MATHEMATICAL PROGRAMMING

CO1

LPP

DECISION MAKING

- The general approach is to **analyze the problem** in economic terms and then **implement the solution** if it does not **aggressive or violent** to other aspects like **human, social and political constraints**.

DECISION MAKING

- **A decision is the conclusion of a process designed to weigh the relative uses or utilities of a set of alternatives on hand, so that decision maker selects the best alternative which is best to his problem or situation and implement it.**
- **Decision Making involves all activities and thinking that are necessary to identify the most optimal or preferred choice among the available alternatives.**
- **The basic requirements of decision-making are**
  - **(i) A set of goals or objectives,**
  - **(ii) Methods of evaluating alternatives in an objective manner,**
  - **(iii) A system of choice criteria and a method of projecting the effects of alternative choices of courses of action.**

OBJECTIVE OF MATHEMATICAL PROGRAMMING

- **"The objective of Mathematical Programming is to provide a scientific basis to the decision maker for solving the problems involving the interaction of various components of an organization by employing a team of scientists from various disciplines, all working together for finding a solution which is in the best interest of the organization as a whole.**
- **The best solution thus obtained is known as optimal decision".**

DEFINITION OF MATHEMATICAL PROGRAMMING

- **The art of winning wars without actually fighting.**
- **The art of giving bad answers to problems where otherwise worse answers are given.**
- **Research into Operations.**
- **Defined as Scientific method for providing executive departments a quantitative basis for decisions regarding the operations under their control.**
- **The application of the theories of Probability, Statistics, Queuing, Games, Linear Programming etc., to the problems of War, Government and Industry.**

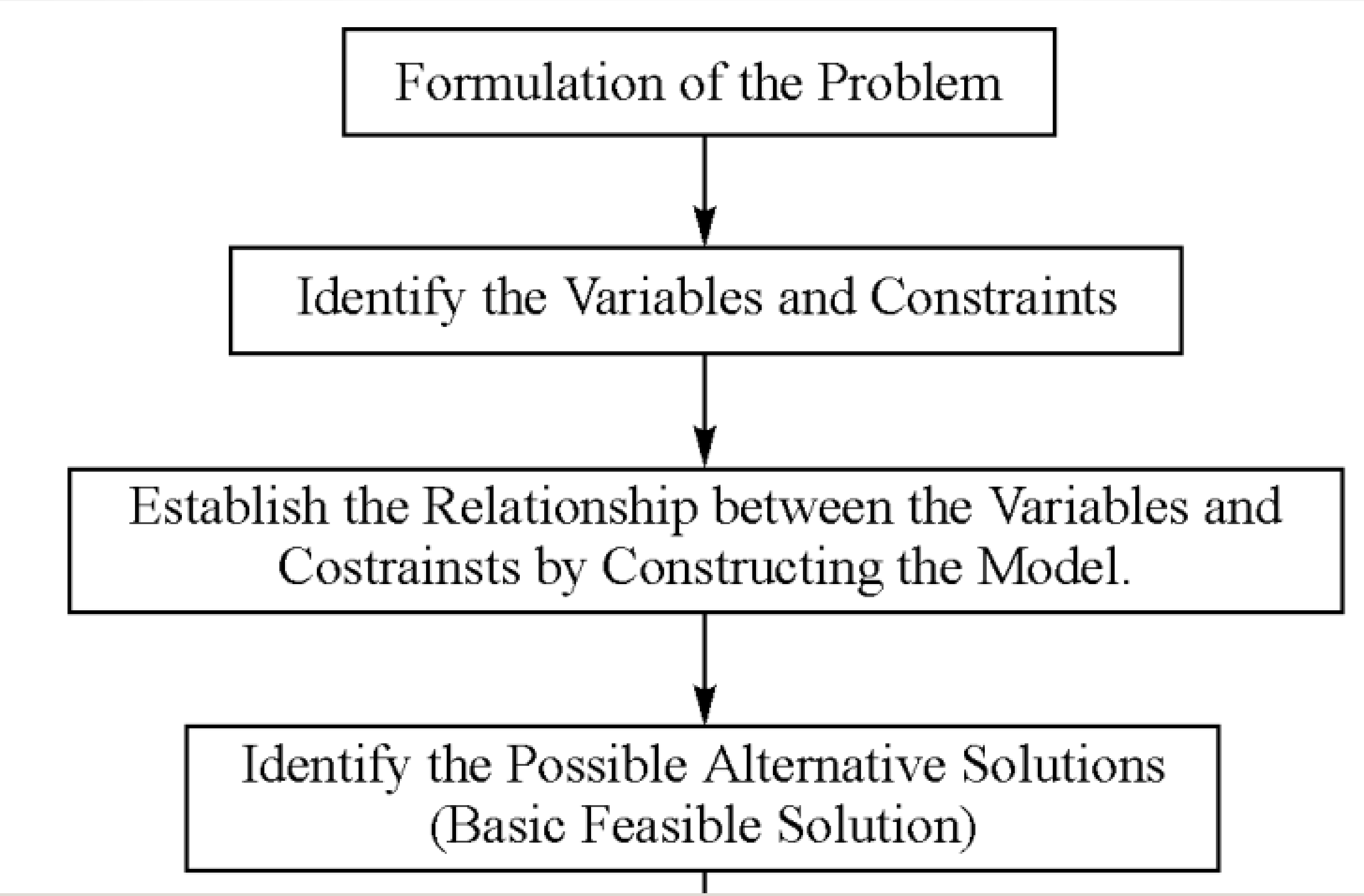
. CHARACTERISTICS OF MATHEMATICAL PROGRAMMING

- **Mathematical Programming is an interdisciplinary team approach.**
- **Mathematical Programming increases the creative ability of the decision maker.**
- **Mathematical Programming is a systems approach.**

.SCOPE OF MATHEMATICAL PROGRAMMING

- In Defense Operations.
- In Industry.
- In Planning For Economic Growth.
- In Agriculture:
- In Traffic control
- In Hospitals

PHASES IN SOLVING MATHEMATICAL PROGRAMMING PROBLEMS
OR
STEPS IN SOLVING MATHEMATICAL PROGRAMMING PROBLEMS



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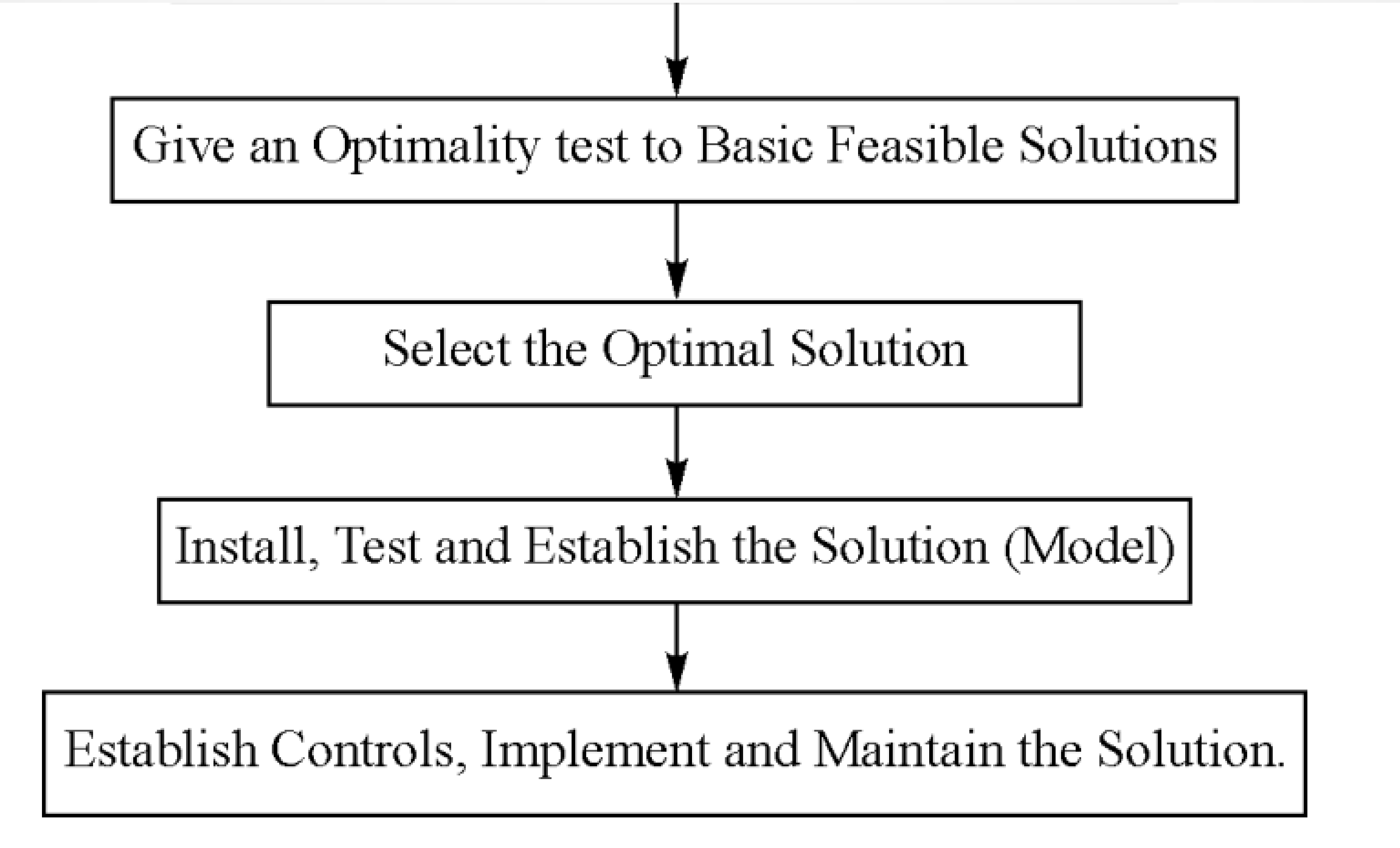
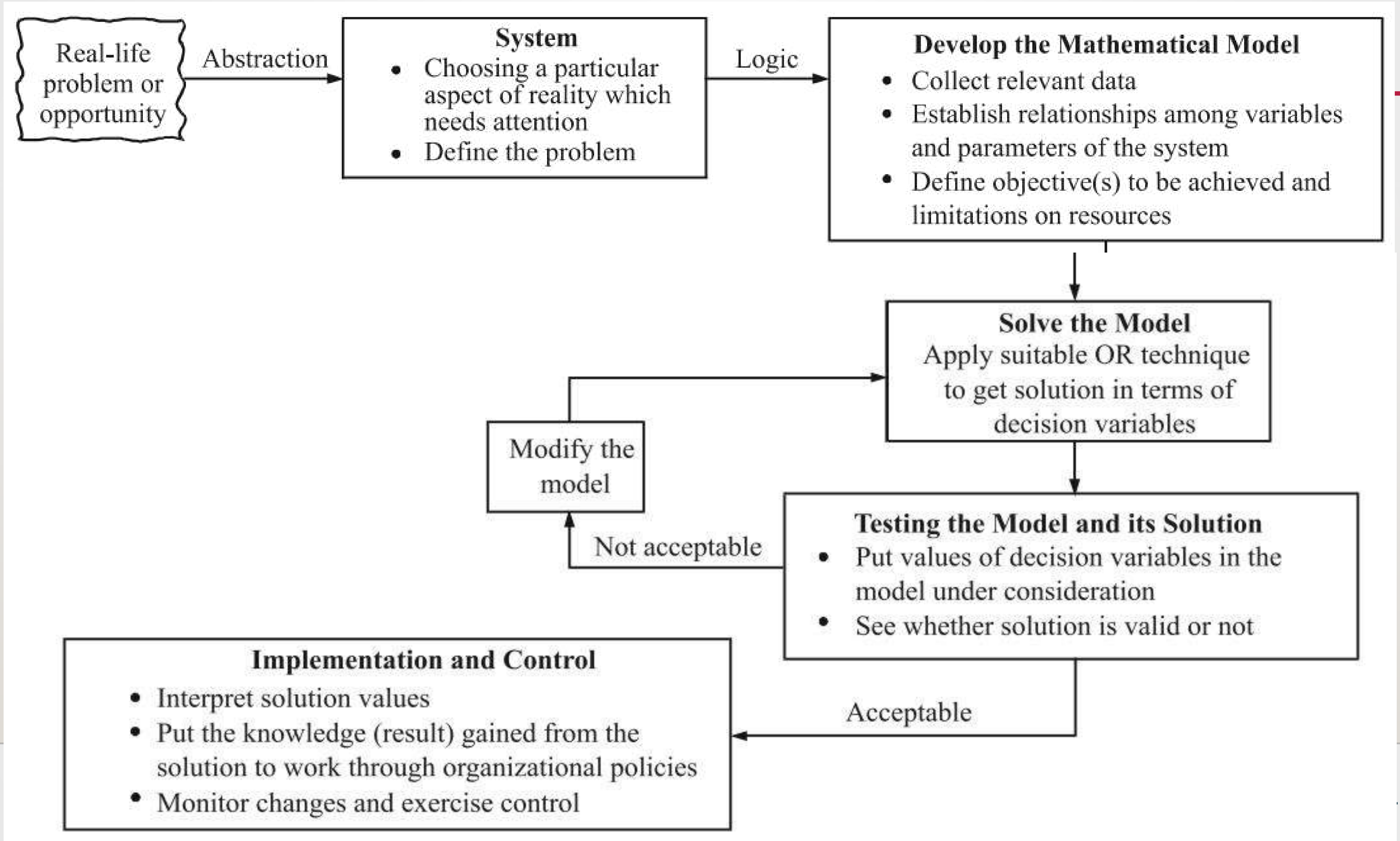
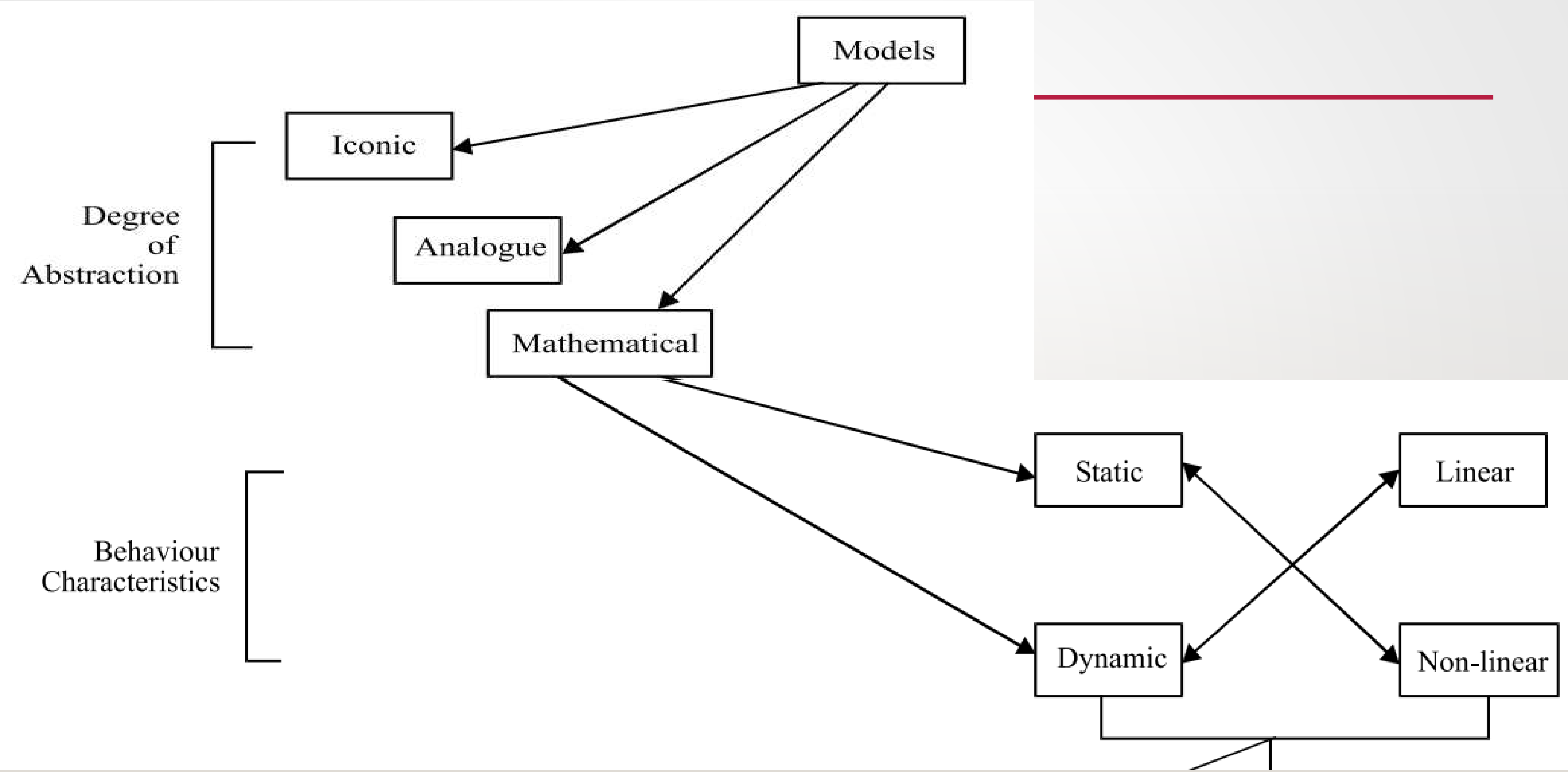
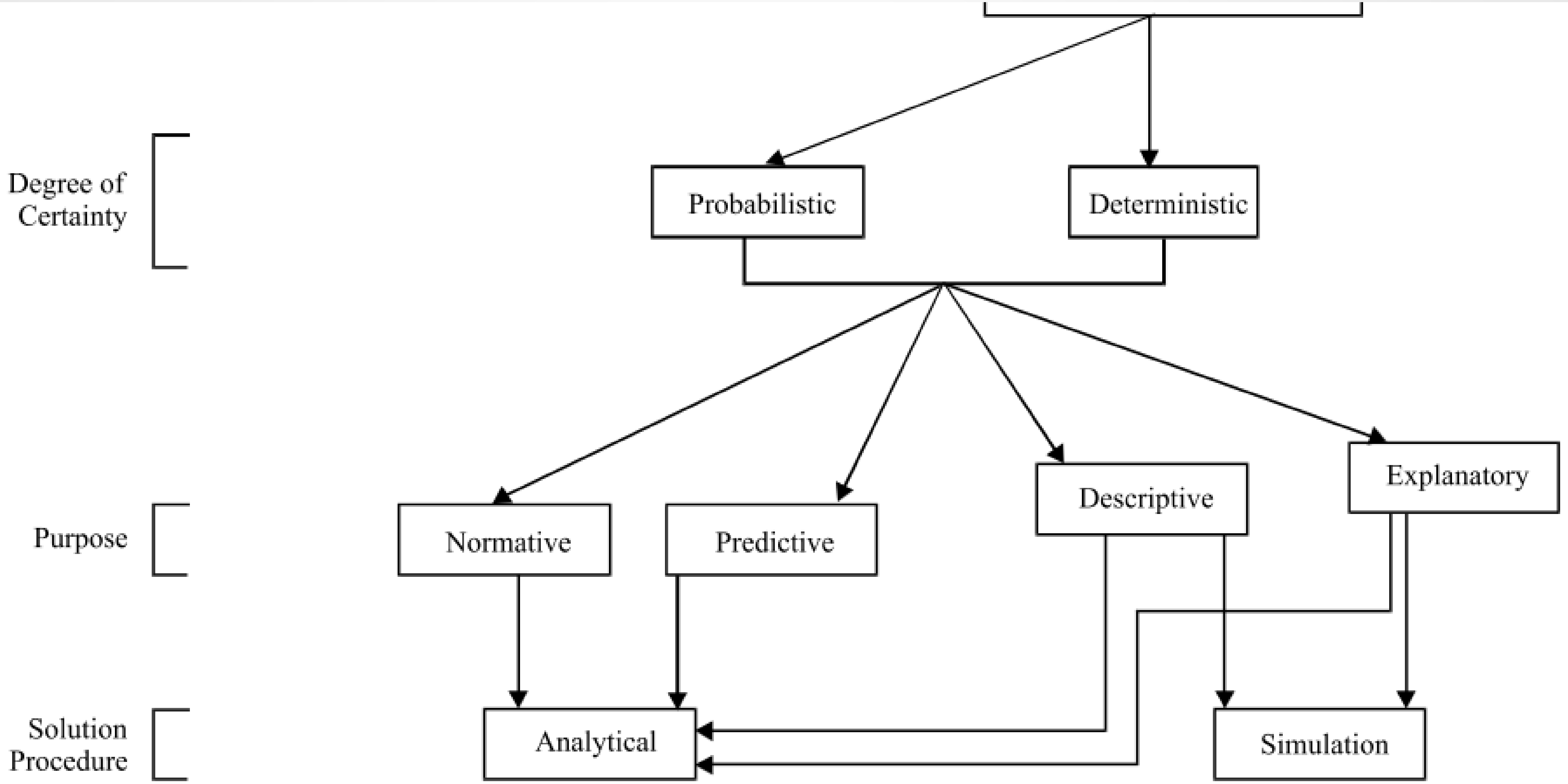


METHODOLOGY OF MATHEMATICAL PROGRAMMING



CLASSIFICATION OF MODELS



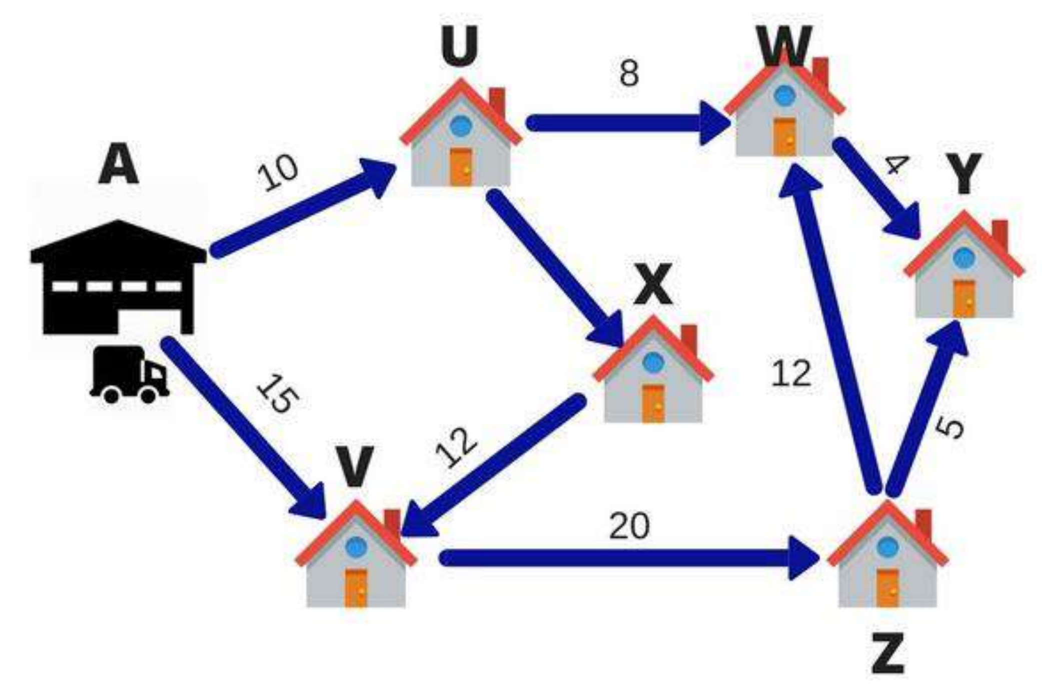

Degree of Certainty
Purpose
Solution Procedure
Probabilistic
Deterministic
Normative
Predictive
Descriptive
Explanatory
Analytical
Simulation

SOLVING THE MATHEMATICAL MODEL

- Once a mathematical model of the problem has been formulated, the next step is to obtain numerical values of decision variables.
  - Analytical Methods.
  - Heuristic Methods.

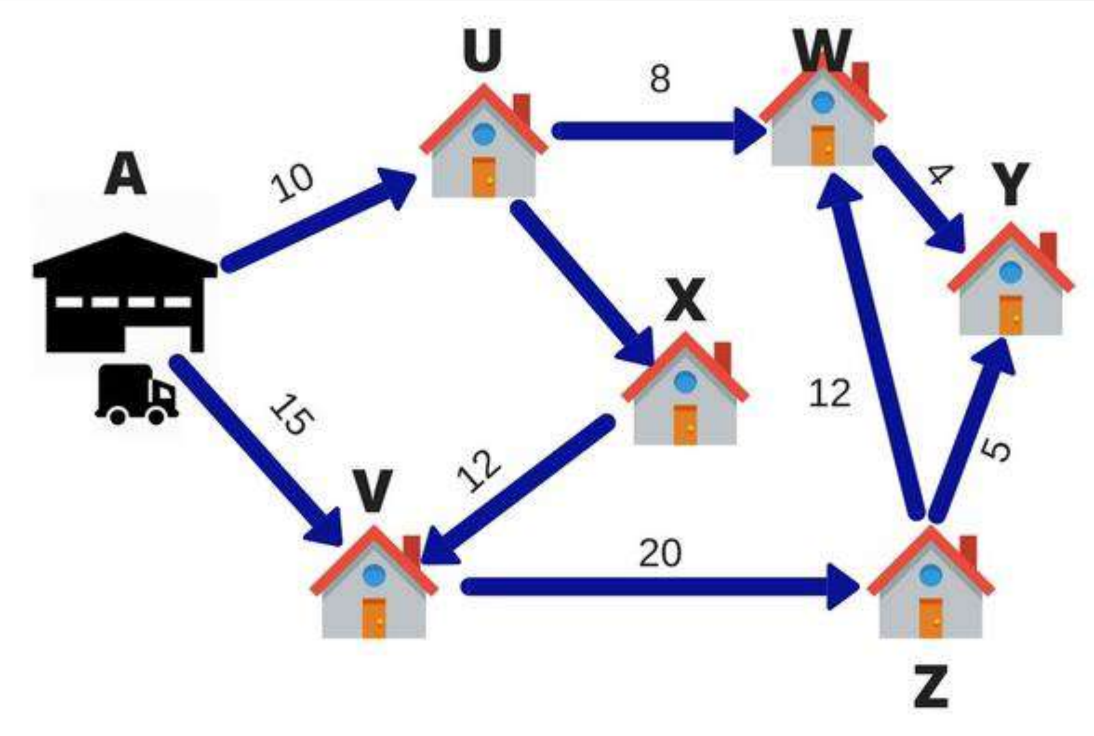
DEFINING THE PROBLEM-TSP

- Let's say a FedEx delivery man has 6 packages to deliver in a day.
- The warehouse is located at point A.
- The 6 delivery destinations are given by U, V, W, X, Y, and Z.
- The numbers on the lines indicate the distance between the cities.
- To save on fuel and time the delivery person wants to take the shortest route.



EXAMPLE OF A LINEAR PROGRAMMING PROBLEM (LPP)

- So, the delivery person will calculate different routes for going to all 6 destinations and then come up with the shortest route.
- This technique of choosing the shortest route is called linear programming.
- In this case, the objective of the delivery person is to deliver the parcel on time at all 6 destinations.
- The process of choosing the best route is called Operation Research.
- Operation research is an approach to decision-making, which involves a set of methods to operate a system.



LINEAR PROGRAMMING PROBLEM(LPP)

- **Linear programming is used for obtaining the most optimal solution for a problem with given constraints.**

- **In linear programming, we formulate our real-life problem into a mathematical model.**

- **Definition: Linear Programming Problem (LPP) is a mathematical technique which is used to optimize (maximize or minimize) the objective function with the limited resources.**

(DEEMED TO BE UNIVERSITY)

LINEAR PROGRAMMING PROBLEM

- **Mathematically, the general linear programming problem (LPP) may be stated as follows:-**
  - **Maximize or Minimize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$**

LINEAR PROGRAMMING PROBLEM

- **Subject to the conditions (constraints)**

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}{*}x_n \leq b_m$$

LINEAR PROGRAMMING PROBLEM

- **Short form of LPP:**

$$\text{Maximize or Minimize} \qquad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij} x_j \leq (or = or \geq)\, b_i \text{ , i = 1,2,3,} \ldots, m \qquad \ldots (1)$$

$$\text{and} \quad x_j \geq 0 \qquad \ldots (2)$$

LINEAR PROGRAMMING APPLICATIONS

- **Engineering Industries :**
  - **Engineering Industries use linear programming to solve design and manufacturing problems and to get the maximum output from a given condition.**
- **Manufacturing Industries:**
  - **Manufacturing Industries use linear programming to maximize the profit of the companies and to reduce the manufacturing cost.**
- **Energy Industries:**
  - **Energy companies use linear programming to optimize their production output.**
- **Transportation Industries:**
  - **Linear programming is also used in transportation industries to find the path to minimize the cost of transportation.**

SOME USEFUL DEFINITIONS

- **Objective function:**
  - **A function $Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$ which is to be optimized (maximized or minimized) is called objective function.**

SOME USEFUL DEFINITIONS

- **Decision variable:**
  - **The decision variables are the variables, which has to be determined x_j , $j = 1,2,3,\ldots,n$, to optimize the objective function.**
- **Constraints:**
  - **There are certain limitations on the use of limited resources called constraints.**

$$\sum_{j=1}^{n} a_{ij}x_j \leq (or = or \geq)\, b_i$$

, $i = 1,2,3,\ldots, m$ are the constraints.

MATHEMATICAL FORMULATION OF A LINEAR PROGRAMMING PROBLEM:

- **The procedure for mathematical formulation of a linear programming problem consists of the following steps.**
  1. **Identify the decision variables.**
  2. **Identify the objective function to be maximized or minimized and express it as a linear function of decision variables.**
  3. **Identify the set of constraint conditions and express them as linear inequalities or equations in terms of the decision variables.**

MATHEMATICAL FORMULATION OF A LINEAR PROGRAMMING PROBLEM: EXAMPLE

- **Case Study / PROBLEM (1):**
  - **A furniture dealer deals only two items viz., tables and chairs. He has to invest Rs.10,000/- and a space to store at most 60 pieces. A table cost him Rs.500/– and a chair Rs.200/–. He can sell all the items that he buys. He is getting a profit of Rs.50 per table and Rs.15 per chair. Formulate this problem as an LPP, so as to maximize the profit.**

CONT…..

- **Solution:**
- (i) **Variables:** Let x_1 and x_2 denote the number of tables and chairs respectively.
- (ii) **Objective function:**
    - **Profit on x_1 tables = 50 x_1**
    - **Profit on x_2 chairs = 15 x_2**
    - **Total profit = 50 x_1+15x_2**
- Let **$Z = 50\ x_1 + 15\ x_2$** , which is the **objective function.**
- Since the total profit is to be **maximized**, we have to **maximize $Z=50\ x_1 +15\ x_2$**

- **A furniture dealer deals only two items viz., tables and chairs. He has to invest Rs.10,000/- and a space to store at most 60 pieces. A table cost him Rs.500/– and a chair Rs.200/–. He can sell all the items that he buys. He is getting a profit of Rs.50 per table and Rs.15 per chair. Formulate this problem as an LPP, so as to maximize the profit.**

CONT…..

- (iii) Constraints:
- The dealer has a space to store at most 60 pieces

$$x_1+x_2 \leq 60$$

- The cost of x_1 tables = Rs.500 x_1
- The cost of x_2 chairs = Rs. 200 x_2
- Total cost = 500 x_1 + 200 x_2 ,which cannot be more than 10000
- $500\, x_1 + 200\, x_2 \leq 10000$
- $5x_1 + 2x_2 \leq 100$

- A furniture dealer deals only two items viz., tables and chairs. He has to invest Rs.10,000/- and a space to store at most 60 pieces. A table cost him Rs.500/– and a chair Rs.200/–. He can sell all the items that he buys. He is getting a profit of Rs.50 per table and Rs.15 per chair. Formulate this problem as an LPP, so as to maximize the profit.

CONT.....

- (iv) **Non-negative restrictions:** Since the number of tables and chairs cannot be negative, we have $x_1 \geq 0, x_2 \geq 0$
- Thus, the mathematical formulation of the LPP is
  - Maximize $Z=50\ x_1+15x_2$
  - Subject to the constrains
  - $x_1+x_2 \leq 60$
  - $5x_1+2x_2 \leq 100$
  - $x_1, x_2 \geq 0$

- A furniture dealer deals only two items viz., **tables and chairs**. He has to invest Rs.10,000/- and a space to store at most 60 pieces. A table cost him Rs.500/– and a chair Rs.200/–. He can sell all the items that he buys. He is getting a profit of Rs.50 per table and Rs.15 per chair. Formulate this problem as an LPP, so as to maximize the profit.

EXAMPLE 2

- **A dietician wishes to mix two types of food F1 and F2 in such a way that the vitamin contents of the mixture contains atleast 6units of vitamin A and 9 units of vitamin B. Food F1 costs Rs.50 per kg and F2 costs Rs 70 per kg. Food F1 contains 4 units per kg of vitamin A and 6 units per kg of vitamin B while food F2 contains 5 units per kg of vitamin A and 3 units per kg of vitamin B. Formulate the above problem as a linear programming problem to minimize the cost of mixture.**

CONT…

- **(i)** **Variables:** **Let the mixture contains x_1 kg of food F_1 and x_2 kg of food F_2**
- **(ii)** **Objective function:**
  - **cost of x_1 kg of food F_1 = 50 x_1**
  - **cost of x_2 kg of food F_2 = $70x_2$**
  - **The cost is to be minimized**
  - **Therefore minimize Z= 50 x_1+$70x_2$**

- A dietician wishes to mix two types of food F1 and F2 in such a way that the vitamin contents of the mixture contains atleast 6units of vitamin A and 9 units of vitamin B. Food F1 costs Rs.50 per kg and F2 costs Rs 70 per kg. Food F1 contains 4 units per kg of vitamin A and 6 units per kg of vitamin B while food F2 contains 5 units per kg of vitamin A and 3 units per kg of vitamin B. Formulate the above problem as a linear programming problem to **minimize the cost of mixture**.

CONT…

- **(iii) Constraints:**
- We make the following table from the given data

Resources	Food (in kg)		Requirement
	F_1 (x_1)	F_2 (x_2)	
Vitamin A (units/kg)	4	5	6
Vitamin B (units/kg)	6	3	9
Cost (Rs/kg)	50	70	

Table 10.1

- $4x_1+5x_2 \geq 6$ (since the mixture contains 'at least 6' units of vitamin A ,we have the inequality of the type $\geq$)
- $6x_1+3x_2 \geq 9$(since the mixture contains 'at least 9' units of vitamin B ,we have the inequality of the type $\geq$)

- A dietician wishes to mix two types of food F1 and F2 in such a way that the vitamin contents of the mixture contains atleast 6units of vitamin A and 9 units of vitamin B. Food F1 costs Rs.50 per kg and F2 costs Rs 70 per kg. Food F1 contains 4 units per kg of vitamin A and 6 units per kg of vitamin B while food F2 contains 5 units per kg of vitamin A and 3 units per kg of vitamin B. Formulate the above problem as a linear programming problem to minimize the cost of mixture.

CONT…

- (iv) **Non-negative restrictions:**
  - **Since the number of kgs of vitamin A and vitamin B are non-negative , we have $x_1, x_2 \geq 0$**
  - **Thus, we have the following linear programming model**
  - **Minimize $Z = 50x_1 + 70x_2$**
  - **subject to $4x_1 + 5x_2 \geq 6$**
  - **$6x_1 + 3x_2 \geq 9$**
  - **and $x_1, x_2 \geq 0$**

- A dietician wishes to mix two types of food F1 and F2 in such a way that the vitamin contents of the mixture contains atleast 6units of vitamin A and 9 units of vitamin B. Food F1 costs Rs.50 per kg and F2 costs Rs 70 per kg. Food F1 contains 4 units per kg of vitamin A and 6 units per kg of vitamin B while food F2 contains 5 units per kg of vitamin A and 3 units per kg of vitamin B. Formulate the above problem as a linear programming problem to minimize the cost of mixture.

LPP BY GRAPHICAL METHOD

LINEAR PROGRAMMING PROBLEM

- **Short form of LPP:**

$$\text{Maximize or Minimize} \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \sum_{j=1}^{n} a_{ij} x_j \leq (or = or \geq)\, b_i \text{, i = 1,2,3,}\ldots, m \qquad \ldots (1)$$

$$\text{and} \quad x_j \geq 0 \qquad \ldots (2)$$

SOME USEFUL DEFINITIONS

- **Solution:**
  - **A set of values of decision variables x_j, j=1,2,3,…, n satisfying all the constraints of the problem is called a solution to that problem.**
- **Feasible solution:**
  - **A set of values of the decision variables that satisfies all the constraints of the problem and non-negativity restrictions is called a feasible solution of the problem.**

SOME USEFUL DEFINITIONS

- **Optimal solution:**
  - **Any feasible solution which maximizes or minimizes the objective function is called an optimal solution.**
- **Feasible region:**
  - **The common region determined by all the constraints including non-negative constraints $x_j \geq 0$ of a linear programming problem is called the feasible region (or solution region) for the problem.**

GRAPHICAL METHOD

- The graphical method is used to optimize the two-variable linear programming.

- If the problem has two decision variables, a graphical method is the best method to find the optimal solution.

SOLUTION OF LPP BY GRAPHICAL METHOD

- After formulating the linear programming problem, our aim is to determine the values of decision variables to find the optimum (maximum or minimum) value of the objective function.
- Linear programming problems which involve only two variables can be solved by graphical method.
- If the problem has three or more variables, the graphical method is impractical.

THE MAJOR STEPS INVOLVED IN GRAPHICAL METHOD

1. State the problem mathematically.
2. Write all the constraints in the form of equations and draw the graph.
3. Find the feasible region.
4. Find the coordinates of each vertex (corner points) of the feasible region.
   - The coordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at the point

THE MAJOR STEPS INVOLVED IN GRAPHICAL METHOD

5. By substituting these corner points in the objective function we can get the values of the objective function.

6. If the problem is maximization then the maximum of the above values is the optimum value.

   If the problem is minimization then the minimum of the above values is the optimum value.

EXAMPLE 1

- **Solve the following LPP**

1. Maximize $Z = 2\,x_1 + 5x_2$
2. Subject to the conditions

$$x_1 + 4x_2 \leq 24$$

$$3x_1 + x_2 \leq 21$$

$$x_1 + x_2 \leq 9 \quad \text{and}$$

3. $x_1, x_2 \geq 0$

SOLUTION

- First we have to find the feasible region using the given conditions.
- Since both the decision variables x_1 and x_2 are non-negative ,the solution lies in the first quadrant.

Maximize $Z = 2\, x_1 + 5x_2$

subject to the conditions $x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$ and $x_1,\, x_2 \geq 0$

(DEEMED TO BE UNIVERSITY)

SOLUTION

Maximize $Z = 2x_1 + 5x_2$

subject to the conditions $x_1 + 4x_2 \leq 24$, $3x_1 + x_2 \leq 21$, $x_1 + x_2 \leq 9$ and $x_1, x_2 \geq 0$

- Write all the inequalities of the constraints in the form of equations.
- Therefore we have the lines $x_1 + 4x_2 = 24$; $3x_1 + x_2 = 21$; $x_1 + x_2 = 9$
- $x_1 + 4x_2 = 24$ is a line passing through the points (0 , 6) and (24 , 0).
- [(0,6) is obtained by taking $x_1 = 0$ in $x_1 + 4x_2 = 24$, (24 , 0) is obtained by taking $x_2 = 0$ in $x_1 + 4x_2 = 24$].
- Any point lying on or below the line $x_1 + 4x_2 = 24$ satisfies the constraint $x_1 + 4x_2 \leq 24$.

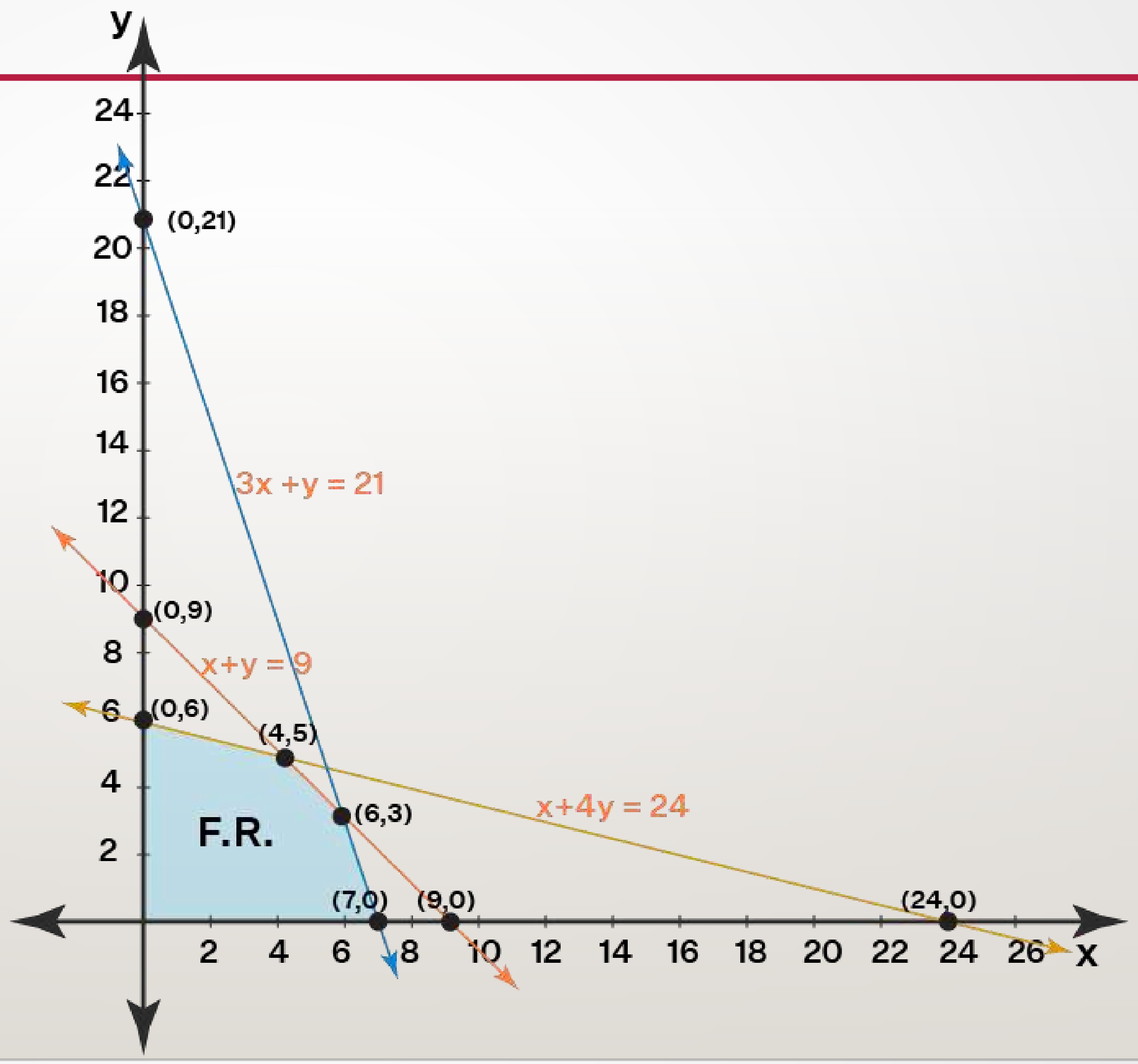
SOLUTION

Maximize $Z = 2\,x_1 + 5x_2$

subject to the conditions $\mathbf{x_1 + 4x_2 \leq 24}$, $3x_1 + x_2 \leq 21$, $x_1 + x_2 \leq 9$ and $x_1, x_2 \geq 0$

- $3x_1 + x_2 = 21$ is a line passing through the points (0, 21) and (7, 0).
- Any point lying on or below the line $3\,x_1 + x_2 = 21$ satisfies the constraint $3\,x_1 + x_2 \leq 21$.
- $x_1 + x_2 = 9$ is a line passing through the points (0 , 9) and (9 , 0) .
- Any point lying on or below the line $x_1 + x_2 = 9$ satisfies the constraint $x_1 + x_2 \leq 9$.

GRAPH.


y
x
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22
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18
16
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12
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16
18
20
22
24
26
(0,21)
3x +y = 21
(0,9)
x+y = 9
(0,6)
(4,5)
(6,3)
x+4y = 24
F.R.
(7,0)
(9,0)
(24,0)

GRAPH.

- The feasible region satisfying all the conditions is OABCD.

- The co-ordinates of the points are O(0,0) A(7,0);B(6,3) [the point B is the intersection of two lines x1+ x2= 9 and 3 x1+ x2= 21];C(4,5) [the point C is the intersection of two lines x1+ x2 = 9 and x1+ 4x2 = 24] and D(0,6).

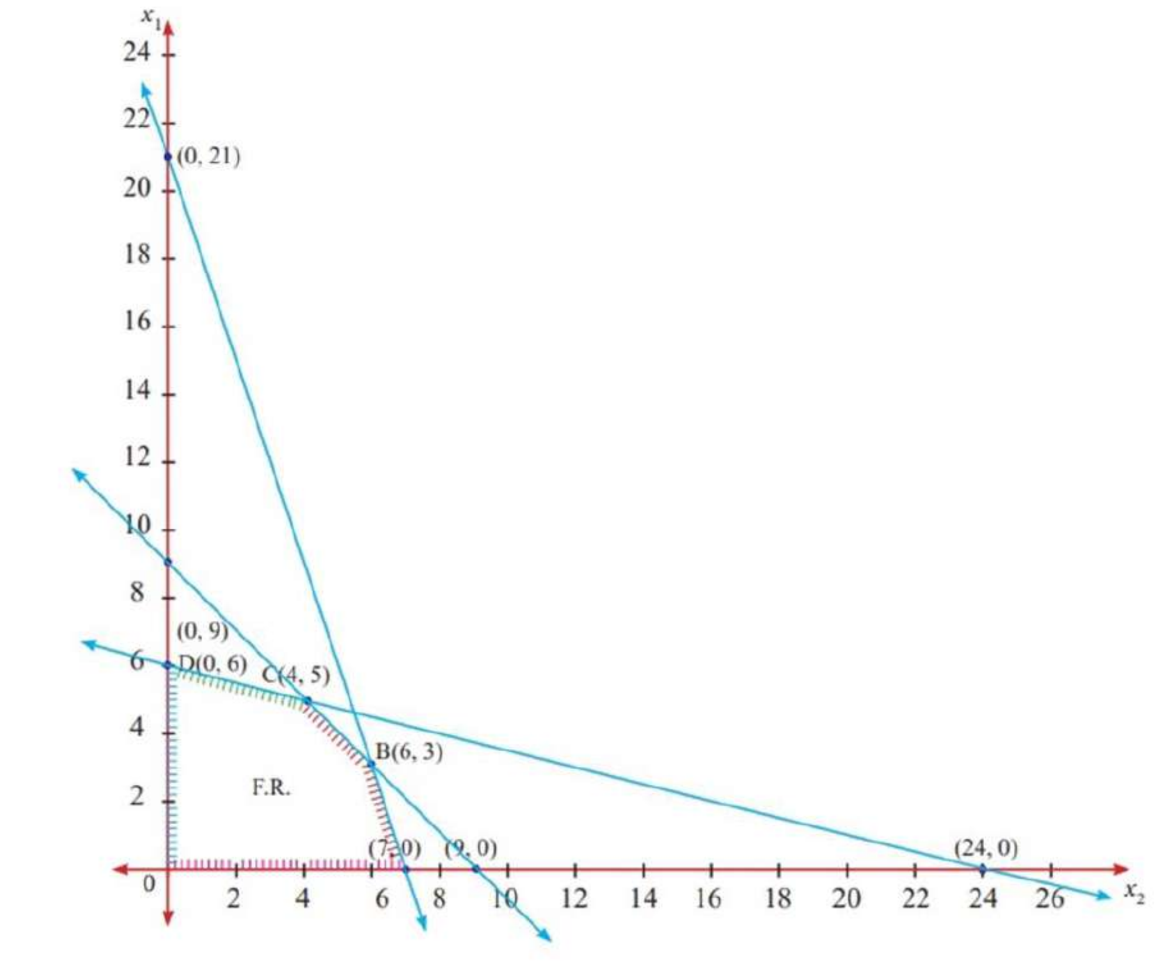


Fig 10.1

GRAPH.

Corner points	$Z = 2x_1 + 5x_2$
O(0,0)	0
A(7,0)	14
B(6,3)	27
C(4,5)	33
D(0,6)	30

Table 10.2

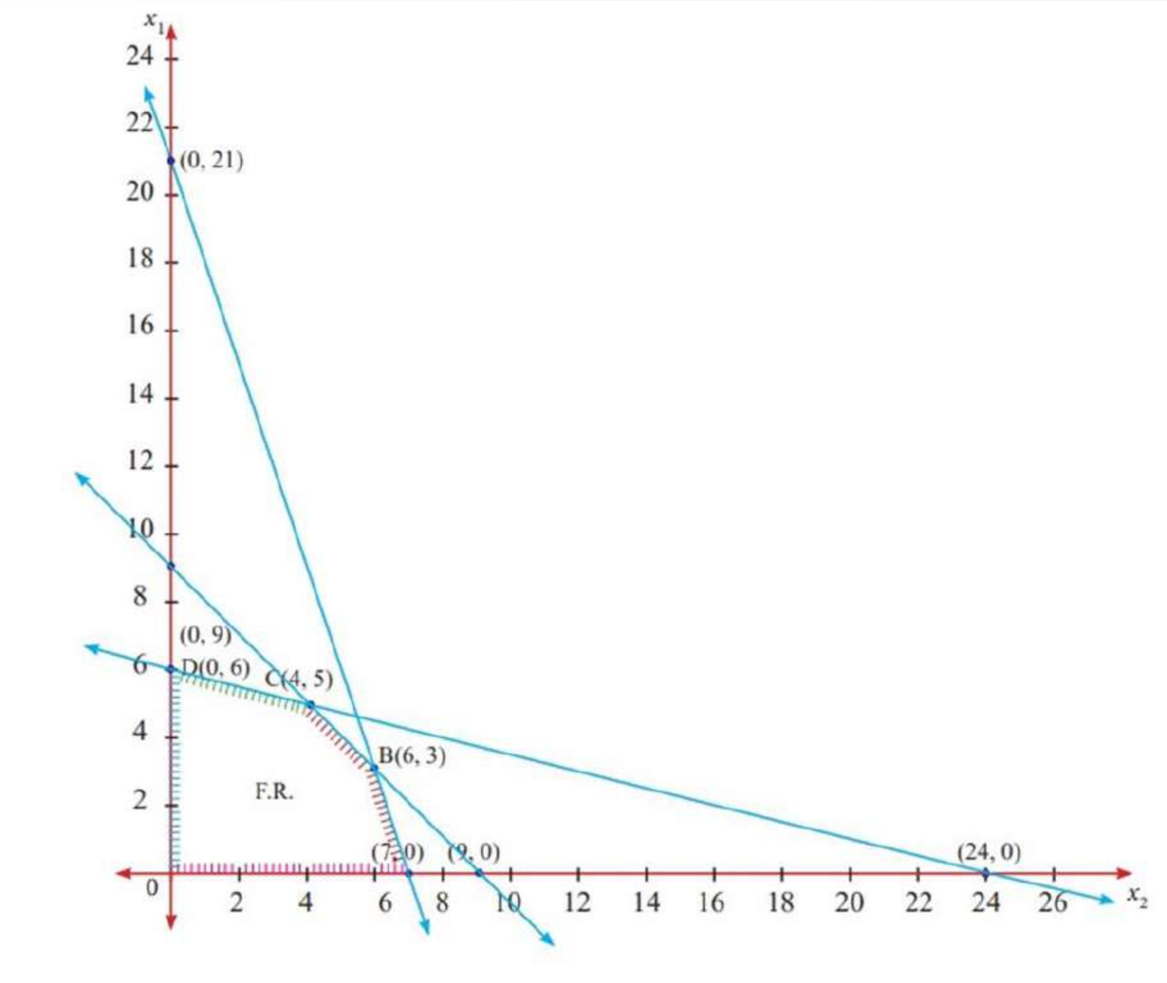


Fig 10.1

- Maximum value of Z occurs at C.
- Therefore the solution is $x1 = 4$, $x2 = 5$, $Z\ max = 33$

EXAMPLE 2

- **Solve the following LPP by graphical method**

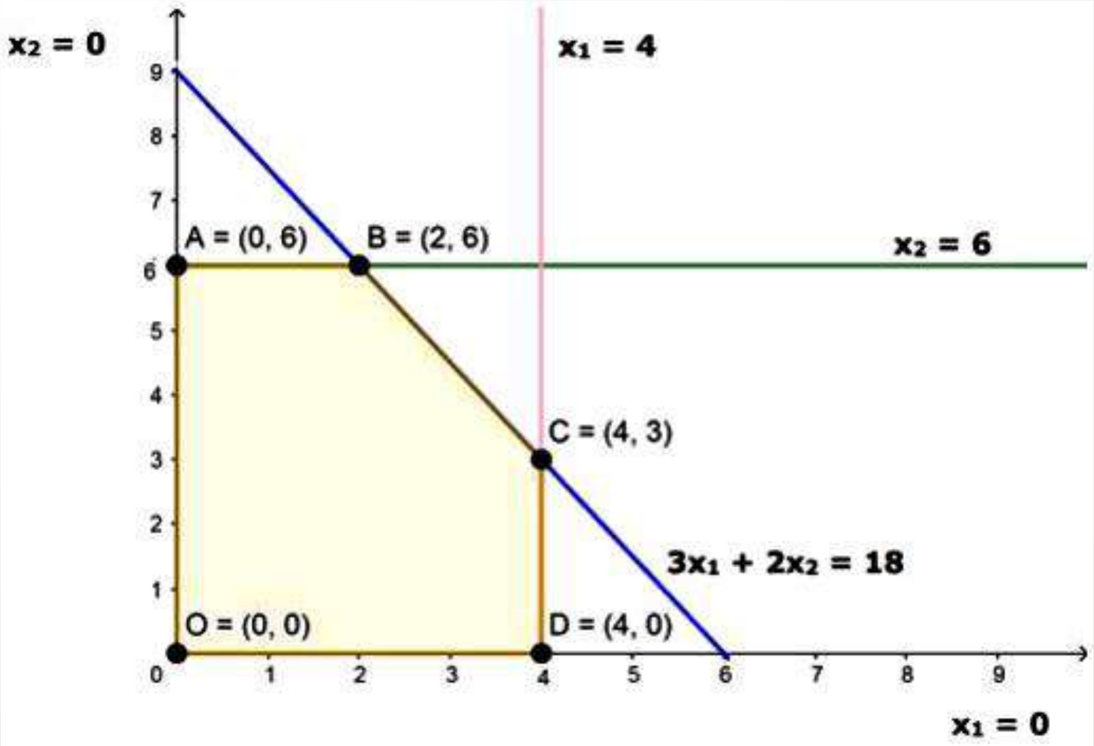
Maximize $z = 3x_1+5x_2$

Subject to constraints

$3x_1+ 2x_2 \leq 18$;

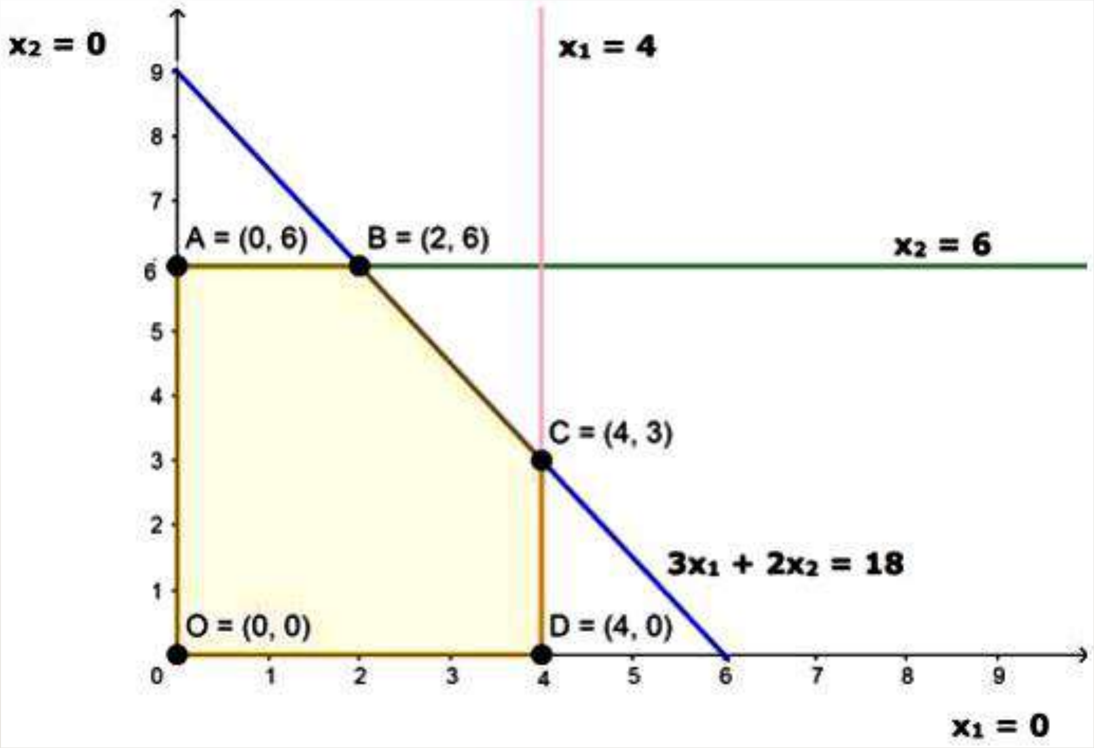
$x_1\leq 4$,

$x_2\leq 6$ and $x_1, x_2 \geq 0$



CONT..

Z at O ($x_1 = 0, x_2 = 0$)	$Z = 3(0) + 5(0) = 0 + 0 = 0$
Z at A ($x_1 = 0, x_2 = 6$)	$Z = 3(0) + 5(6) = 0 + 30 = 30$
Z at B ($x_1 = 2, x_2 = 6$)	$Z = 3(2) + 5(6) = 6 + 30 = 36$
Z at C ($x_1 = 4, x_2 = 3$)	$Z = 3(4) + 5(3) = 12 + 15 = 27$
Z at D ($x_1 = 4, x_2 = 0$)	$Z = 3(4) + 5(0) = 12 + 0 = 12$



- The Z value maximized at point B where $x_1 = 2$ and $x_2 = 6$. Z = 36

EXAMPLE 3

- **Solve the following LPP by graphical method**

Minimize $z = 5x_1+4x_2$

Subject to constraints $4x_1+ x_2 \geq 40$;

$2x_1+3x_2 \geq 90$ and $x_1, x_2 > 0$

SOLUTION

- Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

- Consider the equations $4x_1+x_2 = 40$ and $2x_1+3x_2 = 90$

- $4x_1+x_2 = 40$ is a line passing through the points (0,40) and (10,0). Any point lying on or above the line $4x_1+x_2= 40$ satisfies the constraint $4x_1+x_2 \geq 40$.

- $2x_1+3x_2 = 90$ is a line passing through the points (0,30) and (45,0). Any point lying on or above the line $2x_1+3x_2= 90$ satisfies the constraint $2x_1+3x_2 \geq 90$.

SOLUTION

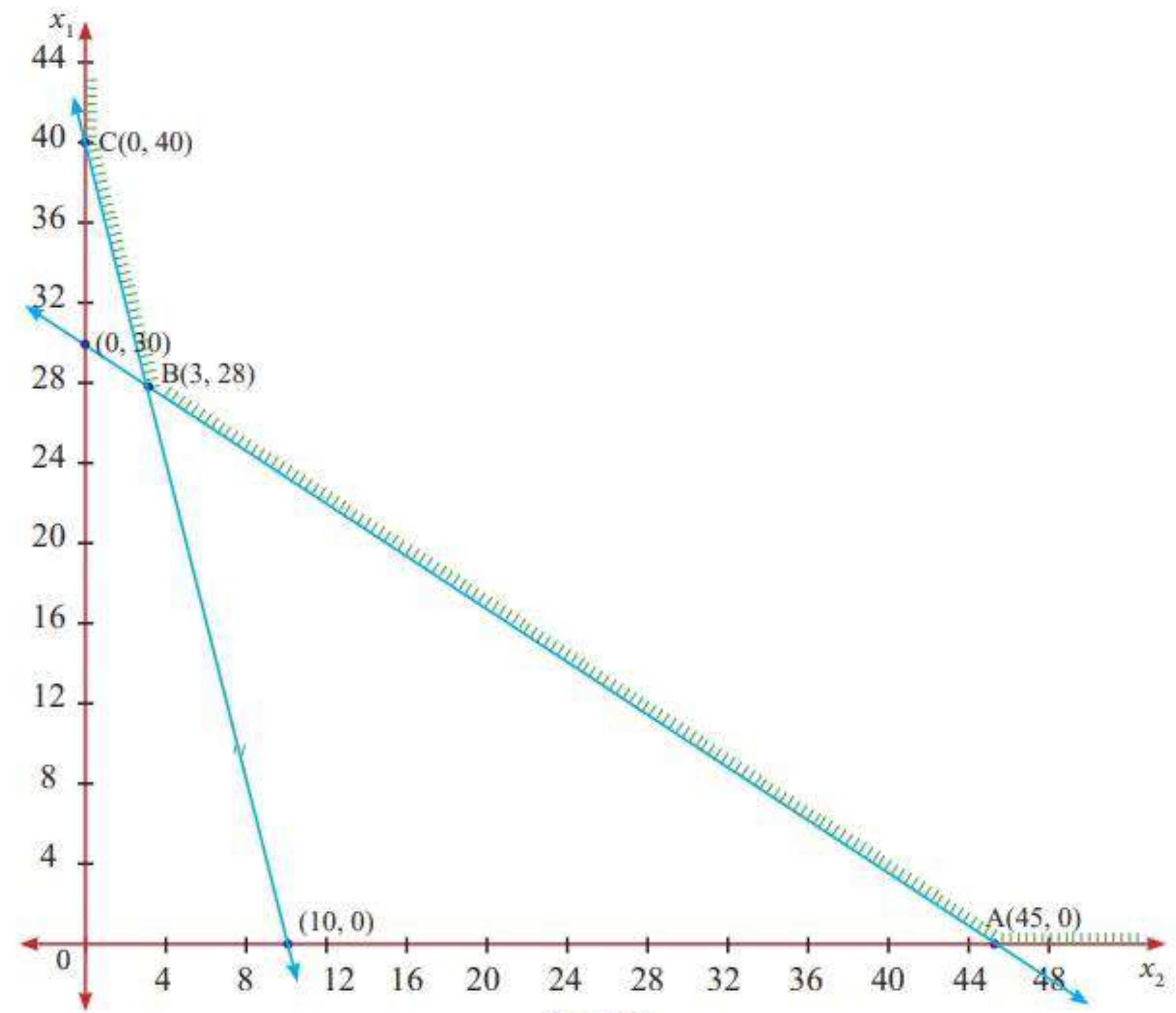


Fig 10.2

SOLUTION

- The feasible region is ABC (since the problem is of minimization type we are moving towards the origin.

Corner points	$z = 5x_1+4x_2$
A(45,0)	225
B(3,28)	127
C(0,40)	160

Table 10.4

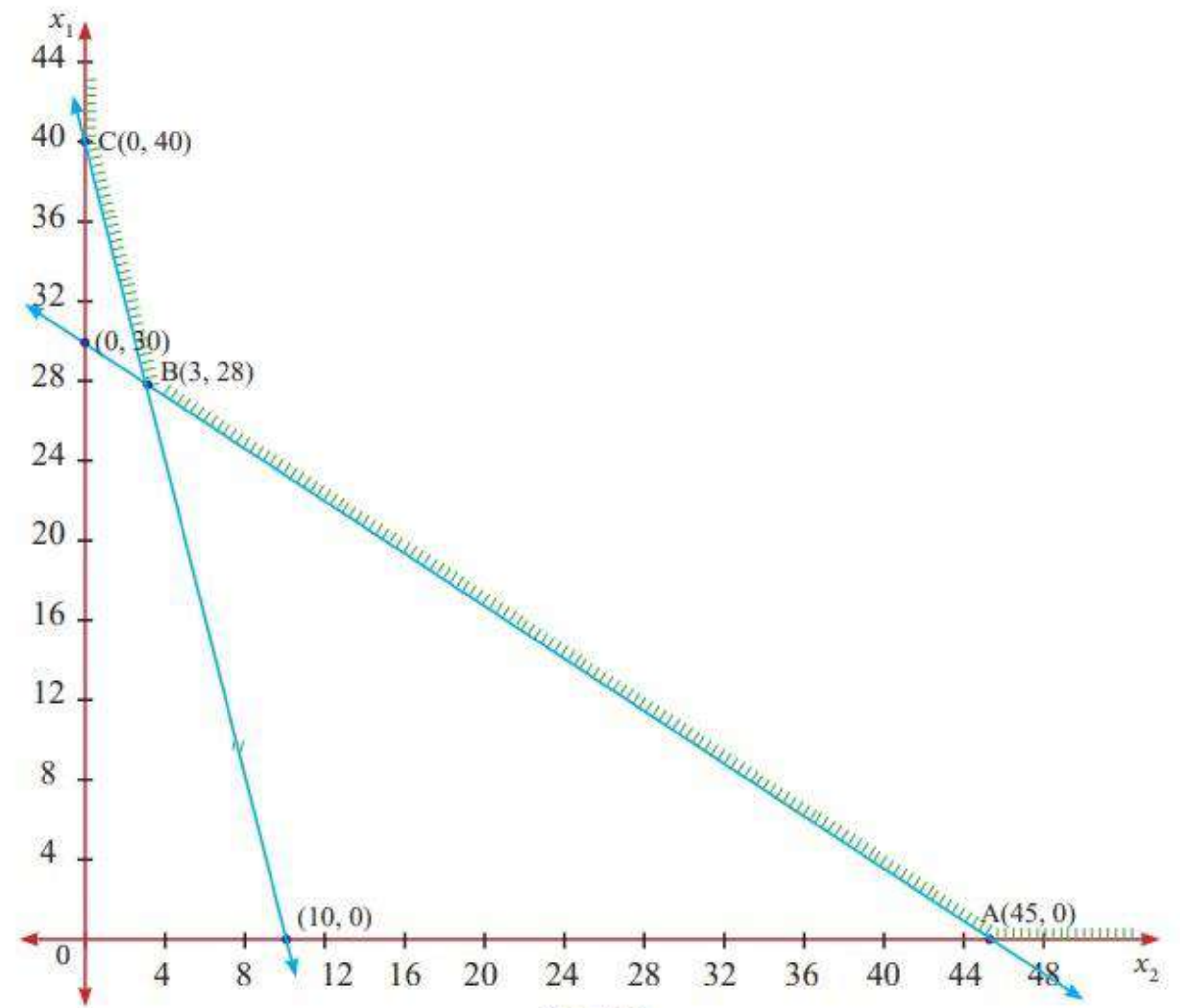


Fig 10.2

SOLUTION

Corner points	$z = 5x_1+4x_2$
A(45,0)	225
B(3,28)	127
C(0,40)	160

Table 10.4

- The minimum value of Z occurs at **B(3,28).**
- Hence the optimal solution is $x_1 = 3, x_2 = 28$ and $Z_{min}=127$

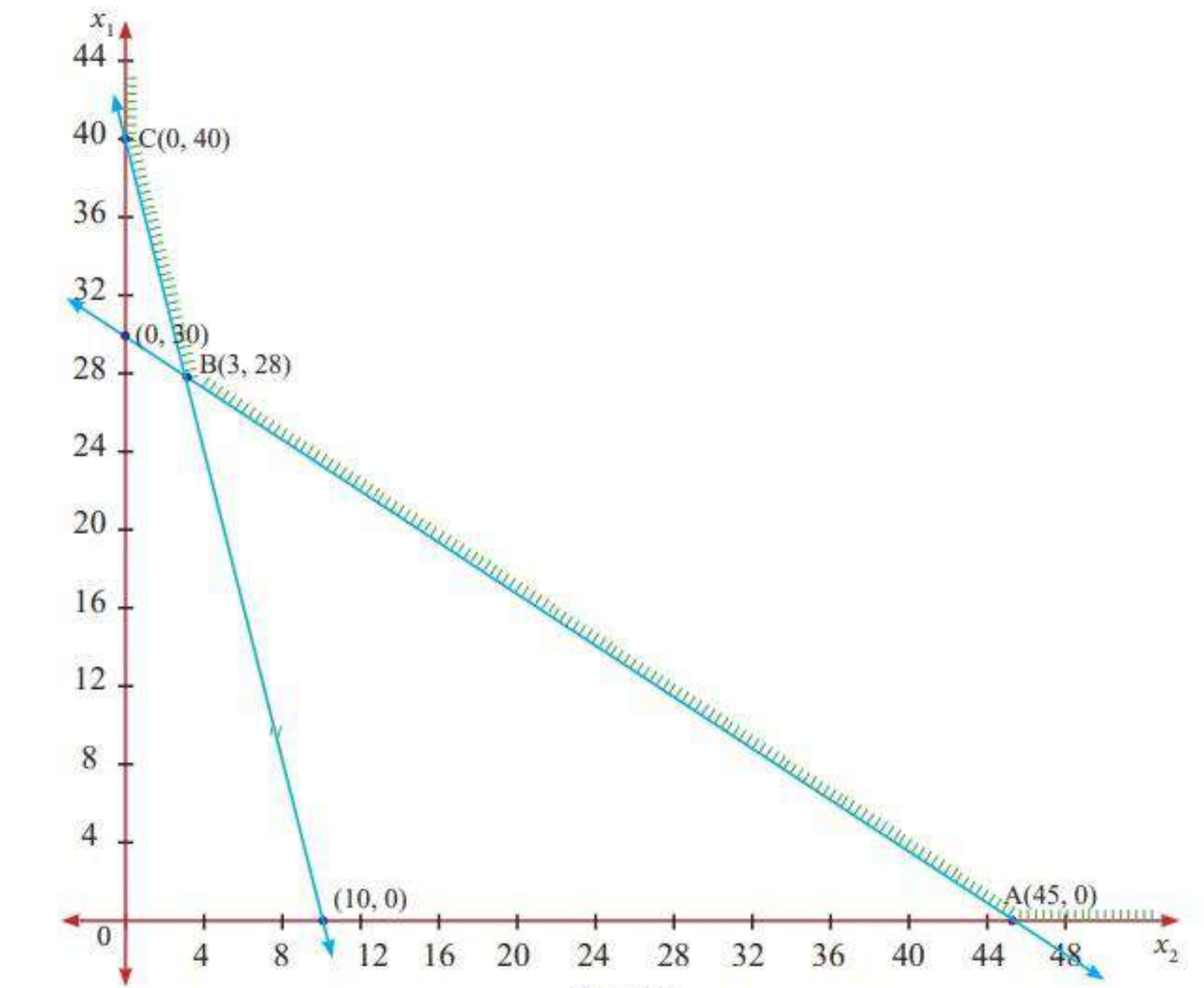


Fig 10.2

LPP BY SIMPLEX METHOD

SIMPLEX METHOD- INTRODUCTION

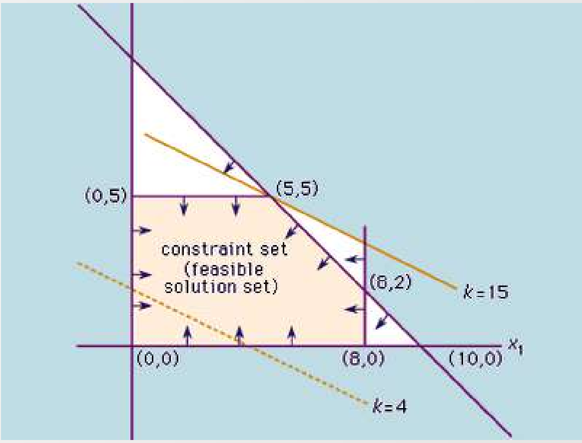
- we used the graphical method to solve linear programming problems, but the graphical approach will not work for problems that have more than two variables.
- In real life situations, linear programming problems consist of literally thousands of variables and are solved by computers.
- We can solve these problems algebraically, but that will not be very efficient.

SIMPLEX METHOD- INTRODUCTION

- Suppose we were given a problem with, say, 5 variables and 10 constraints. By choosing all combinations of five equations with five unknowns, we could find all the corner points, test them for feasibility, and come up with the solution, if it exists. But the trouble is that even for a problem with so few variables, we will get more than 250 corner points, and testing each point will be very boring.
- So we need a method that has a systematic algorithm and can be programmed for a computer.
- The method has to be efficient enough so we wouldn't have to evaluate the objective function at each corner point. We have just such a method, and it is called the simplex method.

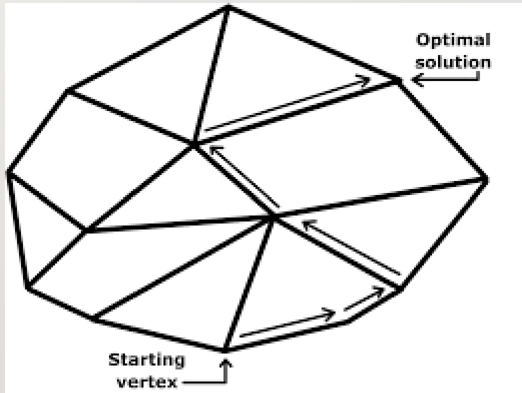
Introduction to Simplex Method

- A linear program (LP) is a method of achieving the best outcome given the task of either maximizing or minimizing a linear objective function, subject to linear constraints.



Source: Encyclopedia Britanica

- The simplex method uses an approach that is very efficient. It begins with a corner point of the feasibility region where all the main variables are zero. Then systematically moves from corner point to corner point, while improving the value of the objective function at each stage. The process continues until the optimal solution is found.



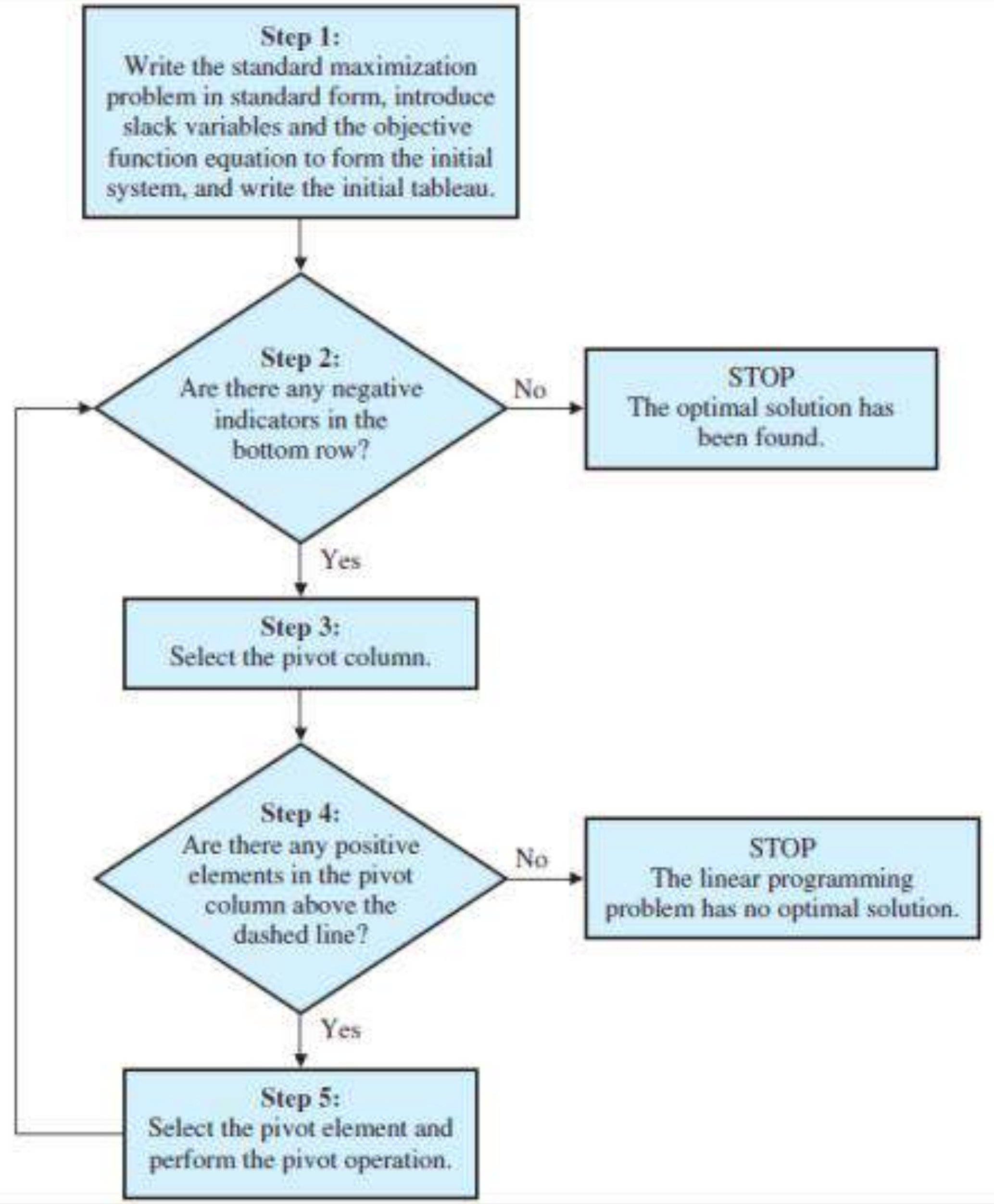
Source: UC Davis Mathematics

Introduction to Simplex Method

- The Simplex method is an approach to solving linear programming problems by hand using **slack variables, tableau**, and **pivot element** as a means of finding the optimal solution of an optimization problem.

- Solving a LP problem using simplex method involves the following steps:
  1. Format the problem in Standard Form
  2. Convert inequality constraints to equations using **slack variables**
  3. Set up the initial simplex **tableau** using the objective function and the slack equations
  4. Check for optimality. If optimality is reached, go to step 8.
  5. Identify the **pivot element**
  6. Create a revised/new tableau through pivoting operations.
  7. Check for optimality. If optimality is NOT reached, go to step 5
  8. Determine the optimal value of the objective function and the decision variables.

- Standard form
- Introducing slack variables
- Creating the tableau
- Pivot variables
- Creating a new tableau
- Checking for optimality
- Identify optimal values



Problem solving using Simplex method

- Let us learn more about the steps of the Simplex method while solving a LP optimization problem shown below. The goal is to minimize the objective function (-Z). This LP problem has 3 decision variables (x1, x2 and x3) and two inequality constraints.

$$Minimize : -z = -8x_1 - 10x_2 - 7x_3$$

$$s.t. : x_1 + 3x_2 + 2x_3 \leq 10$$

$$-x_1 - 5x_2 - x_3 \geq -8$$

$$x_1, x_2, x_3 \geq 0$$

Step 1: Standard Form

Standard form has three requirements:

1. must be a maximization problem
2. all linear constraints must be in a <= inequality
3. all variables are non-negative

$$Minimize: -z = -8x_1 - 10x_2 - 7x_3$$

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$$s.t. : x_1 + 3x_2 + 2x_3 \leq 10$$
$$-x_1 - 5x_2 - x_3 \geq -8$$
$$x_1, x_2, x_3 \geq 0$$

The given problem is to minimize –Z. To satisfy requirement 1, let us transform the minimization problem into a maximization problem, by multiplying both the left and the right sides of the objective function by –1.

$$-1 \times (-z = -8x_1 - 10x_2 - 7x_3)$$
$$z = 8x_1 + 10x_2 + 7x_3$$
$$Maximize : z = 8x_1 + 10x_2 + 7x_3$$

Similarly, let us transform the >= constraint to <= constraint by multiplying both sides of the constraint by –1.

$$-1 \times (-x_1 - 5x_2 - x_3 \geq -8)$$
$$x_1 + 5x_2 + x_3 \leq 8$$

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Step 2: Slack variables

Convert the <= inequalities into equations by adding one slack variable for each inequality.

Slack variables are needed in the constraints to transform them into solvable equalities with one definite answer.

The variable s_1 represents the amount (slack) by which $x_1 + 3x_2 + 2x_3$ falls short of 10.
Similarly, s_2 represents the amount (slack) by which $x_1 + 5x_2 + x_3$ falls short of 8.

$$x_1 + 3x_2 + 2x_3 + \mathbf{s_1} = 10$$

$$x_1 + 5x_2 + x_3 + \mathbf{s_2} = 8$$

$$x_1, x_2, x_3, \mathbf{s_1}, \mathbf{s_2} \geq 0$$

Step 3: Set up the initial Tableau: preparation

A Simplex tableau is used

- to perform row operations on the LP model, and
- to check a solution for optimality.

The current LP model is shown on the right.

$$Maximize: z = 8x_1 + 10x_2 + 7x_3$$

$$s.t.: x_1 + 3x_2 + 2x_3 + s_1 = 10$$

$$x_1 + 5x_2 + x_3 + s_2 = 8$$

The tableau will be created from the coefficients of the 3 equations, (re-)written as shown on the right.

The first equation corresponds to the objective function. The latter 2 equations correspond to the 2 constraints.

$$8x_1 + 10x_2 + 7x_3 + 0s_1 + 0s_2 = z$$

$$1x_1 + 3x_2 + 2x_3 + 1s_1 + 0s_2 = 10$$

$$1x_1 + 5x_2 + 1x_3 + 0s_1 + 1s_2 = 8$$

Step 3: Set up the initial Tableau: C_js

$$8x_1 + 10x_2 + 7x_3 + 0s_1 + 0s_2 = z$$
$$1x_1 + 3x_2 + 2x_3 + 1s_1 + 0s_2 = 10$$
$$1x_1 + 5x_2 + 1x_3 + 0s_1 + 1s_2 = 8$$

In the tableau below,

- the 2nd row of the tableau lists all the variables.
- The top row lists the coefficients of these variables in the objective function:

  C_j: coefficient in the jth column
- The last two rows list the coefficients in the two linear constraints.
- The last column contains the numbers on the RHS of the equations.

Tableau →

C_j ->	8	10	7	0	0	
	x1	x2	x3	s1	s2	b
	1	3	2	1	0	10
	1	5	1	0	1	8

Step 3: Set up the initial Tableau: basic variables

	C_j ->	8	10	7	0	0	solution
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8

basic variables:

A variable is called a basic variable if it's column (in the tableau above) contains 1 in a row and zeros in all other rows.

- Here, the s1 column contains 1 in the 1st row (below the horizontal line) and zeros in all other rows. So, s1 is a basic variable in the 1st row.
- Similarly, s2 is a basic variable in the 2nd row.
- In the objective function (z = 8x1 + 10x2 +7x3 +0s1 +0s2), the coefficients of the basic variables (s1 and s2) are zero. These (C_{Bi}) are written in the left most column above.

Step 3: Set up the initial Tableau: z_js

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8
	z_j	0	0	0	0	0	0
	Cj - zj	8	10	7	0	0	

z_js:

$$z_j = \Sigma_i C_{Bi} a_{ij}$$

1st column: z1 = 0 x1 + 0x1 = 0

Similarly, z_i (I=1 to 6) is zero in all 6 columns. Write these zi values in a row below.

- Now, in each of the 6 columns, compute $C_j - z_j$ and write them in the last row.

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Step 4: Check for optimality

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	1	3	2	1	0	10
0	s2	1	5	1	0	1	8
	z_j	0	0	0	0	0	0
	Cj - zj	8	10	7	0	0	

- A **solution is optimum** (maximum) **if all** (Cj – zj) values in the last row of the tableau are **<= zero**.
- In the above tableau, there are 3 positive values of (Cj – zj). So, this solution is **NOT optimal**.
- So, we will go to step 5.

Step 5: Identify pivot row/column and pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	10/3
0	s2	1	5	1	0	1	8	$\boxed{8/5}$
	z_j	0	0	0	0	0	0	
	Cj - zj	8	$\boxed{10}$	7	0	0		

- **Pivot column**: In the Cj – zj row, there are 3 positive values: 8, 10 and 7. Pick the **largest** value. In our case, it is 10. The column containing this number is called the pivot column. In our case, it is the x2 column (outlined in yellow color).
- **Pivot row**: In each of the linear constraint row, compute the ratio of b to the value in the pivot column of that row. For example, b in the first row is 10 and the value in the pivot (x2) column in that row is 3. So, the ratio for the 1st row is 10/3. The ratio for the 2n row is 8/5. The row containing the **smallest** ratio is called the pivot row. In our case, 2nd row (green) is the pivot row.

Step 5: Identify pivot row/column and pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	10/3
0	s2	1	5	1	0	1	8	$\boxed{8/5}$
	z_j	0	0	0	0	0	0	
	Cj - zj	8	$\boxed{10}$	7	0	0		

- **Pivot element**: The value present in the pivot column (x2 column) **and** in the pivot row (2nd row) is called pivot element. In our case, the value is 5.
- The basic variable in the pivot row is called the **leaving variable**. So, s2 is the leaving variable.
- The variable in the pivot column is called the entering variable. So, x2 is the **entering variable**.
- In the next step (of revision of the tableau), the entering variable (x2) will become the basic variable, replacing the leaving variable (s2).

Step 6: Create a New Tableau via Pivoting

- **Pivoting** is a process (involving row operations) of
  a) obtaining a 1 in the location of the pivot element, and
  b) making all other entries zeros in that (pivot) column.

Old tableau with
revised pivot row

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

a) Obtain a 1 in the location of the pivot element:
The value of the pivot element was 5 in the old tableau. Divide every element of the pivot row (row 2 in our case) by the value of the pivot element, i.e., 5, to obtain a revised pivot row (shown in red colour). Now, the revised pivot has value 1 in the pivot column (x2 column).

Old tableau with revised pivot row

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

b) Via row operations, make all entries zero in the (pivot) column of all remaining (non-pivot) rows.

Let **n** denote the value in the pivot column of a row. The new row is obtained by the following rule:

Value in the **new** row = value in the **old** row – **n** * value in the **pivot** row

In the next slide, the steps of revising the 1st row is illustrated.

Old tableau with revised pivot row

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j							
	Cj - zj							

Value in the **new** row = value in the **old** row – **n** * value in the **pivot** row

Let us revise Row 1 using the above formula.
In Row 1 and pivot (x2) column, the value is 3. So, **n** = 3.
To begin with, let us obtain the new value in column 1 of Row 1.
In Row 1 and **1st** column, the value is 1. It's new value will become

New value = old value – **n** * value in the pivot row of 1st column

= 1 – 3*(1/5) = 2/5

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	1	3	2	1	0	10	
10	x2	1/5	1	1/5	0	1/5	8/5	

New row = old row – **n** * pivot row

n = 3 for Row 1.

In Row 1 and 1st column, the value is 1. Its new value will become

New value = old value – **n** * value in the pivot row of 1st column

= 1 – 3 * (1/5) = 2/5

In Row 1 and **2nd** column, the value is 3. Its new value will become

= 3 – 3 * 1 = 0

Repeat the process for all columns of Row 1. The revised Tableau is shown below:

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	
10	x2	1/5	1	1/5	0	1/5	8/5	

Step 7: Check for optimality

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	
10	x2	1/5	1	1/5	0	1/5	8/5	
	z_j	2	10	2	0	2	16	
	Cj - zj	6	0	5	0	-2		

- Compute zj using the formula $z_j = \Sigma_i \, C_{Bi} \, a_{ij}$
- Compute Cj - zj
- The **solution is optimum if all** values in the last row of the tableau are <= zero.
- In the above tableau, there are 2 positive values of (Cj – zj). So, this solution is **NOT optimal**.
- So, we will return to step 5 to derive a new tableau.

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Step 5: Identify the new pivot element

	C_j ->	8	10	7	0	0		
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b	ratio
0	s1	2/5	0	7/5	1	-3/5	26/5	(26/5)/(2/5)
10	x2	1/5	1	1/5	0	1/5	8/5	(8/5)/(1/5)
	z_j	2	10	2	0	2	16	
	Cj - zj	6	0	5	0	-2		

- The largest value in the bottom row is 6 So, the first (x1) column is the **pivot column,**
- Then we compute the ratios. The smallest ratio is 8 (row 2). So, Row 2 is the **pivot row**.
- The **pivot element** is **1/5**.
- The variable x1 and x2 are incoming and outcoming variables, respectively. So, x1 will replace x2 as the basic variable.

Step 6: Create a New Tableau via Pivoting

Old tableau after revising pivot row

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	2/5	0	7/5	1	-3/5	26/5
8	x1	I	5	5	0	5	8

New row = old row – **n** * pivot row

By carrying out pivoting operations, we get a N**ew Tableau** as shown below.

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

Step 7: Check for optimality

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

- Since all values in the Cj -zj row are <= zero, this is the **optimal solution**.

Step 8: Identify the optimal values

	C_j ->	8	10	7	0	0	
C_{Bi}	Basic variables	x1	x2	x3	s1	s2	b
0	s1	0	-2/5	-3/5	1	-13/5	2
8	x1	I	5	5	0	5	8
	z_j	8	40	40	0	40	64
	Cj - zj	0	-30	-33	0	-40	

- Here, the basic variables are x1 and s1. The non-basic variables are x2, x3 and s2.
- The optimal value of a non-basic variable is zero.
- If a variable is basic, the value of b (in the last column) of the row containing 1 is the optimal value.
- The 1 of the basic variable x1 is found in the 2nd row. The value of b in the second row is 8. This is the optimal value of the variable x1. The optimal value of s1 is 2.
- The optimal objective value is 64 and can be found when x1=8, x2=0, and x3=0.

EXAMPLE 2

Max $z = 12x_1 + 16x_2$

$$10x_1 + 20x_2 \le 120$$
$$8x_1 + 8x_2 \le 80$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Solution: $z = 12x_1 + 16x_2 + 0s_1 + 0s_2$

$$10x_1 + 20x_2 + 1s_1 + 0s_2 <= 120$$
$$8x_1 + 8x_2 + 0s_1 + 1s_2 <= 80$$
$$x_1, x_2, s_1, s_2 >= 0$$

	C_j ->	12	16	0	0		
C_{Bi}	Basic variables	x1	x2	s1	s2	b	ratio
0	s1	10	20	1	0	120	$\boxed{120/20}$
0	s2	8	8	0	1	80	80/8
	zj	0	$\boxed{0}$	0	0	0	
	Cj - zj	12	16	0	0		

	C_j ->	12	16	0	0	
C_{Bi}	Basic variables	x1	x2	s1	s2	b
16	x2	1/2	1	1/20	0	6
0	s2	4	0	-2/5	1	32
	zj	8	16	4/5	0	96
	Cj - zj	4	0	-4/5	0	

Revised tableau
Non-optimal solution

	C_j ->	12	16	0	0		
C_{Bi}	Basic variables	x1	x2	s1	s2	b	ratio
16	x2	1/2	1	1/20	0	6	12
0	s2	4	0	-2/5	1	32	8
	zj	8	16	4/5	0	96	
	Cj - zj	4	0	-4/5	0		

Pivot element = 4

	C_j ->	12	16	0	0	
C_{Bi}	Basic variables	x1	x2	s1	s2	b
16	x2	0	1	1/10	-1/8	2
12	x1	1	0	-1/10	1/4	8
	zj	12	16	4/10	1	128
	Cj - zj	0	0	-4/10	-1	

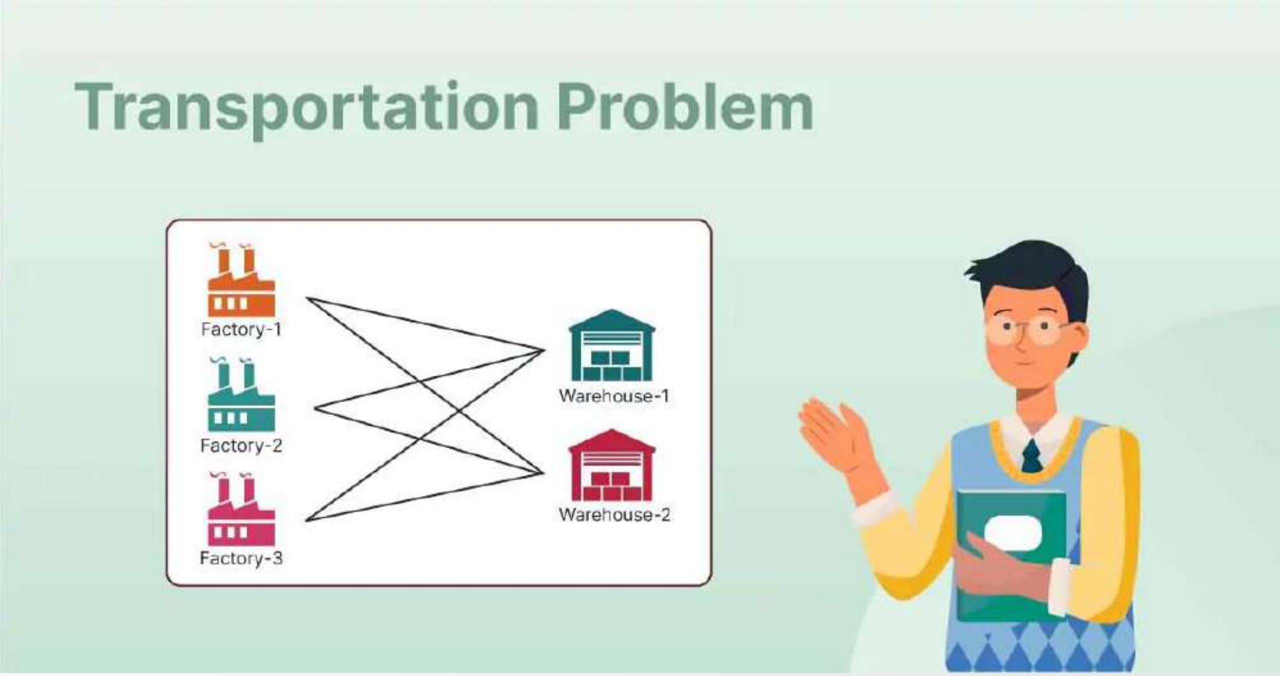
Revised tableau
Optimal solution

Optimal values are

z = 128
x1 = 12
x2 = 16

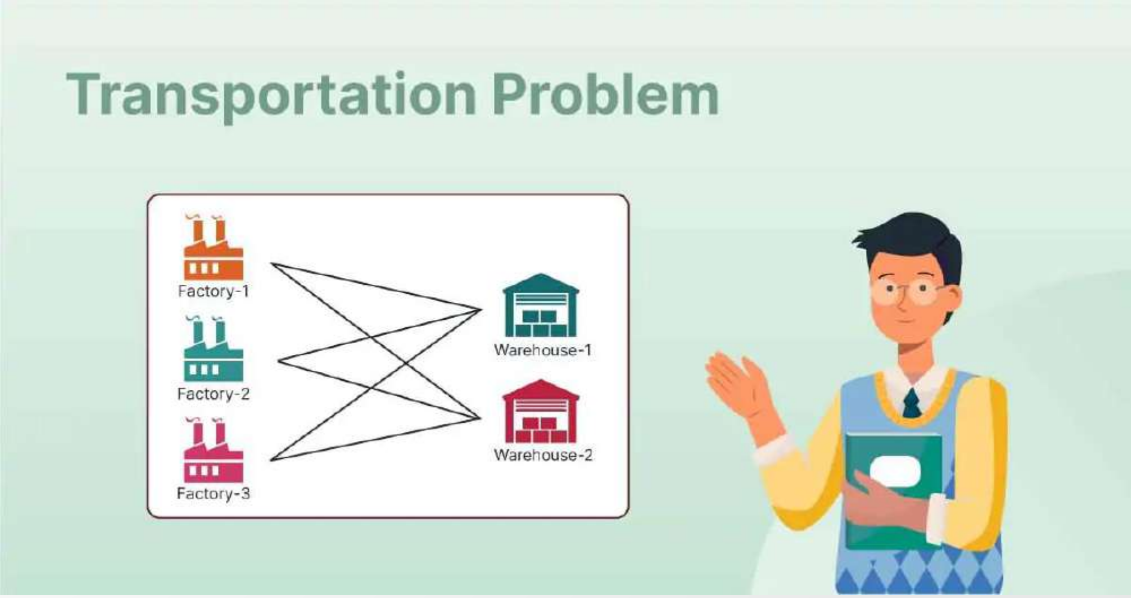
TRANSPORTATION PROBLEM

- Transportation problem works in a way of minimizing the cost function.
- The cost function is the amount of money spent to the logistics provider for transporting the commodities from production or supplier place to the demand place.



TRANSPORTATION PROBLEM

- It includes the distance between the two locations, the path followed, mode of transport, the number of units that are transported, the speed of transport, etc.
- To transport the commodities with minimum transportation cost without any compromise in supply and demand.



INTRODUCTION

- The transportation problem is a special type of linear programming problem, where the **objective is to minimize the cost of distributing a product from a number of sources to a number of destinations.**

- Transportation deals with the transportation of a commodity (single product) from 'm' sources (origin or supply or capacity capacity centers) to 'n' destinations (sinks or demand or requirement centers).

INTRODUCTION

- It is assumed that, level of supply of each source and the amount of demand at each destination are known .
- The unit transportation cost of commodity from each source to each destination are known.
- The **objective** is to determine the amount to be shifted from each source to each destination such that the total transportation cost is **minimum**.
- The goal of minimizing the total transportation cost.

TYPES OF TRANSPORTATION PROBLEM

- **Balanced Transportation Problem:** where the total supply equals to the total demand.

- **Unbalanced Transportation Problem:** where the total supply is not equal to the total demand.

METHODS OF SOLVING TRANSPORTATION PROBLEMS

- To **find the initial basic feasible solution** there are three methods:
  - **1. North West Corner Cell Method.**
  - **2. Least Call Cell Method.**
  - **3. Vogel's Approximation Method (VAM).**

BASIC STRUCTURE OF TRANSPORTATION PROBLEM:

- In the above table **D_1, D_2, D_3 and D_4** are the destinations where the **products/goods** are to be delivered from different sources **S_1, S_2, S_3 and S_4.**
- **S_i** is the **supply** from the **source O_i**.
- **d_j** is the **demand** of the **destination D_j**.
- **C_{ij} is the cost** when the **product is delivered from source S_i to destination D_j.**

		Destination D1	D2	D3	D4	Supply (si)
Source	O1	C_{11}	C_{12}	C_{13}	C_{14}	S1
	O2	C_{21}	C_{22}	C_{23}	C_{24}	S2
	O3	C_{31}	C_{32}	C_{33}	C_{34}	S3
	O4	C_{41}	C_{42}	C_{43}	C_{44}	S4
Demand (di)		d1	d2	d3	d4	

EXAMPLE 1: SOLVING BALANCED TRANSPORTATION PROBLEM BY NORTH WEST CORNER METHOD

- Three **sources (O1, O2, and O3)** and
- Four **destinations (D1, D2, D3, and D4).**
- **what is the best way to solve this problem?**
- The supply for the sources O1, O2, and O3 are **300, 400, and 500**, respectively.
- Demands for the destination D1, D2, D3, and D4 are **250, 350, 400, and 200**, respectively.

Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand	250	350	400	200	1200

CONT.

- The starting point for the **North West Corner technique is (O1, D1),** which is the table's northwest corner.
- The cost of transportation is calculated for each value in the cell.
- As indicated in the diagram, **compare the demand for column D1** with the **supply from source O1** and assign a **minimum of two** to the cell (O1, D1).
- **Column D1's demand has been met**, hence the **entire column will be canceled**.
- The supply from the source O1 is still **300 – 250 = 50.**

Source	Destination D1	D2	D3	D4	Supply
O1	250 3	1	7	4	~~300~~ 50
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand	~~250~~ 0	350	400	200	1200

CONT.

- **(O1, D2),**
- The supply from O1 is 50 and the demand for D2 is 350, allocate 50 to the cell (O1, D2).
- Now, row O1 is canceled because the supply from row O1 has been completed. Hence, the demand for Column D2 has become 350 – 50 = 300.
- **(O2, D2),**
- The shortest supply from source O2 (400) and the demand for column D2 (300) is 300, thus putting 300 in the cell (O2, D2).
- The demand for column D2 has been met, the column can be deleted, and the remaining supply from source O2 is 400 – 300 = 100.

Destination

Source	D1	D2	D3	D4	Supply
O1	250 3	50 1	7	4	~~300~~ ~~50~~ 0
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand	~~250~~ 0	~~350~~ 300	400	200	1200

Destination

Source	D1	D2	D3	D4	Supply
O1	250 3	50 1	7	4	~~300~~ ~~50~~ 0
O2	2	300 6	5	9	~~400~~ 100
O3	8	3	3	2	500
Demand	~~250~~ 0	~~350~~ ~~300~~ 0	400	200	1200

CONT.

- **(O2, D3),**
- compare the O2 supply (i.e. 100) to the D2 demand (i.e. 400) and assign the smaller (i.e. 100) to the cell (O2, D2).
- Column D3 has a leftover demand of 400 – 100 = 300.
- **Continuing in the same manner.**
- Now just multiply the allocated value with the respective cell value (i.e. the cost) and add all of them to get the basic solution i.e. **(250 * 3) + (50 * 1) + (300 * 6) + (100 * 5) + (300 * 3) + (200 * 2) = 4400**

Source \ Destination	D1	D2	D3	D4	Supply
O1	250 / 3	50 / 1	7	4	~~300~~ ~~50~~ 0
O2	2	300 / 6	100 / 5	9	~~400~~ ~~100~~ 0
O3	8	3	3	2	500
Demand	~~250~~ 0	~~350~~ ~~300~~ 0	~~400~~ 300	200	1200

Source \ Destination	D1	D2	D3	D4	Supply
O1	250 / 3	50 / 1	7	4	~~300~~ ~~50~~ 0
O2	2	300 / 6	100 / 5	9	~~400~~ ~~100~~ 0
O3	8	3	300 / 3	200 / 2	~~500~~ ~~200~~ 0
Demand	~~250~~ 0	~~350~~ ~~300~~ 0	~~400~~ ~~300~~ 0	~~200~~ 0	1200

EXAMPLE 2:NORTH WEST CORNER RULE

Source	Destination			Supply
	A	B	C	
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	

- Total Supply = Total Demand = 34

Source	Destination			Supply
	A	B	C	
1	**2**	**7**	**4**	5
2	**3**	**3**	**1**	8
3	**5**	**4**	**7**	7
4	**1**	**6**	**2**	14
Demand	7	9	18	34

EXAMPLE 2:NORTH WEST CORNER RULE

Source	Destination A	Destination B	Destination C	Supply
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

Source	Destination A	Destination B	Destination C	Supply
1	2 $\boxed{5}$	7	4	~~5~~ 0
2	3 $\boxed{2}$	3 $\boxed{6}$	1	~~8~~ ~~6~~ 0
3	5	4 $\boxed{3}$	7 $\boxed{4}$	~~7~~ ~~4~~ 0
4	1	6	2 $\boxed{14}$	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	34

- The IBFS is = (2*5)+(3*2)+(3*6)+(4*3)+(7*4)+(2*14) = 102.

SOLVING UNBALANCED TRANSPORTATION PROBLEM

- The sum of all the supplies, O1, O2, O3, and O4, does not equal the sum of all the demands, D1, D2, D3, D4, and D5, the situation is unbalanced.

Destination

Source	D1	D2	D3	D4	D5	Supply (Si)
O1	5	1	8	7	5	15
O2	3	9	6	7	8	25
O3	4	2	7	6	5	42
O4	7	11	10	4	9	35
Demand (di)	30	20	15	10	20	117 \ 95

CONT.

- The idea of a dummy row or dummy column will be applied in this type of scenario.
- Because the **supply is more than the demand** in this situation, a **fake demand column will be inserted**, with a demand of (total supply – total demand), i.e. 117 – 95 = 22.
- A fake **supply row** would have been **introduced if demand was greater than supply.**

		Destination						Supply (Si)
		D1	D2	D3	D4	D5		
Source	O1	5	1	8	7	5	0	15
	O2	3	9	6	7	8	0	25
	O3	4	2	7	6	5	0	42
	O4	7	11	10	4	9	0	35
Demand (di)		30	20	15	10	20	22	117 / 117

2. LEAST COST METHOD(LCM)

- The least cost method is more **economical than north-west corner rule** , since **it starts with a lower beginning cost.**
- **Step 1: Find the cell with the least(minimum) cost in the transportation table.**
- **Step 2: Allocate the maximum feasible quantity to this cell.**
- **Step:3: Eliminate the row or column where an allocation is made.**
- **Step:4: Repeat the above steps for the reduced transportation table until all the allocations are made.**

EXAMPLE:

- Total Supply = Total Demand = 24
- The given problem is a balanced transportation problem.
- Hence there exists a feasible solution to the given problem.
- Given Transportation Problem is:
  - The least cost is 1 corresponds to the cells (O1, D1) and (O3, D4).
  - Take the Cell (O1, D1) arbitrarily.
  - Allocate min (6,4) = 4 units to this cell.

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	5	8
O_3	5	2	2	1	10
Demand	4	6	8	6	

	D_1	D_2	D_3	D_4	Supply (a_i)
O_1	1	2	3	4	6
O_2	4	3	2	5	8
O_3	5	2	2	1	10
Demand (b_j)	4	6	8	6	

CONT…

	D_1	D_2	D_3	D_4	a_i
O_1	(4) 1	2	3	4	6/2
O_2	4	3	2	5	8
O_3	5	2	2	1	10
b_j	4/0	6	8	6	

- **The reduced table is**

	D_2	D_3	D_4	a_i
O_1	2	3	4	2
O_2	3	2	5	8
O_3	2	2	1	10
b_j	6	8	6	

- The least cost corresponds to the cell (O3, D4). Allocate min (10,6) = 6 units to this cell.

	D_2	D_3	D_4	a_i
O_1	2	3	4	2
O_2	3	2	5	8
O_3	2	2	(6) 1	10/4
b_j	6	8	6/0	

- The reduced table is

	D_2	D_3	a_i
O_1	2	3	2
O_2	3	2	8
O_3	2	2	4
b_j	6	8	

- The least cost is 2 corresponds to the cells (O1, D2), (O2, D3), (O3, D2), (O3, D3)
- Allocate min (2,6) = 2 units to this cell.

	D_2	D_3	a_i
O_1	(2) 2	3	2/0
O_2	3	2	8
O_3	2	2	4
b_j	6/4	8	

- The reduced table is

	D_2	D_3	a_i
O_2	3	2	8
O_3	2	2	4
bj	4	8	

CONT…

- The least cost is 2 corresponds to the cells (O2, D3), (O3, D2), (O3, D3)
- Allocate min (8,8) = 8 units to this cell.

	D_2	D_3	a_i
O_2	3	(8) 2	8/0
O_3	2	2	4
b_j	4	8/0	

- The reduced table is

	D_2	a_i
O_3	2	4
b_j	4	

- **Here allocate 4 units in the cell (O3, D2)**

	D_2	a_i
O_3	(4) 2	4/0
b_j	4/0	

- **Thus we have the following allocations:**

	D_1	D_2	D_3	D_4	a_i
O_1	(4) 1	(2) 2	3	4	6/2/0
O_2	4	3	(8) 2	5	8/0
O_3	5	(4) 2	2	(6) 1	10/4/0
b_j	4/0	6/4/0	8/0	6/0	

- Transportation schedule : O1→ D1, O1→D2, O2→D3, O3→D2, O3→D4
- Total transportation cost : = (4×1)+ (2×2)+(8×2)+(4×2)+(6×1) =Rs. 38.

CONT...

	D_1	D_2	D_3	D_4	a_i
O_1	(4) 1	(2) 2	3	4	6/2/0
O_2	4	3	(8) 2	5	8/0
O_3	5	(4) 2	2	(6) 1	10/4/0
b_j	4/0	6/4/0	8/0	6/0	

- Transportation schedule : O1→ D1, O1→D2, O2→D3, O3→D2, O3→D4
- Total transportation cost : = (4×1)+ (2×2)+(8×2)+(4×2)+(6×1) =Rs. 38.

EXAMPLE 2:

- According to the Least Cost Cell method, the **least cost among all the cells in the table** has to be **found which is 1 (i.e. cell (O1, D2)).**
- Now check the supply from the row O1 and demand for column D2 and allocate the smaller value to the cell.
- The smaller value is 300 so allocate this to the cell. The supply from O1 is completed so cancel this row and the remaining demand for the column D2 is 350 – 300 = 50.

Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand	250	350	400	200	1200

Source	D1	D2	D3	D4	Supply
O1	~~3~~	300 ~~1~~	~~7~~	~~4~~	~~300~~ 0
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	~~350~~ 50	400	200	1200

CONT…

- Now find the cell with the least cost among the remaining cells.
- There are two cells with the least cost i.e. (O2, D1) and (O3, D4) with cost 2. Lets select (O2, D1).
- Now find the demand and supply for the respective cell and **allocate the minimum among them to the cell and cancel the row or column whose supply or demand becomes 0 after allocation.**

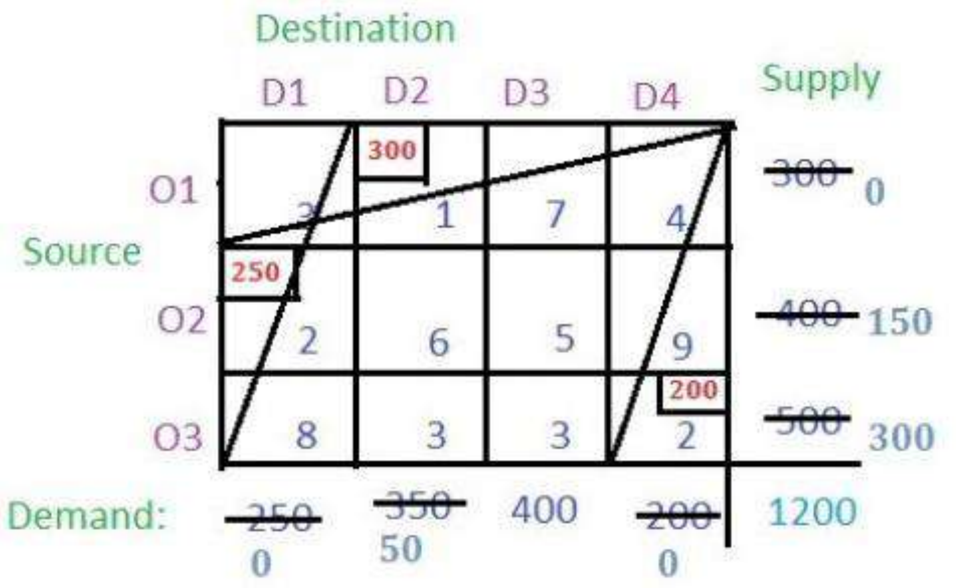
Destination

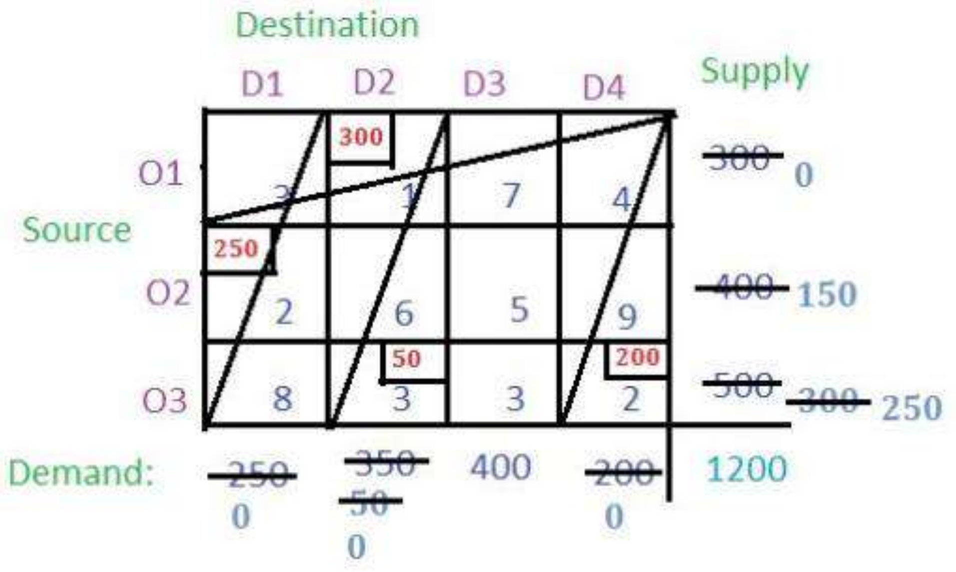
Source	D1	D2	D3	D4	Supply
O1	3	300 1	7	4	~~300~~ 0
O2	250 2	6	5	9	~~400~~ 150
O3	8	3	3	2	500
Demand:	~~250~~ 0	~~350~~ 50	400	200	1200

CONT…

- Now the cell with the least cost is (O3, D4) with cost 2. Allocate this cell with 200 as the demand is smaller than the supply. So the column gets cancelled.

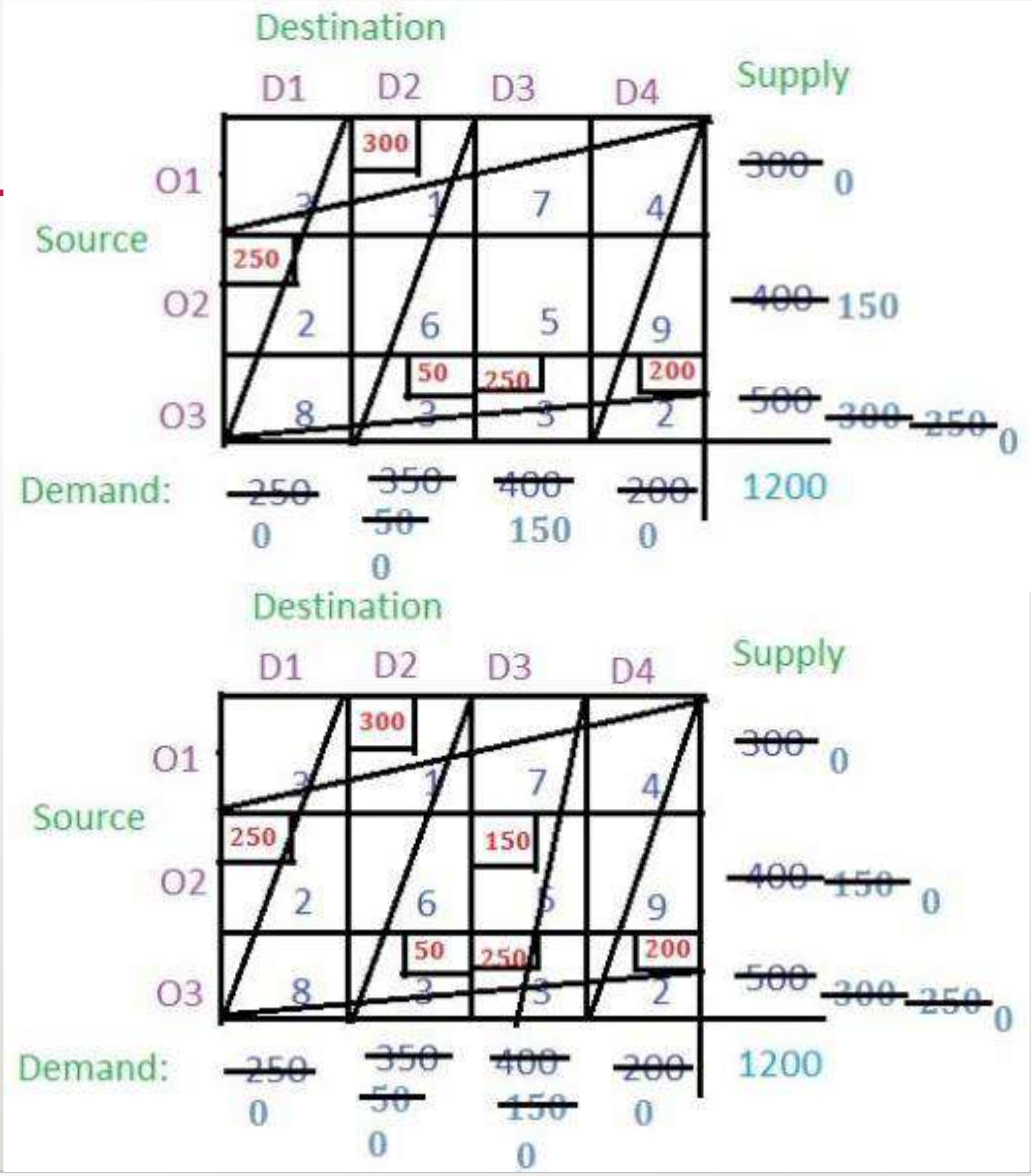
- There are two cells among the unallocated cells that have the least cost. Choose any at random say (O3, D2). Allocate this cell with a minimum among the supply from the respective row and the demand of the respective column. Cancel the row or column with zero value.





CONT…

- Now the cell with the least cost is (O3, D3). Allocate the minimum of supply and demand and cancel the row or column with zero value.
- The only remaining cell is (O2, D3) with cost 5 and its supply is 150 and demand is 150 i.e. demand and supply both are equal. Allocate it to this cell.
- The basic solution i.e. (300 * 1) + (250 * 2) + (150 * 5) + (50 * 3) + (250 * 3) + (200 * 2) = 2850



ROW MINIMA METHOD

- we allocate as much as possible in the lowest cost cell of the first row.
- Subtract this min value from supply and demand .
- If the supply is 0, then cross (strike) that row and If the demand is 0 then cross (strike) that column.
- If min unit cost cell is not unique, then select the cell where maximum allocation can be possible.
- Repeat this process for all uncrossed rows and columns until all supply and demand values are 0.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

CONT..

- In 1st row, The smallest transportation cost is 10

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19~~	~~30~~	~~50~~	~~10(7)~~	0
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	7	

- In 2nd row, The smallest transportation cost is 30

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19~~	~~30~~	~~50~~	~~10(7)~~	0
S_2	70	~~30(8)~~	40	60	1
S_3	40	~~8~~	70	20	18
Demand	5	0	7	7	

CONT..

- In 2nd row, The smallest transportation cost is 40

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19~~	~~30~~	~~50~~	~~10(7)~~	0
~~S_2~~	~~70~~	~~30(8)~~	~~40(1)~~	~~60~~	0
S_3	40	~~8~~	70	20	18
Demand	5	0	6	7	

- In 3rd row, The smallest transportation cost is 20

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19~~	~~30~~	~~50~~	~~10(7)~~	0
~~S_2~~	~~70~~	~~30(8)~~	~~40(1)~~	~~60~~	0
S_3	40	~~8~~	70	~~20(7)~~	11
Demand	5	0	6	0	

CONT..

- In 3rd row, The smallest transportation cost is 40

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10(7)	0
S_2	70	30(8)	40(1)	60	0
S_3	40(5)	8	70	20(7)	6
Demand	0	0	6	0	

- In 3rd row, The smallest transportation cost is 70

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10(7)	0
S_2	70	30(8)	40(1)	60	0
S_3	40(5)	8	70(6)	20(7)	0
Demand	0	0	0	0	

CONT..

- Initial feasible solution is

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10 **(7)**	7
S_2	70	30 **(8)**	40 **(1)**	60	9
S_3	40 **(5)**	8	70 **(6)**	20 **(7)**	18
Demand	5	8	7	14	

- The minimum total transportation cost $=10\times7+30\times8+40\times1+40\times5+70\times6+20\times7=1110$

COLUMN MINIMA METHOD

- we allocate as much as possible in the lowest cost cell of the first Column.
- Subtract this min value from supply and demand .
- If the supply is 0, then cross (strike) that row and If the demand is 0 then cross (strike) that column.
- If min unit cost cell is not unique, then select the cell where maximum allocation can be possible.
- Repeat this process for all uncrossed rows and columns until all supply and demand values are 0.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

CONT..

- In 1st column, The smallest transportation cost is 19.
- In 2nd column, The smallest transportation cost is 8

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10	2
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	0	8	7	14	

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10	2
S_2	70	30	40	60	9
S_3	40	8(8)	70	20	10
Demand	0	0	7	14	

CONT..

- In 3rd column, The smallest transportation cost is 40.

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10	2
S_2	70	30	40(7)	60	2
S_3	40	8(8)	70	20	10
Demand	0	0	0	14	

- In 4th column, The smallest transportation cost is 10.

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19(5)~~	~~30~~	~~50~~	~~10(2)~~	0
S_2	70	30	40(7)	60	2
S_3	40	8(8)	70	20	10
Demand	0	0	0	12	

CONT..

- In 4th column, The smallest transportation cost is 20.

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19(5)~~	~~30~~	~~50~~	~~10(2)~~	0
S_2	70	30	40(7)	60	2
~~S_3~~	~~40~~	~~8(8)~~	~~70~~	~~20(10)~~	0
Demand	0	0	0	2	

- In 4th column, The smallest transportation cost is 60.

	D_1	D_2	D_3	D_4	Supply
~~S_1~~	~~19(5)~~	~~30~~	~~50~~	~~10(2)~~	0
~~S_2~~	~~70~~	~~30~~	~~40(7)~~	~~60(2)~~	0
~~S_3~~	~~40~~	~~8(8)~~	~~70~~	~~20(10)~~	0
Demand	0	0	0	0	

CONT..

- Initial feasible solution is

	D_1	D_2	D_3	D_4	Supply
S_1	19 (5)	30	50	10 (2)	7
S_2	70	30	40 (7)	60 (2)	9
S_3	40	8 (8)	70	20 (10)	18
Demand	5	8	7	14	

- The minimum total transportation cost $=19\times5+10\times2+40\times7+60\times2+8\times8+20\times10=779$

VOGEL'S APPROXIMATION METHOD

Destination

Source	D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	2	6	5	9	400	3
O3	8	3	3	2	500	1
Demand:	250	350	400	200	1200	
Column Difference:	1	2	2	2		

Source	Destination D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	250 2	6	5	9	~~400~~ 150	3
O3	8	3	3	2	500	1
Demand:	~~250~~ 0	350	400	200	1200	
Column Difference:	1	2	2	2		

VOGEL'S APPROXIMATION METHOD

Destination

Source	D1	D2	D3	D4	Supply	Row Difference	
O1	3	1	7	4	300	2	3
O2	250 2	6	5	9	~~400~~ 150	3	1
O3	8	3	3	2	500	1	1
Demand:	~~250~~ 0	350	400	200	1200		
Column Difference:	1	2	2	2			
	-	2	2	2			

VOGEL'S APPROXIMATION METHOD

Destination

Source	D1	D2	D3	D4	Supply	Row Difference	
O1	2	300 1	7	4	~~300~~ 0	2	3
O2	250 2	6	5	9	~~400~~ 150	3	1
O3	8	3	3	2	500	1	1
Demand:	~~250~~ 0	~~350~~ 50	400	200	1200		
Column Difference:	1	2	2	2			
	-	2	2	2			

VOGEL'S APPROXIMATION METHOD

Destination

Source	D1	D2	D3	D4	Supply	Row Difference		
O1	3	300 1	7	4	~~300~~ 0	2	3	-
O2	250 2	6	5	9	~~400~~ 150	3	1	1
O3	8	3	3	200 2	~~500~~ 300	1	1	1
Demand:	~~250~~ 0	~~350~~ 50	400	~~200~~ 0	1200			
Column Difference:	1	2	2	2				
	-	2	2	2				
	-	3	2	7				

Destination

Source	D1	D2	D3	D4	Supply	Row Difference			
O1	3	300 1	7	4	~~300~~ 0	2	3	-	-
O2	250 2	6	5	9	~~400~~ 150	3	1	1	1
O3	8	50 3	3	200 2	~~500~~ ~~300~~ 250	1	1	1	0
Demand:	~~250~~ 0	~~350~~ ~~50~~ 0	400	~~200~~ 0	1200				
Column Difference:	1	2	2	2					
	-	2	2	2					
	-	3	2	7					
	-	3	2	-					

Destination

Source	D1	D2	D3	D4	Supply	Row Difference			
O1	2	300 1	7	4	~~300~~ 0	2	3	-	-
O2	250 2	6	5	9	~~400~~ 150	3	1	1	1
O3	8	50 3	250 3	200 2	~~500~~ ~~300~~ ~~250~~ 0	1	1	1	0
Demand:	~~250~~ 0	~~350~~ ~~50~~ 0	~~400~~ 150	~~200~~ 0	1200				
Column Difference:	1	2	2	2					
	-	2	2	2					
	-	3	2	7					
	-	3	2	-					

VOGEL'S APPROXIMATION METHOD

Destination

Source	D1	D2	D3	D4	Supply	Row Difference			
O1	3	300 / 1	7	4	~~300~~ 0	2	3	-	-
O2	250 / 2	6	150 / 5	9	~~400~~ ~~150~~ 0	3	1	1	1
O3	8	50 / 3	250 / 3	200 / 2	~~500~~ ~~300~~ ~~250~~ 0	1	1	1	0
Demand:	~~250~~ 0	~~350~~ ~~50~~ 0	~~400~~ ~~150~~ 0	~~200~~ 0	1200				
Column Difference:	1	2	2	2					
	-	2	2	2					
	-	3	2	7					
	-	3	2	-					

VOGEL'S APPROXIMATION METHOD

- No balance remains. So multiply the allocated value of the cells with their corresponding cell cost and add all to get the final cost

- i.e. **(300 * 1) + (250 * 2) + (50 * 3) + (250 * 3) + (200 * 2) + (150 * 5) = 2850**

PROBLEM

In the table, three sources A,B and C with the Production Capacity of 50units, 40units, 60 units of product respectively is given. Every day the demand of three retailers P,Q,R is to be furnished with at least 20units, 95units and 35units of product respectively. The transportation costs are also given in the matrix.

Source/Destination	P	Q	R	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

Source/Destination	P	Q	R	Supply	
A	20 5	30 8	4	~~50~~	~~30~~ 0
B	6	40 6	3	~~40~~	0
C	3	25 9	35 6	~~60~~	~~35~~ 0
Demand	~~20~~	~~95~~	~~35~~	150	
	0	~~65~~ ~~25~~ 0	0		

Total Cost can be computed by multiplying the units assigned to each cell with the concerned transportation cost.

Total Cost$= (20 * 5) + (30 * 8) + (40 * 6) + (25 * 9) + (35 * 6) = Rs.1015.$

Source/Destination	D	E	F	Supply
A	5	50 8	4	~~50~~ 0
B	6	5 6	3 35	~~40~~ ~~5~~ 0
C	20 3	40 9	6	~~60~~ ~~40~~ 0
Demand	~~20~~ 0	~~95~~ ~~90~~ ~~40~~ 0	~~35~~ 0	150

$$\text{Total Cost} = 50*8+5*6+35*3+20*3+40*9 = Rs.955.$$

The total demand is 1000, whereas the total supply is 800.
Total supply < total demand.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Here dummy origin is added to the table. The cost associated with the dummy origin will be zero.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200
Demand	250	250	500	1000

Plant	Warehouse			Supply
	W1	W2	W3	
A	28 (50)	17	26 450	~~500~~
B	19	12 250	16 50	~~300~~
Unsatisfied demand	0 200	0	0	~~200~~
Demand	~~250~~	~~250~~	~~500~~	1000

Total

$$\text{Cost} = 28(50) + 26(450) + 12(250) + 16(50) + 0(200) = 16900$$

TUTORIAL PROBLEM-1

Luminous lamps have three factories - F_1, F_2, and F_3 with production capacity 30, 50, and 20 units per week respectively. These units are to be shipped to four warehouses W_1, W_2, W_3, and W_4 with requirement of 20, 40, 30, and 10 units per week respectively. The transportation costs (in Rs.) per unit between factories and warehouses are given below.

Factory	**Warehouse**				**Supply**
	W_1	W_2	W_3	W_4	
F_1	1	2	1	4	30
F_2	3	3	2	1	50
F_3	4	2	5	9	20
Demand	20	40	30	10	

TUTORIAL PROBLEM-2

- The Ushodaya departmental store has three plants located throughout a state with production capacity 80, 60 and 70 kilo grams of rice. Each day the firm must furnish its four retail shops R_1, R_2, R_3, & R_4 with at least 40, 60, 50, and 60 gallons respectively. The transportation costs (in Rs.) are given below.

Store	Retail Shop				Supply
	1	2	3	4	
1	3	5	7	6	80
2	2	5	8	2	60
3	3	6	9	2	70
Demand	40	60	50	60	

TUTORIAL PROBLEM-3

KL University branches located at Vijayawada, Hyderabad, and Chennai. KL University provides course material in printed form at these locations with capacities 15, 30 and 20 units at Vijayawada, Hyderabad, and Chennai respectively. The university distributes the course materialto students located at three locations Bangalore, Hyderabad and Coimbatore. The demand of the students is 5, 20 and 40 units for Bangalore, Hyderabad and Coimbatore respectively. The cost of transportation per unit varies between different supply points and destination points. The transportation costs are given in the table. The management of KL University would like to determine minimum transportation cost.

U/S	BGR	HYD	CON	Supply
BZA	15	60	35	15
HYD	45	30	60	30
CHE	30	90	20	20
Demand	5	20	40	

TUTORIAL PROBLEM-4

The distribution manager of a company needs to minimize global transport costs between a set of three factories (supply points) S1, S2, and S3, and a set of four distributors (demand points) D1, D2, D3, and D4. The following table shows the transportation cost from each supply point to every demand point, the supply of the product at the supply points, and the demand of the product at the demand points.

F/D	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

DUALITY IN LPP

- **Primal Problem :**
  - **It is the original formulation of a linear programming problem.**
- **Dual Problem :**
  - **It is a derived linear programming problem that is associated with the primal formulation.**

PRIMAL	CONVERSION	DUAL
Maximization Problem	⇔	Minimization Problem
Minimization Problem	⇔	Maximization Problem
Objective Coefficients	⇔	Right Hand Side (RHS) values
Right Hand Side (RHS) values	⇔	Objective Coefficients
Number of Variables	⇔	Number of Constraints
Number of Constraints	⇔	Number of Variables
Variables are in terms of X n	⇔	Variables are in terms of Y n

CONT..

- **Variables = 2**
- **Constraints = 2**

Minimize $z = 3x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 40$
$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$

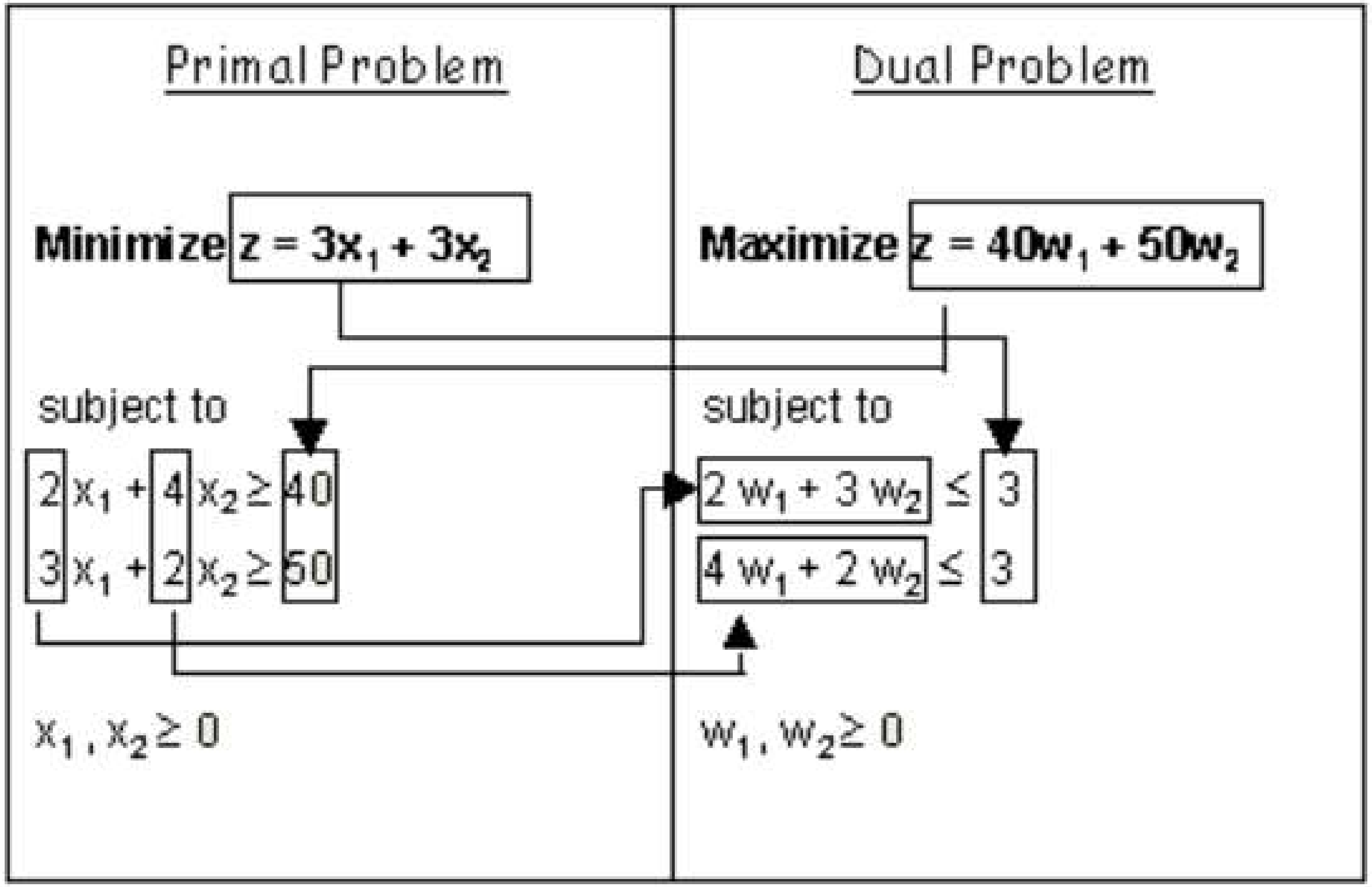
Solution.

Maximize $z = 40w_1 + 50w_2$

subject to

$2w_1 + 3w_2 \leq 3$
$4w_1 + 2w_2 \leq 3$

$w_1, w_2 \geq 0$



CONT…

Q. Construct the dual for the following minimization problem.

Min. $Z = 5X_1 + 8X_2 + 4X_3$

Constraints:-

$2X_1 + 3X_2 - X_3 \geq 10$

$X_1 + X_2 + 3X_3 \geq 5$

$2X_1 + 5X_2 - 2X_3 \leq 7$

$3X_1 - X_2 + 3X_3 \geq 9$

$X_1 + 5X_2 \geq 6$

$X_1, X_2, X_3 \geq 0$

(n) variable= 3
(m) Constraints = 5

$-2X_1 - 5X_2 + 2X_3 \geq -7$

SOLUTION:-

DUAL

Max. $Z^* = 10Y_1 + 5Y_2 - 7Y_3 + 9Y_4 + 6Y_5$

Constraints:-

$2Y_1 + Y_2 - 2Y_3 + 3Y_4 + Y_5 \leq 5$

$3Y_1 + Y_2 - 5Y_3 - Y_4 + 5Y_5 \leq 8$

$-Y_1 + 3Y_2 + 2Y_3 + 3Y_4 \leq 4$

$Y_1, Y_2, Y_3, Y_4, Y_5 \geq 0$

EXAMPLE

- Example. Write the dual of the following primal:
- Minimum $Z = 3x1 - 2x2 + 4x3$
- Subject to the constraints:
    - $3x1 + 5x2 + 4x3 \geq 7$
    - $6\ x1 - x2 + 3x3 \geq 4$
    - $7x1 + 2x2 - 3x3 \leq 10$
    - $x1 - 2x2 + 5x3 \geq 3$
    - $4\ x1 + 7\ x2 - 2x3 \geq 2$
- And $x1, x2, x3 \geq 0$

FARKAS'S LEMMA

- **Lemma :** Let **A be a matrix** and **x and b vectors**.

Then **the system** **$Ax=b$, $x\geq 0$** **has no solution(infeasible)**

if and only if (<=>) the system **$A^T y \geq 0$, $b^T y<0$** **has a solution**, where **y is a vector**.

Farkas' lemma — Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following two assertions is true:

1. There exists an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$.
2. There exists a $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{A}^\top \mathbf{y} \geq 0$ and $\mathbf{b}^\top \mathbf{y} < 0$.

VARIANTS OF FARKAS LEMMA

The Farkas Lemma has several variants with different sign constraints (the first one is the original version):[7]:92

- Either the system $\mathbf{Ax} = \mathbf{b}$ has a solution with $\mathbf{x} \geq 0$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} \geq 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$.
- Either the system $\mathbf{Ax} \leq \mathbf{b}$ has a solution with $\mathbf{x} \geq 0$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} \geq 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$ and $\mathbf{y} \geq 0$.
- Either the system $\mathbf{Ax} \leq \mathbf{b}$ has a solution with $\mathbf{x} \in \mathbb{R}^n$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} = 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$ and $\mathbf{y} \geq 0$.
- Either the system $\mathbf{Ax} = \mathbf{b}$ has a solution with $\mathbf{x} \in \mathbb{R}^n$, or the system $\mathbf{A}^\mathsf{T}\mathbf{y} = 0$ has a solution with $\mathbf{b}^\mathsf{T}\mathbf{y} \neq 0$.

FARKAS LEMMA : CERTIFICATE OF INFEASIBILITY

Farkas Lemma : Certificate of Infeasibility

Let $A \in R^{m \times n}, b \in R^m$ and $x = [x_1, x_2, ..., x_n]^T$.

Then $Ax \geq b$ has no solution or is inconsistent iff

there exists $y \in R^m$ such that:

1. $y \geq 0$,
2. $A^T y = 0$ and
3. $b^T y < 0$.

The IDEA is if we could find a vector y, such that above conditions are satisfied, then we can conclude that the given system of linear equations have no solution.

EXAMPLE ON FARKAS LEMMA

Example:

Consider the following system of linear equations:

$$x_1 + x_2 + 2x_3 \geq 1 \tag{1}$$

$$-x_1 + x_2 + x_3 \geq 2 \tag{2}$$

$$x_1 - x_2 + x_3 \geq 1 \tag{3}$$

$$-x_2 - 3x_3 \geq 0 \tag{4}$$

Here, $\mathrm{A} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$, $\mathrm{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

SOLUTION…

Next we go for elimination of x_1:

$$(1) + (2) \implies \qquad 2x_2 + 3x_3 \geq 3 \qquad (5)$$

$$(2) + (3) \implies \qquad 2x_3 \geq 3 \qquad (6)$$

$$-x_2 - 3x_3 \geq 0 \qquad (4)$$

Next we go for elimination of x_2:

$$\frac{1}{2} \times (5) \implies \qquad x_2 + \frac{3}{2}x_3 \geq \frac{3}{2} \qquad (7)$$

$$2x_3 \geq 3 \qquad (6)$$

$$-x_2 - 3x_3 \geq 0 \qquad (4)$$

SOLUTION…

Now,

$$(4) + (7) \implies \qquad \frac{-3}{2}x_3 \geq \frac{3}{2} \qquad (8)$$

$$2x_3 \geq 3 \qquad (6)$$

Next we go for elimination of x_3:

$$\frac{3}{4} \times (6) \implies \qquad \frac{3}{2}x_3 \geq \frac{9}{4} \qquad (9)$$

$$\frac{-3}{2}x_3 \geq \frac{3}{2} \qquad (8)$$

If we add (8) and (9), we have

$$0 \geq \frac{15}{4} \qquad \text{(this is a contradiction)}$$

(DEEMED TO BE U N I V E R S I T Y)

SOLUTION…

The contradiction is due to

$$\begin{aligned}(9)+(8) &= \frac{3}{4}(6)+(8)\\ &= \frac{3}{4}[(2)+(3)]+[(4)+(7)]\\ &= \frac{3}{4}[(2)+(3)]+(4)+\frac{1}{2}(5)\\ &= \frac{3}{4}[(2)+(3)]+(4)+\frac{1}{2}[(1)+(2)]\\ &= \frac{3}{4}(2)+\frac{3}{4}(3)+(4)+\frac{1}{2}(1)+\frac{1}{2}(2)\\ &= \frac{1}{2}(1)+\frac{5}{4}(2)+\frac{3}{4}(3)+(4)\end{aligned}$$

Therefore, the y vector is the set of the coefficients in the above equation, i.e., $y= \begin{bmatrix}1/2\\5/4\\3/4\\1\end{bmatrix}$

SOLUTION…

Now, $A^T y = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 2 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} (1)\cdot\left(\frac{1}{2}\right) + (-1)\cdot\left(\frac{5}{4}\right) + (1)\cdot\left(\frac{3}{4}\right) + (0)\cdot(1) \\ (1)\cdot\left(\frac{1}{2}\right) + (1)\cdot\left(\frac{5}{4}\right) + (-1)\cdot\left(\frac{3}{4}\right) + (-1)\cdot(1) \\ (2)\cdot\left(\frac{1}{2}\right) + (1)\cdot\left(\frac{5}{4}\right) + (1)\cdot\left(\frac{3}{4}\right) + (-3)\cdot(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

SOLUTION…

$$\text{Also, } b^T y = \begin{bmatrix} 1 & -2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 5/4 \\ 3/4 \\ 1 \end{bmatrix}$$

$$= \left[(1) \cdot \left(\tfrac{1}{2}\right) + (-2) \cdot \left(\tfrac{5}{4}\right) + (1) \cdot \left(\tfrac{3}{4}\right) + (0) \cdot (1) \right]$$

$$= \left[-\tfrac{5}{4} \right] < 0$$

As the conditions are satisfied, we can conclude that the given system of equations is inconsistent.

ELLIPSOID METHOD

- In mathematical optimization, the ellipsoid method is an **iterative method** for **minimizing convex functions**.
- When specialized to solving feasible linear optimization problems with rational data, the ellipsoid method is an algorithm which **finds an optimal solution in a number of steps that is polynomial in the input size.**
- The ellipsoid method generates a sequence of ellipsoids whose volume uniformly decreases at every step, thus enclosing a minimizer of a convex function.

KARMARKAR'S ALGORITHM

- Consider a linear programming problem in matrix form:
  - maximize $c^T x$
  - subject to $Ax \le b$.
- Karmarkar's algorithm determines the next feasible direction toward optimality and scales back by a factor $\mathbf{0 < \gamma \le 1}$. It is described in a number of sources. Karmarkar also has extended the method to solve problems with integer constraints and non-convex problems.

$$
\begin{aligned}
\text{Minimize} \quad & Z = \mathbf{C}^{\mathrm{T}}\mathbf{X} \\
\text{subject to :} \quad & \mathbf{A}\mathbf{X} = \mathbf{0} \\
& \mathbf{1}\mathbf{X} = 1 \\
\text{with :} \quad & \mathbf{X} \geq \mathbf{0}
\end{aligned}
$$

where $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $\mathbf{1} = [1 \quad 1 \quad \cdots \quad 1]_{(1\times n)}$, $\mathbf{A} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$ and $n \geq 2$. It is also assumed that $\mathbf{X}_0 = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$ is a feasible solution and $Z_{\min} = 0$. The two other variables are defined as $r = \dfrac{1}{\sqrt{n(n-1)}}$, $\alpha = \dfrac{(n-1)}{3n}$.