23MT2014

THEORY OF COMPUTATION

Topic:

THE PUMPING LEMMA APPLICATIONS

Session – 16-b

The Pumping Lemma for Context-Free Languages

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications
of
The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w=a^m b^m c^m$$

$$w=uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Pumping Lemma says:

$$uv^i xy^i z \in L$$ for all $$i \geq 0$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$\boxed{\textbf{Case 1:} \quad vxy \text{ is within } a^m}$

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aa...a}_{vxy}\underbrace{aa\;bbb...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: v and y consist from only a

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aa\ldots a}_{vxy}\underbrace{a\,bbb\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\,\underbrace{aaaaa...aaaaa}_{v^2xy^2}\,\underbrace{a\,bbb...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aaaaa...aaaaa}_{v^2xy^2}\underbrace{a\;bbb...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bb}_{u}\underbrace{b...bb}_{vxy}\underbrace{b\ ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bb}_{u}\underbrace{b...bb}_{vxy}\underbrace{b\ ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m

$$\underbrace{\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;c}_{u}\underbrace{cc\ldots c}_{vxy}\,\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: Similar analysis with case 1

$$\underbrace{\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,c}_{u}\;\underbrace{cc...c}_{vxy}\;\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaa\,bbb}_{vxy}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w=a^m b^m c^m$$

$$w=uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\,bbb}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa...aaaaaaa}^{m+k_1}\ \overbrace{bbbbbbb...bbb}^{m+k_2}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaaaaaa\ bbbbbbb..}_{v^2xy^2}\underbrace{.bbb\ ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \ge 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \ge 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\ \overbrace{bbbbbbb\ldots bbb}^{m+k_2}\ \overbrace{ccc\ldots ccc}^{m}$$

$$u \qquad\qquad v^2xy^2 \qquad\qquad z$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$u \qquad vxy \qquad z$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b

$$k_1 + k_2 + k \geq 1$$

$$\overbrace{aaa\ldots aaaaa}^{m}\overbrace{bb}^{k_1}\overbrace{aa}^{k_2}\overbrace{bb\ bbbbbbb\ldots bbb}^{m+k}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaabbaabb\ bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\ ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 + k \geq 1$$

$$\overbrace{aaa\ldots aaaaa}^{m}\overbrace{bb}^{k_1}\overbrace{aa}^{k_2}\overbrace{bb\ bbbbbbb\ldots bbb}^{m+k}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaabbaabb\ bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\ ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\,bbb}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^m and c^m

$$\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa\ldots aaa\;bbb\ldots}_{u}\underbrace{bbb\;ccc\ldots}_{vxy}\underbrace{ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: Similar analysis with case 4

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bbb...}_{u}\underbrace{bbb\ ccc...}_{vxy}\underbrace{ccc}_{z}$$

There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

More Applications of The Pumping Lemma

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\{vv : v \in \{a,b\}\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{vv : v \in \{a,b\}^*\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{vv : v \in \{a,b\}^*\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{vv : v \in \{a,b\}^*\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^m b^m a^m b^m \in L$

$$L = \{vv : v \in \{a,b\}^*\}$$

We can write: $a^m b^m a^m b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine <u>**all**</u> the possible locations of string vxy in $a^m b^m a^m b^m$

$$L=\{vv : v\in\{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy|\le m \qquad |vy|\ge 1$$

Case 1: vxy is within the first a^m

$$v=a^{k_1} \qquad y=a^{k_2} \qquad k_1+k_2\ge 1$$

$$\overbrace{a\,\ldots\ldots\,a}^{m}\;\overbrace{b\,\ldots\ldots\,b}^{m}\;\overbrace{a\,\ldots\ldots\,a}^{m}\;\overbrace{b\,\ldots\ldots\,b}^{m}$$

$$u \quad v \quad x \quad y \qquad\qquad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots a}^{m+k_1+k_2} \overbrace{b \ldots\ldots b}^{m} \overbrace{a \ldots\ldots a}^{m} \overbrace{b \ldots\ldots b}^{m}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$\boxed{\textbf{Case 1:} \quad vxy \text{ is within the first } a^m}$

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1 + k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m
y is in the first b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$$

$$\overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$$u \quad v \quad x \quad y \quad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: v is in the first a^m
y is in the first b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots a}^{m+k_1}\overbrace{b \ldots\ldots\ldots\ldots b}^{m+k_2}\overbrace{a \ldots\ldots a}^{m}\overbrace{b \ldots\ldots b}^{m}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m

y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1 + k_2 \ge 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

$$L=\{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$$

$$\overbrace{a \ldots\ldots a}^{m}\;\overbrace{b \ldots\ldots b}^{m}\;\overbrace{a \ldots\ldots a}^{m}\;\overbrace{b \ldots\ldots b}^{m}$$

$u \quad v \quad x \quad y \quad z$

$$L=\{vv: v\in\{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$$

$$\overbrace{a\ldots\ldots a}^{m}\,\overbrace{b\ldots b}^{k_2}\,\overbrace{a\ldots a}^{k_1}\,\overbrace{b\ldots\ldots\ldots b}^{m+k_3}\,\overbrace{a\ldots\ldots a}^{m}\,\overbrace{b\ldots\ldots b}^{m}$$

$$u \qquad v^2 \qquad x \qquad y^2 \qquad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1, k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 4: v in the first a^m

y Overlaps the first $a^m b^m$

Analysis is similar to case 3

$$\overbrace{a\ldots\ldots a}^{m}\,\overbrace{b\ldots\ldots b}^{m}\,\overbrace{a\ldots\ldots a}^{m}\,\overbrace{b\ldots\ldots b}^{m}$$

$$u \mid v \mid x \mid y \mid z$$

Other cases: vxy is within $a^m \boxed{b^m} a^m b^m$

or

$a^m b^m \boxed{a^m} b^m$

or

$a^m b^m a^m \boxed{b^m}$

Analysis is similar to case 1:

$\boxed{a^m} b^m a^m b^m$

More cases:

vxy overlaps $a^m \boxed{b^m a^m} b^m$

or

$a^m b^m \boxed{a^m b^m}$

Analysis is similar to cases 2,3,4:

$\boxed{a^m b^m} a^m b^m$

There are no other cases to consider

Since $|vxy| \leq m$, it is impossible vxy to overlap:

$$\boxed{a^m b^m a^m} b^m$$

nor

$$a^m \boxed{b^m a^m b^m}$$

nor

$$\boxed{a^m b^m a^m b^m}$$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{vv : v \in \{a,b\}^*\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$

$\{ww : w \in \{a,b\}\}$

$\{a^{n!} : n \geq 0\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$

$\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{a^{n!} : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n!} : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^{m!} \in L$

$$L = \{a^{n!} : n \geq 0\}$$

We can write: $a^{m!} = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **<u>all</u>** the possible locations of string vxy in $a^{m!}$

There is only one case to consider

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots a}^{m!}$$

$$\underbrace{}_{u}\underbrace{}_{v}\underbrace{}_{x}\underbrace{}_{y}\underbrace{}_{z}$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m! + k_1 + k_2}$$

$$u \qquad v^2 \qquad x \qquad y^2 \qquad z$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m!+k} \qquad k = k_1 + k_2$$

$$\underbrace{\quad}_{u}\underbrace{\quad}_{v^2}\underbrace{\quad}_{x}\underbrace{\quad}_{y^2}\underbrace{\quad}_{z}$$

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$a^{m!+k} = uv^2xy^2z$$

$$1 \leq k \leq m$$

Since $1 \le k \le m$, for $m \ge 2$ we have:

$$m! + k \le m! + m$$
$$< m! + m!m$$
$$= m!(1+m)$$
$$= (m+1)!$$

$$\Downarrow$$

$$m! < m! + k < (m+1)!$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m! < m! + k < (m+1)!$$

$$\Downarrow$$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

Contradiction!!!

We obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n!} : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{ww : w \in \{a,b\}\}$$

$$\{a^{n^2} b^n : n \geq 0\}$$

$$\{a^{n!} : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

Theorem: The language

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n^2} b^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n^2} b^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^{m^2} b^m \in L$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

We can write: $a^{m^2} b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations

of string vxy in $a^{m^2} b^m$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated case: v is in a^m

y is in b^m

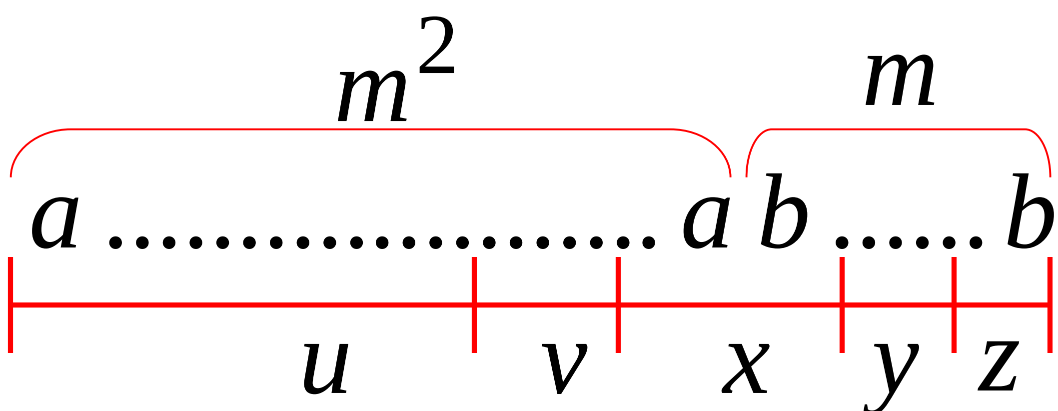
$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m^2}\,\overbrace{b \ldots\ldots b}^{m}$$

$$\underbrace{\quad}_{u}\;\underbrace{\quad}_{v}\;\underbrace{\quad}_{x}\;\underbrace{\quad}_{y}\;\underbrace{\quad}_{z}$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$



$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ **and** $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots a}^{m^2}\,\overbrace{b \ldots\ldots b}^{m}$$

$$\underbrace{\qquad\qquad}_{u}\underbrace{\qquad}_{v}\underbrace{\qquad\quad}_{x}\underbrace{\quad}_{y}\underbrace{\quad}_{z}$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots a}^{m^2 - k_1}\ \overbrace{b \ldots b}^{m - k_2}$$

$$u \quad v^0 \quad x \quad y^0 \quad z$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

$k_1 \neq 0$ and $k_2 \neq 0$ $\quad 1 \leq k_1 + k_2 \leq m$

$$\Downarrow$$

$$(m - k_2)^2 \leq (m-1)^2$$

$$= m^2 - 2m + 1$$

$$< m^2 - k_1$$

$$\Downarrow$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$\Downarrow$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^0xy^0z \in L$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0xy^0z \notin L$$

Contradiction!!!

When we examine the rest of the cases we also obtain a contradiction

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

SELF ASSESSMENT QUESTIONS

Q.1. Which of the following statements is true about Pushdown Automata (PDA)?

a) PDA can only recognize regular languages.

b) PDA can recognize context-free languages but not context-sensitive languages.

c) PDA can recognize context-free languages and some context-sensitive languages.

d) PDA can recognize any recursively enumerable language.

Answer: c) PDA can recognize context-free languages and some context-sensitive languages.

SELF ASSESSMENT QUESTIONS

Q.2 In a PDA, the stack serves the purpose of:

a) Storing the input symbols.

b) Storing the current state of the automaton.

c) Providing additional memory for computation.

d) Determining the next transition of the automaton.

Answer: a c) Providing additional memory for computation.

SELF ASSESSMENT QUESTIONS

Q.3 Which of the following is a limitation of a deterministic PDA (DPDA) compared to a non-deterministic PDA (NPDA)?

a) DPDA can recognize a larger class of languages.

b) DPDA cannot recognize any context-free languages.

c) DPDA can only recognize regular languages.

d) DPDA cannot recognize languages with nested structures.

Answer: d) DPDA cannot recognize languages with nested structures.

TERMINAL QUESTIONS

Q.1. Explain the concept of pushdown automata (PDA) and its key components.

Q.2.Describe the process of accepting a string by a pushdown automaton. How does the PDA handle the interactions between the input string and the stack?

Q.3.Discuss the difference between deterministic pushdownautomata (DPDA) and non-deterministic pushdown automata (NPDA). How do they differ in terms of language recognition and the behavior of the stack?