MATHEMATICAL PROGRAMMING

CO3

NON-LINEAR PROGRAMMING : WOLF METHOD

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Wolfe's Method

It is used for solving the Quadratic programming problem (QPP)

The general form of the QPP is

$$f(x) = cx + \frac{1}{2}x^T Q\, x$$

linear

$$\text{s.t. } Ax \leq b \text{ and } x \geq 0$$

where Q is a symmetric matrix and b, c are the real vectors.

Important features:

1) The function $x^T Q x$ defines a quadratic form.

2) The constraints are assumed to be LINEAR, which ensures the convex solution space.

Kuhn-Tucker conditions for QPP

Consider a QPP

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) \leq 0, \quad x \geq 0$$

Convert all constraints into equality.

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) + S^2 = 0 \longrightarrow \text{Slack}$$

$$x \geq 0$$

Kuhn-Tucker conditions for QPP

Consider a QPP

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) \le 0, \quad x \ge 0$$

Convert all constraints into equality.

$$\text{Maximize } Z = f(x)$$

$$\text{s.t. } g(x) + S^2 = 0$$

$$x \ge 0$$

The Lagrangian function is

$$L(x, \lambda, S) = f(x) - \lambda(g(x) + S^2)$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow \nabla f - \lambda \nabla g = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow g(x) + S^2 = 0$$

$$\frac{\partial L}{\partial S} = 0 \quad \Rightarrow \boxed{\lambda S = 0}$$

$$\lambda, S, x \ge 0$$

Steps involved in Wolfe's Method

Step 1: Write all constraints in $\leq$ sign

Step 2: Convert all constraints into Equality by adding slack variables S_i^2 in the i^{th} constraints.

Step 3: Obtain Kuhn-Tucker conditions:

Construct the Lagrangian function

$$L = f(x) - \lambda(g(x) + S^2) - \mu(-x + S^2)$$

The necessary and sufficient conditions are:

$$\frac{\partial L}{\partial X} = 0; \frac{\partial L}{\partial \lambda} = 0 ; \frac{\partial L}{\partial \mu} = 0 ; \frac{\partial L}{\partial S} = 0$$

Take $s_i = S_i^2$, and derive the Kuhn-Tucker conditions.

Step 4: Construct the modified LPP using artificial variables.

Step 5: Solve by Simplex algorithm by Two-phase method.

Step 6: The optimal solution obtained in Step 5 is an optimal solution to the given QPP also.

Example: Apply Wolfe's method, solve the following quadratic programming problem

$$\text{Maximize } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

$$\text{s.t. } \boxed{x_1 + 2x_2 \le 2}\ ;\ \ x_1, x_2 \boxed{\ge} 0$$

Solution:

Step 1: Write all constraints in $\boxed{\le}$ sign

$$x_1 + 2x_2 \le 2\ ;$$
$$-x_1 \le 0\ ;$$
$$-x_2 \le 0$$

Step 2: Convert into Equality:

$$\text{Maximize } Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

s.t.

$$x_1 + 2x_2 + S_1^2 = 2 \quad \lambda_1$$
$$-x_1 + S_2^2 = 0\ ; \quad \mu_1$$
$$-x_2 + S_3^2 = 0 \quad \mu_2$$
$$x_i, S_i \ge 0$$

Step 3: Obtain Kuhn-Tucker conditions:

Construct the Lagrangian function

$$L(x_1, x_2, S_1, S_2, S_3, \lambda_1, \mu_1, \mu_2)$$
$$= (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$$
$$- \lambda_1(x_1 + 2x_2 + S_1^2 - 2) - \mu_1(-x_1 + S_2^2)$$
$$- \mu_2(-x_2 + S_3^2)$$

Step 3: Obtain Kuhn-Tucker conditions:

Construct the Lagrangian function

$$L(x_1, x_2, S_1, S_2, S_3, \lambda_1, \mu_1, \mu_2)$$
$$= (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$$
$$- \lambda_1(x_1 + 2x_2 + S_1^2 - 2) - \mu_1(-x_1 + S_2^2)$$
$$- \mu_2(-x_2 + S_3^2)$$

The necessary and sufficient conditions are:

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -x_1 - 2x_2 - S_1^2 + 2 = 0$$

Take $s_i = S_i^2$, we have

K.T.

$$\boxed{\lambda_1 s_1 = 0, \quad \mu_1 x_1 = 0, \quad \mu_2 x_2 = 0}$$

and $x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2 \geq 0$

Step 4: Construct the modified LPP

$$\text{Maximize } Z = -A_1 - A_2$$

$$\text{s.t. } 4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$$

$$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$$

$$x_1 + 2x_2 + s_1 = 2$$

$$\text{And } \lambda_1 s_1 = 0, \quad \mu_1 x_1 = 0, \mu_2 x_2 = 0$$

Step 3: Obtain Kuhn-Tucker conditions:

Construct the Lagrangian function

$$L(x_1, x_2, S_1, S_2, S_3, \lambda_1, \mu_1, \mu_2)$$
$$= (4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$$
$$- \lambda_1(x_1 + 2x_2 + S_1^2 - 2) - \mu_1(-x_1 + S_2^2)$$
$$- \mu_2(-x_2 + S_3^2)$$

The necessary and sufficient conditions are:

$$\frac{\partial L}{\partial x_1} = 4 - 4x_1 - 2x_2 - \lambda_1 + \mu_1 = 0$$

$4x_1 + 2x_2 + \lambda - \mu_1 = 4$

$$\frac{\partial L}{\partial x_2} = 6 - 2x_1 - 4x_2 - 2\lambda_1 + \mu_2 = 0$$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 = 6$

$$\frac{\partial L}{\partial \lambda_1} = -x_1 - 2x_2 - S_1^2 + 2 = 0$$

Take $s_i = S_i^2$, we have

K.T.

$$\boxed{\lambda_1 s_1 = 0, \quad \mu_1 x_1 = 0, \quad \mu_2 x_2 = 0}$$

and $x_1, x_2, s_1, \lambda_1, \mu_1, \mu_2 \geq 0$

Step 5: Solve by Simplex algorithm

c_j	0	0	0	0	0	-1	-1	0	
BV	x_1	x_2	λ_1	μ_1	μ_2	A_1	A_2	s_1	Sol
$z_j - c_j$	-6	-6	-3	1	1	0	0	0	-10
A_1	$\boxed{4}$	2	1	-1	0	1	0	0	4
A_2	2	4	2	0	-1	0	1	0	6
s_1	2	0	0	0	0	0	0	1	2
$z_j - c_j$	0				1		0	0	-4
x_1	1	1/2	1/4	-1/4	0	1/4	0	0	1
A_2	0			1/2			1	0	
s_1	0	•					0	1	

Maximize $Z = -A_1 - A_2$

s.t. $4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$

$x_1 + 2x_2 + s_1 = 2$

and $\lambda_1 s_1 = 0,\ \mu_1 x_1 = 0, \mu_2 x_2 = 0$

$\begin{vmatrix} \boxed{4} & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = \frac{2}{4}$

$\begin{vmatrix} -6 & 1 \\ \boxed{4} & 0 \end{vmatrix} = 4 + 0 = 4$

$\begin{vmatrix} -6 & -10 \\ \boxed{4} & 4 \end{vmatrix} = -40 + 24 = -16/4$

Step 5: Solve by Simplex algorithm

c_j	0	0	0	0	0	-1	-1	0	
BV	x_1	x_2	λ_1	μ_1	μ_2	A_1	A_2	s_1	Sol
$z_j - c_j$	-6	-6	-3	1	1	0	0	0	-10
A_1	$\boxed{4}$	2	1	-1	0	1	0	0	4
A_2	2	4	2	0	-1	0	1	0	6
s_1	2	0	0	0	0	0	0	1	2
$z_j - c_j$	0	-3	-3/2	-1/2	1	3/2	0	0	-4
x_1	1	½	¼	-1/4	0	¼	0	0	1
A_2	0	3	3/2	½	-1	-1/2	1	0	4
s_1	0	$\boxed{3/2}$	-1/4	1/4	0	-1/4	0	1	1

2
4/3
2/3

Maximize $Z = -A_1 - A_2$

s.t. $4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$

$x_1 + 2x_2 + s_1 = 2$

and $\boxed{\lambda_1 s_1 = 0,\ \mu_1 x_1 = 0, \mu_2 x_2 = 0}$

Step 5: Solve by Simplex algorithm

c_j	0	0	0	0	0	-1	-1	0	
BV	x_1	x_2	λ_1	μ_1	μ_2	A_1	A_2	s_1	Sol
$z_j - c_j$	-6	-6	-3	1	1	0	0	0	-10
A_1	$\boxed{4}$	2	1	-1	0	1	0	0	4
A_2	2	4	2	0	-1	0	1	0	6
s_1	2	0	0	0	0	0	0	1	2
$z_j - c_j$	0	-3	-3/2	-1/2	1	3/2	0	0	-4
x_1	1	½	¼	-1/4	0	¼	0	0	1
A_2	0	$\boxed{3}$	3/2	½	-1	-1/2	1	0	4
s_1	0	3/2	-1/4	1/4	0	-1/4	0	1	1
$z_j - c_j$	0	0	-2	0	1	1	0	2	-2
x_1	1	0	1/3	-1/3	0	1/3	0	-1/3	2/3
A_2	0	0	2	0	-1	0	1	-2	2
x_2	0	1	-1/6	1/6	0	-1/6	0	2/3	2/3

Maximize $Z = -A_1 - A_2$

s.t. $4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$

$x_1 + 2x_2 + s_1 = 2$

and $\boxed{\lambda_1 s_1 = 0,\ \mu_1 x_1 = 0, \mu_2 x_2 = 0}$

Step 5: Solve by Simplex algorithm

c_j	0	0	0	0	0	-1	-1	0	
BV	x_1	x_2	λ_1	μ_1	μ_2	A_1	A_2	s_1	Sol
$z_j - c_j$	-6	-6	-3	1	1	0	0	0	-10
A_1	$\boxed{4}$	2	1	-1	0	1	0	0	4
A_2	2	4	2	0	-1	0	1	0	6
s_1	2	0	0	0	0	0	0	1	2
$z_j - c_j$	0	-3	-3/2	-1/2	1	3/2	0	0	-4
x_1	1	½	¼	-1/4	0	¼	0	0	1
A_2	0	$\boxed{3}$	3/2	½	-1	-1/2	1	0	4
s_1	0	3/2	-1/4	1/4	0	-1/4	0	1	1
$z_j - c_j$	0	0	-2	0	1	1	0	2	-2
x_1	1	0	1/3	-1/3	0	1/3	0	-1/3	2/3
A_2	0	0	$\boxed{2}$	0	-1	0	1	-2	2
x_2	0	1	-1/6	1/6	0	-1/6	0	2/3	2/3

Maximize $Z = -A_1 - A_2$

s.t. $4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$

$x_1 + 2x_2 + s_1 = 2$

and $\boxed{\lambda_1 s_1 = 0,\ \mu_1 x_1 = 0, \mu_2 x_2 = 0}$

$z_j - c_j$	0	0	0	0	0	1	1	0	0
x_1	1	0	0	+1/3	1/6	1/3	+1/6	0	1/3
λ_1	0	0	1	0	-1/2	0	½	-1	1
x_2	0	1	0	1/6	-1/2	-1/6	1/2	1/2	5/6

$x_1 = 1/3 \quad x_2 = 5/6$

Since all $z_j - c_j \geq 0$. Thus, it is optimal and solution is

$$\boxed{x_1 = \frac{1}{3}, x_2 = \frac{5}{6}}$$

and Value of Z is

$$Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

$$= \frac{25}{6}$$