Department of AI & DS
CSE and CS&IT

COURSE NAME: PROBABILITY, STATISTICS AND QUEUING THEORY

COURSE CODE: 23MT2005

Topic

Queuing Model-I

Session – 21

AIM OF THE SESSION

To familiarize students with the basic concept of Birth and death model of Queuing theory

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

1. Define Model 1
2. Describe the Characteristics of Queuing model1
3. Describe the performance measures of birth and death process

LEARNING OUTCOMES

At the end of this session, you should be able to:

1. Define Queue and its characteristics
2. Describe the characteristics of queuing theory and queue discipline
3. Summarize the performance measures

Model 1

In this system arrivals are Poisson, service times are exponential, single server and unlimited waiting space

To find steady state probability distribution of this model we write the steady state equations of the model.

The probability that there will be 'n' units (n>0) in the system at a time (t+Δt) may be expressed as the sum of three independent compound probabilities.

$\rho = \frac{\lambda}{\mu}$ <1 is the system utilization factor or traffic intensity or the fraction of time the server is busy.

steady state probability of 'n' customers in the system $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0,$ if n>0

$$P_0 = 1 - \frac{\lambda}{\mu}$$

Model 1

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = P_n(1 - \rho)$$

Expected number of customer in the system (Ls)

$$\text{Ls} = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\rho}{(1-\rho)}$$

Expected number of customers in the queue (L_q)

$$\text{Lq} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1 - \frac{\lambda}{\mu}} = \frac{\rho^2}{(1-\rho)}$$

Model 1

$$P(w = 0) = P\text{ (no unit in the system)} = P_0 = 1 - \frac{\lambda}{\mu}$$

Hence the complete distribution for waiting time is partly continuous and partly discrete.

$$\therefore \quad \psi(w)dw = \begin{cases} \lambda\left(1 - \dfrac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}dw, & w > 0 \\ 1 - \dfrac{\lambda}{\mu} & , w = 0 \end{cases}$$

Expected waiting time in the queue $W_q = \int_0^\infty w\,\varphi(w)dw = \int_0^\infty \lambda(1 - \frac{\lambda}{\mu})e^{-(\mu-\lambda)w}dw =$

$$= \lambda/\mu(\mu - \lambda)$$

Model 1

Probability distribution of busy period

When the server is busy, a new arrival has to wait (w > 0)

The probability distribution is given by

$$\psi(w/w>0)dw = \frac{\psi(w)dw}{P(w>0)} = \frac{\psi(w)dw}{\int_0^\infty \psi(w)dw}$$

$$= \frac{\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}}{\int_0^\infty \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}dw} = \frac{\lambda(\mu-\lambda)e^{-(\mu-\lambda)w}dw}{\lambda/\mu} \qquad = (\mu-\lambda)\,e^{-(\mu-\lambda)w}dw$$

This is the probability density function for the busy period.

Model 1

Expected waiting time of a customer in the queue who has to wait

$$E(W/W>0)=\int_0^\infty w\psi(w>0)\,dw=\int_0^\infty w(\mu-\lambda)e^{-(\mu-\lambda)w}dw \;=\frac{1}{\mu-\lambda}\quad \text{(or)}$$

$$E(W/W>0)=\frac{W_q}{\rho(W>0)}=\frac{\dfrac{\lambda}{\mu(\mu-\lambda)}}{\lambda/\mu}=\frac{1}{\mu-\lambda}$$

Expected length of a non-empty queue, (L/L >0)

$$E(L/L>0)=L_s/\Pr of\;\;(\text{an arrival has to wait, L} > 0)$$

$$=L_S/(1-P_0)=\frac{1}{1-P}$$

Measures of performance of a queueing system:

Typical measures of performance include server utilization (% of time a server is busy), length of waiting lines, and waiting times of customers. Queueing model is used to predict these measures of system performance as a function of one or more of input parameters.

The input parameters include

- the arrival rate of customers,
- the service demands of customers,
- the rate at which server works,
- and the number and arrangement of customers.

Queueing Notation

Kendall proposed notational system for parallel service system which has been widely adopted. The abridged form of the notation is A/B/C/N/K.

A – Represents the inter arrival time distribution

B – Represents the service time distribution

C – Represents the number of parallel servers

N – Represents the system capacity

K – Represents the size of the calling population

For example, M/M/1/∞/∞indicates a single-server system, that has unlimited queue capacity and an infinite population of potential arrivals. The inter arrival times and service times are exponentially distributed.

M/M/1/∞/∞ is often shortened as M/M/1.

Transient and steady States:

A system is said be in **transient state** when the behavior of the system is dependent on time.

A system is said to be in **steady state** when the behavior of the system is independent of time. In this topic we study only the steady state analysis.

A list of Symbols:

n = number of customers in the queuing system

$P_n(t)$ = steady state probability of having 'n' customers in the system.

P_n = transient state probability that exactly 'n' customers are in the system at time t.

ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION

λ= Mean arrival rate ($1/\lambda$ is the inter arrival time)

μ=mean service rate ($1/\mu$ is the mean service time)

s= number of parallel service stations

$\rho=\lambda/(\mu s)$=Traffic intensity (or utilization factor) for the service facility, ie., the expected fraction of time server is busy.

L_s=Expected system length, ie., expected member of customers in the system (number of customers waiting in the queue+ number of customers in service.

L_q= Expected queue length, ie., expected number of customers waiting in the queue.

ACTIVITIES/ CASE STUDIES/ IMPORTANT FACTS RELATED TO THE SESSION

W_s=Expected waitiing time of an arriving customers in the system.

W_q=Expected waiting time of an arriving customer in the queue (Expected waiting time in the system-expected service tune.

(W/W>0)=Expected waiting time of a customer who has to wait.

(L/L>0)=Expected length of a non-empty queue

P(W>0)= Probability of an arriving customer has to wait.

Example

Example-1 : Customers arrive to a hair cut salon according to a Poisson process with a mean arrival rate of 5/hr. Because of the reputation of the salon, customers were always willing to wait. Customer processing time was exponentially distributed with an average of 10 min. Answer the following questions

a) Average no. of customers in the shop and average no. of customers waiting for hair cut.

b) Average no. of waiting when there is at least one person waiting.

c) What percentage of time a customer can walk right in without having to wait at all?

d) If the salon has only four seats, what is the probability that a customer, upon arrival, will not be able to find a seat and have to stand?

Example

e) How much time customers spend waiting in the queue?

f) What is the average system waiting time?

g) What is the probability that line delay is more than 45 min.

Solution: Here we have λ= 5/hour, μ=1/10 per minute=6 per hour and $\rho = \frac{\lambda}{\mu}$=5/6

a) Average number of customers in the shop

$$=L_s=\frac{\lambda}{\mu-\lambda} = 5$$

Average number of customers waiting for hair cut

$$=L_q=\frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{25}{6(6-5)} = 4\frac{1}{6}$$

Example

b) Average number waiting where there is at least one person waiting

$P(n>1)=(P^2+P^3+\dots)P_0=P^2$

$$=\frac{L_q}{P(n>1)}=\frac{\frac{\rho^2}{1-\rho}}{\rho^2}=\frac{1}{1-\rho}=\frac{\mu}{\mu-\lambda}=6$$

C) Probability that a customer can walk right in =Probability that there is no unit in the system=$P_0=1-\frac{\lambda}{\mu}$=1/6➔% of customers can walk in =$100P_0$=100/6=16.7%

d) Probability that an arrival will not be able to find a seat=$P(n\geq5)=\rho^5$

e) Waiting time of the customer in the queue $W_q=\frac{L_q}{\lambda}=\frac{\lambda}{\mu(\mu-\lambda)}$ =5/6hour

f) Average waiting time in the system =$W_s=\frac{1}{\mu-\lambda}=1hour$

Example

g) Probability that the line delay is more than 45 minutes

$$=P(W>3/4)=\int_{3/4}^{\infty} \frac{\lambda}{\mu(\mu-\lambda)} e^{-(\mu-\lambda)w} dw$$

$$=\frac{\lambda}{\mu(\mu-\lambda)}\left[\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)}\right]_{3/4}^{\infty} = (5/6)e^{-3/4}$$

Example 2

Example-2 : In a factory, the machine breakdown occur on an average rate of 10 machines per hour. The idle time cost of a machine is estimated to be Rs. 20 per hour. The factory works 8 hours a day. The factory manager is considering 2 mechanics for repairing the machines. The first mechanic A takes about 5 minutes on an average to repair a machine and demands wages Rs. 10 per hour. The second mechanic B takes about 4 minutes in repairing a machine and demands wages at the rate of Rs. 15 per hour. Assuming that the rate of machine breakdown is Poisson-distributed and the repair rate is exponentially distributed, which of the two mechanics should be engaged?

Example 2

Solution : Here we shall compare the expected daily cost viz., total wages paid plus cost due to machine breakdown for both the repairman.

Total wages for First mechanic = Hourly rate x No. of hours.

10 x 8 = Rs. 80.

Total wages for second mechanic =15 x 8 =Rs. 120.

Cost of non-productive time = (Average number of machines in the system) X (Cost of idle machine hour) X (Number of Hours)

$$=\frac{\lambda}{\mu-\lambda} \text{ X } 20 \text{ x } 8.$$

For First mechanic: λ=10 machines per hour; μ=12 machines per hour (Given (1/μ)=5; μ=1/5=60/5=12)

Total cost= $80+\frac{10}{12-10}$ X 20 x 8=80+880=Rs. 880.

For Second mechanic: λ=10 machines per hour; μ=12 machines per hour (Given (1/μ)=4; μ=1/4=60/4=15)

Total cost= $120+\frac{10}{15-10}$ X 20 x 8=120+320=Rs. 440.

Obviously, the second mechanic should be employed by the company.

SUMMARY

In this session, Queuing models and its performance measures have discussed.

1. Performance measures of Model 1 using the notations.
2. Performance measures of queuing system
3. Difference between Transient state and Steady state.

TERMINAL QUESTIONS

1. Arrivals of machinists at a tool crib are considered to be Poisson distributed at an average rate of 6 per hour. The length of time the machinists must remain at the tool crib is exponentially distributed with the average time being 0.05 hour.

a) What is the average number of machinists in the queue?
b) What is the average number of machinists at the tool crib?
c) What is the probability that a machinist arriving at the tool crib will have to wait?
d) What is the average length of the queue that form time to time?
e) What is the probability that there are more than 2 machinists at the tool crib?
f) What is the probability that no machinist waiting to be served?
g) What is the expected length of non-empty queue?
h) Estimate the fraction of time that there is no machinist at the tool crib?

2. At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes. It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

TERMINAL QUESTIONS

3. In the production shop of a company, the breakdown of the machinists is found to be Poisson distributed with an average rate of 3 machines per hour. Breakdown time at one machine costs Rs. 40 per hour to the company. There are two choices before the company for hiring the repairman. One of the repairmen is slow but cheap the other fast but expensive. The slow-cheap repairman demands Rs. 20 per hour and will repair the breakdown machines exponentially at the rate of 4 per hour. The fast-expensive repairman demands Rs. 30 per hour and will repair machines exponentially at the rate of 6 per hour, which repairmen should be hired?

REFERENCES FOR FURTHER LEARNING OF THE SESSION

Reference Books:

1. D. Gross, J.F.Shortle, J.M. Thompson, and C.M. Harris, Fundamentals of Queueing Theory, 4th Edition, Wiley, 2008

2. William Feller, An Introduction to Probability Theory and Its Applications: Volime 1, Third Edition, 1968 by John Wiley & Sons,Inc.

Sites and Web links:

1. https://www.khanacademy.org/math/statistics-probability/significance-tests-one-sample/more-significance-testing-videos/v/small-sample-hypothesis-test

2. J.F. Shortle, J.M. Thompson, D. Gross and C.M. Harris, Fundamentals of Queueing Theory, 5th Edition, Wiley, 2018.

3. https://onlinecourses.nptel.ac.in/noc22_ma17/previ ew 3.

4. https://www.youtube.com/watch?v=Wo75G99F9f M&list=PLwdnzlV3ogoX2OHyZz3QbEYFhbqM 7x275&index=3

THANK YOU

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