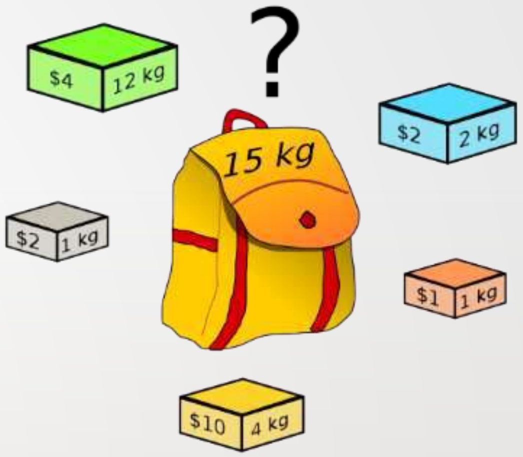
DESIGN AND ANALYSIS OF ALGORITHMS

SESSION-21

KNAPSACK PROBLEM

You are given the following-

- A knapsack (kind of shoulder bag) with limited weight capacity.
- Few items each having some weight and value.



The problem states-

Which items should be placed into the knapsack such that-

- The **value or profit** obtained by putting the items into the knapsack is **maximum.**
- And the **weight limit** of the knapsack does **not exceed.**

KNAPSACK PROBLEM VARIANTS & ITS DIFFERENCES

Variants:

- 0/1 Knapsack.
  - not allowed to break items. We either take the whole item or don't take it.

- Fractional Knapsack
  - can break items for maximizing the total value of knapsack

0/1 KNAPSACK PROBLEM

In 0/1 Knapsack Problem,

- As the name suggests, items are indivisible here.
- We can not take the fraction of any item.
- We have to either take an item completely or leave it completely.

Example:

Consider the knapsack instance n = 3, $(w_1, w_2, w_3) = (2, 3, 4)$, $(p_1, p_2, p_3) = (1, 2, 5)$ and m = 6.

Probability of **Chosen Items (x_i)** = [{0, 0, 0}, {0, 0, 1}, ……., {1, 1, 1}]

No. of Possible Solutions **(2^n)** = $2^3 = 8$.

The problem is to find the **Best Optimal Solution** among the 8 for the 0/1 Knapsack.

0/1 KNAPSACK PROBLEM

1. Let $f_i(y_j)$ be the values of optimal solution. Then S^i is a pair (p,w) where $p = f(y_j)$ and $w = y_j$

   Initially $S^0 = \{ (0,0) \}$. We can compute S^{i+1} from S^i. The Computations of S^i are sequence of decisions made for obtaining optimal solution.

2. Let x_n be the optimal sequence. Then there are two instances $\{x_n\}$ and $\{x_{n-1},\ldots,x_1\}$. So from $\{x_{n-1},\ldots,x_1\}$ will choose optimal sequence with respect to x_n.

0/1 KNAPSACK PROBLEM

3. The formulas that are used while solving 0/1 knapsack problem.

Let $f_n(m)$ be the value of an optimal solution, then

$$f_n(m) = \max \{ f_{n-1}(m),\ f_{n-1}(m - w_n) + p_n \}$$

General formula

$$f_i(y) = \max \{ f_{i-1}(y),\ f_{i-1}(y - w_i) + p_i \}$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, $(w_1, w_2, w_3) = (2, 3, 4)$, $(p_1, p_2, p_3) = (1, 2, 5)$ and m = 6.

$$f_n(m) = \max\{ f_{n-1}(m),\ f_{n-1}(m - w_n) + p_n \}$$

$$f_3(6) = \max\{ f_2(6),\ f_2(6 - w_3) + p_3 \}$$

$$f_3(6) = \max\{ f_2(6),\ f_2(6 - 4) + 5 \}$$

$$f_3(6) = \max\{ f_2(6),\ f_2(2) + 5 \}$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = \max\{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = \max\{ f_1(6),\ f_1(6-3) + 2 \}$$

$$f_2(6) = \max\{ f_1(6),\ f_1(3) + 2 \}$$

$$f_2(2) = \max\{ f_1(2),\ f_1(2-3) + 2 \}$$

$$f_2(2) = \max\{ f_1(2),\ f_1(-1) + 2 \}$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$f_3(6) = \max\{ f_2(6),\ f_2(2) + 5 \}$

$f_2(6) = \max\{ f_1(6),\ f_1(3) + 2 \}$

$f_2(2) = \max\{ f_1(2),\ f_1(-1) + 2 \}$

$f_1(6) = \max\{ f_0(6),\ f_0(6-2) + 1 \}$

$f_1(6) = \max\{ f_0(6),\ f_0(4) + 1 \}$

$f_1(6) = \max\{ 0, 0 + 1 \} = 1$

$f_1(3) = \max\{ f_0(3),\ f_0(3-2) + 1 \}$

$f_1(3) = \max\{ f_0(3),\ f_0(1) + 1 \}$

$f_1(3) = \max\{ 0,\ 0 + 1 \} = 1$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = \max\{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = \max\{ 1,\ 1 + 2 \}$$

$$f_2(2) = \max\{ f_1(2),\ f_1(-1) + 2 \}$$

$$f_1(6) = \max\{ f_0(6),\ f_0(6-2) + 1 \}$$
$$f_1(6) = \max\{ f_0(6),\ f_0(4) + 1 \}$$
$$f_1(6) = \max\{ 0,\ 0 + 1 \} = 1$$

$$f_1(3) = \max\{ f_0(3),\ f_0(3-2) + 1 \}$$
$$f_1(3) = \max\{ f_0(3),\ f_0(1) + 1 \}$$
$$f_1(3) = \max\{ 0,\ 0 + 1 \} = 1$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = max\ \{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = max\ \{ 1,\ 3 \} = 3$$

$$f_2(2) = max\ \{ f_1(2),\ f_1(-1) + 2 \}$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = max\{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = max\{ 1,\ 1 + 2 \}$$

$$f_2(2) = max\{ f_1(2),\ f_1(-1) + 2 \}$$

$$f_1(2) = max\{ f_0(2),\ f_0(2-2) + 1 \}$$

$$f_1(2) = max\{ 0,\ 0 + 1 \} = 1$$

$$f_1(-1)$$

$$-INF$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = \max \{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = \max \{ 1,\ 1 + 2 \}$$

$$f_2(2) = \max \{ 1,\ -INF + 2 \}$$

$$f_1(2) = \max \{ f_0(2),\ f_0(2-2) + 1 \}$$
$$f_1(2) = \max \{ 0,\ 0 + 1 \} = 1$$

$$f_1(-1)$$
$$-INF$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = \max\{ f_2(6),\ f_2(2) + 5 \}$$

$$f_2(6) = \max\{ 1,\ 1 + 2 \}$$

$$f_2(2) = \max\{ 1,\ -INF + 2 \}$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = max\ \{\ f_2(6),\ \ f_2(2)\ \ +5\ \}$$

$$f_2(6) = max\ \{\ 1,\ \ 3\ \} = 3$$

$$f_2(2) = max\ \{\ 1,\ \ -INF\} = 1$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$f_3(6) = max\{3, 1 + 5\}$$

$$f_2(6) = max\{1, 3\} = 3$$

$$f_2(2) = max\{1, -INF\} = 1$$

0/1 Knapsack Problem – Solution

Consider the knapsack instance n = 3, m=6

$(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$$\boxed{f_3(6) = max\{\,3,\ 1\ + 5\,\} = 6}$$

0/1 Knapsack Problem – Solution: Set Method

Initially $S^0 = \{(0,0)\}$

$$S_1^i = \{ (P,W) \,/\, (P - p_{i+1}, W - w_{i+1}) \in S^i \}$$

S^{i+1} can be computed by merging from S^i and S_1^i

- Purging or dominance rule: if S^{i+1} contains two pairs (p_j, w_j) and (p_k, w_k) with the property that $p_j \le p_k$ and $w_j \ge w_k$ then the pair (p_j, w_j) can be discarded.
- When generating S^i, we can also purge all pairs (p, w) with $w > m$ as these pairs determine the value of $f_n(x)$ only for $x > m$.
- The optimal solution $f_n(m)$ is given by the *highest profit pair*.

0/1 Knapsack Problem – Solution: Set Method

Consider the knapsack instance $n = 3$, $m=6$ $(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

Initially $S^0 = \{(0,0)\}$

$S_1^i = \{ (P,W) / (P- p_{i+1}, W- w_{i+1}) \in S^i \}$

S^{i+1} can be computed by merging from S^i and S_1^i

0/1 Knapsack Problem – Solution: Set Method

Consider the knapsack instance n = 3, m=6 $(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$S^0 = \{(0,0)\}$ $S_1^0 = \{ (1, 2) \}$

$S^1 = \{(0, 0), (1, 2)\}$ $S_1^1 = \{ (2, 3), (3, 5) \}$

$S^2 = \{(0, 0), (1, 2), (2, 3), (3, 5) \}$ $S_1^2 = \{ (5, 4), (6, 6), (7, 7), (8, 9) \}$

$S^3 = \{(0, 0), (1, 2), (2, 3), (3, 5), (5, 4), (6, 6), \boldsymbol{(7, 7)}, \boldsymbol{(8, 9)} \}$

$$S_1^i = \{ (P,W) / (P - p_{i+1}, W - w_{i+1}) \in S^i \}$$

0/1 Knapsack Problem – Solution: Set Method

Consider the knapsack instance n = 3, m=6 $(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$S^0 = \{(0,0)\}$ $S_1^0 = \{ (1, 2) \}$

$S^1 = \{(0, 0), (1, 2)\}$ $S_1^1 = \{ (2, 3), (3, 5) \}$

$S^2 = \{(0, 0), (1, 2), (2, 3), (3, 5) \}$ $S_1^2 = \{ (5, 4), (6, 6), (7, 7), (8, 9) \}$

$S^3 = \{(0, 0), (1, 2), (2, 3), \mathbf{(3, 5)}, \mathbf{(5, 4)}, (6, 6)\}$

$S_1^i = \{ (P,W) / (P - p_{i+1}, W - w_{i+1}) \in S^i \}$

S^{i+1} can be computed by merging from S^i and S_1^i

0/1 Knapsack Problem – Solution: Set Method

Consider the knapsack instance n = 3, m=6 $(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$S^0 = \{(0,0)\}$ $S_1^0 = \{ (1, 2) \}$

$S^1 = \{(0, 0), (1, 2)\}$ $S_1^1 = \{ (2, 3), (3, 5) \}$

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$S^3 = \{(0, 0), (1, 2), (2, 3), (5, 4), (6, 6)\}$

$S_1^i = \{ (P,W) / (P - p_{i+1}, W - w_{i+1}) \in S^i \}$

S^{i+1} can be computed by merging from S^i and S_1^i

0/1 Knapsack Problem – Solution: Set Method

Consider the knapsack instance n = 3, m=6 $(w_1, w_2, w_3) = (2, 3, 4)$ & $(p_1, p_2, p_3) = (1, 2, 5)$.

$S^0 = \{(0,0)\}$ $S_1^0 = \{ (1, 2) \}$

$S^1 = \{(0, 0), (1, 2)\}$ $S_1^1 = \{ (2, 3), (3, 5) \}$

$S^2 = \{(0, 0), (1, 2), (2, 3), (3, 5) \}$ $S_1^2 = \{ (5, 4), (6, 6), (7, 7), (8, 9) \}$

$S^3 = \{(0, 0), (1, 2), (2, 3), (5, 4), \boldsymbol{(6, 6)}\}$

$S_1^i = \{ (P,W) / (P - p_{i+1}, W - w_{i+1}) \in S^i \}$

S^{i+1} can be computed by merging from S^i and S_1^i

Example 1: Consider the knapsack instance n = 3,

$(w_1,w_2,w_3) = (2,3,4)$, $(p_1,p_2,p_3) = (1,2,5)$, and m = 6.

Initially

$S^0 = \{(0,0)\}$

Include 1st object

$S_1^0 = \{ (0+1,0+2) \} = \{ (1,2) \}$

Next Stage can be obtained S^{0+1} (S^1)can be computed by merging from S^0 and S_1^0

$S^1 = \{(0,0)(1,2)\}$

Include 2nd object

$S_1^1 = \{ (0+2,0+3) (1+2, 2+3) \} = \{ (2,3) (3,5) \}$

Next Stage can be obtained S^{1+1} (S^2)can be computed by merging from S^1 and S_1^1

$S^2 = \{(0,0)(1,2)(2,3)(3,5)\}$

Include 3rd object

$S_1^2 = \{ (0+5,0+4) (1+5, 2+4)(2+5,3+4) (3+5,5+4) \}$

$= \{(5,4)(6,6)(7,7)(8,9)\}$

Next Stage can be obtained S^{2+1} (S^3)can be computed by merging from S^2 and S_1^2

$S^3 = \{(0,0)(1,2)(2,3)(3,5)\ (5,4)(6,6)(7,7)(8,9)\}$

Apply Purging rule

Pairs(3,5) (7,7)(8,9) will be discarded

Therefore ,

$S^3 = \{(0,0)(1,2)(2,3)(5,4)(6,6)\}$

$X=(1,0,1)$

0/1 KNAPSACK ALGORITHM

Algorithm DKP(p,w,n,m)

{

S^0 := {(0,0)};

for i := 1 to n-1 do

{

$S^{i-1} = \{(P,W) | (P-p_i, W-w_i) \in S^{i-1}$ and $W <= m\}$;

S^i = MergePurge(S^{i-1} , S_1^{i-1});

}

(PX,WX) = last pair in S^{n-1} ;

(PY,WY)=(P^1+ p_n ,W^1+w_n) where W^1 is the largest W in any pair in S^{n-1} such that W+ w_n <= m;

// Trace back for x_n ,x_{n-1},…..,x_1.

if (PX > PY) then x_n = 0;

else x_n=1;

TraceBackFor(x_{n-1},…..,x_1);

}

COMPLEXITY ANALYSIS

- The complexity of the 0/1 knapsack algorithm

The complexity of the algorithm is $O(n^2)$.

Example 4.3 : *Solve Knapsack instance $M = 8$, and $n = 4$. Let P_i and W_i are as shown below.*

i	P_i	W_i
1	1	2
2	2	3
3	5	4
4	6	5

Solution : Let us build the sequence of decision S^0, S^1, S^2.

$$S^0 = \{(0, 0)\} \text{ initially}$$

$$S_1^0 = \{(1, 2)\}$$

That means while building S_1^0 we select the next i^{th} pair. For S_1^0 we have selected first (P, W) pair which is (1, 2).

Now

$$S^1 = \{\text{Merge } S^0 \text{ and } S_0^1\}$$

$$= \{(0, 0), (1, 2)\}$$

$$S_1^1 = \{\text{ Select next (P, W) pair and add it with } S^1 \}$$

$$= \{ (2, 3), (2+0, 3+0), (2+1, 3+2) \}$$

$S_1^1 = \{(2, 3), (3, 5)\}$ $\because$ Repetition of (2, 3) is avoided.

$S^2 = \{\text{Merge candidates from } S^1 \text{ and } S_1^1\}$

$= \{(0, 0), (1, 2), (2, 3),(3, 5)\}$

$\therefore$ $S_1^2 = \{\text{Select next (P, W) pair and add it with } S^2\}$

$= \{(5, 4), (6, 6), (7, 7), (8, 9)\}$

Now $S^3 = \{\text{Merge candidates from } S^2 \text{ and } S_1^2\}$

$S^3 = \{(0, 0), (1, 2), (2, 3), (5, 4), (6, 6), (7, 7), (8, 9)\}$

Note that the pair (3, 5) is purged from S^3. This is because, let us assume $(P_j, W_j) = (3, 5)$ and $(P_k, W_k) = (5, 4)$. Here $P_j \le P_k$ and $W_j > W_k$ is true hence we will eliminate pair (P_j, W_j) i.e. (3, 5) from S^3.

$S_1^3 = \{(6, 5), (7, 7), (8, 8), (11, 9), (12, 11), (13, 12), (14, 14)\}$

$$S^4 = \begin{array}{l}\{(0,0), (1,2), (2,3), (5,4), (6,6), (7,7), (8,9) \\ (6,5), (8,8), (11,9), (12,11), (13,12), (14,14)\}\end{array}$$

Now we are interested in M = 8. We get pair (8, 8) in S^4. Hence we will set $x_4 = 1$. Now to select next object $(P - P_4)$ and $(W - W_4)$.

i.e (8 – 6) and (8 – 5).

i.e. (2, 3)

Pair $(2, 3) \in S^2$. Hence set $x_2 = 1$. So we get the final solution as (0, 1, 0, 1).

Consider the knapsack instance $n = 3$, $m=50$, $(w_1, w_2, w_3) = (10,20,30)$ & $(p_1, p_2, p_3) = (60, 100, 120)$.

SAMPLE QUESTIONS

- Differentiate between fractional knapsack and 0/1 knapsack
- State o/1 knapsack problem
- Consider the knapsack instance n = 3, m=50, (w1, w2, w3) = (10,20,30) & (p1, p2, p3) = (60, 100, 120).
- Consider the knapsack instance n = 5, m=100, (w1, w2, w3,w4,w5) = (10,20,30,40,50) & (p1, p2, p3,p4,p5) = (60, 100, 120,140,150).