BEALE'S METHOD

- Beale's Method In Beale's method we solve Quadratic Programming problem and in this method we does not use the Kuhn-Tucker condition. At each iteration the objective function is expressed in terms of non basic variables only. Let the QPP be given in the form

$$Maximize\ f(X) = CX + \frac{1}{2}X^TQX$$

subject to $AX = b,\ X \geq 0.$

Where

$$X = (x_1, x_2, ..., x_{n+m})$$

$$c \quad is \quad 1 \times n$$

$$A \quad is \quad m \times (n+m)$$

and Q *is symmetric and every QPP with linear constraints.*

Algorithm

Step 1

First express the given QPP with Linear constraints in the above form by introducing slack and surplus variable.

Step 2

Now select arbitrary m variables as basic and remaining as non-basic.

Now the constraints equation $AX = b$ can be written as

$$BX_B + RX_{NB} = b \Rightarrow X_B = B^{-1}b - B^{-1}RX_{NB}$$

where

X_B-basic vector $\quad X_{NB}$-non-basic vector

and the matrix A is partitioned to submatrices B and R corresponding to X_B and X_{NB} respectively.

Step 3

Express the basis X_B in terms of non-basic X_{NB} only, using the given additional constraint equations, if any.

Step 4

Express the objective function $f(x)$ in terms of X_{NB} only using the given and additional constraint , if any . Thus we observe that by increasing the value of any of the non-basic variables, the value of the objective function can be improved. Now the constraints on the new problem become

$$B^{-1}RX_NB \leqslant B^{-1}b \qquad (since \;\; X_B \geqslant 0)$$

Thus, any component of X_{NB} can increase only until $\dfrac{\partial f}{\partial x_{NB}}$ becomes zero or one or more components of X_B are reduced to zero.

Step 5

Now we have $m+1$ non-zero variables and $m+1$ constraints which is a basic solution to the extended set of constraints.

Step 6

We go on repeating the above procedure until no further improvement in the objective function may be obtain by increasing one of the non-basic variables.

Quadratic Programming problem (QPP)

The general form of the QPP is

$$f(x) = cx + \frac{1}{2}x^T Q\, x$$

s.t. $Ax \le b$ and $x \ge 0$

where Q is a symmetric matrix and b, c are the real vectors.

Beale's method

Beale developed a technique of solving the QPP that Does Not Use the Kuhn-Tucker conditions.

- His technique involves partitioning the variables into basic (x_B) and non-basic variables (x_{NB}).
- Express x_B & objective function f in terms of x_{NB} variables.

Example 1: Use Beale's method to solve QPP:

Maximize $f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

s.t. $x_1 + 2x_2 \leq 4$;

$x_1, x_2 \geq 0$

Solution: Write in Standard form:

Maximize $f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

s.t. $x_1 + 2x_2 + s_1 = 4$;

$x_1, x_2, s_1 \geq 0$

Here, $m = 1$; $n = 3$.

x_B x_{NB}

$m=1$ $n-m$

$$\text{Max } f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$$\text{s.t. } x_1 + 2x_2 + s_1 = 4;$$

$$\boxed{x_1, x_2, s_1} \geq 0$$

Let us choose arbitrary s_1 as initial basic variable,

i.e., $x_B = (s_1)$; $x_{NB} = (x_1, x_2)$.

Working Rule

Express x_B & f in terms of x_{NB} as

To find the entering variable:

To find the leaving variable:

$$\text{Max } f(x) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$$\text{s.t. } x_1 + 2x_2 + s_1 = 4;$$

$$x_1, x_2, s_1 \geq 0$$

$$x_B = (s_1);\ x_{NB} = (x_1, x_2)$$

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Iteration 1:

Express x_B & f in terms of x_{NB} as

$$s_1 = 4 - x_1 - 2x_2$$

and $$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

To find the entering variable:

Partial derivative of f w.r.t. $x_{NB} = (x_1, x_2)$

$$\frac{\partial f}{\partial x_1} = -2x_1 + 4; \quad \frac{\partial f}{\partial x_2} = -6x_2 + 6$$

At the point $x_{NB} = (x_1, x_2) = 0$,

$\frac{\partial f}{\partial x_1} = 4$ & $\frac{\partial f}{\partial x_2} = 6$ Since $\frac{\partial f}{\partial x_2}$ is the most positive

so x_2 is ENTERING variables in the basis.

$$x_B = (s_1)\ ;\ x_{NB} = (x_1, \cancel{x_2})$$

$$x_B = (s_1, x_2)\ ;\ x_{NB} = (x_1, u_1)$$

To find the leaving variable:

$$\text{Min}\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\gamma_{20}}{|\gamma_{22}|}\right\} = \min\left\{\frac{4}{|-2|}, \frac{6}{|-6|}\right\}$$

$$= 1 \text{ corresponding to } \frac{\gamma_{20}}{|\gamma_{22}|}.$$

Thus, introduce a free non-basic variable

$$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2} \quad \text{i.e., } \boxed{u_1 = 3 - 3x_2}$$

Iteration 2:

$x_2 = \frac{3-u_1}{3}$

$s_1 = 4 - x_1 - \frac{2}{3}(3 - u_1)$

$= 2 - x_1 + \frac{2}{3}u_1$

And $f = 4x_1 + 6\left(\frac{3-u_1}{3}\right) - x_1^2 - 3\left(\frac{3-u_1}{3}\right)^2$

$= 3 - x_1^2 - \frac{u_1^2}{3} + 4x_1$ ✓

$s_1 = 4 - x_1 - 2x_2$

$f = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$

$u_1 = 3 - 3x_2$

$x_B = (s_1, x_2);\ x_{NB} = (x_1, u_1)$

Iteration 2:

$$x_B = (s_1, x_2)\,;\; x_{NB} = (x_1, u_1)$$

$x_2 = \frac{3-u_1}{3}$; $s_1 = 2 - x_1 + \frac{2}{3}u_1$

And $f = 3 - x_1^2 - \frac{u_1^2}{3} + 4x_1$

To find the entering variable:

Partial derivative of f w.r.t. x_1, u_1:

$$\frac{\partial f}{\partial x_1} = -2\,x_1 + 4 \;;\; \frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1$$

At the point $x_{NB} = (x_1, u_1) = 0$, we get

$$\frac{\partial f}{\partial x_1} = 4\,;\, \frac{\partial f}{\partial u_1} = 0$$

Thus, ENTERING variables is $\boxed{x_1.}$

To find the leaving variable:

$$Min\left\{\frac{\alpha_{10}}{|\alpha_{11}|}, \frac{\alpha_{20}}{|\alpha_{21}|}, \frac{\gamma_{10}}{|\gamma_{11}|}\right\} = Min\left\{\frac{2}{|-1|}, \frac{1}{|0|}, \frac{4}{|-2|}\right\}$$

$= 2$ corresponding to $\frac{\gamma_{10}}{|\gamma_{11}|}$.

Thus, define a free non-basic variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_1}$$

i.e., $\boxed{u_2 = -x_1 + 2}$

Iteration 3:

Express x_B & f in terms of x_{NB} as

$$x_1 = 2 - u_2\ ; \quad x_2 = 1 - \frac{1}{3}u_1\ ;$$
$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

$x_B = (s_1, x_2, x_1);\ \ x_{NB} = (u_1, u_2)$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1\ ;\ \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0\ ;\ \frac{\partial f}{\partial u_2} = 0$$

Thus, current x_B is the optimal.

Answer:

$x_1 = 2\ ; x_2 = 1\ ; s_1 = 0$

& Max $f = 7$

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Iteration 3:

Express x_B & f in terms of x_{NB} as

$x_B = (s_1, x_2, x_1)$; $x_{NB} = (u_1, u_2)$

$$x_1 = 2 - u_2\ ; \quad x_2 = 1 - \frac{1}{3}u_1\ ;$$

$$s_1 = u_2 + \frac{2}{3}u_1 \quad f = 7 - u_2^2 - \frac{u_1^2}{3}$$

To find the entering variable:

$$\frac{\partial f}{\partial u_1} = -\frac{2}{3}u_1\ ; \quad \frac{\partial f}{\partial u_2} = -2u_2$$

At the point $x_{NB} = (u_1, u_2) = 0$, we get

$$\frac{\partial f}{\partial u_1} = 0 \quad ; \quad \frac{\partial f}{\partial u_2} = 0$$

≤ 0

Thus, current x_B is the optimal.

Answer:

$x_1 = 2\ ; x_2 = 1\ ; s_1 = 0$

& Max $f = 7$

Solution:

Step:1

$$Max.\ Z = 4x_1 + 6x_2 - 2{x_1}^2 - 2x_1x_2 - 2{x_2}^2 \tag{1}$$

subject to

$$x_1 + 2x_2 + x_3 = 2 \tag{2}$$

and $x_1, x_2, x_3 \geq 0$

taking $X_B = (x_1); X_{NB} = \binom{x_2}{x_3}$

and $x_1 = 2 - 2x_2 - x_3$ (3)

Step:2

put (3) in (1), we get

$$Max.\ f(x_2, x_3) = 4(2 - 2x_2 - x_3) + 6x_2 - 2(2 - 2x_2 - x_3)^2 - 2(2 - 2x_2 - x_3)x_2 - 2{x_2}^2$$

$$\frac{\partial f}{\partial x_2} = -2 + 8(2 - 2x_2 - x_3) + 8x_2 - 4x_2 - 2(2 - x_3)$$

$$\frac{\partial f}{\partial x_3} = -4 + 4(2 - 2x_2 - x_3) + 2x_2$$

Now $\displaystyle \frac{\partial f}{\partial x_2}_{(0,0)} = 10$

$$\frac{\partial f}{\partial x_3}_{(0,0)} = 4$$

Here '+ve' value of $\frac{\partial f}{\partial x_i}$ indicates that the objective function will increase if x_i increased . Similarly '-ve' value of $\frac{\partial f}{\partial x_i}$ indicates that the objective function will decrease if x_i is decrease . Thus, increase in x_2 will give better improvement in the objective function.

Step:3

$f(x)$ will increase if x_2 increased .

If x_2 is increased to a value greater then 1, x_1 will be negative.

Since $x_1 = 2 - 2x_2 - x_3$

$x_3 = 0; \dfrac{\partial f}{\partial x_2} = 0$

$$\Rightarrow 10 - 12x_2 = 0$$

$$\Rightarrow x_2 = \frac{5}{6}$$

$Min. (1, \dfrac{5}{6}) = \dfrac{5}{6}$

The new basic variable is x_2.

Second Iteration:

Step:1

let $X_B = (x_2)$, $X_{NB} = \binom{x_1}{x_3}$

$$x_2 = 1 - \frac{1}{2}(x_1 + x_3)$$

Step:2

Substitute (4) in (1)

$$Max.\ f(x_1, x_3) = 4x_1 + 6(1 - \frac{1}{2}(x_1 + x_3)) - 2{x_1}^2 - 2x_1(1 - \frac{1}{2}(x_1 + x_3)) - 2(1 - \frac{1}{2}(x_1 + x_3))^2$$

$$\frac{\partial f}{\partial x_1} = 1 - 3x_1, \qquad \frac{\partial f}{\partial x_2} = -1 - x_3$$

$$\frac{\partial f}{\partial x_1}_{(0,0)} = 1$$

$$\frac{\partial f}{\partial x_3}_{(0,0)} = -1$$

This indicates that x_1 can be introduce to increased objective function.

Step:3

- $$x_2 = 1 - \frac{1}{2}(x_1 + x_3) \ \ and \ \ x_3 = 0$$

If x_1 is increased to a value greater then 2, x_2 will become negative.

$$\frac{\partial f}{\partial x_1} = 0$$
$$\Rightarrow 1 - 3x_1 = 0$$
$$\Rightarrow x_1 = \frac{1}{3}$$
$$Min.\ (2, \frac{1}{3}) = \frac{1}{3}$$
$$\text{Therefore } x_1 = \frac{1}{3}$$
$$\text{Hence } x_1 = \frac{1}{3}, \qquad x_2 = \frac{5}{6}, \qquad x_3 = 0$$
$$\text{and} \qquad Max.\ f(x) = \frac{25}{6} \quad \square$$