Department of AI & DS
CSE and CS&IT

COURSE NAME: PROBABILITY, STATISTICS AND QUEUING THEORY

COURSE CODE: 23MT2005

Topic

Queuing Model-3 and 4

Session – 23

AIM OF THE SESSION

To familiarize students with the basic concept of Queuing Model 3 and 4

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Define Model 3 and model 3
2. Describe the Characteristics of Queuing model 3
3. Describe the performance measures of Model 4

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Differentiate between the models 1 and 3 and also 2 and 4
2. Summarize the performance measures of Model 3 and Model 4.

Model 3-M/M/S/∞/FCFS

In this model arrivals follow a Poisson process; the service times are i.i.d. (independent and identically distributed) and follow an exponential distribution. There are *s* servers ($s \geq 1$). In the M/M/s model there is no balking or reneging, so all arrivals eventually receive service.

This is a more realistic system. There are 's' numbers of counters arranged in parallel and a customer can go to any one of the free counters for his service, where the service time in each counter is identical and follows the same exponential distribution

(i) when $n \leq s$, all the customers may be served simultaneously and there is no queue.

$$\mu_n = n\mu,\ n = 0,1,2,\ldots\ldots, s\ .$$

(ii) If $n \geq s$, all servers are busy, maximum no. of customers waiting in the queue is (n-s)

$$\mu_n = s\,\mu$$

The steady state difference questions are given by

Model 3-M/M/S/∞/FCFS

$$P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ 1 \le n \le s$$

$$\& \ P_n = P_{s+(n-s)} = \frac{1}{s^{n-s}} \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^n P_0,\ n \ge s.$$

Using the normalizing condition, $\sum_{n=0}^{\infty} P_n = 1$.

$$\Rightarrow P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\frac{\rho^s}{1-\rho}\right)\right]^{-1},\ \text{where } \rho = \frac{\lambda}{s\mu}$$

$$\therefore P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, & n = 0, 1, 2, \ldots, s-1 \\ \frac{1}{s!}\frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, & n = s, s+1, \ldots \end{cases}$$

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n = \frac{\rho (s\rho)^s}{s!(1-p)^2} P_0 = \frac{\rho}{(1-\rho)^2} P_s$$

$$L_s = L_q + \frac{\lambda}{\mu},\quad W_q = \frac{L_q}{\lambda} = P_s \frac{1}{s\mu(1-\rho)^2},\ W_s = W_q + \frac{1}{\mu}$$

Model 3-M/M/S/∞/FCFS

The mean number of waiting individuals who actually wait

$$(L / L > 0) = \frac{\sum_{n=s+1}^{\infty}(n-s)P_n}{\sum_{n=s+1}^{\infty}P_n} = \frac{1}{(1-\rho)}$$

The mean waiting time in queue for those who actually wait is given by

$$(W / W > 0) = \frac{1}{\rho - 1}.\frac{1}{s\mu} = \frac{1}{s\mu - \lambda}, \quad \rho = \frac{\lambda}{s\,\mu}$$

Probability that an arriving customer finds a server free $= \sum_{n=0}^{s-1} P_n = P_0 \sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n$

Model 3-M/M/S/∞/FCFS

Example 1. City hospital's eye clinic offers free vision tests every Wednesday evening. There are three ophthalmologists on duty. A test takes, on the average, 20 min. And the actual time is found to be approximately exponentially distributed around this average. Clients arrive according to a Poisson process with a mean of 6 /hr, and patients are taken on a first-come, first-served basis. The hospital planners are interested in knowing: (i) what is the average number of people waiting. (ii) the average amount of time a patient spends at the clinic and (iii) the average percentage ideal time of each of the doctors.

We have $\lambda=6/\text{hr}$, $\mu=\frac{1}{20}/\text{min}=3/\text{hr}$ and $s=3$, $\rho=\frac{\lambda}{s\mu}=\frac{2}{3}$

$$P_0=\left[\sum_{n=0}^{s-1}\frac{(\rho s)^n}{n!}+\sum_{n=s}^{\infty}\frac{(\rho s)^n}{s^{n-s}s!}\right]^{-1}$$

$$=\left[1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{3.3!}+\frac{2^5}{3^2\ 3!}+\ldots\right]^{-1}$$

$$=\left[5+\frac{2^3}{3!}\ \frac{1}{1-\frac{2}{3}}\right]^{-1}=\frac{1}{9}$$

Model 3-M/M/S/∞/FCFS

Avverage number of people waiting in the queue

$$L_q = \frac{\left(\frac{\lambda}{\mu}\right)^s \rho P_0}{s!\,(1-\rho)^2} = \frac{2^3\left(\frac{2}{3}\right)}{3!\left(1-\frac{2}{3}\right)^2}\left(\frac{1}{9}\right) = \frac{8}{9}$$

Average amount of time a patient spends at the clinic

$$= W_s = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{8/9}{6} + \frac{1}{3} = 28.9 \text{ min.}$$

Long term average fraction of idle time for any server in an M / M / s is equal to

$$1 - \rho = 1 - \frac{2}{3} = \frac{1}{3}.$$

Therefore, each physician is idle $\frac{1}{3}$ of the time, given that 3 servers on duty, two of them will be busy at any time (on average), since $\frac{\lambda}{\mu} = 2$.

Furthermore, the fraction of time that there is at least one idle doctor can be computed as

$$P_0 + P_1 + P_2 = P(W_q = 0) = 5/9.$$

i. e. 55.5% of time there is at least one idle doctor.

Model-4 M/M/S/N/FCFS MODELS

In the parallel server model with s servers, there is a limit N placed on the number allowed in the system.

$$P_n = \begin{cases} \dfrac{(s\rho)^n P_0}{n!}, & 0 \le n \le s \\ \dfrac{s^s \rho^n P_0}{s!}, & s \le n \le N \\ 0, & n > N \end{cases}$$

Where $\rho = \dfrac{\lambda}{s\mu}$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \sum_{n=s}^{N} \frac{(s\rho)^n}{s!\, s^{n-s}} \right]^{-1}$$

$$= \begin{cases} \left[\displaystyle\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} (1-\rho^{N-s+1}) \right]^{-1}, & \rho = \dfrac{\lambda}{s\mu} \ne 1 \\ \left[\displaystyle\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!} (N-s+1) \right]^{-1}, & \rho = \dfrac{\lambda}{s\mu} = 1 \end{cases}$$

$$L_q = \sum_{n=s+1}^{N} (n-s) P_n = \sum_{n=s}^{N} \frac{(n-s)(s\rho)^n}{s!\, s^{n-s}} P_0$$

$$= \frac{(s\rho)^s P_0 \rho}{s!(1-\rho)^2} \left[(1-\rho^{N-s+1}) - (1-\rho)(N-s+1)\rho^{N-s} \right] \quad \text{where } P_0 = \left[\sum_{n=0}^{s} \frac{(s\rho)^n}{n!} \right]^{-1}$$

Model-4 M/M/S/N/FCFS MODELS

Average number in the service facility:

$$= L_s - L_q = \sum_{n=0}^{N} n\,P_n - \sum_{n=s}^{N} (n-s)\,P_n = s + \sum_{n=0}^{s-1} (n-s)\,P_n = s + \sum_{n=s}^{N} (n-s)\,\frac{(\rho\, s)^n}{n\,!}$$

Average waiting time in the system:

$W_s = \dfrac{L_s}{\lambda_{eff}}$, where $\lambda_{eff} = \lambda\,(1 - P_N)$ and is called the effective arrival rate,

and $W_q = W_s - \dfrac{1}{\mu} = \dfrac{L_q}{\lambda_{eff}}$, $L_s = Lq + \dfrac{\lambda_{eff}}{\mu} = Lq + \dfrac{\lambda(1 - P_N)}{\mu}$

Model 4- EXAMPLES

Example 1. Consider an automobile emission inspection station with three inspection stalls, each with room for only one car. It is reasonable to assume that cars wait in such a way that when a stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most four cars waiting (seven in the station) at one time. The arrival pattern is Poisson with a mean of one car every minute during the peak periods. The service time is exponential with mean 6 min. The chief inspector, wishes to know the average number in the system during peak periods, the average wait (including services), and the expected number per hour that cannot enter the station because of full capacity.

$\lambda = 1$, $\mu = 1/6$, $s = 3$ and $N = 7$

Thus $\rho = \dfrac{\lambda}{s\mu} = 2$

Model 4- EXAMPLES

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(\rho s)^n}{n!} + \frac{(\rho s)^s}{s!} \frac{1-\rho^{N-s+1}}{1-\rho} \right]^{-1}$$

$$= \left[\sum_{n=0}^{2} \frac{(6)^n}{n!} + \frac{(6)^3}{3!} \frac{1-2^5}{1-2} \right]^{-1} = \frac{1}{1141} = 0.00088$$

Average number in the queue (L_q) $= \dfrac{P_0 (\rho s)^s \rho}{s!(1-\rho)^2} \left[1-\rho^{N-s+1} - (1-\rho)(N-s+1)\rho^{N-s} \right]$

$$= \frac{P_0 (6^3)(2)}{3!} \left[1-2^5-(-1)(5)2^4 \right] = 3.09 \text{ cars.}$$

Average number of cars in the system

$$(L_s) = L_q + \frac{\lambda(1-P_N)}{\mu} = 3.09 + 6\left[1 - \frac{6^7}{(3)^4 (3!)(1141)} \right] = 6.06 \text{ cars}$$

Model 4- EXAMPLES

The average wait during peak periods

$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{L_s}{\lambda(1\text{-}P_N)} = \frac{L_s}{1\text{-}6^7 P_0/(3^4 3!)} = 12.3 \ \text{min.}$$

Expected number of cars that cannot enter the station

$$= 60 \times 1 \times P_N = 60P_7 = \frac{60P_0 6^7}{3^4 3!} = 30.4 \text{ cars/hour.}$$

This suggests an additional step up for inspection.

QUEUEING NETWORKS

A network consisting of several interconnected queues

Network of queues Examples

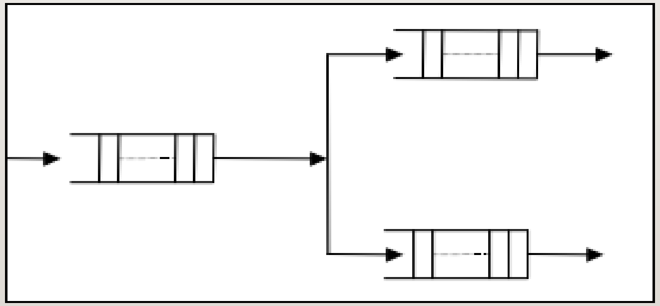
- Customers go form one queue to another in post office, bank, supermarket etc
- Data packets traverse a network moving from a queue in a router to the queue in another router

Queuing network with examples

Interconnected queues with jobs flowing from one queue to another

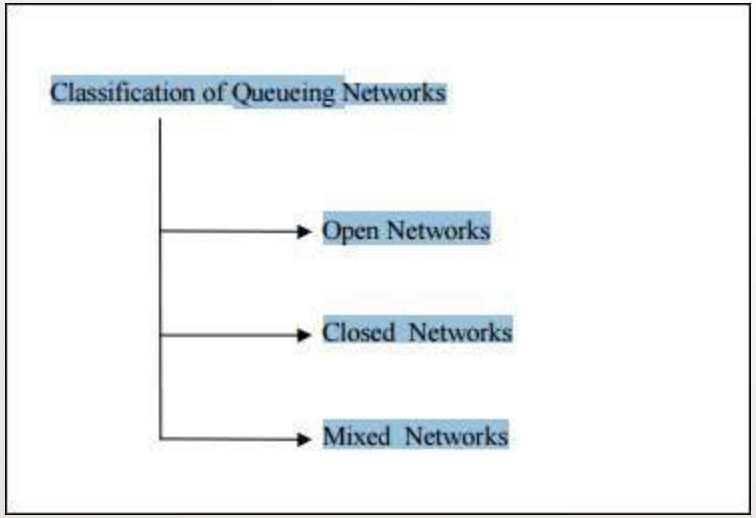
Examples: Machine shop, Communication network, Computer system

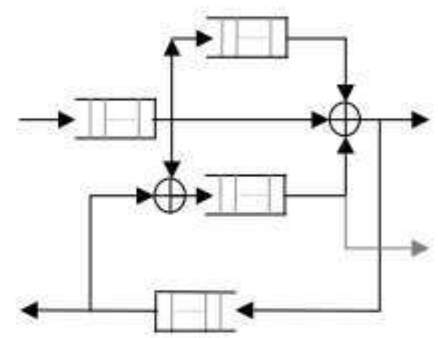
Routing jobs in a queuing network



Probabilistic Routing (Single/Multiple classes)

Class-Based deterministic Routing

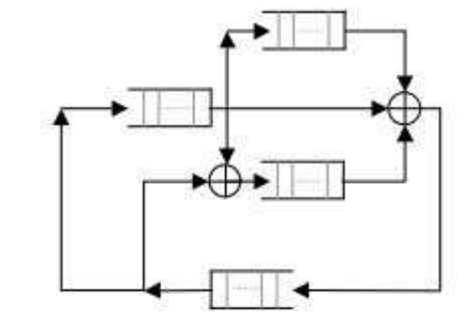




Open Queueing Network

If the network has multiple job classes then it must be open for each class of jobs.

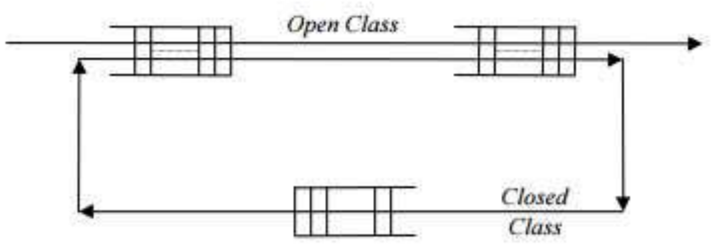
ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION



Closed Queueing Network

If the network has multiple job classes then it must be closed for each class of jobs.

Mixed Network



Network has multiple job classes and is open with respect to some classes but closed with respect to the others.

SUMMARY

In this session, Queuing models and its performance measures have discussed.

1. Performance measures of Model 3 and Model 4.

2. Difference between Model 1 and Model 3 and as well as Model 2 and Model 4.

SELF-ASSESSMENT QUESTIONS

1. One of the characteristics of the queuing system is

a) Customers feedback
b) Hungarian Mechanism
c) Customer experience
d) Service mechanism

2. In a shop customer arrive at the rate of 20 per hour and are getting served at rate of 30 per hour. The waiting time of customer in the queue is

(a) 4 minutes
(b) 4.5 minutes
(c) 5.5 minutes
(d) 6 minutes

TERMINAL QUESTIONS

5. A computer center is equipped with four identical mainframe computers. The number of users at any time is 25. Each user is capable of submitting a job through a terminal every 15 minutes. On the average, but the actual time between submissions is exponential. Arriving jobs will automatically go to the first available computer. The execution time per submission is exponential with mean 2 minutes. Compute the following:

i) Probability that a job is not executed immediately on submission.
ii) Average time until the output of a job is returned to the user.
iii) Average number of jobs waiting for execution.
iv) Average number of idle computers.

6. A car servicing station has two bays where service can be offered simultaneously. Due to space limitation, only four cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time in both the bays is exponentially distributed with μ=8 cars per day per bar. Find the average number of cars in the service station, the average number of cars waiting to be serviced and the average time a car spends in the system.

TERMINAL QUESTIONS

1. Describe in detail about the Model 3

2. List out the performance measures of Model 3 and Model 4

3. Compare the performance measures of Model 1 and Model 3

4. The Golden Muffler Shop has decided to open a second garage bay and hire a second mechanic to handle installations. Customers, who arrive at the rate of about $\lambda = 2$ per hour, will wait in a single line until 1 of the 2 mechanics is free. Each mechanic installs mufflers at the rate of about $\mu = 3$ per hour. To find out how this system compares with the old single-channel waiting-line system, we will compute several operating characteristics for the $M = 2$ channel system and compare the results with those found in single-channel.

TERMINAL QUESTIONS

7. A barber shop has two barbers and three chairs for waiting customers. Assume that customers arrive in a Poisson fashion at a rate of 5 per hour and that each barber services customers according to an exponential distribution with mean of 15 minutes. Further, if a customer arrives and there are no empty chairs in the shop he will leave. Find the steady state probabilities. What is the probability that the shop is empty? What is the expected number of customers in the shop?

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. D. Gross, J.F.Shortle, J.M. Thompson, and C.M. Harris, Fundamentals of Queueing Theory, 4th Edition, Wiley, 2008

2. William Feller, An Introduction to Probability Theory and Its Applications: Volime 1, Third Edition, 1968 by John Wiley & Sons,Inc.

Sites and Web links:

1. https://www.khanacademy.org/math/statistics-probability/significance-tests-one-sample/more-significance-testing-videos/v/small-sample-hypothesis-test

2. J.F. Shortle, J.M. Thompson, D. Gross and C.M. Harris, Fundamentals of Queueing Theory, 5th Edition, Wiley, 2018.

3. https://onlinecourses.nptel.ac.in/noc22_ma17/previ ew 3.

4. https://www.youtube.com/watch?v=Wo75G99F9f M&list=PLwdnzlV3ogoX2OHyZz3QbEYFhbqM 7x275&index=3

THANK YOU

Team – PSQT EVEN SEMESTER 2024-25