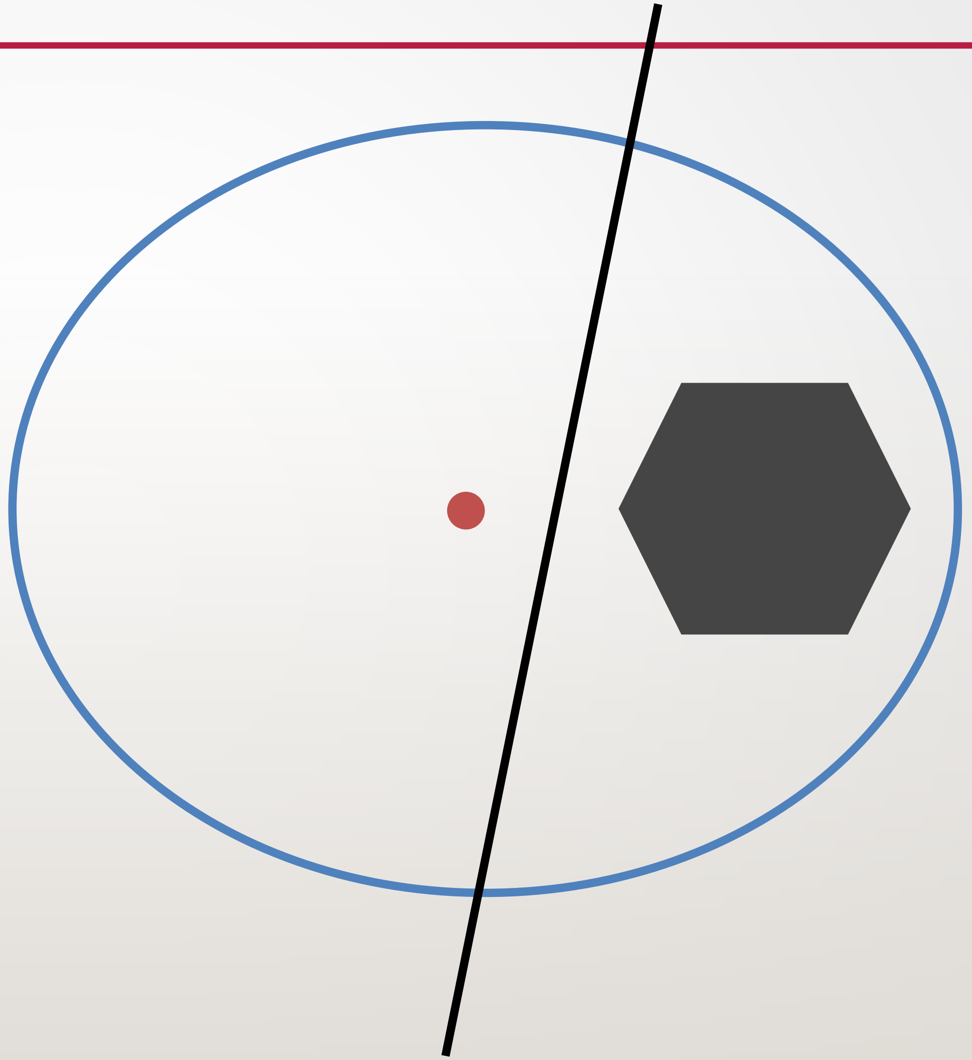
Feasibility via Ellipsoid Method



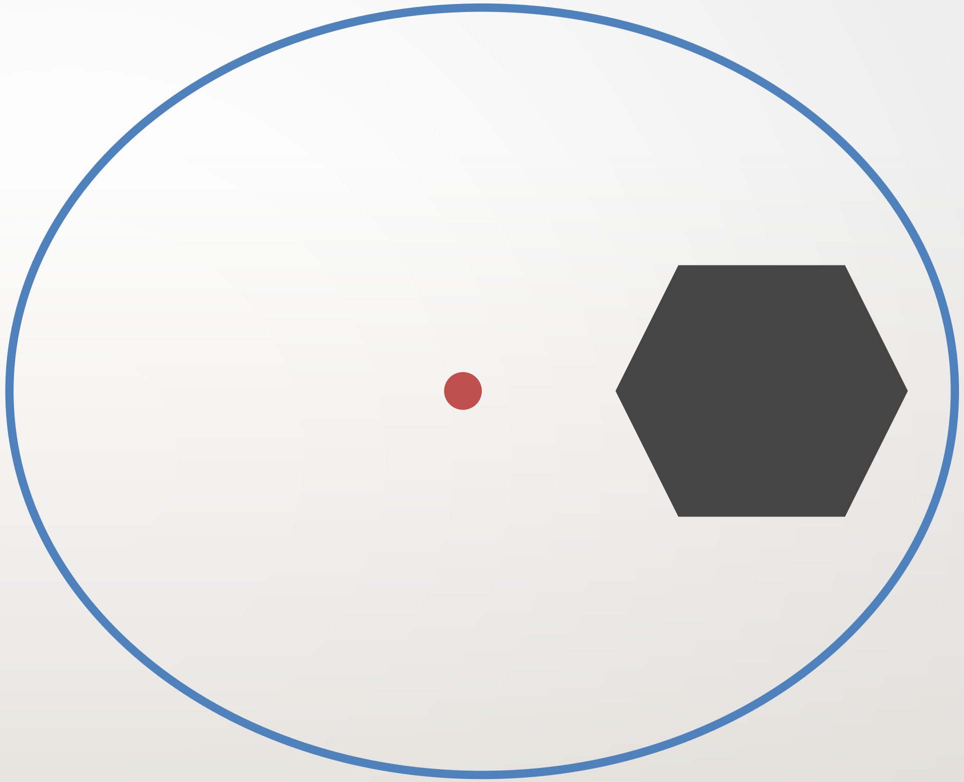
Smallest ellipsoid containing "truncated" ellipsoid

Feasibility via Ellipsoid Method



Separating hyperplane for centroid

Feasibility via Ellipsoid Method



Centroid of ellipsoid

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Algorithm of Ellipsoid Method

$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$ s.t. $A\,\mathbf{x} \leq \mathbf{b}$ or to $x \in \mathcal{K}$.

Ellipsoid $\equiv$ squashed sphere

Start with ball containing (polytope) $\mathcal{K}$.

y_i = center of current ellipsoid.

If $y_i \in \mathcal{K}$, use objective function cut $c \cdot x \geq c \cdot y_i$ to chop off $\mathcal{K}$, half-ellipsoid.

If $y_i \notin \mathcal{K}$, use separating hyperplane to chop off infeasible half-ellipsoid.

New ellipsoid = min. volume ellipsoid containing "unchopped" half-ellipsoid.

$x_1, x_2, \ldots, x_k$: points lying in $\mathcal{P}$. $c \cdot x_k$ is a close to optimal value.

ELLIPSOID METHOD

- The ellipsoid method has polynomial time complexity for solving certain convex optimization problems. However, it is not always the most efficient algorithm for practical use, and other methods like interior-point methods are often preferred for large-scale problems.
- The ellipsoid method has been foundational in the development of interior-point methods and other optimization algorithms. It is often used in theoretical discussions and proofs related to convex optimization.
- The ellipsoid method is mainly used for educational purposes and theoretical analysis, and in practical applications, other optimization algorithms are often more efficient and widely adopted.

SELF-ASSESSMENT QUESTIONS

1. Ellipsoid Method is the first algorithm to solve linear programing problem in:

(a) Exponential time
(b) Polynomial time
(c) Algorithmic time
(d) Factorial time

2. Which of the following is related to Ellipsoid method:

(a) Hypercircle
(b) Hyperplane
(c) Strait line
(d) Hyper ellips

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TERMINAL QUESTIONS

1. Describe Ellipsoid method.

2. List out the components in Ellipsoid method.

3. What Ellipsoid method is used for?

4. Construct an example to examine the Ellipsoid method.

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REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. Operations Research by S.D.Sharma, Kedar Nath Ram Nath, Meerut, 2019.

2. Frederick S. Hillier, Linear and Nonlinear Programming, Series Editor, Stanford University.

3. Mathematical Programming for MIT OpenCourseware.

Sites and Web links:

1. https://www.cs.princeton.edu/courses/archive/fall14/cos521/lecnotes/lec18.pdf

2. https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-251j-introduction-to-mathematical-programming-fall-2009/

3. https://nptel.ac.in/courses/111/105/111105039/

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