TUTORAIL- 4

1. 1.A company manufactures electronic devices, and the lifetime of a product follows an exponential distribution with an average lifetime of 5 years. The company is interested in understanding the probability of a product lasting a certain duration. Analyze the continuous probability distribution of product lifetimes and calculate the probability of a product lasting at least 8 years. Solution:

To analyze the probability of a product lasting at least 8 years given that the lifetime follows an **exponential distribution**, let's proceed step by step.

1. Exponential Distribution Formula:

The probability density function (PDF) of an exponential distribution is:

$$f(t;\lambda) = \lambda e^{-\lambda t}, \quad t \geq 0$$

where:

- λ is the rate parameter ($\lambda = 1/\text{mean lifetime}$),
- t is the lifetime of the product.

The cumulative distribution function (CDF) is:

$$F(t;\lambda) = 1 - e^{-\lambda t}, \quad t \geq 0$$

Thus, the probability of a product lasting at least t years is:

$$P(T \geq t) = 1 - F(t;\lambda) = e^{-\lambda t}.$$

2. Mean Lifetime and Rate Parameter:

The average lifetime is given as 5 years. The rate parameter (λ) is:

$$\lambda = \frac{1}{\text{mean lifetime}} = \frac{1}{5} = 0.2.$$

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3. Calculate the Probability:

To find the probability that the product lasts at least 8 years:

$$P(T \geq 8) = e^{-\lambda \cdot 8}.$$

Substitute $\lambda = 0.2$:

$$P(T \geq 8) = e^{-0.2 \cdot 8} = e^{-1.6}.$$

4. Compute the Value:

Using $e^{-1.6}$:

$$e^{-1.6} \approx 0.2019.$$

5. Final Answer:

The probability of a product lasting at least 8 years is approximately:

$$P(T \geq 8) \approx 0.2019 \text{ or } 20.19\%.$$

2. 2. The Heights of 1000 students are normally distributed with a mean of 174.5cm and a standard deviation of 6.9 cm. assuming that the heights are recorded to the nearest half-cm, how many of these students would you expect to have heights a) less than 160.0 cms? b) between 171.5 and 182.0 cms inclusive? c) equal to 175.0cm? d) greater than or equal to 188.0cms.

Solution:

Given:

- $\mu = 174.5, \sigma = 6.9, N = 1000$

a) Less than 160.0 cm:

$Z = -2.10, P(Z < -2.10) = 0.0179$

Expected students: $0.0179 \times 1000 = 18.$

b) Between 171.5 and 182.0 cm:

$Z = -0.43$ to $Z = 1.09, P = 0.5285$

Expected students: $0.5285 \times 1000 = 529.$

c) Equal to 175.0 cm:

Probability: $P = 0.0291$

Expected students: $0.0291 \times 1000 = 29.$

d) Greater than or equal to 188.0 cm:

$Z = 1.96, P(Z \geq 1.96) = 1 - 0.9750 = 0.0250$

Expected students: $0.0250 \times 1000 = 25.$

3. . Scores on a particular test are normally distributed with a mean of 30 and standard deviation of
What is the probability of anyone scoring less than 40?

SOL:-

1. Test Scores Problem

Given:

- $\mu = 30, \sigma = 4, X = 40$

Find $P(X < 40)$:

$$Z = \frac{40 - 30}{4} = 2.5$$

From the standard normal table, $P(Z < 2.5) = 0.9938$.

Answer: $P(X < 40) = 0.9938$ or **99.38%.**

: 4. Butterfly-style valves used in heating and ventilating industries have high flow coefficient. Flow coefficient can be modelled by a normal distribution with mean 496 Cv and standard deviation 25Cv. Find the probability that a valve will have a flow coefficient of a) atleast 450Cv b). between 445.5 and 522Cv

SOL:-

2. Valve Flow Coefficient Problem

Given:

- $\mu = 496, \sigma = 25$

a) At least 450 Cv

Find $P(X \geq 450)$:

$$Z = \frac{450 - 496}{25} = -1.84$$

From the standard normal table, $P(Z < -1.84) = 0.0332$.

$$P(X \geq 450) = 1 - P(Z < -1.84) = 1 - 0.0332 = 0.9668$$

Answer: $P(X \geq 450) = 0.9668$ or **96.68%**.

b) Between 445.5 and 522 Cv

Find $P(445.5 \leq X \leq 522)$:

1. $Z_1 = \frac{445.5-496}{25} = -2.02$, $P(Z_1 < -2.02) = 0.0217$
2. $Z_2 = \frac{522-496}{25} = 1.04$, $P(Z_2 < 1.04) = 0.8508$

$$P(445.5 \leq X \leq 522) = P(Z_2) - P(Z_1) = 0.8508 - 0.0217 = 0.8291$$

Answer: $P(445.5 \leq X \leq 522) = 0.8291$ or **82.91%**.

5. The daily amount of coffee, in liters, is dispensed by a machine located in an airport lobby is a
random

variable 'X' having a continuous uniform distribution with A=7 and B=10. Obtain the probability that
on

a given day the amount of coffee dispensed by this machine will be

i) at most 8.8 liters

ii) more than 7.4 liters but less than 9.5 liters

iii) At least 8.5 liters

SOL:-

The daily amount of coffee dispensed follows a **continuous uniform distribution** with:

- Minimum (A) = 7 liters
- Maximum (B) = 10 liters

The PDF of a uniform distribution is:

$$f(x) = \frac{1}{B - A}, \quad A \leq x \leq B$$

$$f(x) = \frac{1}{10 - 7} = \frac{1}{3}, \quad 7 \leq x \leq 10$$

The CDF is:

$$F(x) = \frac{x - A}{B - A}, \quad A \leq x \leq B$$

i) Probability that $X \leq 8.8$:

$$P(X \leq 8.8) = F(8.8) = \frac{8.8 - 7}{10 - 7} = \frac{1.8}{3} = 0.6$$

Answer: $P(X \leq 8.8) = 0.6$ or **60%**.

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ii) Probability that $7.4 < X < 9.5$:

$$P(7.4 < X < 9.5) = F(9.5) - F(7.4)$$

1. $F(9.5) = \frac{9.5-7}{3} = \frac{2.5}{3} = 0.8333$
2. $F(7.4) = \frac{7.4-7}{3} = \frac{0.4}{3} = 0.1333$

$$P(7.4 < X < 9.5) = 0.8333 - 0.1333 = 0.7$$

Answer: $P(7.4 < X < 9.5) = 0.7$ or **70%**.

iii) Probability that $X \geq 8.5$:

$$P(X \geq 8.5) = 1 - P(X < 8.5) = 1 - F(8.5)$$

1. $F(8.5) = \frac{8.5-7}{3} = \frac{1.5}{3} = 0.5$

$$P(X \geq 8.5) = 1 - 0.5 = 0.5$$

Answer: $P(X \geq 8.5) = 0.5$ or **50%**.

6. In an industrial process the diameter of a ball bearing is an important component part. The buyer sets specifications on the diameter to be 3.0±0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean μ=3.0 and standard deviation σ =0.005. On the average, what % of manufactured ball bearings will be scraped?

Solution:

The diameter of ball bearings follows a **normal distribution** with:

- Mean (μ) = $3.0\,\text{cm}$
- Standard deviation (σ) = $0.005\,\text{cm}$

Specifications: $3.0 \pm 0.01\,\text{cm}$ ($2.99 \leq X \leq 3.01$)

The percentage of ball bearings **scraped** corresponds to the percentage of ball bearings **outside this range**.

Step 1: Standardize the limits using the Z-score formula

$$Z = \frac{X - \mu}{\sigma}$$

1. For $X = 2.99$:

$$Z = \frac{2.99 - 3.0}{0.005} = -2.0$$

2. For $X = 3.01$:

$$Z = \frac{3.01 - 3.0}{0.005} = 2.0$$

Step 2: Find the cumulative probabilities

Using standard normal distribution tables:

- $P(Z < -2.0) = 0.0228$
- $P(Z < 2.0) = 0.9772$

The probability within the range $2.99 \leq X \leq 3.01$:

$$P(-2.0 \leq Z \leq 2.0) = P(Z < 2.0) - P(Z < -2.0)$$

$$P(-2.0 \leq Z \leq 2.0) = 0.9772 - 0.0228 = 0.9544$$

Step 3: Probability of being scraped

The probability of being **outside** the range:

$$P(\text{scraped}) = 1 - P(-2.0 \leq Z \leq 2.0)$$

$$P(\text{scraped}) = 1 - 0.9544 = 0.0456$$

Percentage scraped:

$$P(\text{scraped}) \times 100 = 0.0456 \times 100 = 4.56\%$$

7. In a test on 2000 electric bulbs it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2150 hours (b) less than 1950 hours (c) more than 1920 hours and but less than 2160 hours. (d) exactly 1960 hours.

Solution:

The life of electric bulbs follows a **normal distribution** with:

- Mean (μ) = 2040 hours
- Standard deviation (σ) = 60 hours
- Total bulbs (N) = 2000

(a) More than 2150 hours

1. Calculate the Z-score:

$$Z = \frac{2150 - 2040}{60} = \frac{110}{60} = 1.8333$$

2. From the standard normal table, $P(Z < 1.83) = 0.9664$.

$$P(Z > 1.83) = 1 - P(Z < 1.83) = 1 - 0.9664 = 0.0336$$

3. Expected number of bulbs:

$$N \times P(Z > 1.83) = 2000 \times 0.0336 = 67.2 \approx 67$$

(b) Less than 1950 hours

1. Calculate the Z-score:

$$Z = \frac{1950 - 2040}{60} = \frac{-90}{60} = -1.5$$

2. From the standard normal table, $P(Z < -1.5) = 0.0668$.
3. Expected number of bulbs:

$$N \times P(Z < -1.5) = 2000 \times 0.0668 = 133.6 \approx 134$$

(c) More than 1920 hours but less than 2160 hours

1. Calculate the Z-scores:
   - For 1920:

   $$Z = \frac{1920 - 2040}{60} = \frac{-120}{60} = -2.0$$

   From the table, $P(Z < -2.0) = 0.0228$.
   - For 2160:

   $$Z = \frac{2160 - 2040}{60} = \frac{120}{60} = 2.0$$

   From the table, $P(Z < 2.0) = 0.9772$.
2. Probability within the range:

$$P(-2.0 \leq Z \leq 2.0) = P(Z < 2.0) - P(Z < -2.0) = 0.9772 - 0.0228 = 0.9544$$

3. Expected number of bulbs:

$$N \times 0.9544 = 2000 \times 0.9544 = 1908.8 \approx 1909$$

(d) Exactly 1960 hours

For a continuous probability distribution, the probability of a variable taking an **exact value** is **0**.

Answer: $P(X = 1960) = 0.$

8. The life of a certain type of device has an advertised failures rate of 0.01 per hour. The failure rate is

constant, and the exponential distribution applies.

a) What is the mean time to failure?

b) What is the probability that 200 hours will pass before a failure is observed?

Solution:

The life of the device follows an **exponential distribution** with a constant failure rate (λ):

$$\lambda = 0.01 \, \text{failures/hour.}$$

The PDF of the exponential distribution is:

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

The CDF is:

$$P(T \leq t) = 1 - e^{-\lambda t}.$$

(a) Mean time to failure (MTTF):

The mean of an exponential distribution is given by:

$$\text{Mean (MTTF)} = \frac{1}{\lambda}.$$

$$\text{MTTF} = \frac{1}{0.01} = 100 \, \text{hours.}$$

(b) Probability that 200 hours will pass before a failure is observed:

$$P(T > 200) = 1 - P(T \leq 200) = e^{-\lambda t}.$$

Substitute $\lambda = 0.01$ and $t = 200$:

$$P(T > 200) = e^{-0.01 \times 200} = e^{-2}.$$

Using $e^{-2} \approx 0.1353$:

$$P(T > 200) = 0.1353.$$

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9. Suppose that a study of a certain computer system reveals that the response time, in seconds, has an

exponential distribution with a mean of 3 seconds.

a) What is the probability that response time exceeds 5 seconds?

b) What is the probability that the response time is less than 10 seconds?

c) What are the mean and variance of response time?

Solution:

The response time follows an **exponential distribution** with:

$$\text{Mean} = 3\,\text{seconds}, \quad \lambda = \frac{1}{\text{Mean}} = \frac{1}{3} \approx 0.3333.$$

(a) Probability that response time exceeds 5 seconds:

$$P(T > 5) = 1 - P(T \leq 5) = e^{-\lambda t}.$$

Substitute $\lambda = \frac{1}{3}$ and $t = 5$:

$$P(T > 5) = e^{-\frac{1}{3}\times 5} = e^{-\frac{5}{3}}.$$

Using $e^{-\frac{5}{3}} \approx 0.1889$:

$$P(T > 5) = 0.1889.$$

(b) Probability that response time is less than 10 seconds:

$$P(T \leq 10) = 1 - e^{-\lambda t}.$$

Substitute $\lambda = \frac{1}{3}$ and $t = 10$:

$$P(T \leq 10) = 1 - e^{-\frac{1}{3}\times 10} = 1 - e^{-\frac{10}{3}}.$$

Using $e^{-\frac{10}{3}} \approx 0.0498$:

$$P(T \leq 10) = 1 - 0.0498 = 0.9502.$$

(c) Mean and variance of response time:

For an exponential distribution:

- Mean = $\frac{1}{\lambda} = 3\,\text{seconds}.$
- Variance = $\frac{1}{\lambda^2} = 3^2 = 9\,\text{seconds}^2.$

VIVA:

1. Define Memory less property of exponential.

A.

The **memoryless property** of the exponential distribution states that the probability of an event occurring in the future is independent of how much time has already elapsed. Formally:

For a random variable T with an exponential distribution and rate parameter λ:

$$P(T > t + s \mid T > t) = P(T > s).$$

2. Conditions for the approximation of Binomial to Normal

A.

The binomial distribution $B(n, p)$ can be approximated by a normal distribution $N(\mu, \sigma^2)$ when the following conditions are met:

1. **Large n:**
   The number of trials n should be sufficiently large.
2. **$np \geq 5$ and $n(1-p) \geq 5$:**
   The expected number of successes (np) and failures ($n(1-p)$) should both be at least 5 to ensure the normal approximation is accurate.
3. **Mean and variance are defined:**
   - Mean $\mu = np$
   - Variance $\sigma^2 = np(1-p)$.

Additionally, a **continuity correction** is applied to improve accuracy:
For $P(X \leq k)$, use $P(X \leq k + 0.5)$ with the normal approximation.

3. Define Joint random variables

A.

Joint random variables describe the relationship between two or more random variables considered together.

For two random variables X and Y:

- **Joint Probability Distribution:**
  Describes the probability of X and Y taking specific values simultaneously, denoted by $P(X = x, Y = y)$ for discrete variables or the joint PDF $f_{X,Y}(x, y)$ for continuous variables.
- **Joint Cumulative Distribution Function (CDF):**

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y).$$