SUBJECTCODE: 23MT2005

PROBABILITY STATISTICS AND QUEUING THEORY

Tutorial 2:

- **Make use of Random Variables to determine the probability functions.**
- **Calculate the Expected value and variance of a random variable**

Date of the Session: //______________________________Time of the Session:______to________

<u>Learning outcomes:</u>

- **Classify the random variables and their probability functions**
- **Calculate the mean and variance of a random variable.**

1. Determine the value of c so that each of the following functions can serve as a probability distribution of the discrete random variable X

a) $f(x) = c(x^2 + 4),\ \ for\ x = 0,1,2,3;$

b) $f(x) = c\binom{2}{x}\binom{3}{3-x},\ \ for\ x = 0,1,2.$

Solution

Part (a)

The probability function is:

$$f(x) = c(x^2 + 4) \quad \text{for } x = 0, 1, 2, 3$$

Step 1: Calculate $f(x)$ for each value of x

- For $x = 0$:

$$f(0) = c(0^2 + 4) = c(0 + 4) = 4c$$

- For $x = 1$:

$$f(1) = c(1^2 + 4) = c(1 + 4) = 5c$$

- For $x = 2$:

$$f(2) = c(2^2 + 4) = c(4 + 4) = 8c$$

- For $x = 3$:

$$f(3) = c(3^2 + 4) = c(9 + 4) = 13c$$

Step 2: Sum the probabilities

Now, sum the probabilities for $x = 0, 1, 2, 3$:

$$f(0) + f(1) + f(2) + f(3) = 4c + 5c + 8c + 13c = 30c$$

Step 3: Set the sum equal to 1

To ensure this is a valid probability distribution, the sum of all probabilities must be equal to 1:

$$30c = 1$$

Step 4: Solve for c

$$c = \frac{1}{30}$$

Part (b)

The probability function is:

$$f(x) = c \cdot \frac{2}{x} \cdot (3 - x) \quad \text{for } x = 0, 1, 2$$

Step 1: Check $f(0)$

For $x = 0$, the term $\frac{2}{0}$ makes the probability undefined. So, we exclude $x = 0$ from the distribution.

Step 2: Calculate $f(x)$ for $x = 1$ and $x = 2$

- For $x = 1$:

$$f(1) = c \cdot \frac{2}{1} \cdot (3 - 1) = c \cdot 2 \cdot 2 = 4c$$

- For $x = 2$:

$$f(2) = c \cdot \frac{2}{2} \cdot (3 - 2) = c \cdot 1 \cdot 1 = c$$

Step 3: Sum the probabilities

Now, sum the probabilities for $x = 1$ and $x = 2$:

$$f(1) + f(2) = 4c + c = 5c$$

Step 4: Set the sum equal to 1

To ensure this is a valid probability distribution:

$$5c = 1$$

Step 5: Solve for c

$$c = \frac{1}{5}$$

Final Answer Summary:

- For **Part (a)**, $c = \frac{1}{30}$.
- For **Part (b)**, $c = \frac{1}{5}$, with the exclusion of $x = 0$ since $f(0)$ is undefined.

2. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \end{cases}$$

Determine i) a ii) Compute P(X≤1.5).

Solution

Step 1: Ensure the PDF integrates to 1

The total probability for a continuous random variable X is:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

The given piecewise function is:

$$f(x) = \begin{cases} ax, & 0 \le x \le 1, \\ a, & 1 \le x \le 2, \\ -ax + 3a, & 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Integrate the PDF across the defined intervals $[0,1]$, $[1,2]$, and $[2,3]$:

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax + 3a)\,dx = 1.$$

Step 1.1: Integrate each part

1. For $[0, 1]$:

$$\int_0^1 ax\,dx = \left[\frac{ax^2}{2}\right]_0^1 = \frac{a(1)^2}{2} - \frac{a(0)^2}{2} = \frac{a}{2}.$$

2. For $[1, 2]$:

$$\int_1^2 a\,dx = [ax]_1^2 = a(2) - a(1) = a.$$

3. For $[2, 3]$:

$$\int_2^3 (-ax + 3a)\,dx = \int_2^3 -ax\,dx + \int_2^3 3a\,dx.$$

$$\int_2^3 -ax\,dx = \left[-\frac{ax^2}{2}\right]_2^3 = -\frac{a(3)^2}{2} + \frac{a(2)^2}{2} = -\frac{9a}{2} + \frac{4a}{2} = -\frac{5a}{2}.$$

$$\int_2^3 3a\,dx = [3ax]_2^3 = 3a(3) - 3a(2) = 9a - 6a = 3a.$$

So, the total for $[2, 3]$ is:

$$-\frac{5a}{2} + 3a = \frac{a}{2}.$$

Step 1.2: Combine and solve for a

The total integral is:

$$\frac{a}{2} + a + \frac{a}{2} = 1.$$

Simplify:

$$\frac{a}{2} + \frac{a}{2} + a = 1 \quad \Longrightarrow \quad 2a = 1 \quad \Longrightarrow \quad a = \frac{1}{2}.$$

Step 2: Compute $P(X \leq 1.5)$

To compute $P(X \leq 1.5)$, split the integral at $x = 1$:

$$P(X \leq 1.5) = \int_0^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx.$$

1. For $\int_0^1 f(x)\,dx$:

$$\int_0^1 ax\,dx = \int_0^1 \frac{1}{2}x\,dx = \left[\frac{1}{4}x^2\right]_0^1 = \frac{1}{4}(1)^2 - \frac{1}{4}(0)^2 = \frac{1}{4}.$$

2. For $\int_1^{1.5} f(x)\,dx$: Here, $f(x) = a$ for $1 \leq x \leq 2$. So:

$$\int_1^{1.5} f(x)\,dx = \int_1^{1.5} \frac{1}{2}\,dx = \frac{1}{2} \cdot (1.5 - 1) = \frac{1}{2} \cdot 0.5 = \frac{1}{4}.$$

Step 3: Combine results

$$P(X \leq 1.5) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Final Answers:

1. $a = \frac{1}{2}$,
2. $P(X \leq 1.5) = \frac{1}{2}$.

3. A continuous random variable X that can assume values between x=2 and x=5 has a density function given by $f(x) = \dfrac{2(1+x)}{27}$. Find

i) Verify the validation of the density functions
ii) Find the cumulative distribution function
iii) P(X<4)
iv) P(3≤X≤4).

Solution

Part i) Verify the validation of the density function

The total probability for a valid PDF must equal 1:

$$\int_2^5 f(x)\,dx = 1.$$

Substitute $f(x) = \frac{2(1+x)}{27}$:

$$\int_2^5 \frac{2(1+x)}{27}\,dx = \frac{2}{27}\int_2^5 (1+x)\,dx.$$

Split the integral:

$$\frac{2}{27}\int_2^5 (1+x)\,dx = \frac{2}{27}\left[\int_2^5 1\,dx + \int_2^5 x\,dx\right].$$

1. First integral:

$$\int_2^5 1\,dx = [x]_2^5 = 5 - 2 = 3.$$

2. Second integral:

$$\int_2^5 x\,dx = \left[\frac{x^2}{2}\right]_2^5 = \frac{(5)^2}{2} - \frac{(2)^2}{2} = \frac{25}{2} - \frac{4}{2} = \frac{21}{2}.$$

Combine the results:

$$\frac{2}{27}\left(3 + \frac{21}{2}\right) = \frac{2}{27}\left(\frac{6}{2} + \frac{21}{2}\right) = \frac{2}{27}\cdot\frac{27}{2} = 1.$$

Thus, the density function is valid.

Part ii) Find the cumulative distribution function (CDF)

The cumulative distribution function $F(x)$ is defined as:

$$F(x) = P(X \le x) = \int_2^x f(t)\,dt, \quad \text{for } x \in [2,5].$$

Substitute $f(t) = \frac{2(1+t)}{27}$:

$$F(x) = \int_2^x \frac{2(1+t)}{27}\,dt = \frac{2}{27}\int_2^x (1+t)\,dt.$$

Split the integral:

$$\frac{2}{27}\int_2^x (1+t)\,dt = \frac{2}{27}\left[\int_2^x 1\,dt + \int_2^x t\,dt\right].$$

1. First integral:

$$\int_2^x 1\,dt = [t]_2^x = x - 2.$$

2. Second integral:

$$\int_2^x t\,dt = \left[\frac{t^2}{2}\right]_2^x = \frac{x^2}{2} - \frac{(2)^2}{2} = \frac{x^2}{2} - 2.$$

Combine the results:

$$F(x) = \frac{2}{27}\left((x-2) + \frac{x^2}{2} - 2\right) = \frac{2}{27}\left(\frac{x^2}{2} + x - 4\right).$$

Simplify:

$$F(x) = \frac{1}{27}x^2 + \frac{2}{27}x - \frac{8}{27}.$$

For $x < 2$, $F(x) = 0$, and for $x > 5$, $F(x) = 1$.

Thus, the CDF is:

$$F(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{27}x^2 + \frac{2}{27}x - \frac{8}{27}, & 2 \le x \le 5, \\ 1, & x > 5. \end{cases}$$

Part iii) Find $P(X < 4)$

$$P(X < 4) = F(4).$$

Substitute $x = 4$ into the CDF:

$$F(4) = \frac{1}{27}(4)^2 + \frac{2}{27}(4) - \frac{8}{27}.$$

Simplify:

$$F(4) = \frac{1}{27}(16) + \frac{2}{27}(4) - \frac{8}{27} = \frac{16}{27} + \frac{8}{27} - \frac{8}{27}.$$

$$F(4) = \frac{16}{27}.$$

So:

$$P(X < 4) = \frac{16}{27}.$$

Part iv) Find $P(3 \leq X \leq 4)$

$$P(3 \leq X \leq 4) = F(4) - F(3).$$

Substitute $x = 4$ and $x = 3$ into the CDF:

1. For $F(4)$:

$$F(4) = \frac{16}{27}.$$

2. For $F(3)$:

$$F(3) = \frac{1}{27}(3)^2 + \frac{2}{27}(3) - \frac{8}{27} = \frac{1}{27}(9) + \frac{2}{27}(3) - \frac{8}{27}.$$

$$F(3) = \frac{9}{27} + \frac{6}{27} - \frac{8}{27} = \frac{7}{27}.$$

Now:

$$P(3 \leq X \leq 4) = F(4) - F(3) = \frac{16}{27} - \frac{7}{27} = \frac{9}{27} = \frac{1}{3}.$$

Final Answers:

i) The PDF is valid.

ii) The CDF is:

$$F(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{27}x^2 + \frac{2}{27}x - \frac{8}{27}, & 2 \leq x \leq 5, \\ 1, & x > 5. \end{cases}$$

iii) $P(X < 4) = \frac{16}{27}$.

iv) $P(3 \leq X \leq 4) = \frac{1}{3}$.

4. A Random variable X can assume 0,1,2,3,4. A Probability distribution is shown here then find

a) P(X=3) b) P(X≥ 2) c) CDF

Solution

Given Assumed Probability Distribution:

$$P(X=0)=0.1$$
$$P(X=1)=0.2$$
$$P(X=2)=0.3$$
$$P(X=3)=0.25$$
$$P(X=4)=0.15$$

a) Find $P(X=3)$

- From the assumed distribution, we have $P(X=3)=0.25$.
- **Correct answer**: $P(X=3)=0.25$.

b) Find $P(X\geq 2)$

To find $P(X\geq 2)$, sum the probabilities for $X=2,3$, and 4:

$$P(X\geq 2)=P(X=2)+P(X=3)+P(X=4)=0.3+0.25+0.15=0.7$$

- **Correct answer**: $P(X\geq 2)=0.7$.

c) Find the Cumulative Distribution Function (CDF)

The CDF $F(x)$ is calculated as:

- $F(0)=0.1$
- $F(1)=0.3$
- $F(2)=0.6$
- $F(3)=0.85$
- $F(4)=1.0$
- For $x>4$, $F(x)=1$.

5. Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2. Or 3power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.

Solution

Given Information:

The probabilities for $X = 0, 1, 2, 3$ are:

- $P(X = 0) = 0.4$
- $P(X = 1) = 0.3$
- $P(X = 2) = 0.2$
- $P(X = 3) = 0.1$

Step 1: Calculate the mean $E(X)$

$$E(X) = (0 \cdot 0.4) + (1 \cdot 0.3) + (2 \cdot 0.2) + (3 \cdot 0.1)$$

$$E(X) = 0 + 0.3 + 0.4 + 0.3 = 1.0$$

So, the **mean** is:

$$E(X) = 1.0.$$

Step 2: Calculate $E(X^2)$

$$E(X^2) = (0^2 \cdot 0.4) + (1^2 \cdot 0.3) + (2^2 \cdot 0.2) + (3^2 \cdot 0.1)$$

$$E(X^2) = (0 \cdot 0.4) + (1 \cdot 0.3) + (4 \cdot 0.2) + (9 \cdot 0.1)$$

$$E(X^2) = 0 + 0.3 + 0.8 + 0.9 = 2.0$$

So, $E(X^2) = 2.0$.

Step 3: Calculate the variance $\mathrm{Var}(X)$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 2.0 - (1.0)^2 = 2.0 - 1.0 = 1.0$$

So, the **variance** is:

$$\mathrm{Var}(X) = 1.0.$$

Final Answers:

- The **mean** of X is 1.0.
- The **variance** of X is 1.0.

6. The random variable X representing the number of errors per 100 lines of software code has the following probability distribution:

X	2	3	4	5	6
f(x)	0.01	0.25	0.4	0.3	0.04

i) Find Mean, standard deviation and Variance of X.

ii) Obtain the mean and variance of discrete random variable Z=3X-2, when X represents the number of errors per 100 lines of code

Solution

Part i) Find Mean, Standard Deviation, and Variance of X

The probability distribution for the random variable X is given as:

X	$f(X)$
2	0.01
3	0.25
4	0.4
5	0.3
6	0.04

To find the **mean, variance**, and **standard deviation** of X, we use the following formulas:

1. Mean (Expected Value) of X:

$$E(X) = \sum_x P(X = x) \cdot x$$

$$E(X) = (2 \cdot 0.01) + (3 \cdot 0.25) + (4 \cdot 0.4) + (5 \cdot 0.3) + (6 \cdot 0.04)$$

$$E(X) = 0.02 + 0.75 + 1.6 + 1.5 + 0.24 = 4.11$$

So, the **mean** $E(X) = 4.11$.

2. Variance of X:

The variance $\mathrm{Var}(X)$ is given by:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

First, we calculate $E(X^2)$:

$$E(X^2) = (2^2 \cdot 0.01) + (3^2 \cdot 0.25) + (4^2 \cdot 0.4) + (5^2 \cdot 0.3) + (6^2 \cdot 0.04)$$

$$E(X^2) = (4 \cdot 0.01) + (9 \cdot 0.25) + (16 \cdot 0.4) + (25 \cdot 0.3) + (36 \cdot 0.04)$$

$$E(X^2) = 0.04 + 2.25 + 6.4 + 7.5 + 1.44 = 17.63$$

Now, we can find the variance:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 17.63 - (4.11)^2$$

$$\mathrm{Var}(X) = 17.63 - 16.8921 = 0.7379$$

So, the **variance** $\mathrm{Var}(X) = 0.7379$.

3. Standard Deviation of X:

The standard deviation is the square root of the variance:

$$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{0.7379} \approx 0.858$$

So, the **standard deviation** $\mathrm{SD}(X) \approx 0.858$.

Part ii) Find the Mean and Variance of the Discrete Random Variable $Z = 3X - 2$

For the random variable $Z = 3X - 2$, we use the following properties of expectations and variances:

1. Mean of Z:

$$E(Z) = E(3X - 2) = 3E(X) - 2$$

Substitute $E(X) = 4.11$:

$$E(Z) = 3 \times 4.11 - 2 = 12.33 - 2 = 10.33$$

So, the **mean** of Z is $E(Z) = 10.33$.

2. Variance of Z:

$$\text{Var}(Z) = \text{Var}(3X - 2) = 3^2 \cdot \text{Var}(X)$$

Substitute $\text{Var}(X) = 0.7379$:

$$\text{Var}(Z) = 9 \times 0.7379 = 6.6411$$

So, the **variance** of Z is $\text{Var}(Z) = 6.6411$.

Final Answers:

For Part i):

- Mean of X: $E(X) = 4.11$
- Variance of X: $\text{Var}(X) = 0.7379$
- Standard Deviation of X: $\text{SD}(X) \approx 0.858$

For Part ii):

- Mean of $Z = 3X - 2$: $E(Z) = 10.33$
- Variance of $Z = 3X - 2$: $\text{Var}(Z) = 6.6411$

7. If A dealer's profit in units of $5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Find the average profit per automobile and its standard deviation

PDF:

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Step 1: Find the Mean (Expected Value) of X

The expected value $E(X)$ is computed as:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Since $f(x) = 0$ for $x \notin (0, 1)$, we only need to integrate over the interval $[0, 1]$:

$$E(X) = \int_0^1 x \cdot 2(1-x)\, dx$$

First, expand the integrand:

$$E(X) = \int_0^1 2x(1-x)\, dx = \int_0^1 (2x - 2x^2)\, dx$$

Now, integrate each term:

$$E(X) = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = (1^2 - \frac{2(1^3)}{3}) - (0^2 - \frac{2(0^3)}{3})$$

$$E(X) = (1 - \frac{2}{3}) = \frac{1}{3}.$$

So, the **mean** profit per automobile is $\frac{1}{3} \times 5000 = 1666.67$ dollars.

Step 2: Find the Variance of X

The variance $\mathrm{Var}(X)$ is given by:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2.$$

We first need to compute $E(X^2)$:

$$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)\,dx = \int_0^1 2x^2(1-x)\,dx$$

Expand the integrand:

$$E(X^2) = \int_0^1 2x^2(1-x)\,dx = \int_0^1 (2x^2 - 2x^3)\,dx$$

Now, integrate each term:

$$E(X^2) = \left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1 = \left(\frac{2(1^3)}{3} - \frac{2(1^4)}{4}\right) - \left(\frac{2(0^3)}{3} - \frac{2(0^4)}{4}\right)$$

$$E(X^2) = \left(\frac{2}{3} - \frac{2}{4}\right) = \frac{2}{3} - \frac{1}{2} = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}.$$

Now, compute the variance:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9}$$

Find a common denominator:

$$\mathrm{Var}(X) = \frac{3}{18} - \frac{2}{18} = \frac{1}{18}.$$

So, the variance of X is $\frac{1}{18}$.

Step 3: Find the Standard Deviation of X

The standard deviation is the square root of the variance:

$$\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{\frac{1}{18}} \approx 0.2357.$$

Now, to find the standard deviation in terms of dollars, multiply by 5000:

$$\text{SD}(X) = 0.2357 \times 5000 \approx 1178.53 \text{ dollars.}$$

Final Answers:

- **Mean profit per automobile:** 1666.67 dollars.
- **Standard deviation of the profit:** 1178.53 dollars.

VIVA QUESTIONS:

1. What is the key difference between a discrete and continuous random variable?

- **Discrete:** Takes specific, countable values (e.g., 0, 1, 2).
- **Continuous:** Takes any value within a range, including fractions (e.g., 0.5, 1.23).

2. Define the probability mass function (PMF) for a discrete random variable.

For a discrete random variable, the PMF gives the probability that the variable takes a specific value x. It must satisfy $P(X = x) \geq 0$ and $\sum P(X = x_i) = 1$.

3. How do you calculate the cumulative distribution function (CDF) for a random variable?

The CDF gives the probability that a random variable X is less than or equal to a specific value x:

- **Discrete:** $F(x) = \sum_{i \leq x} P(X = x_i)$
- **Continuous:** $F(x) = \int_{-\infty}^{x} f(t)\, dt$

4. Explain the concept of expected value for a random variable?

The expected value (mean) of a random variable is the long-run average value:

- **Discrete:** $E(X) = \sum_i x_i P(X = x_i)$
- **Continuous:** $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

(For Evaluators use only)

Comment of the Evaluator (if Any)	Evaluator's Observation
	Marks Secured:____________out of____________
	Full Name of the Evaluator:
	Signature of the Evaluator:
	Date of Evaluation