23MT2004 - Mathematical Programming

Topic:Geometric Programming: Problems with one-degree of difficulty with positive coefficients

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Geometric Programming: Problems with one-degree of difficulty with positive coefficients

AIM OF THE SESSION

- To familiarize students with the basic concept of Geometric Programming.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Introduce Posynomials, and arithmetic mean - geometric mean inequality.
2. Teach optimization of posynomial objective functions using Geometric Programming.
3. Introduce the concept of degree of difficulty.
4. Sketch the method of solving problem with one degree of difficulty.

Geometric Programming: Problems with one-degree of difficulty with positive coefficients (contd.)

LEARNING OUTCOMES

At the end of this session, the students should be able to:

1. Optimize unconstrained posynomial objective functions using Geometric Programming.
2. Solving optimization problem with one degree of difficulty.

Geometric Programming: an optimization problem

- A constrained optimization problem:

$$\begin{aligned} \text{Minimise } & f(x), x = (x_1, x_2, \ldots, x_n) \\ \text{subject to } & h_1(x) \leq \alpha_1, h_2(x) \leq \alpha_2, \ldots, h_m(x) \leq \alpha_m \\ & g(x) = 0 \end{aligned}$$

- A Geometric Programming (GP) is a type of mathematical optimization problem characterized by posynomial objective function (f) and constraint functions (h and g).

Different types of equations

Type	Function	Comments
Linear	mx +c	-
Quadratic	$cx^2 + bx + a$	Non-linear
Polynomial	$c_0 + c_1x^1 + c_2x^2 + c_3x^3 + \ldots + c_nx^n$	Non-negative integer exponents
	$c_1x_1^3x_2^5 + \ldots + c_nx_1^2x_2^1$	Multiple variables
Posynomial	$c_1x_1^{0.1}x_2^{-0.5} + \ldots + c_nx_1^{2.5}x_2^{1/3}x_3^2$	real exponents $c_j \geq 0, x_i \geq 0$

Posynomial

Type	Function	Comments
Posynomial	$c_1 x_1^{0.1} x_2^{-0.5} + \ldots + c_n x_1^{2.5} x_2^{1/3} x_3^2$	real exponents $c_j \geq 0, x_i \geq 0$

The functions can be written in a symbolic form:

$$f(x) = \sum_{i=1}^{j=n} c_i u_i(x)$$

$$c_j \geq 0, x_i \geq 0 \quad j = 1, 2, \ldots n$$

$$u_i(x) = \prod_{i=1}^{m} (x_i)^{a_{ij}} \ a_{ij} \text{ are real numbers}$$

Geometric Programming

- A Geometric Program (GP) is a type of mathematical optimization problem characterized by posynomial objective and constraint functions.
- For certain highly nonlinear and nonconvex problems, GP is an extremely efficient and reliable solution technique, that scales gracefully to large-scale problems.
- The local optimum of a GP is the global optimum; this is similar to convex programming.
- GPs have numerous applications such as
  - component sizing in IC design
  - aircraft design
  - maximum likelihood estimation for logistic regression in statistics
  - parameter tuning of positive linear systems in control theory.
- The technique is based on arithmetic mean - geometric mean inequality; hence the name Geometric Programming.

Arithmetic-Geometric mean inequality

$$\frac{u_1 + u_2 + \ldots + u_n}{n} \geq (u_1 * u_2 * \ldots * u_n)^{\frac{1}{n}}$$

$$u_i \geq 0 \ \forall i \ \ n \in N$$

$$\sum_{i=1}^{n} \frac{u_i}{n} \geq \prod_{i=1}^{n} (u_i)^{\frac{1}{n}}$$

$$\text{Let } \delta_i = \frac{1}{n} \ \forall i$$

$$\sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} (u_i)^{\delta_i}$$

Arithmetic-Geometric Mean Inequality (contd.)

A generalized version

$$\sum_{i=1}^{n} \delta_i u_i \geq \prod_{i=1}^{n} u_i^{\delta_i}$$

where $u_i, \delta_i \geq 0$, $\forall i$ and $\sum_{i=1}^{n} \delta_i = 1$.
Let $U_i = \delta_i u_i, \forall i$

$$\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} (\frac{U_i}{\delta_i})^{\delta_i}$$

Equality holds good when

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \ldots = \frac{U_n}{\delta_n}$$

GP: Optimization of Posynomial Function

$$\text{Minimise} \quad f(x) = 5x_1 + 20x_2 + 10x_1^{-1}x_2^{-1}, x_1, x_2 \geq 0 \tag{1}$$
$$= U_1 + U_2 + U_3 \tag{2}$$
$$\tag{3}$$

We know that, when $U_i = \delta_i u_i, \forall i$, $\sum_{i=1}^{n} U_i \geq \prod_{i=1}^{n} (\frac{U_i}{\delta_i})^{\delta_i}$ and $\sum_{i=1}^{n} \delta_i = 1$ with $\delta_i \geq 0$

$$\geq (\frac{u_1}{\delta_1})^{\delta_1} (\frac{u_2}{\delta_2})^{\delta_2} (\frac{u_3}{\delta_3})^{\delta_3} \tag{4}$$
$$= (\frac{5x_1}{\delta_1})^{\delta_1} (\frac{20x_2}{\delta_2})^{\delta_2} (\frac{10x_1^{-1}x_2^{-1}}{\delta_3})^{\delta_3} \tag{5}$$
$$= x_1^{\delta_1 - \delta_3} x_2^{\delta_2 - \delta_3} (\frac{5}{\delta_1})^{\delta_1} (\frac{20}{\delta_2})^{\delta_2} (\frac{10}{\delta_3})^{\delta_3} \tag{6}$$

GP: Optimization of Posynomial Function (contd.)

$$f(x) \geq x_1^{\delta_1 - \delta_3} x_2^{\delta_2 - \delta_3} (\frac{5}{\delta_1})^{\delta_1} (\frac{20}{\delta_2})^{\delta_2} (\frac{10}{\delta_3})^{\delta_3} \tag{7}$$

Minimum value of f(x) $\geq$ max value of RHS
So, choose those values of δ_i, such that RHS is independent of x
Let $\delta_1 - \delta_3 = 0$; $\delta_2 - \delta_3 = 0$, $\delta_1 + \delta_2 + \delta_3 = 1$
$\Longrightarrow \delta_1 = \delta_2 = \delta_3 = \frac{1}{3}$

$$f(x) = (\frac{5}{\frac{1}{3}})^{\frac{1}{3}} (\frac{20}{\frac{1}{3}})^{\frac{1}{3}} (\frac{10}{\frac{1}{3}})^{\frac{1}{3}} \tag{8}$$

$$f_{min}(x) = 30 \tag{9}$$

We know that, the equality holds good when Eq. 7,

$$\frac{U_1}{\delta_1} = \frac{U_2}{\delta_2} = \ldots = \frac{U_n}{\delta_n}$$

So, we have

$$\frac{5x_1}{1/3} = \frac{20x_2}{1/3} = \frac{10x_1^{-1}x_2^{-1}}{1/3}$$

comparing first and second term, we get

$$x_1 = 4x_2$$

comparing second and third term we get

$$2x_2 = \frac{1}{x_1x_2}$$

$$\implies 2x_1x_2^2 = 1$$

GP: Optimization of Posynomial Function (contd.)

substituting $x_1 = 4x_2$ in the above equation we get

$$8x_2^3 = 1$$

$$x_2 = \frac{1}{2}$$

now substituting, $x_2 = \frac{1}{2}$ in $x_1 = 4x_2$, we get,

$$x_1 = 2$$

GP: degree of difficulty

Let m denote the number of terms in f(x) and
Let n denote the number of variables.
Then,
d = m-n-1 is defined as the degree of difficulty of Geometric Programming.
If $d < 0$ GP is not applicable.
If $d = 0$ unique solution for deltas.
If $d \geq 1$ multiple solutions.

GP: when degree of difficulty is 1

$$
\begin{aligned}
\text{Minimize} \quad f(x) &= x^2 + x + \frac{3}{x},\ x > 0 &(10)\\
&= U_1 + U_2 + U_3 &(11)\\
&\geq (\frac{U_1}{\delta_1})^{\delta_1}(\frac{U_2}{\delta_2})^{\delta_2}(\frac{U_3}{\delta_3})^{\delta_3}, \text{where, } \sum_{i=1}^{3}\delta_i = 1 \text{ and } \delta_i \geq 0\ \forall i &(12)\\
&= (\frac{x^2}{\delta_1})^{\delta_1}(\frac{x}{\delta_2})^{\delta_2}(\frac{3}{x\delta_3})^{\delta_3} &(13)\\
&= x^{2\delta_1+\delta_2-\delta_3}(\frac{1}{\delta_1})^{\delta_1}(\frac{1}{\delta_2})^{\delta_2}(\frac{1}{\delta_2})^{\delta_2} &(14)
\end{aligned}
$$

GP: when degree of difficulty is 1 (contd.)

$$= x^{2\delta_1+\delta_2-\delta_3}(\frac{1}{\delta_1})^{\delta_1}(\frac{1}{\delta_2})^{\delta_2}(\frac{1}{\delta_2})^{\delta_2} \tag{15}$$

Now equating the powers of x to zero and the constant terms, we get

$$2\delta_1 + \delta_2 - \delta_3 = 0 \tag{16}$$

$$\delta_1 + \delta_2 + \delta_3 = 1 \tag{17}$$

solving the above 2 equations, we get,

$$\delta_1 - 2\delta_3 = -1 \tag{18}$$

GP: when degree of difficulty is 1 (contd.)

substituting the above equation in $\delta_1 + \delta_2 + \delta_3 = 1$, we get,

$$2\delta_3 - 1 + \delta_2 + \delta_3 = 1 \tag{19}$$

$$\delta_2 = 2 - 3\delta_3 \tag{20}$$

$$f \geq (\frac{1}{\delta_1})^{\delta_1}(\frac{1}{\delta_2})^{\delta_2}(\frac{1}{\delta_3})^{\delta_3} \tag{21}$$

$$g(\delta_3) = (\frac{1}{2\delta_3 - 1})^{2\delta_3 - 1}(\frac{1}{2 - 3\delta_3})^{2-3\delta_3}(\frac{3}{\delta_3})^{\delta_3} \tag{22}$$

- We need to find the maximum of RHS.
- RHS is a function of a single variable (δ_3).
- Take logarithm on both sides, differentiate w.r.t. δ_3, and set the derivative to zero to find the value of δ_3 for which the RHS is maximum.
- Then compute δ_1 and δ_2. Then compute f_{min}. Then compute x_{min}.

SELF-ASSESSMENT QUESTIONS

1. Posynomial functions are characterized by
   - A. $x \geq 0$, real valued coefficients and positive exponents.
   - B. $x \geq 0$, positive coefficients and positive exponents.
   - C. $x \geq 0$, positive coefficients and real valued exponents.
   - D. None of the above
2. Geometric programming gives unique solution when the degree of difficulty is
   - A. Negative
   - B. Zero
   - C. Positive
   - D. A negative fraction

1. Define a posynomial function.
2. List the constraints under which GP provides a viable solution.
3. Using GP, compute the minimum of the following function:
   - A. $40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1x_3$
   - B. $f(x_1, x_2, x_3) = x_1x_2x_3 + \frac{x_1}{x_2x_3} + \frac{3x_2}{x_1x_2}$

REFERENCES FOR FURTHER LEARNING OF THE SESSION [Hyperlinks are embedded titles]

Tutorials on Geometric Programming:

1. Geometric Programming Tutorial
2. Chapter 3: Geometric Programming
3. GP Kit

Web links to videos:

1. NPTEL online course, Dr. S. K. Gupta, IIT Roorke
2. NPTEL online course, Dr. D. Chakraborthy, IITKgp