MATHEMATICAL PROGRAMMING
22MT2004

Topic:

LP: Simplex method

CO1; Session - 03

AIM OF THE SESSION

To familiarize students with the method of solving linear programming problems using Simplex method.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Introduce the Standard form of LP for simplex method.
2. Introduce the concept of Slack variables, Tableau, Pivot element.
3. Teach creation of new Tableau to maximize the objective function.
4. Check whether optimal solution is achieved.

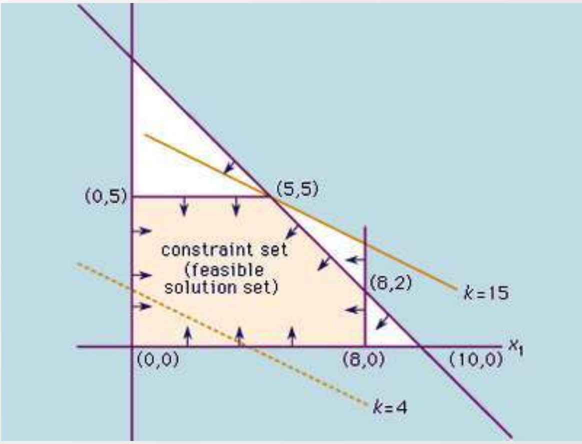
LEARNING OUTCOMES

At the end of this session, the students should be able to:

1. Formulate a linear programming problem in a standard format.
2. Compute the optimal value of the objective function and decision variables using Simplex method.

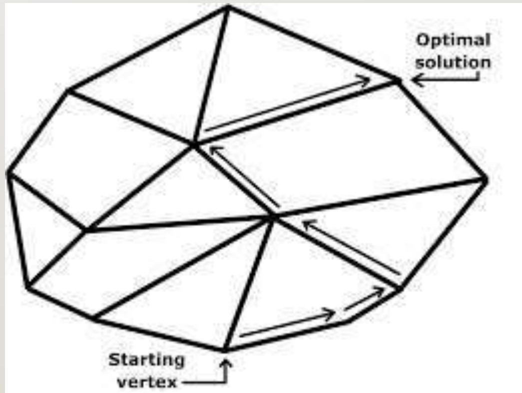
Introduction to Simplex Method

- A linear program (LP) is a method of achieving the best outcome given the task of either maximizing or minimizing a linear objective function, subject to linear constraints.



Source: Encyclopedia Britanica

- The simplex method uses an approach that is very efficient. It begins with a corner point of the feasibility region where all the main variables are zero. Then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.



Source: UC Davis Mathematics

Introduction to Simplex Method

- The Simplex method is an approach to solving linear programming problems by hand using **slack variables, tableau**, and **pivot element** as a means of finding the optimal solution of an optimization problem.

- Solving a LP problem using simplex method involves the following steps:
  1. Format the problem in Standard Form
  2. Convert inequality constraints to equations using **slack variables**
  3. Set up the initial simplex **tableau** using the objective function and the slack equations
  4. Check for optimality. If optimality is reached, go to step 8.
  5. Identify the **pivot element**
  6. Create a revised/new tableau through pivoting operations.
  7. Check for optimality. If optimality is NOT reached, go to step 5
  8. Determine the optimal value of the objective function and the decision variables.

Problem solving using Simplex method

- Let us learn more about the steps of the Simplex method while solving a LP optimization problem shown below. The goal is to minimize the objective function (-Z). This LP problem has 3 decision variables (x1, x2 and x3) and two inequality constraints.

$$Minimize : -z = -8x_1 - 10x_2 - 7x_3$$

$$s.t. : x_1 + 3x_2 + 2x_3 \leq 10$$

$$-x_1 - 5x_2 - x_3 \geq -8$$

$$x_1, x_2, x_3 \geq 0$$

Step 1: Standard Form

Standard form has three requirements:

1. must be a maximization problem
2. all linear constraints must be in a <= inequality
3. all variables are non-negative

$$Minimize: -z = -8x_1 - 10x_2 - 7x_3$$

$$s.t.: x_1 + 3x_2 + 2x_3 \leq 10$$

$$-x_1 - 5x_2 - x_3 \geq -8$$

$$x_1, x_2, x_3 \geq 0$$

Step 1: Standard Form

Standard form has three requirements:

1. must be a maximization problem
2. all linear constraints must be in a <= inequality
3. all variables are non-negative

$$Minimize: -z = -8x_1 - 10x_2 - 7x_3$$
$$s.t.: x_1 + 3x_2 + 2x_3 \leq 10$$
$$-x_1 - 5x_2 - x_3 \geq -8$$
$$x_1, x_2, x_3 \geq 0$$

The given problem is to minimize –Z. To satisfy requirement 1, let us transform the minimization problem into a maximization problem, by multiplying both the left and the right sides of the objective function by –1.

$$-1 \times (-z = -8x_1 - 10x_2 - 7x_3)$$
$$z = 8x_1 + 10x_2 + 7x_3$$
$$Maximize: z = 8x_1 + 10x_2 + 7x_3$$

Similarly, let us transform the >= constraint to <= constraint by multiplying both sides of the constraint by –1.

$$-1 \times (-x_1 - 5x_2 - x_3 \geq -8)$$
$$x_1 + 5x_2 + x_3 \leq 8$$

Step 2: Slack variables

Convert the <= inequalities into equations
by adding one slack variable for each inequality.

Slack variables are needed in the constraints to transform them into solvable equalities with one definite answer.

The variable s_1 represents the amount (slack) by which $x_1 + 3x_2 + 2x_3$ falls short of 10.
Similarly, s_2 represents the amount (slack) by which $x_1 + 5x_2 + x_3$ falls short of 8.

$$x_1 + 3x_2 + 2x_3 + \mathbf{s_1} = 10$$

$$x_1 + 5x_2 + x_3 + \mathbf{s_2} = 8$$

$$x_1, x_2, x_3, \mathbf{s_1}, \mathbf{s_2} \geq 0$$

Step 3: Set up the initial Tableau: preparation

A Simplex tableau is used

- to perform row operations on the LP model, and
- to check a solution for optimality.

The current LP model is shown on the right.

$$Maximize: z = 8x_1 + 10x_2 + 7x_3$$
$$s.t.: x_1 + 3x_2 + 2x_3 + s_1 = 10$$
$$x_1 + 5x_2 + x_3 + s_2 = 8$$

The tableau will be created from the coefficients of the 3 equations, (re-)written as shown on the right.

The first equation corresponds to the objective function. The latter 2 equations correspond to the 2 constraints.

$$8x_1 + 10x_2 + 7x_3 + 0s_1 + 0s_2 = z$$
$$1x_1 + 3x_2 + 2x_3 + 1s_1 + 0s_2 = 10$$
$$1x_1 + 5x_2 + 1x_3 + 0s_1 + 1s_2 = 8$$

Initial Simplex tableau

Cbi ↓	Cj →		Solution	Ratio
	Basic variables			
Zj				
Cj - Zj				

Note:
C_j = Coefficient of objective function
C_{bi} = Coefficient of basic variables
$Z_J = \sum_{i=1}^{n}(C_{bi})(a_{ij})$ (sum of cost of basic variables multiplied by coefficients of constraint variables)

Step 3: Set up the initial Tableau: C_js

$$8x_1 + 10x_2 + 7x_3 + 0s_1 + 0s_2 = z$$
$$1x_1 + 3x_2 + 2x_3 + 1s_1 + 0s_2 = 10$$
$$1x_1 + 5x_2 + 1x_3 + 0s_1 + 1s_2 = 8$$

In the tableau below,

- the 2nd row of the tableau lists all the variables.
- The top row lists the coefficients of these variables in the objective function:

  C_j: coefficient in the jth column
- The last two rows list the coefficients in the two linear constraints.
- The last column contains the numbers on the RHS of the equations.

Tableau

C_j ->	8	10	7	0	0	
	x1	x2	x3	s1	s2	b
	1	3	2	1	0	10
	1	5	1	0	1	8

Step 3: Set up the initial Tableau: basic variables

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8

basic variables:

A variable is called a basic variable if it's column (in the tableau above) contains 1 in a row and zeros in all other rows.

- Here, the s1 column contains 1 in the 1st row (below the horizontal line) and zeros in all other rows. So, s1 is a basic variable in the 1st row.
- Similarly, s2 is a basic variable in the 2nd row.
- In the objective function (z = 8x1 + 10x2 +7x3 +0s1 +0s2), the coefficients of the basic variables (s1 and s2) are zero. These (C_{Bi}) are written in the left most column above.

Step 3: Set up the initial Tableau: z_js

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8
	z_j	0	0	0	0	0	0
	Cj - zj	8	10	7	0	0	

z_js:

$$z_j = \Sigma_i \, C_{Bi} \, a_{ij}$$

1st column: z1 = 0 x1 + 0x1 = 0

Similarly, z_i (I=1 to 6) is zero in all 6 columns. Write these zi values in a row below.

- Now, in each of the 6 columns, compute $C_j - z_j$ and write them in the last row.

Step 4: Check for optimality

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8
	z_j	0	0	0	0	0	0
	Cj - zj	8	10	7	0	0	

- A **solution is optimum** (maximum) **if all** (Cj – zj) values in the last row of the tableau are **<= zero**.
- In the above tableau, there are 3 positive values of (Cj – zj). So, this solution is **NOT optimal**.
- So, we will go to step 5.

Step 5: Identify pivot row/column and pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	10/3
0	s2	1	5	1	0	1	8	8/5
	z_j	0	0	0	0	0	0	
	Cj - Zj	8	10	7	0	0		

- **Pivot column**: In the Cj – zj row, there are 3 positive values: 8, 10 and 7. Pick the **largest** value. In our case, it is 10. The column containing this number is called the pivot column. In our case, it is the x2 column (outlined in yellow color).
- **Pivot row**: In each of the linear constraint row, compute the ratio of b to the value in the pivot column of that row. For example, b in the first row is 10 and the value in the pivot (x2) column in that row is 3. So, the ratio for the 1st row is 10/3. The ratio for the 2n row is 8/5. The row containing the **smallest** ratio is called the pivot row. In our case, 2nd row (green) is the pivot row.

Step 5: Identify pivot row/column and pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	10/3=3.33
0	s2	1	5	1	0	1	8	$\boxed{8/5=1.6}$
	z_j	0	0	0	0	0	0	
	Cj - zj	8	$\boxed{10}$	7	0	0		

- **Pivot element**: The value present in the pivot column (x2 column) **and** in the pivot row (2nd row) is called **pivot element**. In our case, the value is 5.
- The basic variable in the pivot row is called the **leaving variable**. So, s2 is the leaving variable.
- The variable in the pivot column is called the entering variable. So, **x2 is the entering variable**.
- In the next step (of revision of the tableau), the entering variable (x2) will become the basic variable, replacing the **leaving variable (s2)**.

Step 6: Create a New Tableau via Pivoting

- **Pivoting** is a process (involving row operations) of
  a) obtaining a 1 in the location of the pivot element, and
  b) making all other entries zeros in that (pivot) column.

Old tableau with revised pivot row

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

a) Obtain a 1 in the location of the pivot element:
The value of the pivot element was 5 in the old tableau. Divide every element of the pivot row (row 2 in our case) by the value of the pivot element, i.e., 5, to obtain a revised pivot row (shown in red colour). Now, the revised pivot has value 1 in the pivot column (x2 column).

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

Old tableau with revised pivot row

b) Via row operations, make all entries zero in the (pivot) column of all remaining (non-pivot) rows.

Let **n** denote the value in the pivot column of a row. The new row is obtained by the following rule:

Value in the **new** row = value in the **old** row – **n** * value in the **pivot** row

In the next slide, the steps of revising the 1st row is illustrated.

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

Old tableau with revised pivot row

Value in the **new** row = value in the **old** row – **n** * value in the **pivot** row

Let us revise Row 1 using the above formula.
In Row 1 and pivot (x2) column, the value is 3. So, **n** = 3.
To begin with, let us obtain the new value in column 1 of Row 1.
In Row 1 and **1st** column, the value is 1. It's new value will become

New value = old value – **n** * value in the pivot row of 1st column

= 1 – 3*(1/5) = 2/5

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	

New row = old row – **n** * pivot row

n = 3 for Row 1.

In Row 1 and 1st column, the value is 1. Its new value will become

New value = old value – **n** * value in the pivot row of 1st column

= 1 – 3 * (1/5) = 2/5

In Row 1 and **2nd** column, the value is 3. Its new value will become

= 3 – 3 * 1 = 0

Repeat the process for all columns of Row 1. The revised Tableau is shown below:

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	
10	x2	1/5	1	1/5	0	1/5	8/5	

Step 7: Check for optimality

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j	2	10	2	0	2	16	
	Cj - zj	6	0	5	0	-2		

- Compute zj using the formula $Z_j = \Sigma_i C_{Bi} a_{ij}$
- Compute Cj - zj
- The **solution is optimum if all** values in the last row of the tableau are <= zero.
- In the above tableau, there are 2 positive values of (Cj – zj). So, this solution is **NOT optimal**.
- So, we will return to step 5 to derive a new tableau.

Step 5: Identify the new pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	(26/5)/(2/5)=13
10	x2	1/5	1	1/5	0	1/5	8/5	(8/5)/(1/5)=8
	z_j	2	10	2	0	2	16	
	Cj - zj	6	0	5	0	-2		

- The largest value in the bottom row is 6 So, the first (x1) column is the **pivot column,**
- Then we compute the ratios. The smallest ratio is 8 (row 2). So, Row 2 is the **pivot row**.
- The **pivot element** is **1/5**.
- The variable x1 and x2 are incoming and outcoming variables, respectively. So, x1 will replace x2 as the basic variable.

Step 6: Create a New Tableau via Pivoting

Old tableau after revising pivot row

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	2/5	0	7/5	1	-3/5	26/5
8	x1	I	5	5	0	5	8

New row = old row – **n** * pivot row

here **n=2/5**

By carrying out pivoting operations, we get a N**ew Tableau** as shown below.

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

Step 7: Check for optimality

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

- Since all values in the Cj -zj row are <= zero, this is the **optimal solution**.

Step 8: Identify the optimal values

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2/5	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

- Here, the basic variables are x1 and s1. The non-basic variables are x2, x3 and s2.
- The optimal value of a non-basic variable is zero.
- If a variable is basic, the value of b (in the last column) of the row containing 1 is the optimal value.
- The 1 of the basic variable x1 is found in the 2nd row. The value of b in the second row is 8. This is the optimal value of the variable x1. The optimal value of s1 is 2.
- The optimal objective value is 64 and can be found when x1=8, x2=0, and x3=0.

Another solved problem

problem

Max $z = 12x_1 + 16x_2$ — objective function

$$\left.\begin{aligned} 10x_1 + 20x_2 &\le 120 \\ 8x_1 + 8x_2 &\le 80 \\ x_1, x_2 &\ge 0 \end{aligned}\right\} \text{constraints.}$$

Answer

$z = 12x_1 + 16x_2 + 0S_1 + 0S_2$

$10x_1 + 20x_2 + S_1 = 120$

$8x_1 + 8x_2 + S_2 = 80$

$x_1, x_2, S_1, S_2 \ge 0.$

$$z = 12x_1 + 16x2 + 0s1 + 0s2$$

$$10x1 + 20x2 + 1s1 + 0s2 = 120$$

$$8x1 + 8x2 + 0s1 + 1s2 = 80$$

$$x1, x2, s1, s2 >= 0$$

	C_j ->	12	16	0	0		
C_{Bi}	Basic variables	x1	x2	s1	s2	b	ratio
0	s1	10	20	1	0	120	$\boxed{120/2}$ 0
0	s2	8	8	0	1	80	80/8
	zj	0	$\boxed{0}$	0	0	0	
	Cj - zj	12	16	0	0		

	C_j ->	12	16	0	0	
C_{Bi}	Basic variables	x1	x2	s1	s2	b
16	x2	1/2	1	1/20	0	6
0	s2	4	0	-2/5	1	32
	zj	8	16	4/5	0	96
	Cj - zj	4	0	-4/5	0	

Revised tableau
Non-optimal solution

	C_j ->	12	16	0	0		
C_{Bi}	Basic variables	x1	x2	s1	s2	b	ratio
16	x2	1/2	1	1/20	0	6	12
0	s2	4	0	-2/5	1	32	8
	zj	8	16	4/5	0	96	
	Cj - zj	4	0	-4/5	0		

Pivot element = 4

	C_j ->	12	16	0	0	
C_{Bi}	Basic variables	x1	x2	s1	s2	b
16	x2	0	1	1/10	-1/8	2
12	x1	1	0	-1/10	1/4	8
	zj	12	16	4/10	1	128
	Cj - zj	0	0	-4/10	-1	

Revised tableau
Optimal solution

Optimal values are

z = 128
x1 = 8
x2 = 2

SELF-ASSESSMENT QUESTIONS

A standard form of LP problem is required to satisfy the following condition

A) The problem is a maximization problem.
B) All constraints should be of <= zero type.
C) All variables should take non-negative values.
D) All of the above

Inequalities can be converted into equality using the following type of variable

A. Basic
B. Non-basic
C. Slack
D. Any of the above

TERMINAL QUESTIONS

1. Write the steps of converting a minimization problem to a maximization problem.
2. List the steps of optimization following the Simplex method.
3. Define the following terms:
   a) Pivot element
   b) Slack variable
   c) Basic variable
4. In simplex method, what is the criterion to decide whether the optimality is reached?

TERMINAL QUESTIONS

5) Solve the following problem using Simplex method:

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 + 3x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

6) A farmer has 100 acres of land on which she plans to grow wheat and corn. Each acre of wheat requires 4 hours of labor and $20 of capital, and each acre of corn requires 16 hours of labor and $40 of capital. The farmer has at most 800 hours of labor and $2400 of capital available. If the profit from an acre of wheat is $80 and from an acre of corn is $100, how many acres of each crop should she plant to maximize her profit? Determine using the Simplex method.

KL UNIVERSITY (DEEMED TO BE UNIVERSITY)

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Tutorials on Simplex method:

1. https://math.libretexts.org/Bookshelves/Applied_Mathematics/Applied_Finite_Mathematics_(Sekhon_and_Bloom)/04%3A_Linear_Programming_The_Simplex_Method/4.02%3A_Maximization_By_The_Simplex_Method
2. http://recursos.pearson.es/castroman/tutor_chaps/ct03.pdf
3. https://www.imse.iastate.edu/files/2015/08/Explanation-of-Simplex-Method.docx

Web links to videos:

- NPTEL online course, "**Linear Programming Solutions - Simplex Algorithm**", Prof. G. Srinivasan, IIT Madras, https://www.youtube.com/watch?v=qxls3cYg8to