23MT2014

THEORY OF COMPUTATION

Topic:

TURING MACHINES

Session - 17

AIM OF THE SESSION

The aim of this session is to provide an understanding of Turing Machines, their theoretical basis, and their significance in the field of computer science.

INSTRUCTIONAL OBJECTIVES

This Session is designed to:

- To introduce the concept of Turing Machines and their key components.
- To explain the working principles of Turing Machines and their computational capabilities.
- To discuss the importance of Turing Machines in the context of computability theory and the foundation of modern computing.

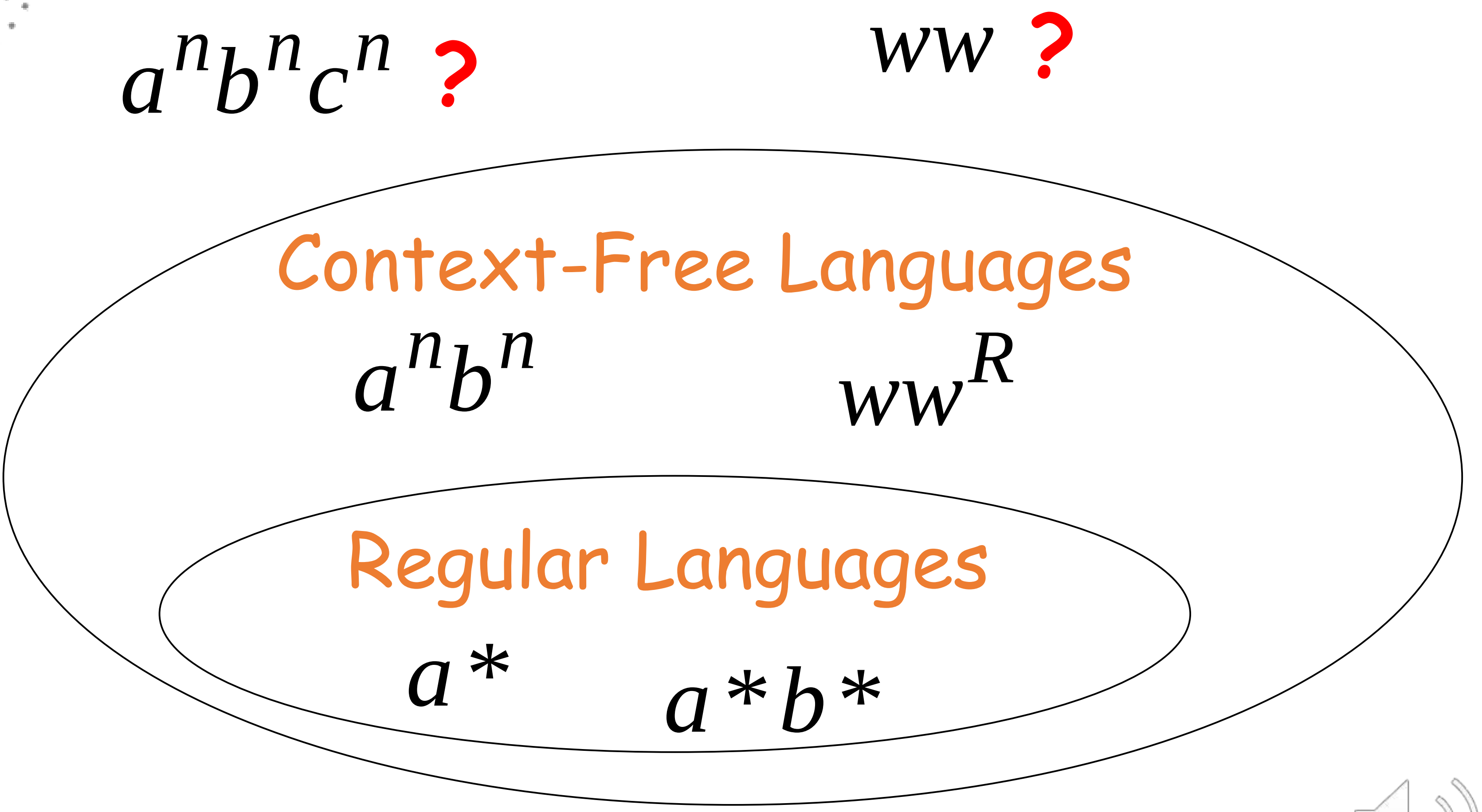
LEARNING OUTCOMES

At the end of this session, you should be able to:

- Define what a Turing Machine is and identify its essential components.
- Describe the operation of a Turing Machine and explain how it can simulate various computational tasks.
- Discuss the significance of Turing Machines in the field of computability theory and the development of computer science.
- Analyze problems and determine whether they are solvable or undecidable based on the concepts learned about Turing Machines.

Turing Machines

The Language Hierarchy



$a^n b^n c^n$?

ww ?

Context-Free Languages

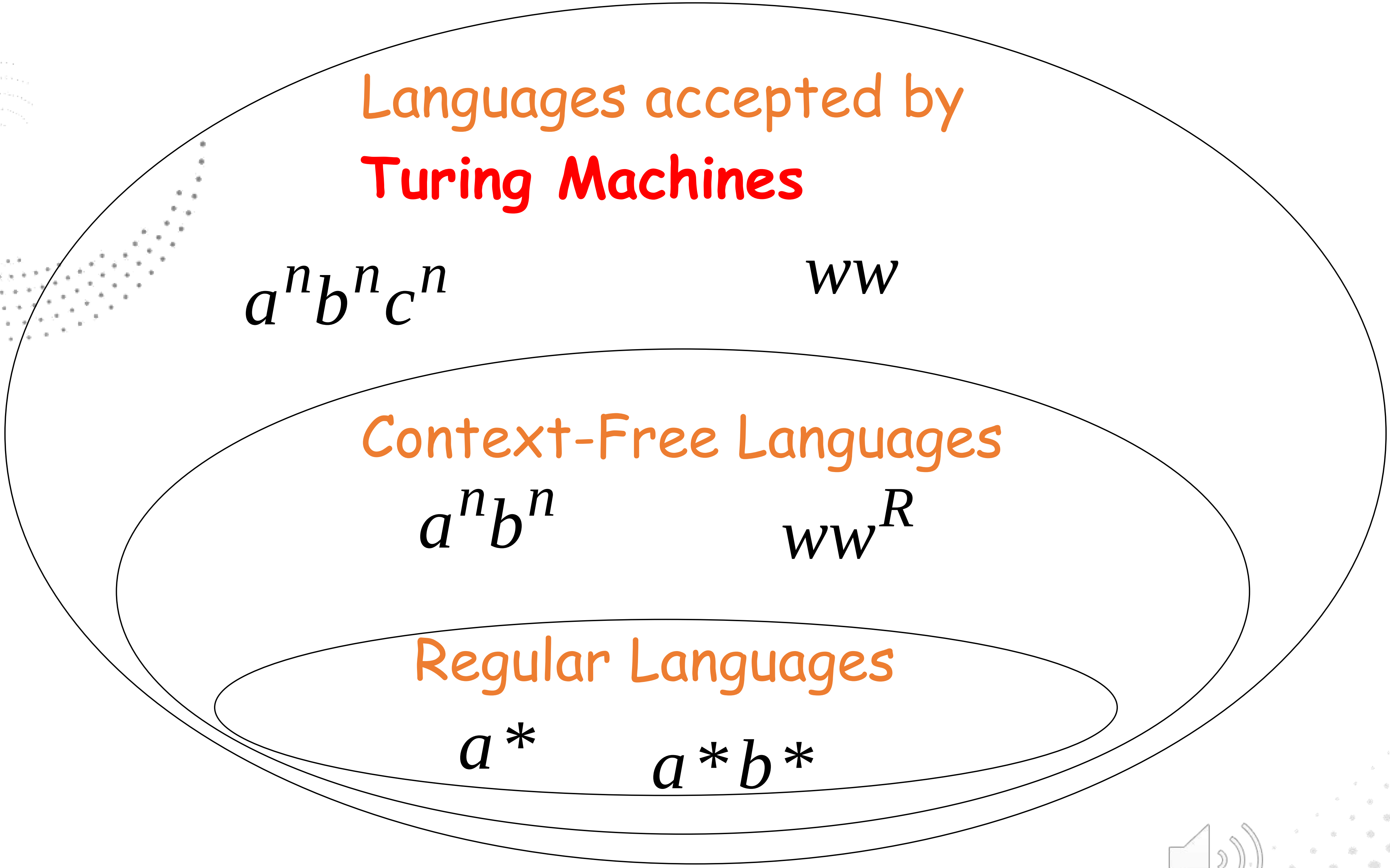
$a^n b^n$

ww^R

Regular Languages

a^*

$a^* b^*$



Languages accepted by **Turing Machines**

$a^n b^n c^n$ ww

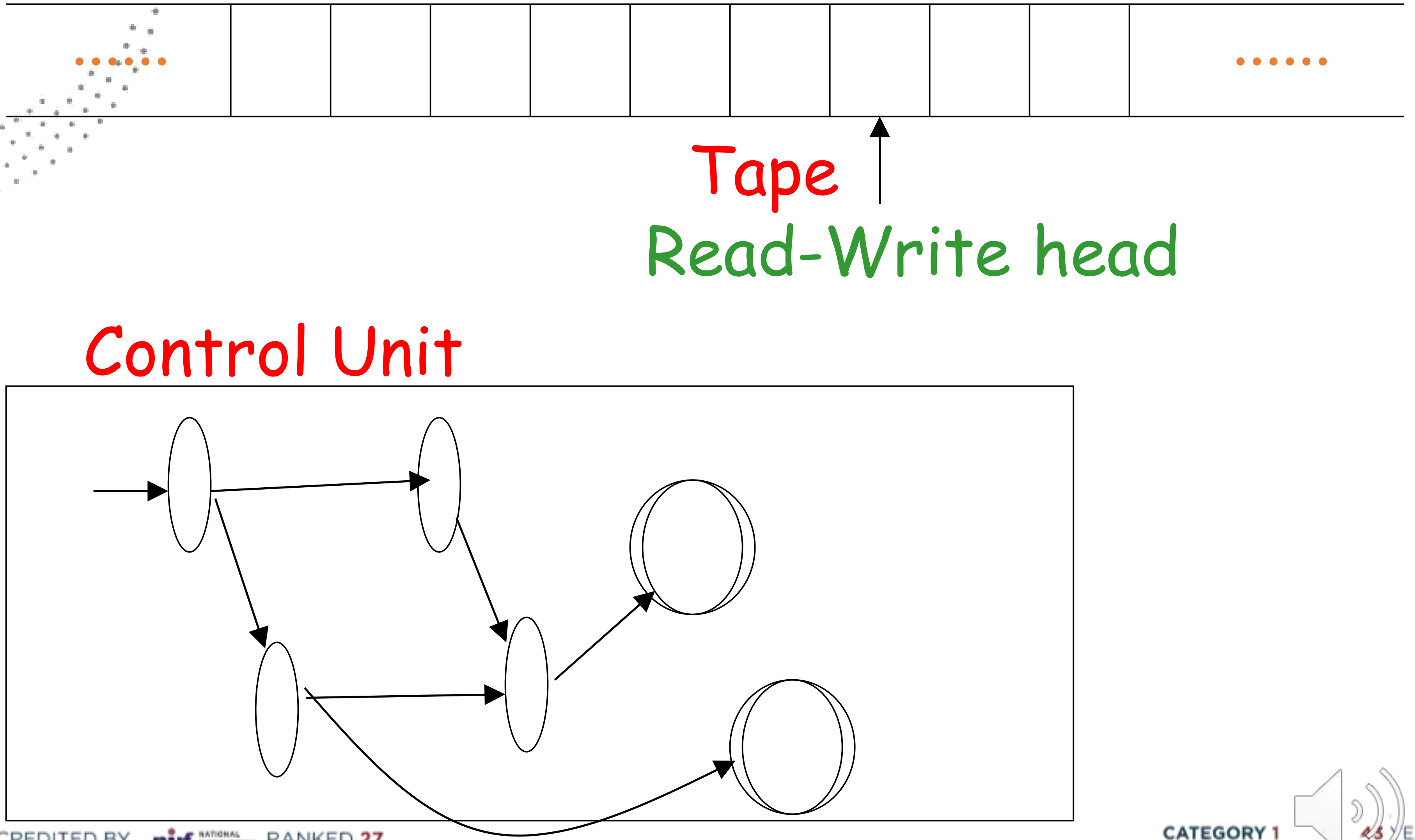
Context-Free Languages

$a^n b^n$ ww^R

Regular Languages

a^* a^*b^*

A Turing Machine



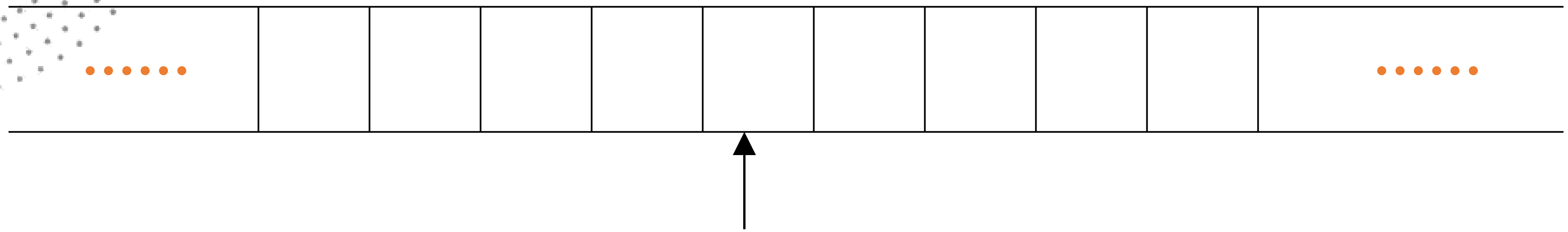
Tape

Read-Write head

Control Unit

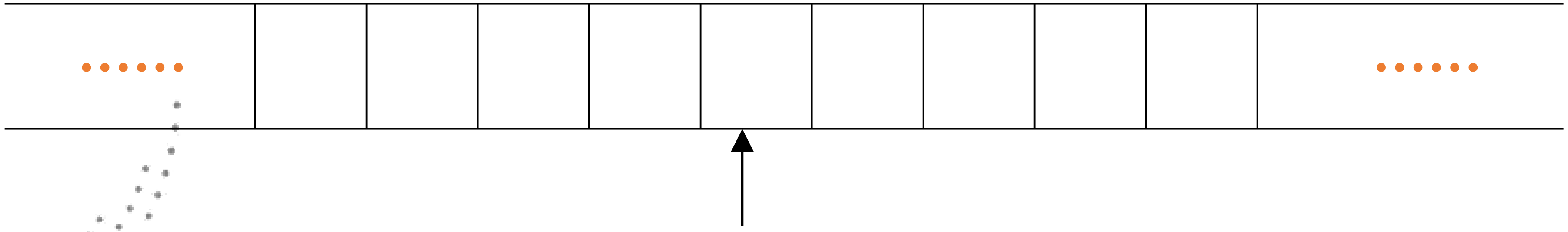
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right



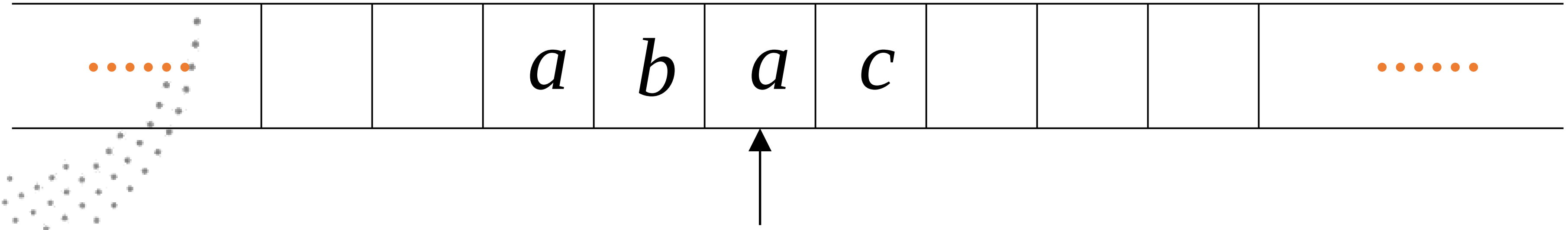
Read-Write head

The head at each time step:

1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

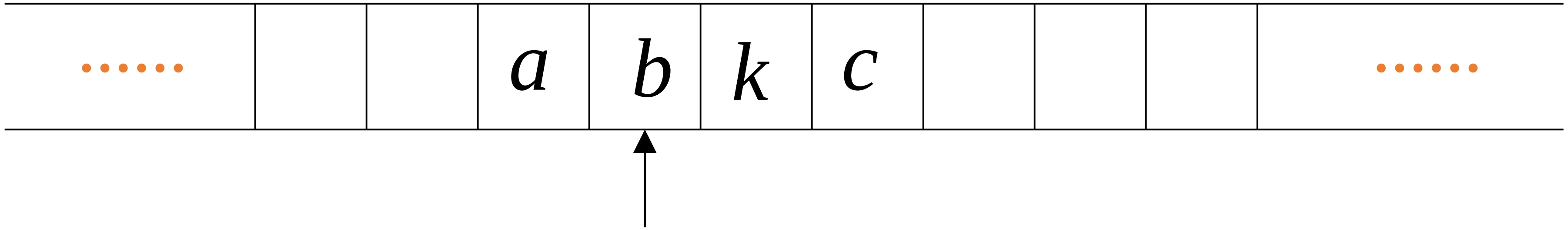
Example:

Time 0



......			a	b	a	c				......

Time 1



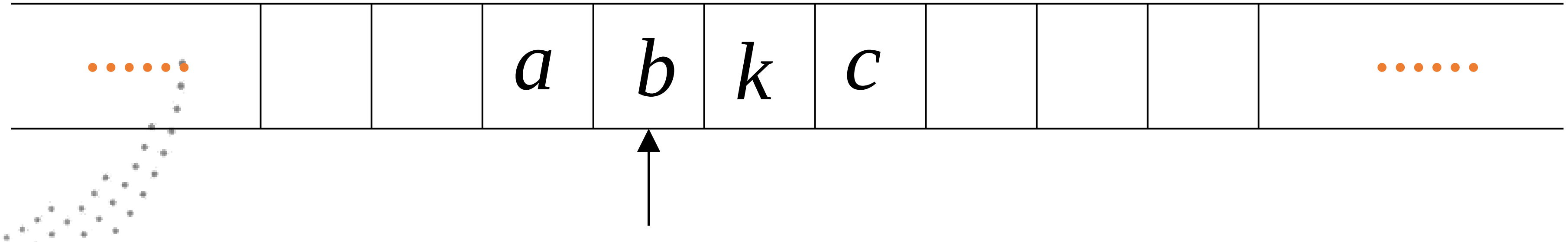
......			a	b	k	c				......

1. Reads a

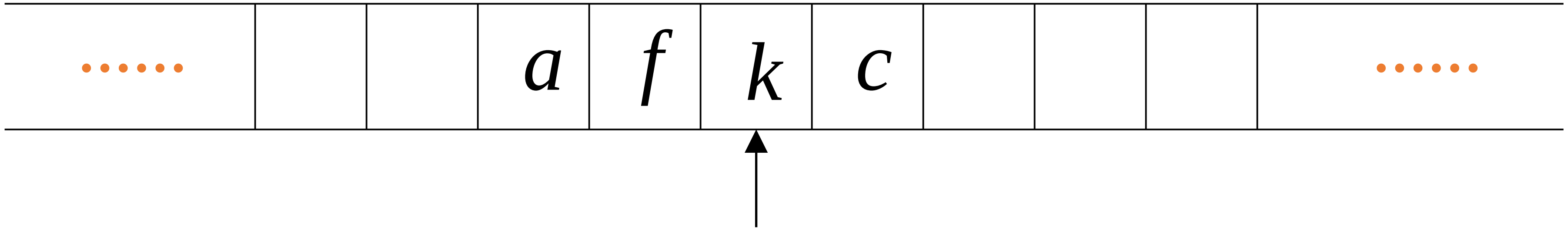
2. Writes k

3. Moves Left

Time 1

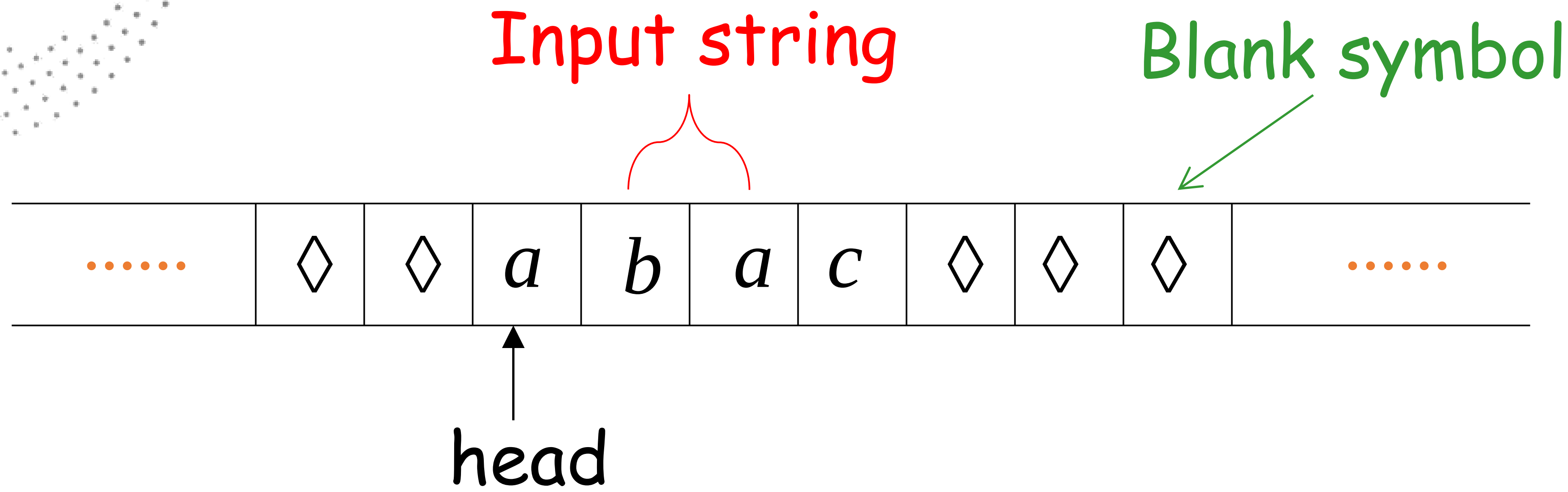


Time 2



1. Reads b
2. Writes f
3. Moves Right

The Input String

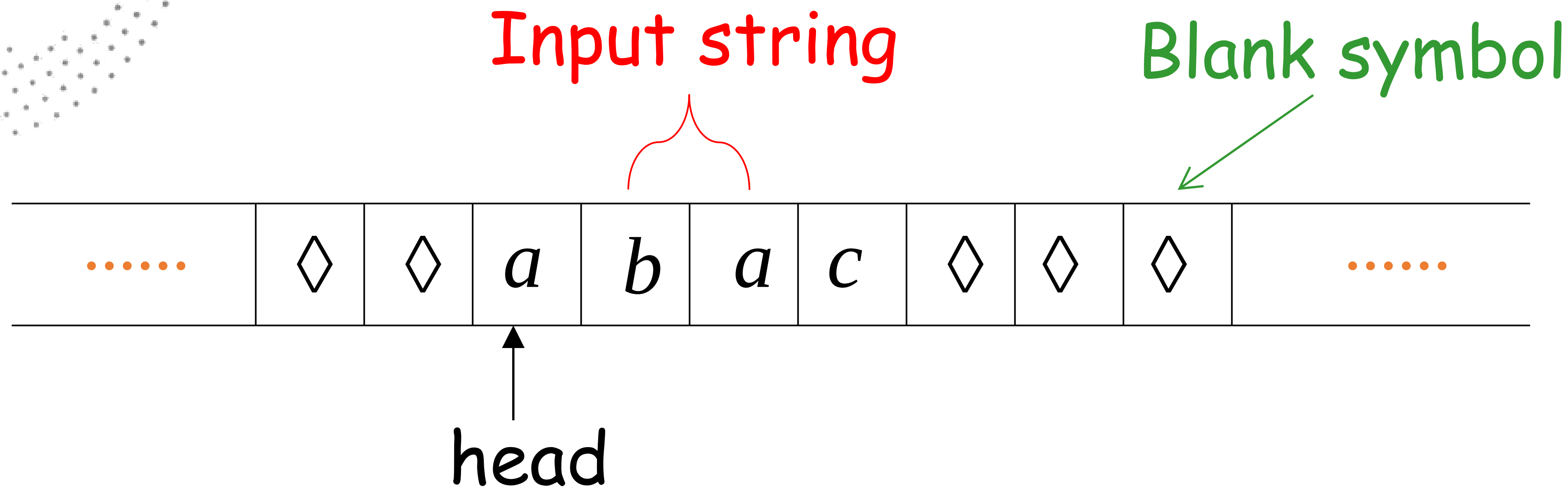


Input string

Blank symbol

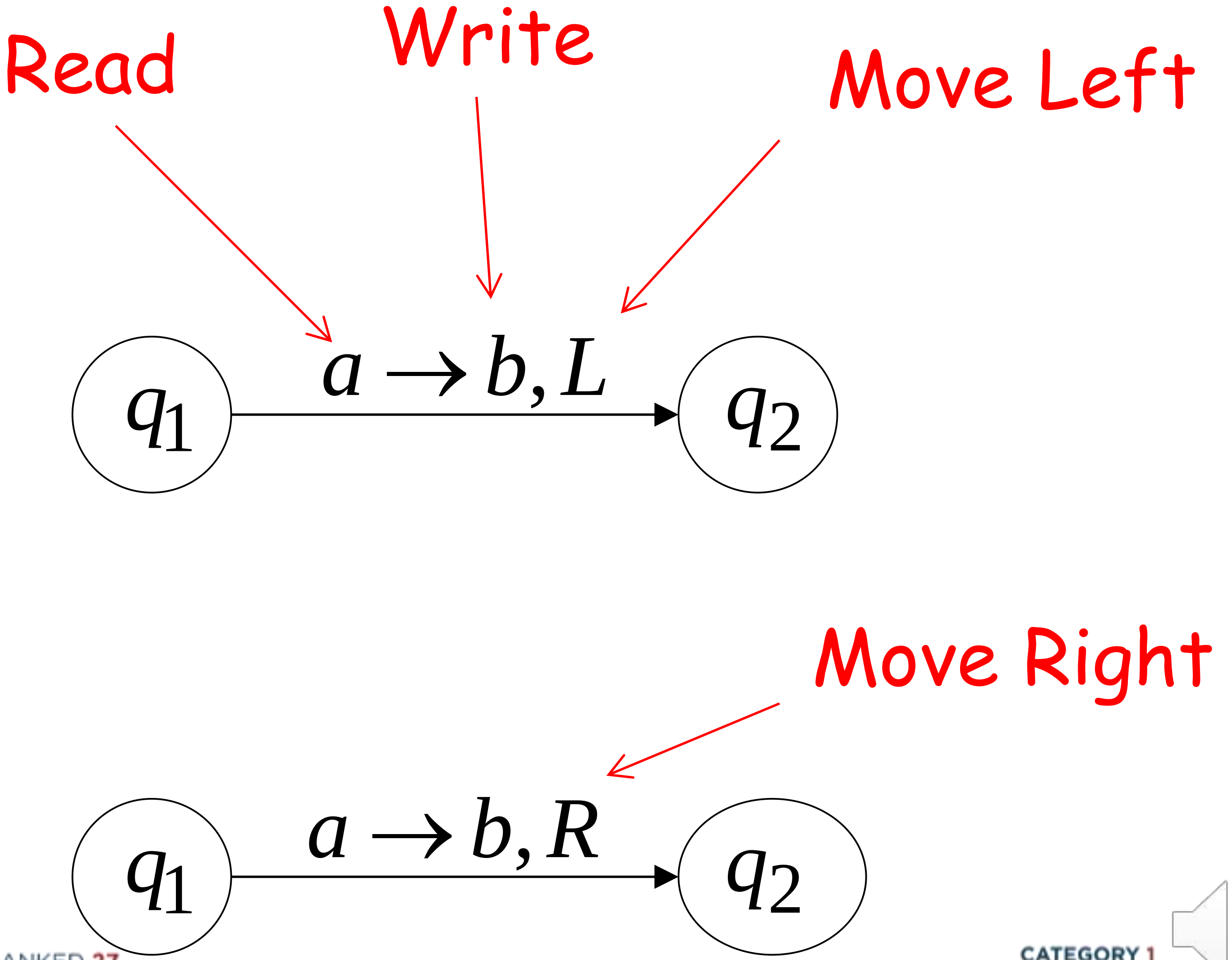
head

Head starts at the leftmost position of the input string



Remark: the input string is never empty

States & Transitions


Read

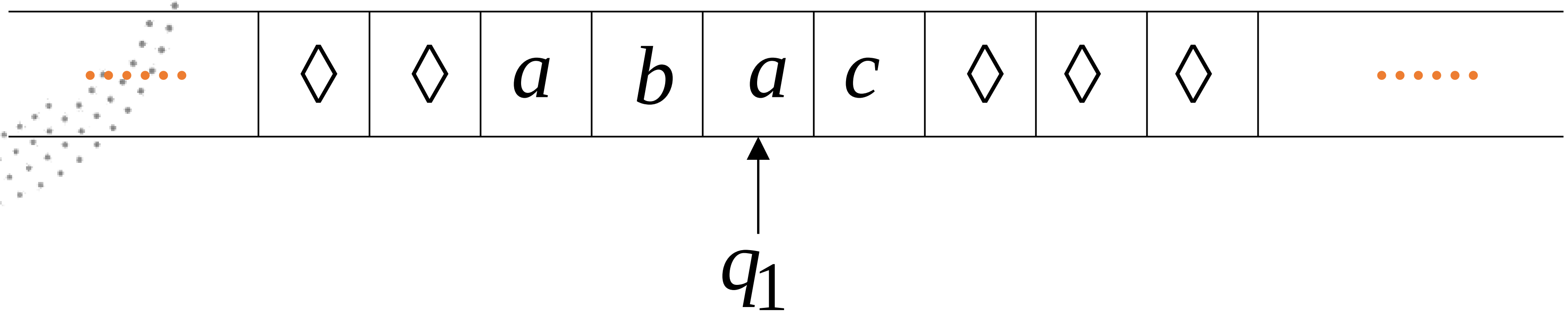
Write

Move Left

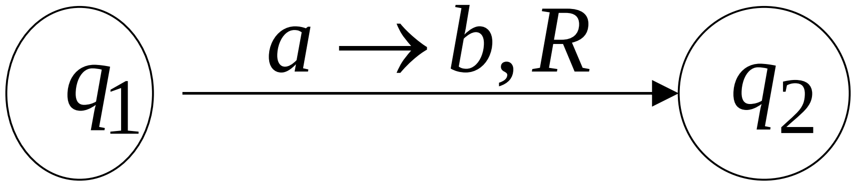
Move Right

Example:

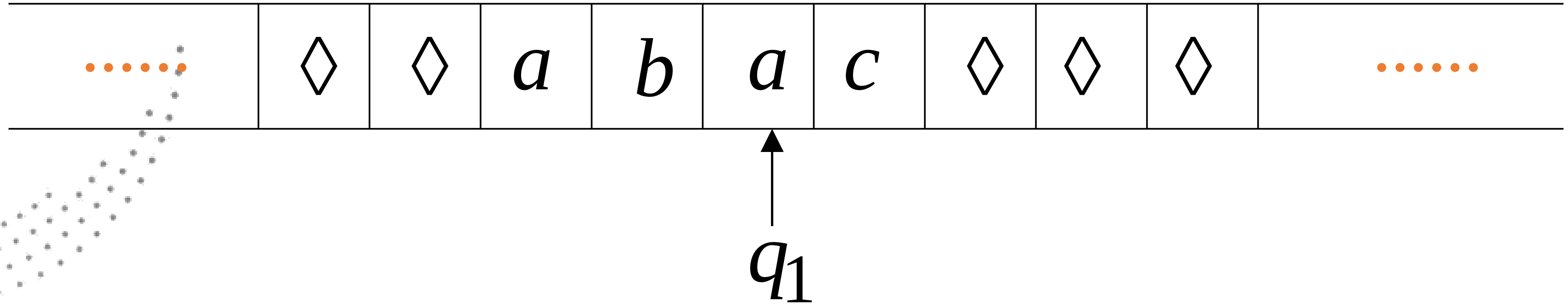
Time 1



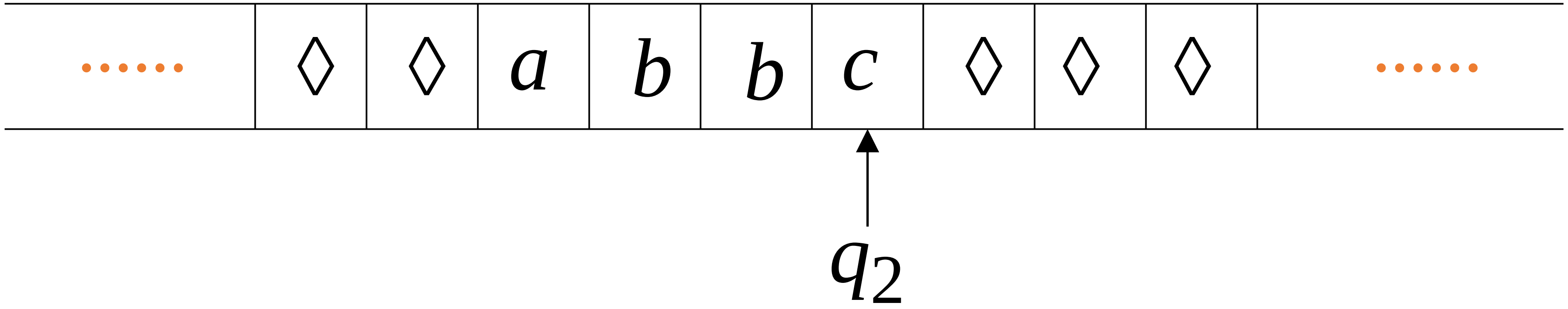
current state

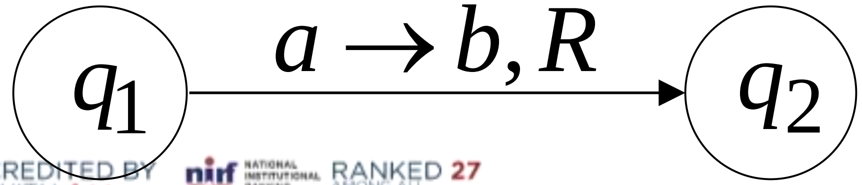


Time 1



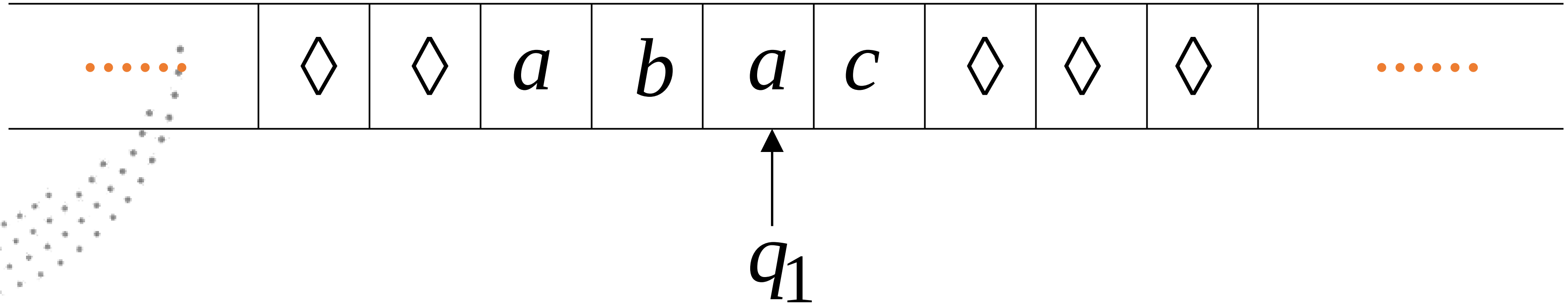
Time 2



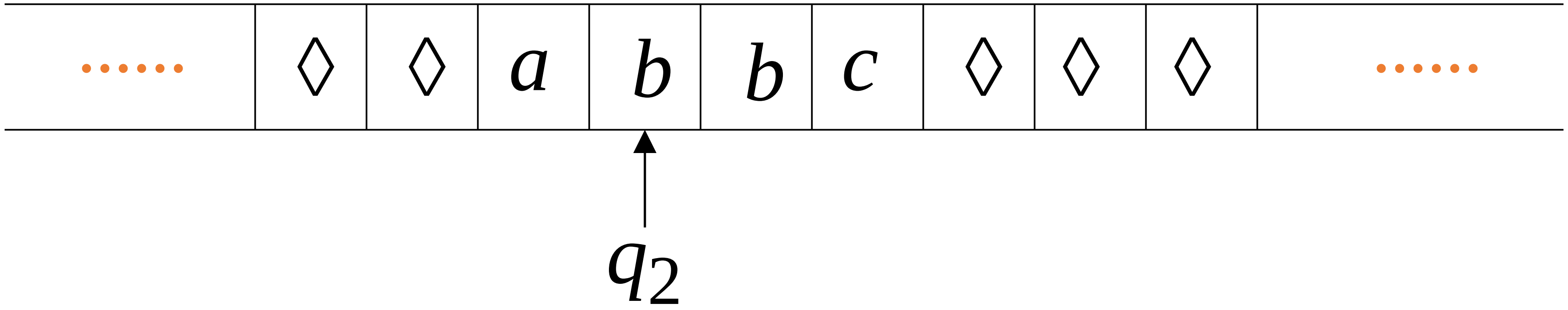


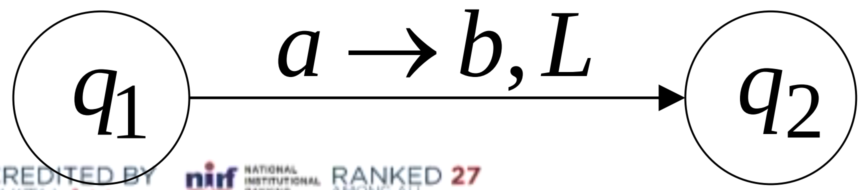
Example:

Time 1



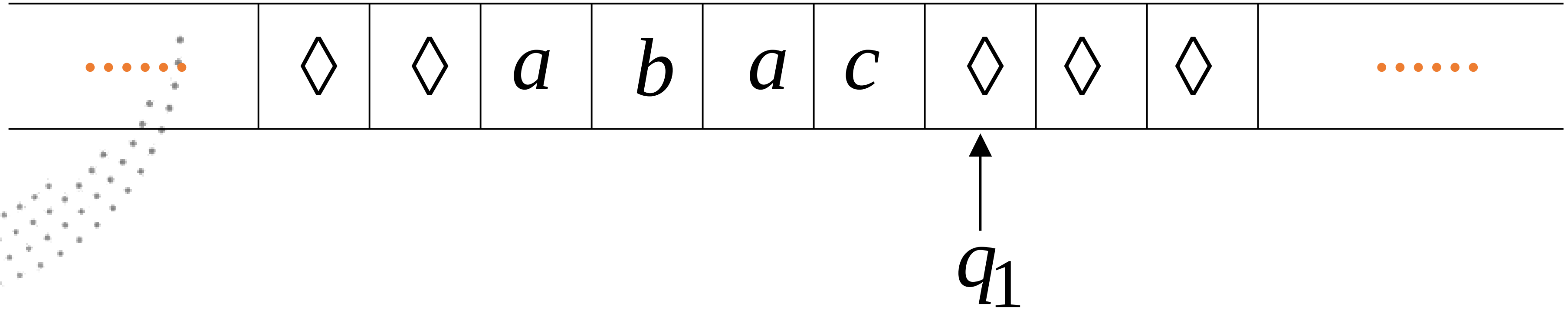
Time 2



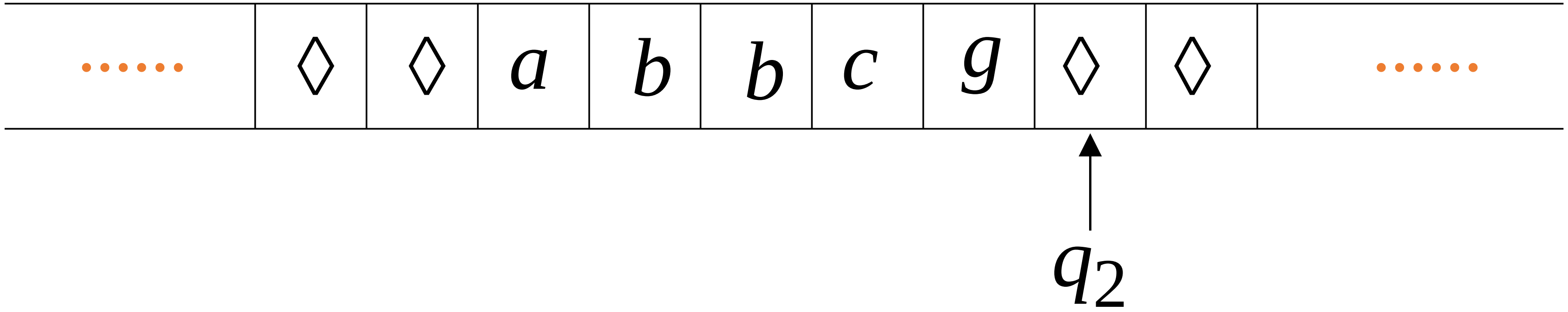


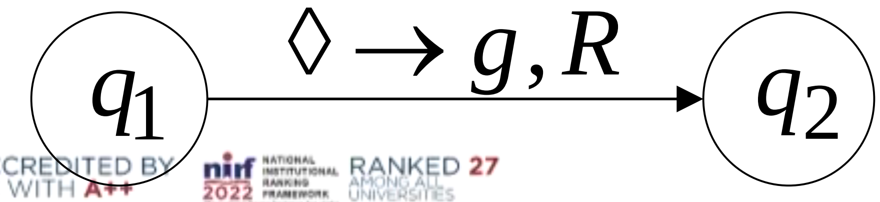
Example:

Time 1



Time 2

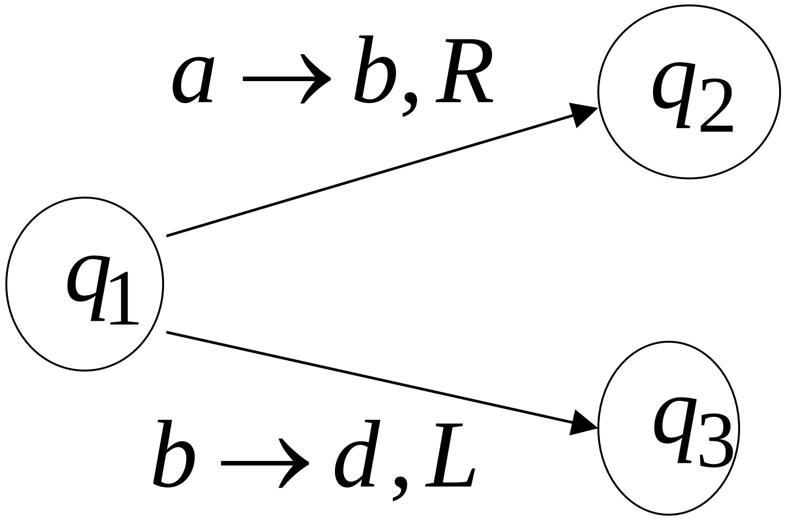




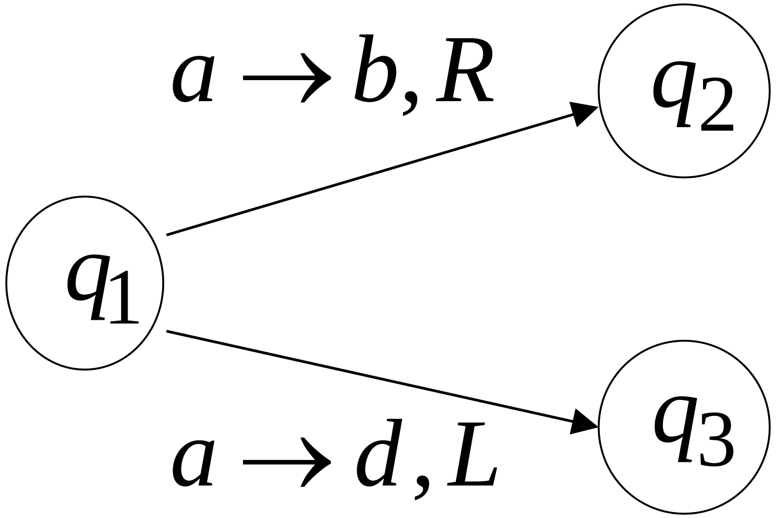
Determinism

Turing Machines are deterministic

Allowed



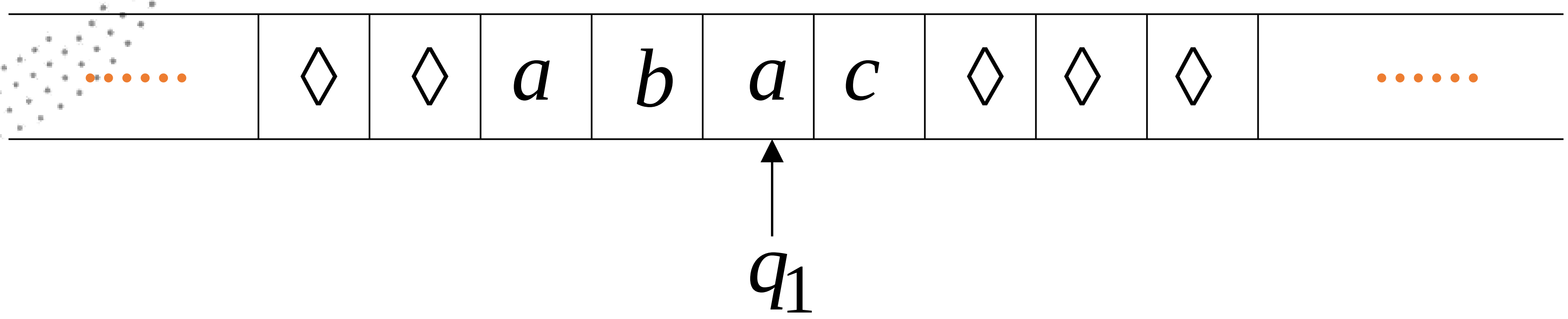
Not Allowed

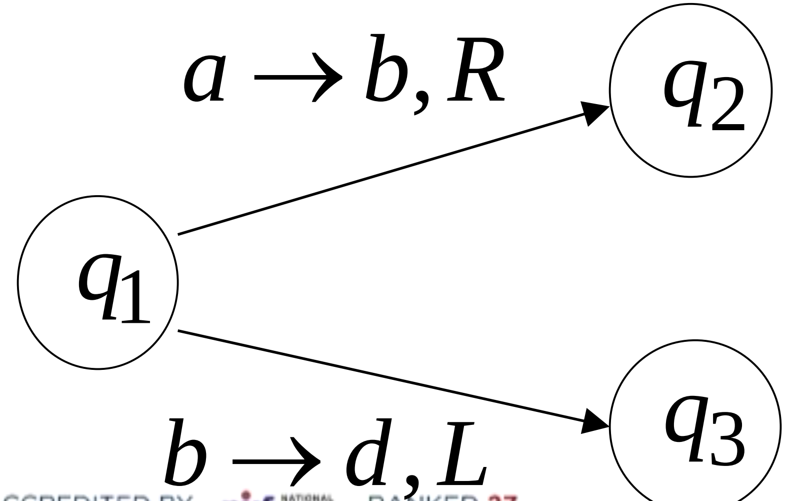


No lambda transitions allowed

Example:

Partial Transition Function





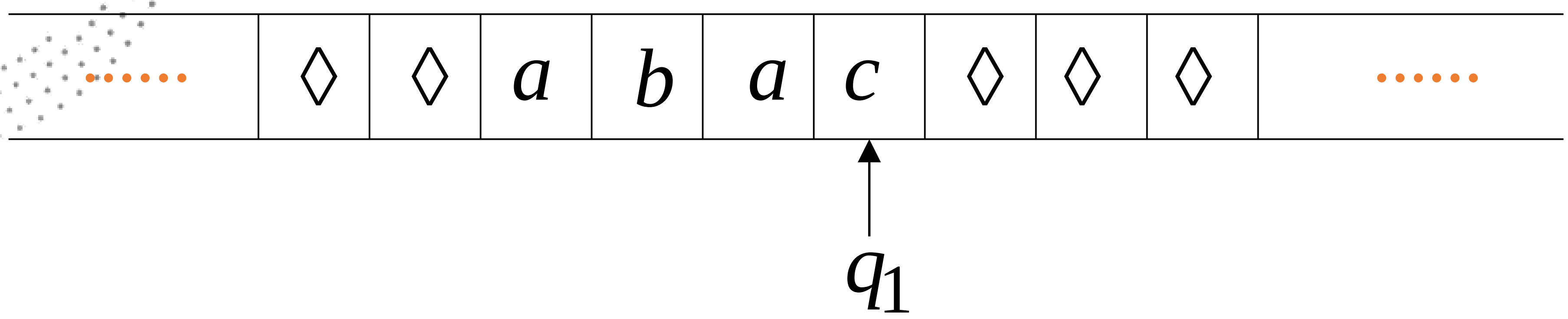
Allowed:

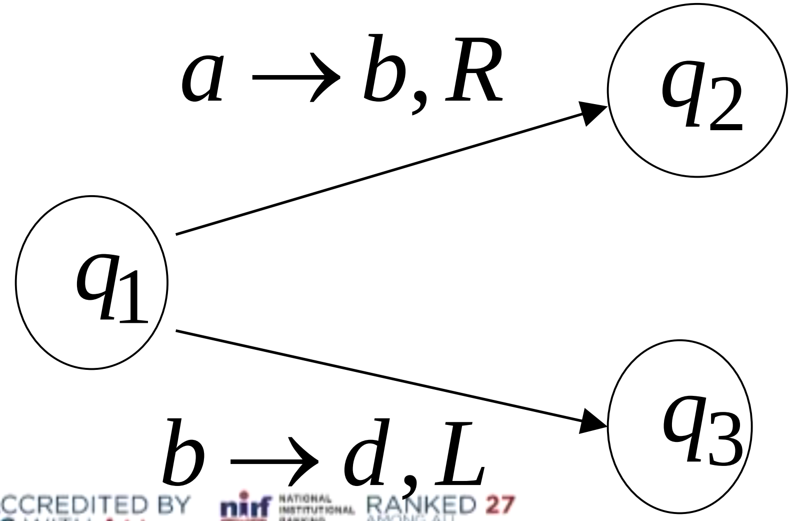
No transition for input symbol c

Halting

The machine ***halts*** if there are no possible transitions to follow

Example:

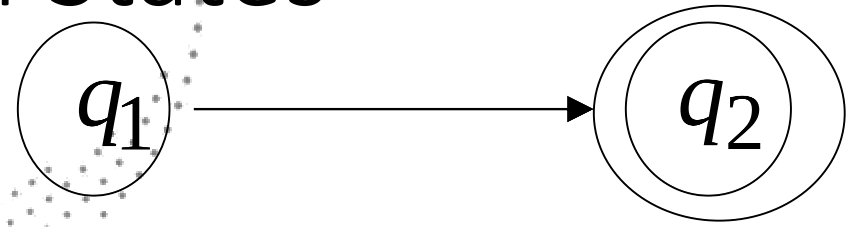




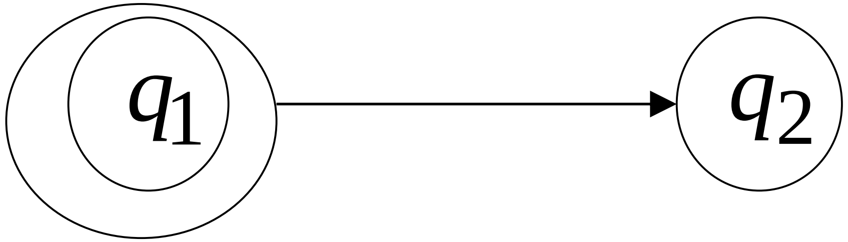
No possible transition

HALT!!!

Final States



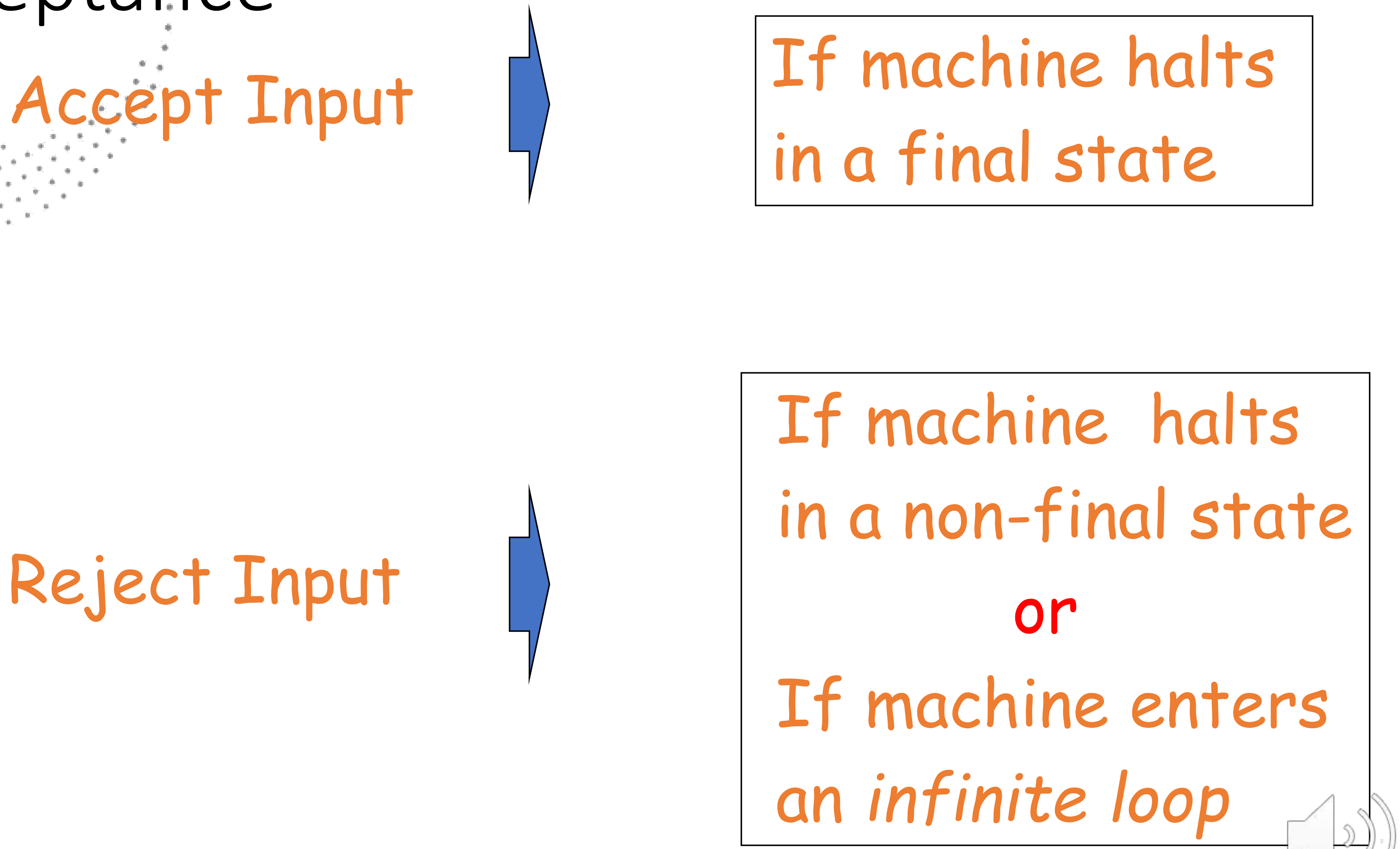
Allowed



Not Allowed

- Final states have no outgoing transitions
- In a final state the machine halts

Acceptance



Accept Input → If machine halts in a final state

Reject Input → If machine halts in a non-final state

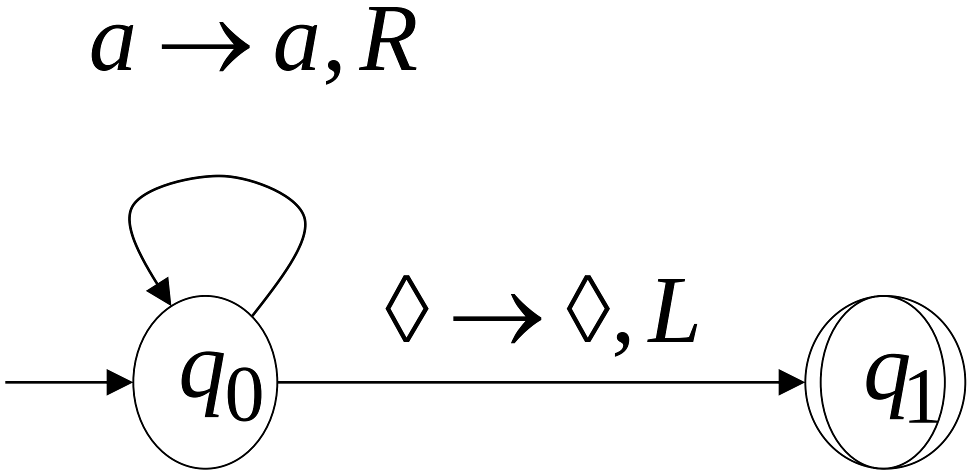
or

If machine enters an *infinite loop*

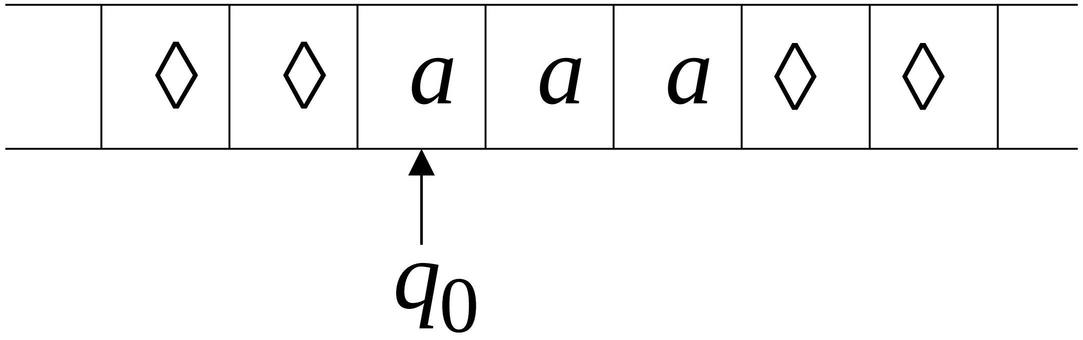
Turing Machine Example

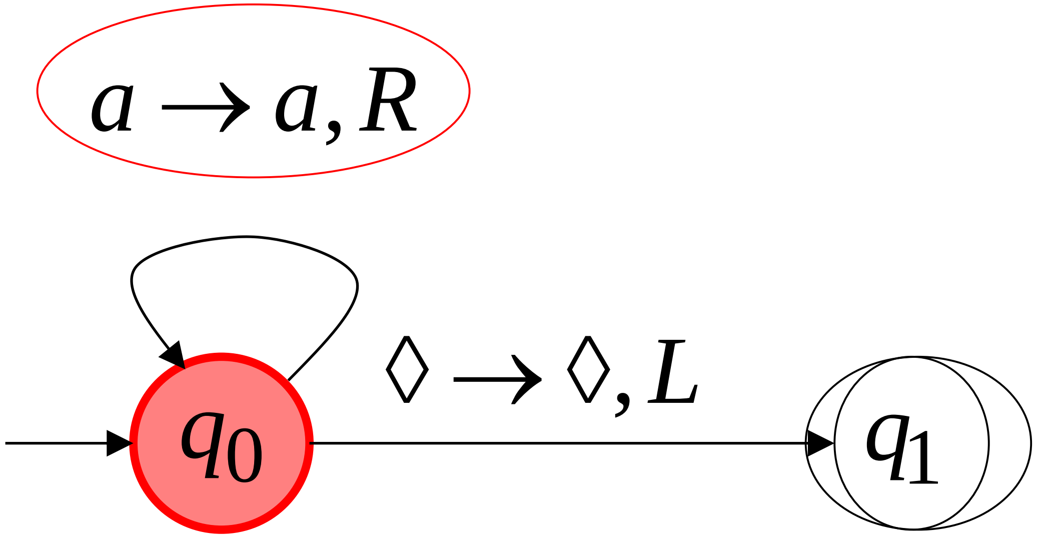
A Turing machine that accepts the language:

$$aa^*$$

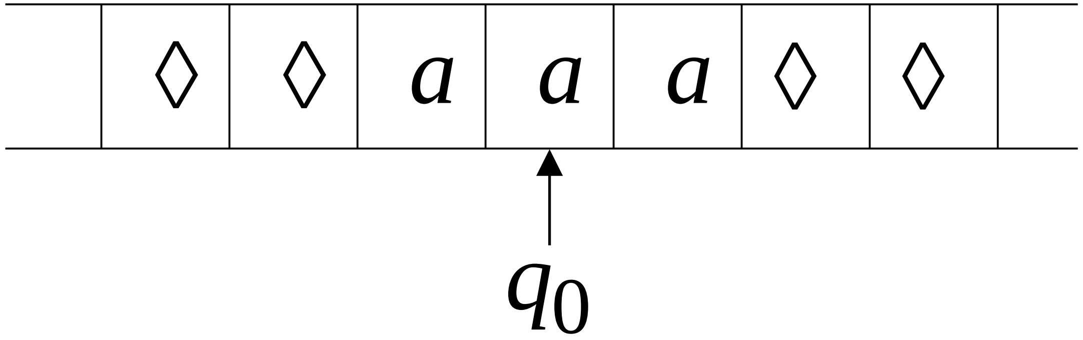


Time 0

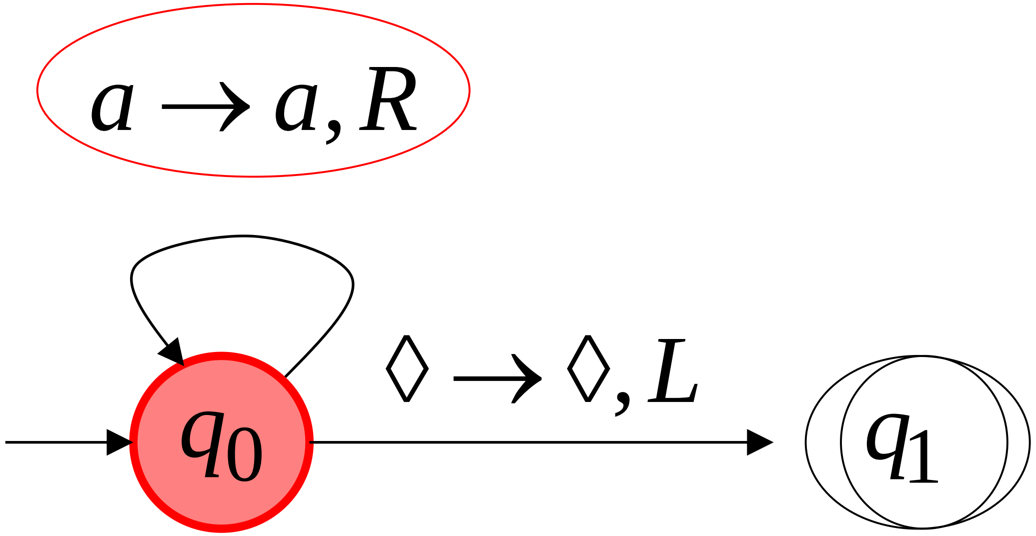




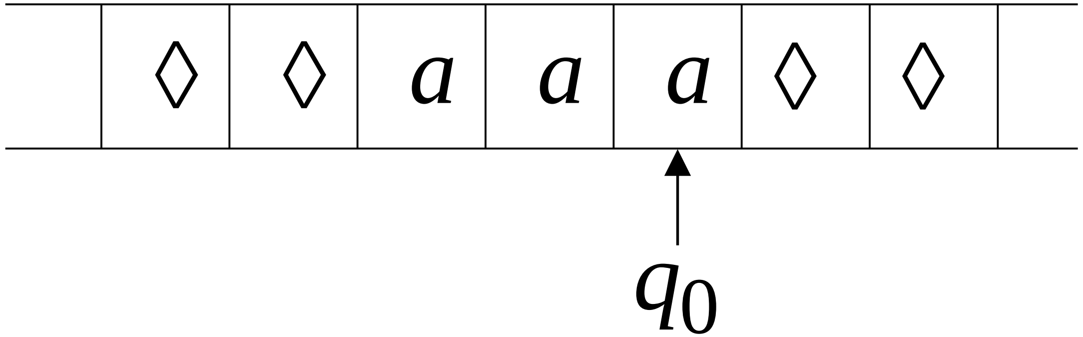
Time 1

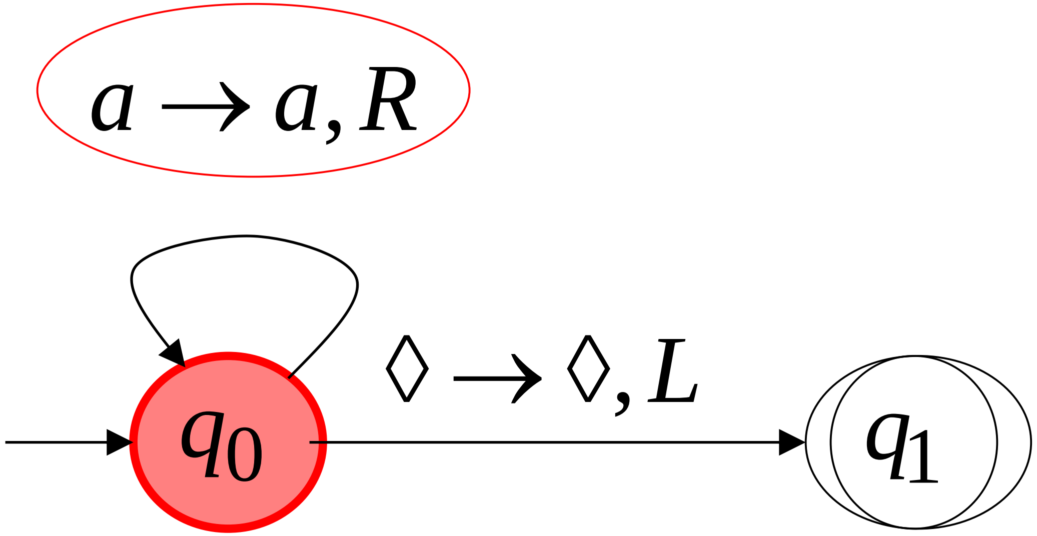


	◊	◊	a	a	a	◊	◊

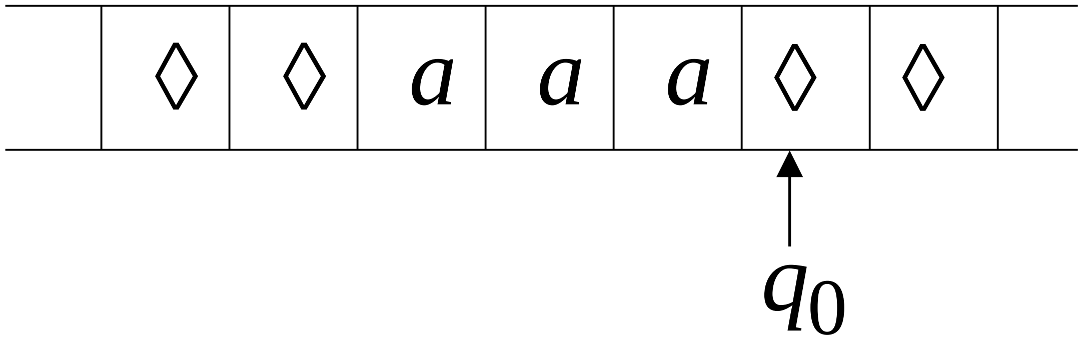


Time 2

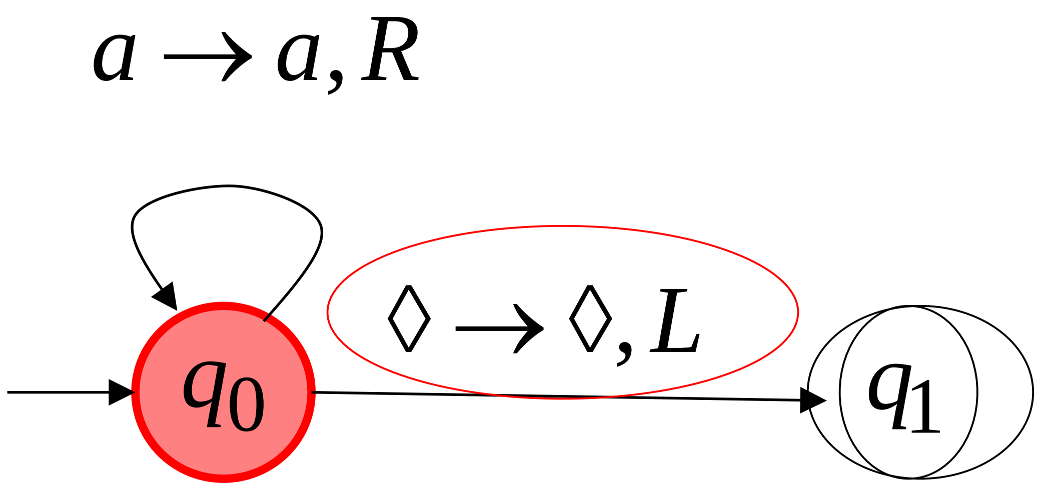




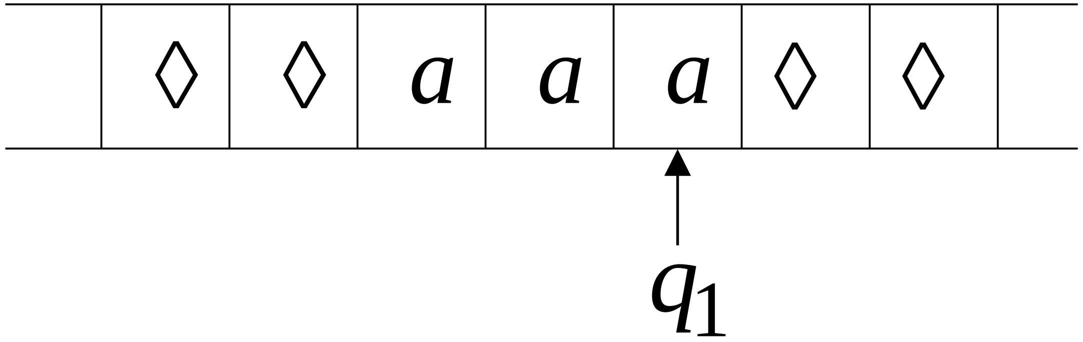
Time 3



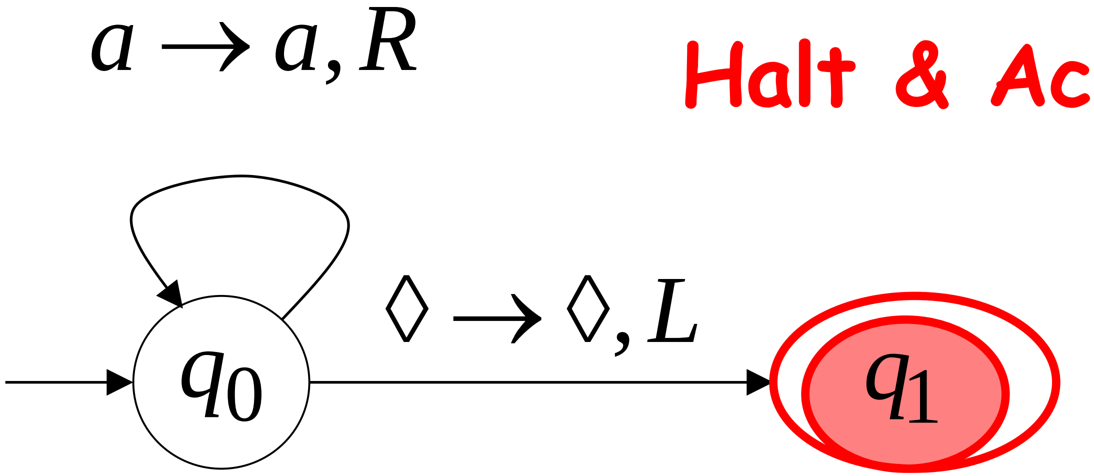
	◊	◊	a	a	a	◊	◊	



Time 4

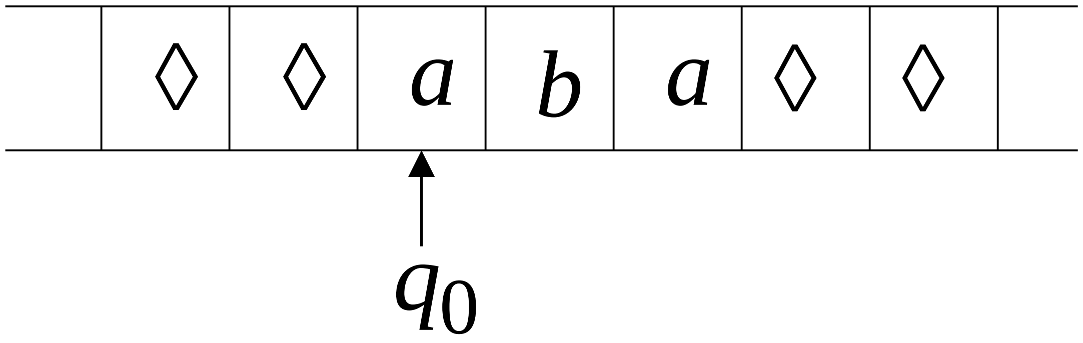


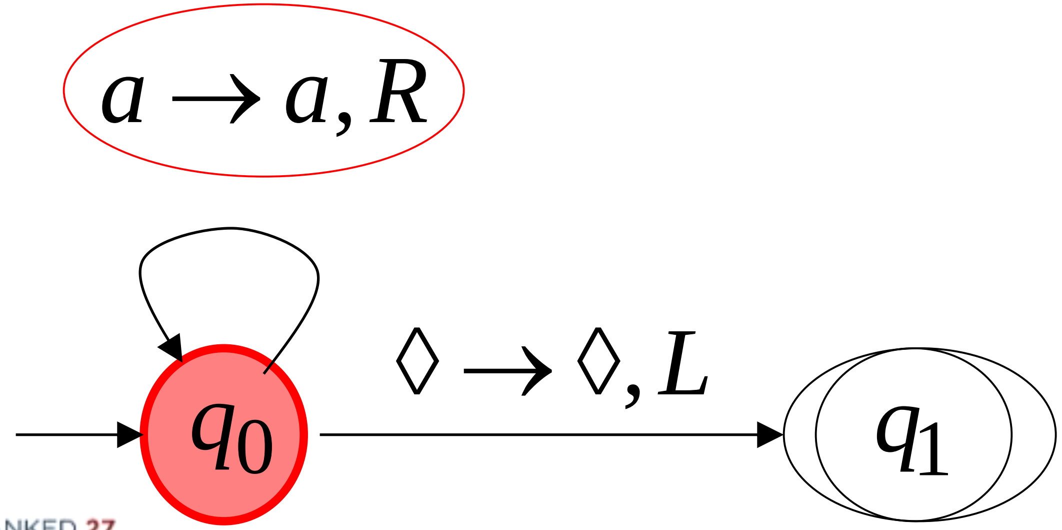
Halt & Accept



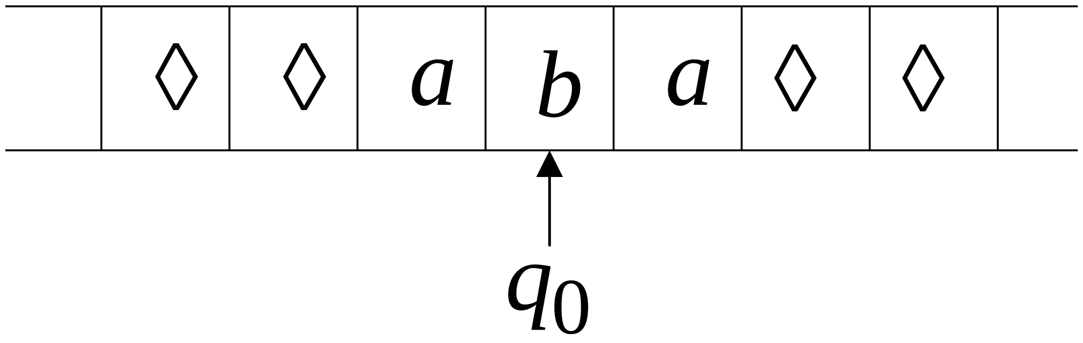
Rejection Example

Time 0



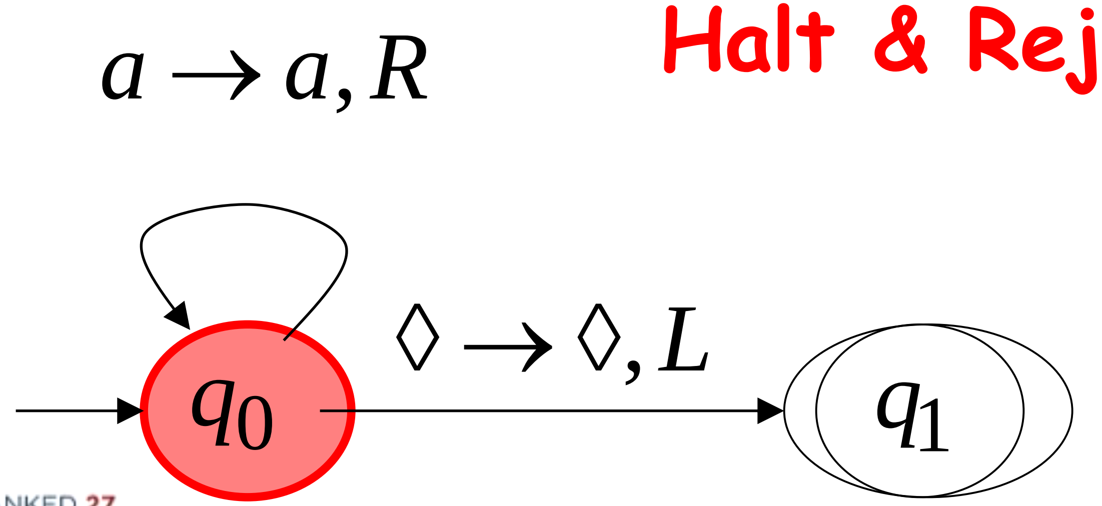


Time 1



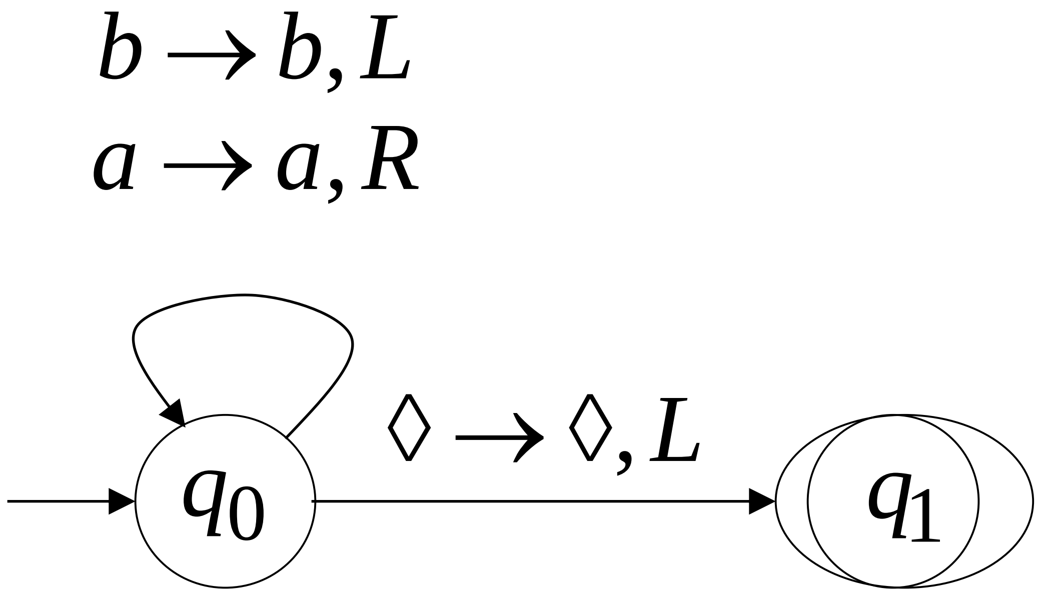
No possible Transition

Halt & Reject

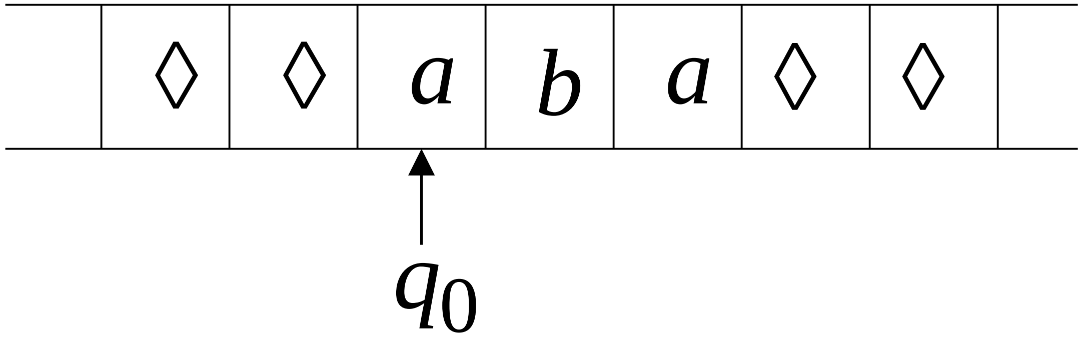


Infinite Loop Example

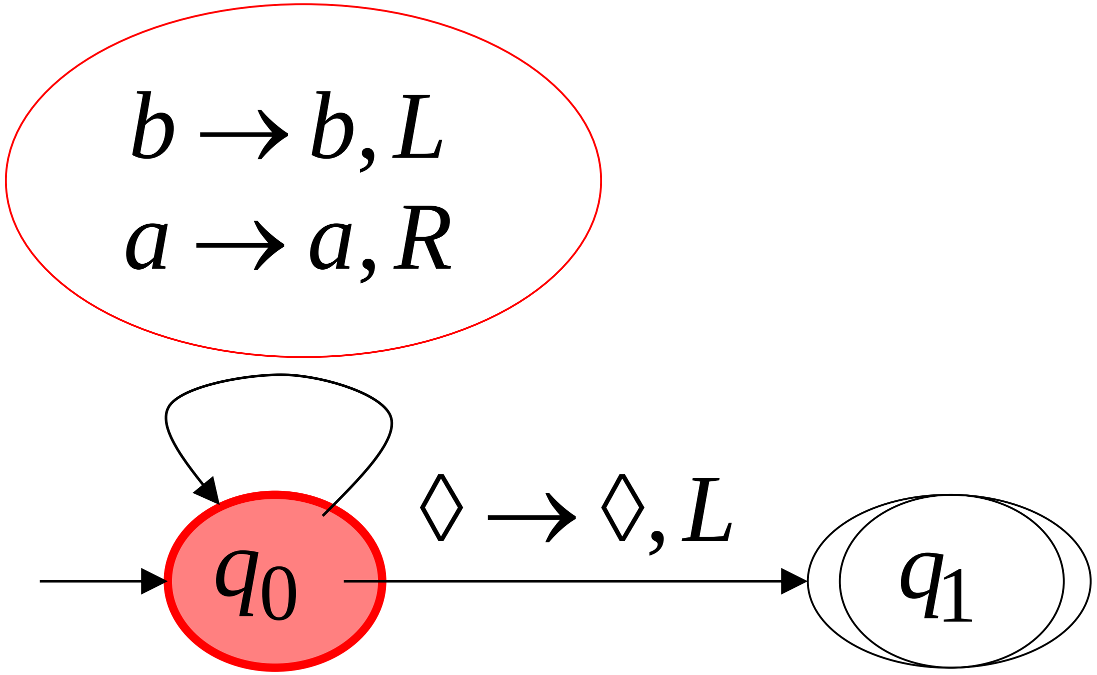
A Turing machine for language $aa^*+b(a+b)^*$



Time 0

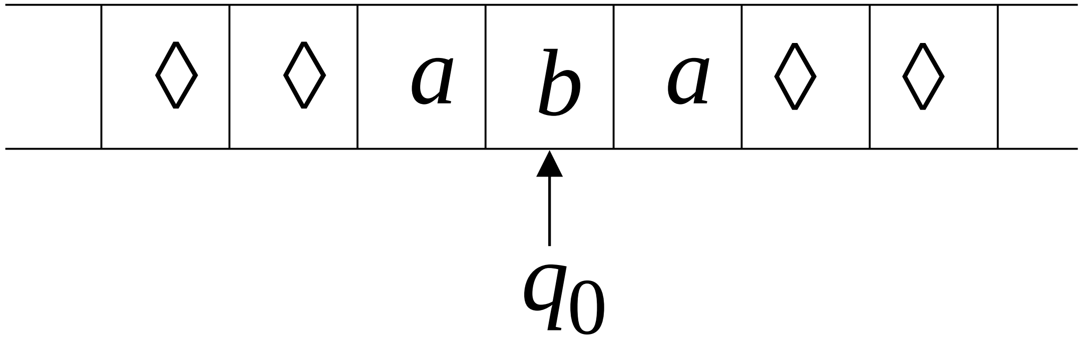


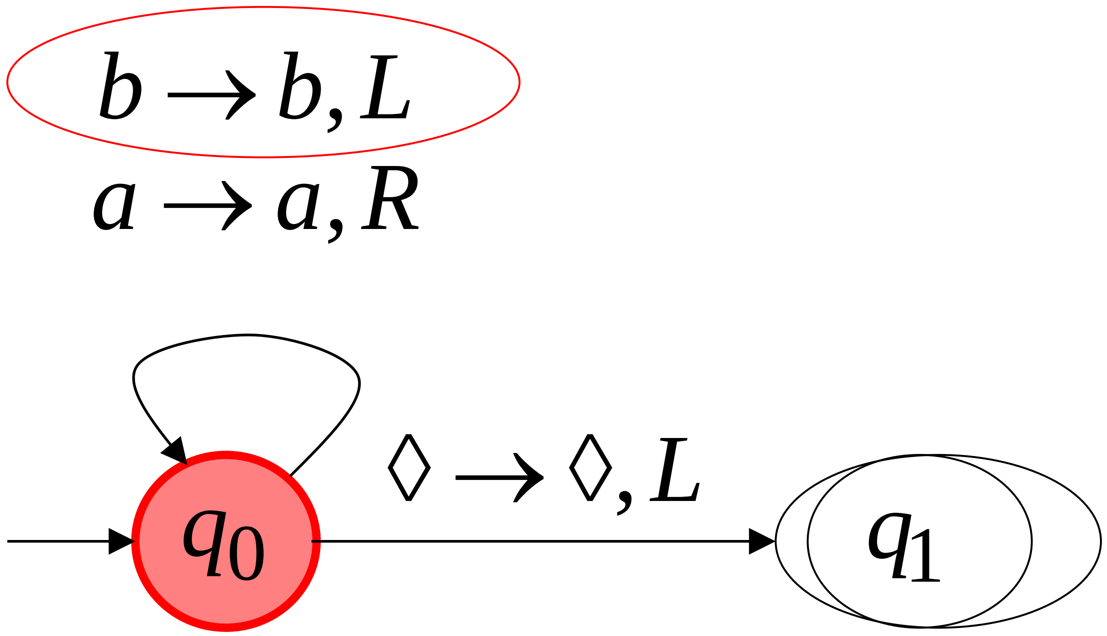
◊
◊
a
b
a
◊
◊
q_0



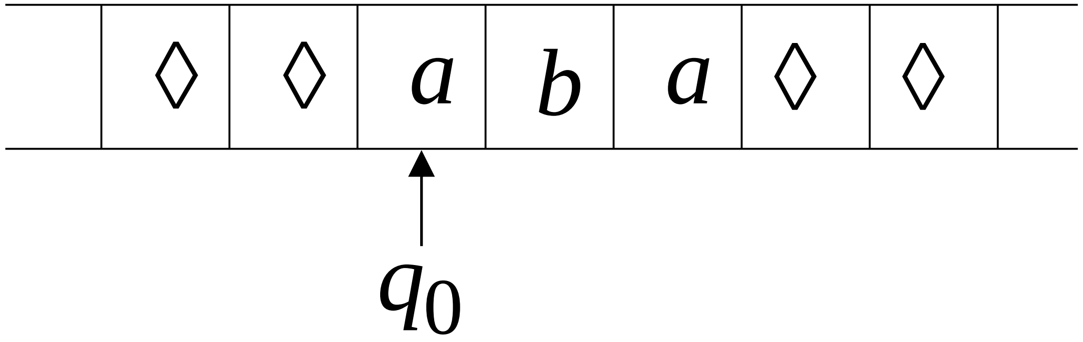
$b \rightarrow b, L$
$a \rightarrow a, R$
q_0
$\lozenge \rightarrow \lozenge, L$
q_1

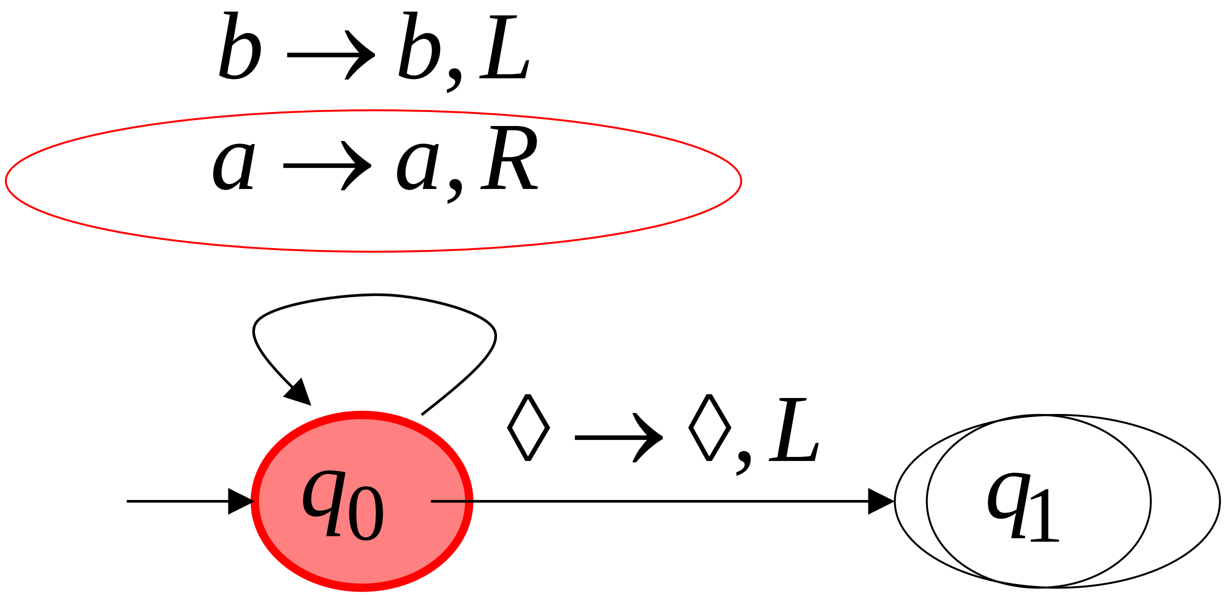
Time 1


◊
◊
a
b
a
◊
◊
q_0

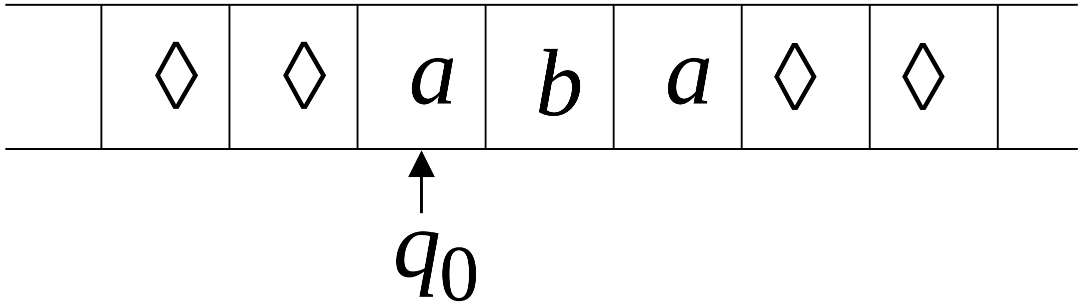

$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 2

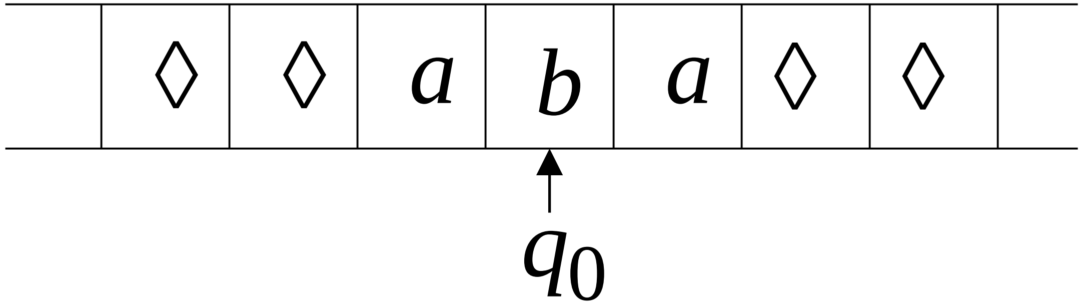




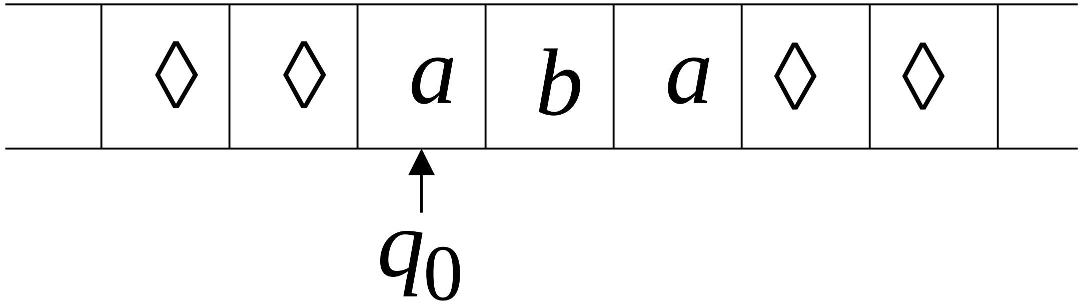
Time 2



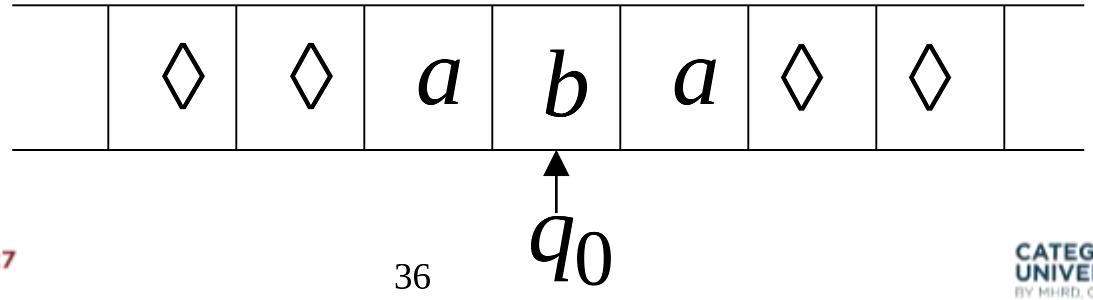
Time 3



Time 4



Time 5



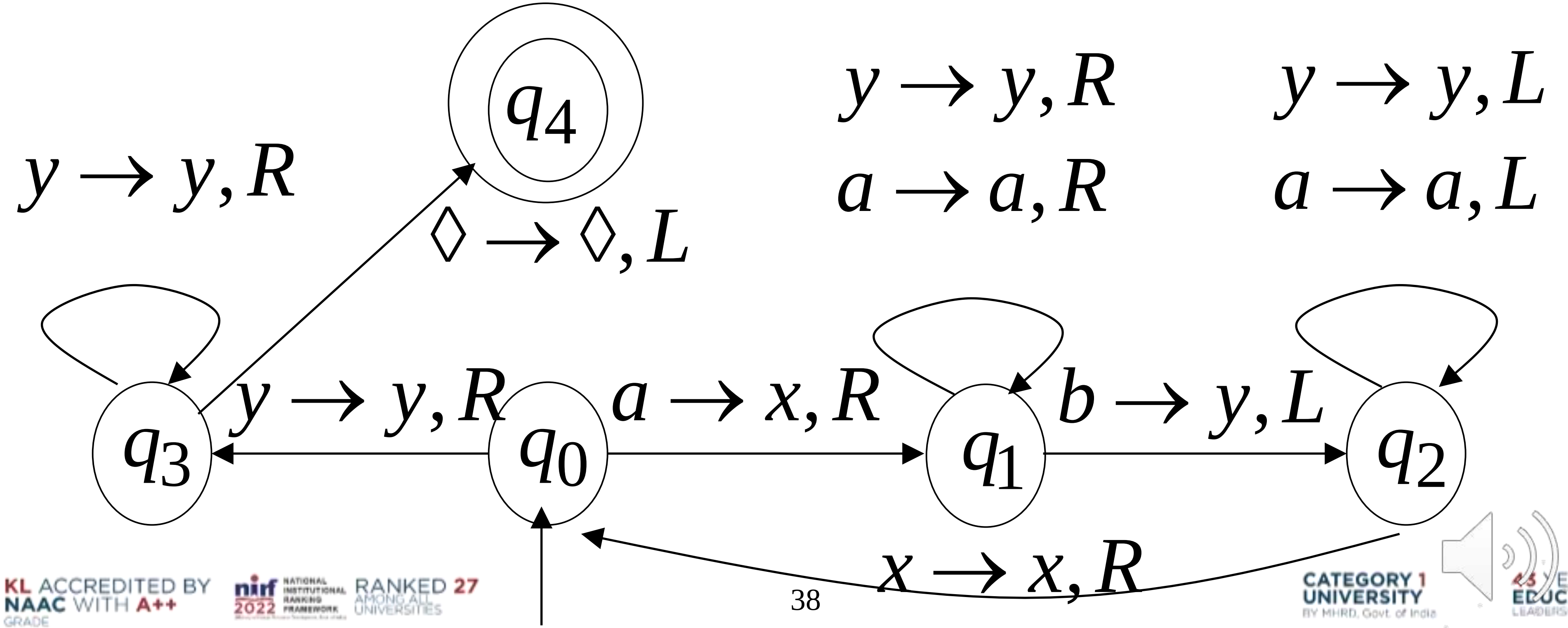
Infinite loop

Because of the infinite loop:

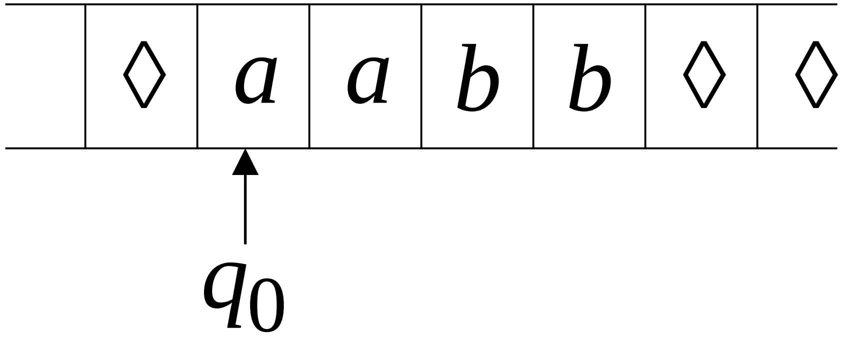
- The final state cannot be reached
- The machine never halts
- The input is not accepted

Another Turing Machine Example

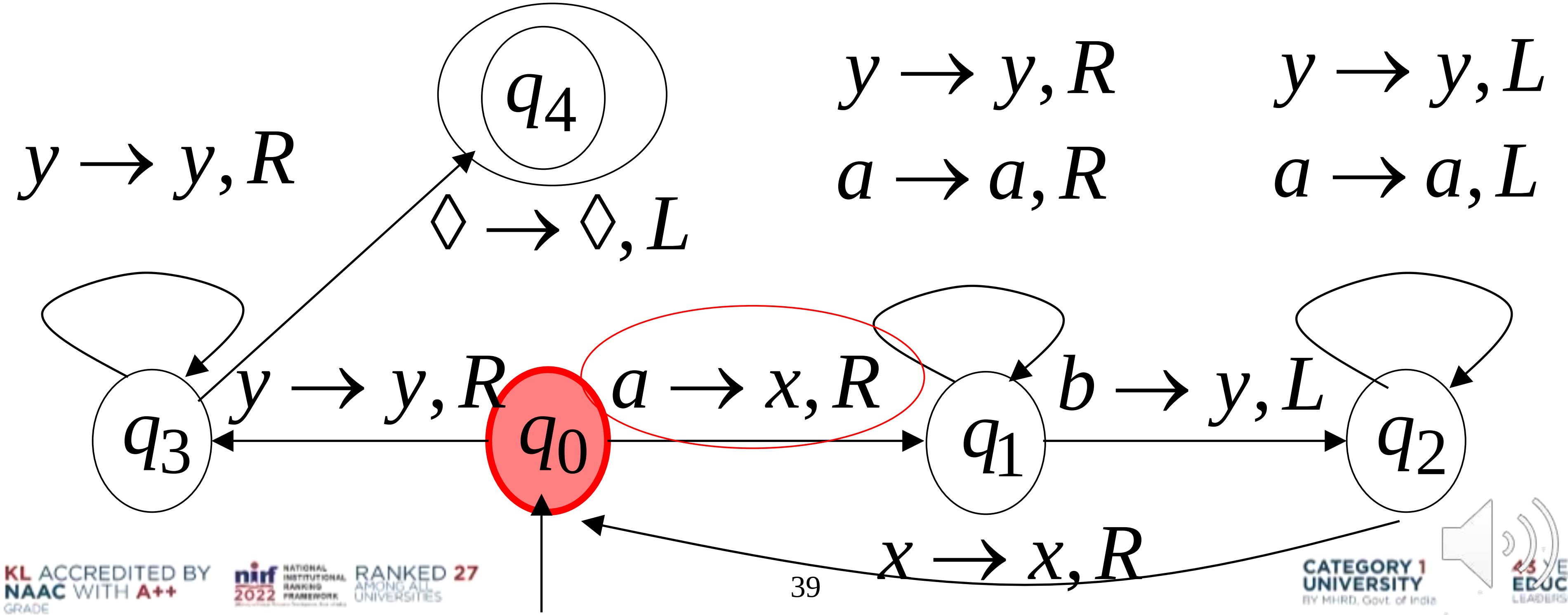
Turing machine for the language $\{a^n b^n\}$



Time 0

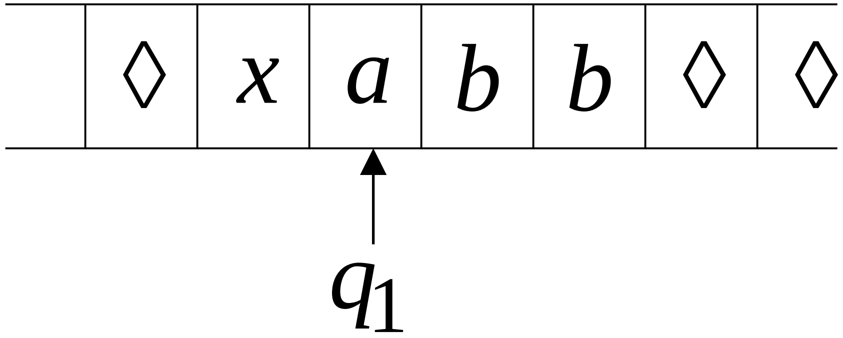


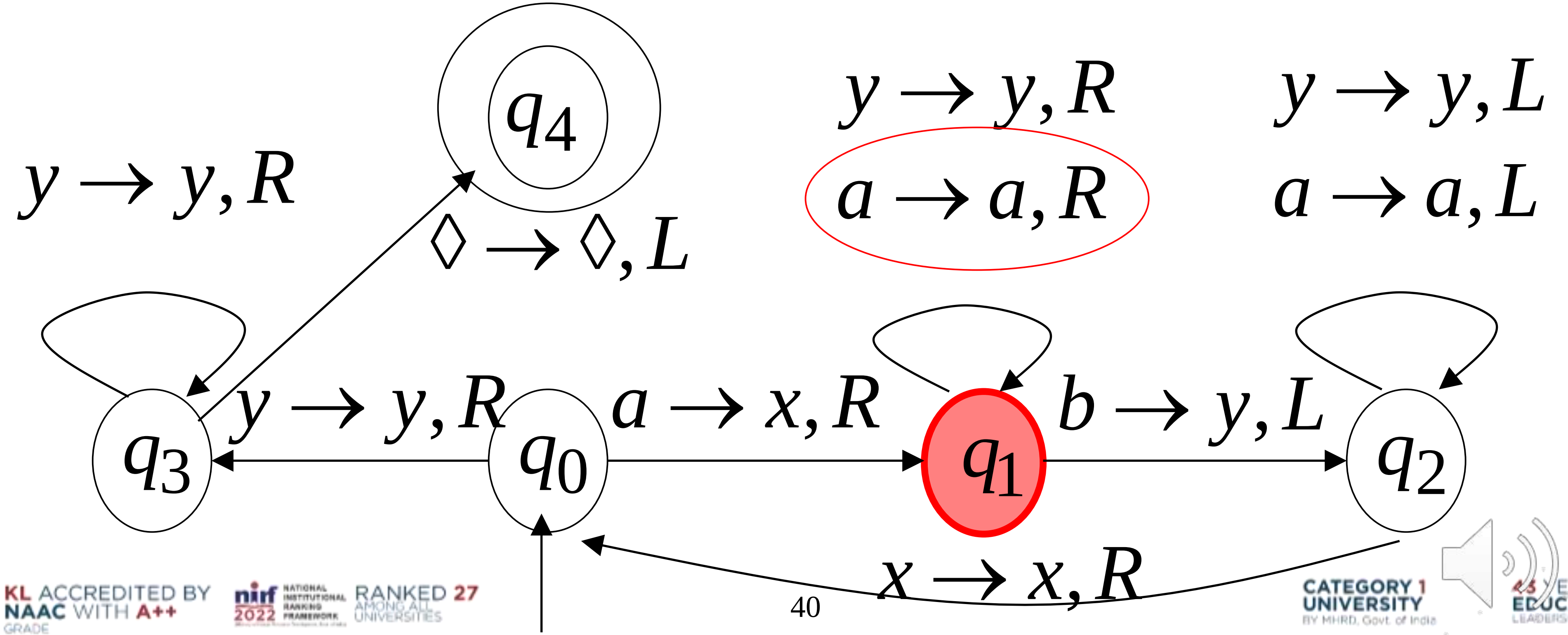
◊
a
a
b
b
◊
◊
q_0



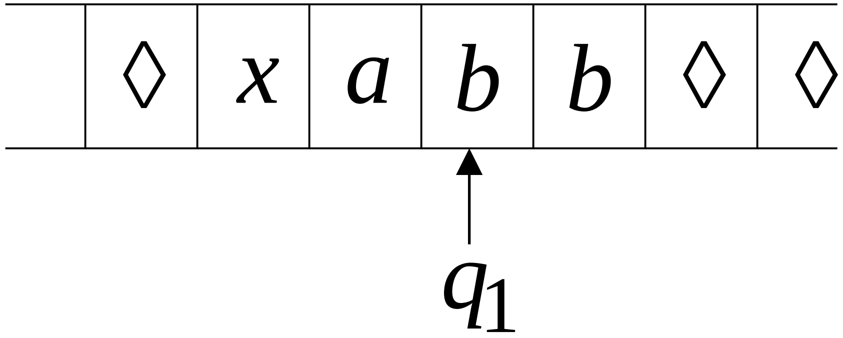
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 1

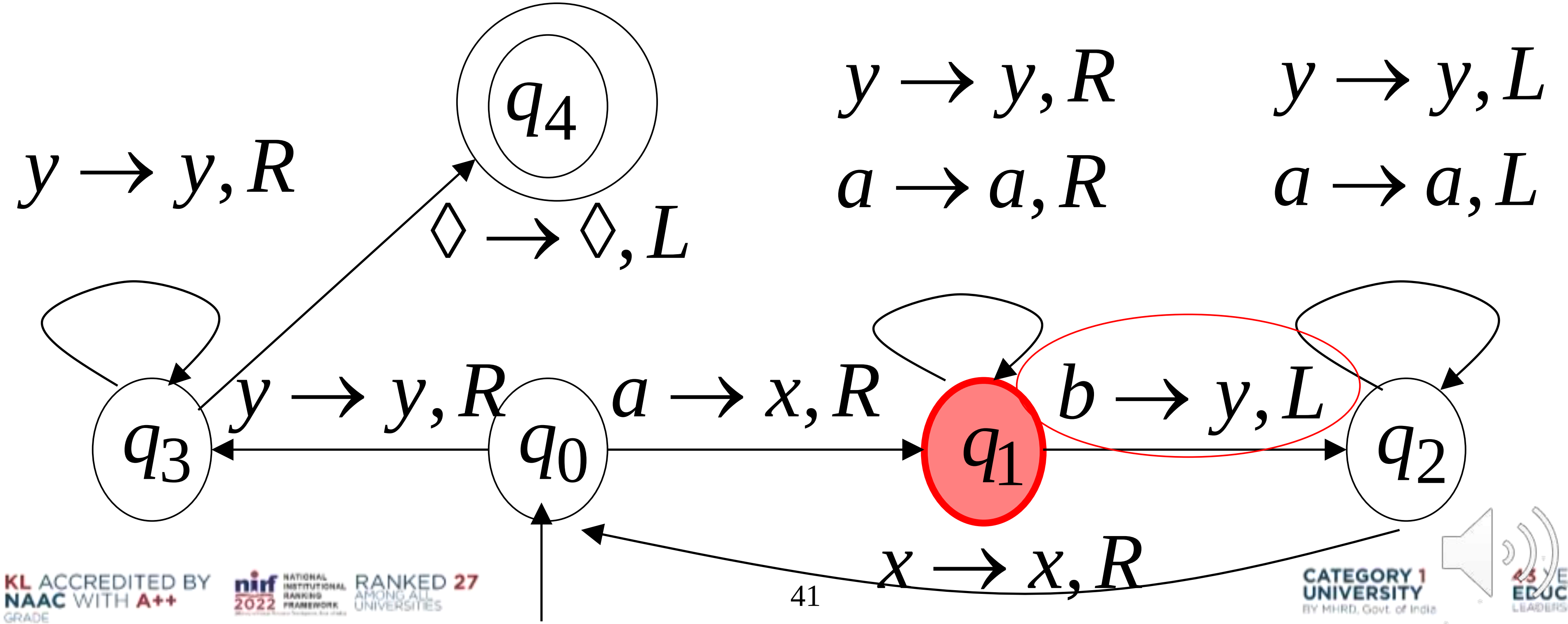

◊
x
a
b
b
◊
◊
q_1


q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

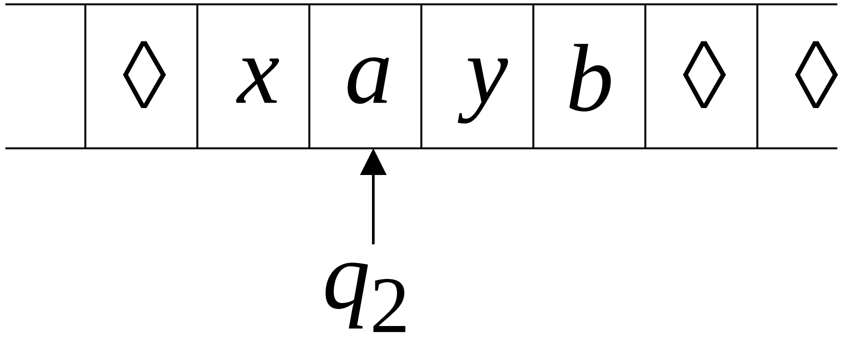
Time 2

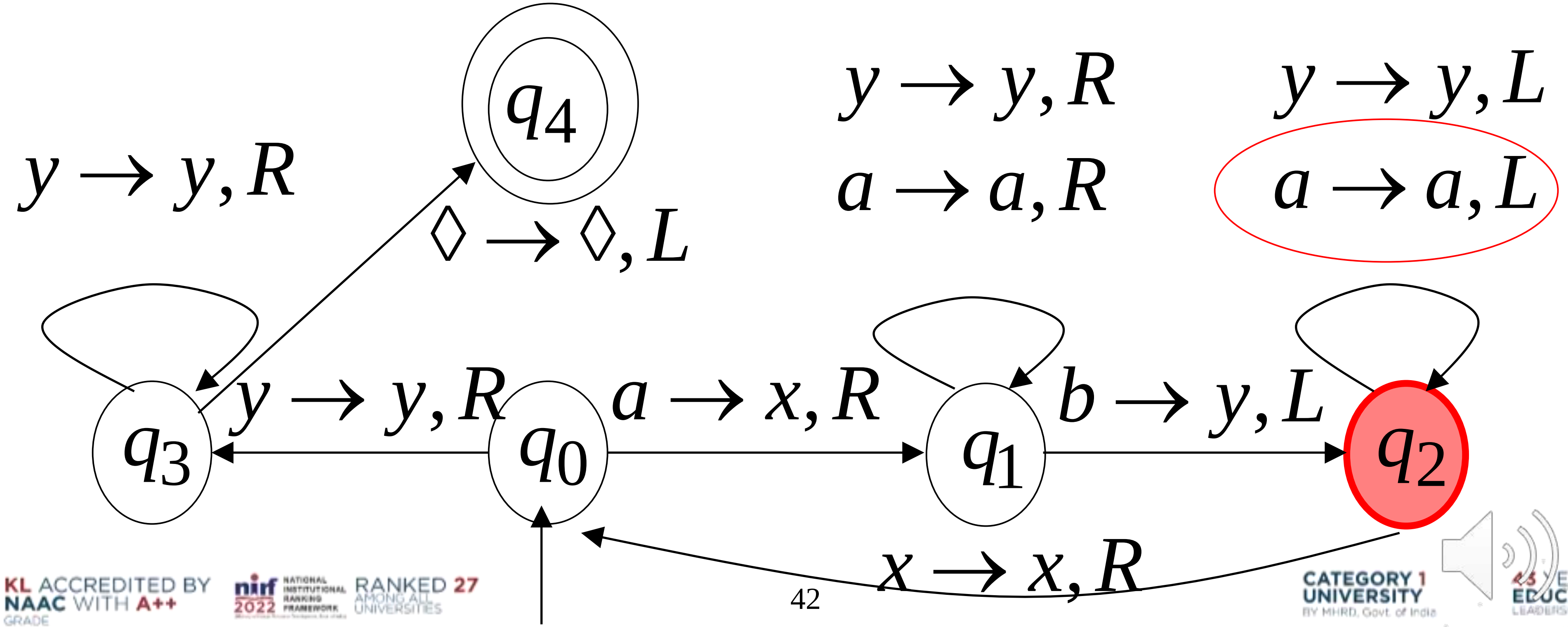


$\Diamond$	x	a	b	b	$\Diamond$	$\Diamond$

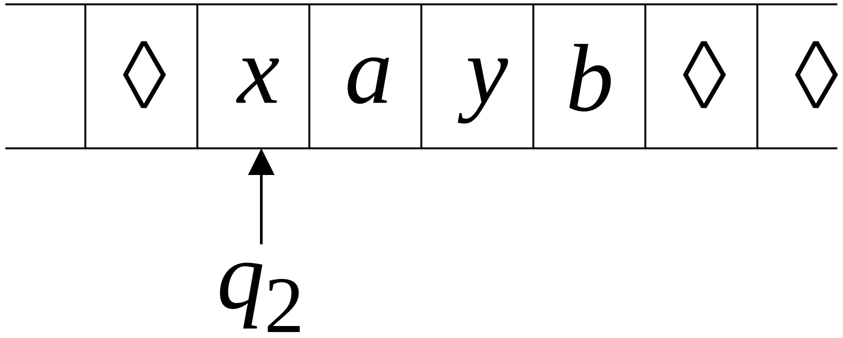


Time 3


◊
x
a
y
b
◊
◊
q_2

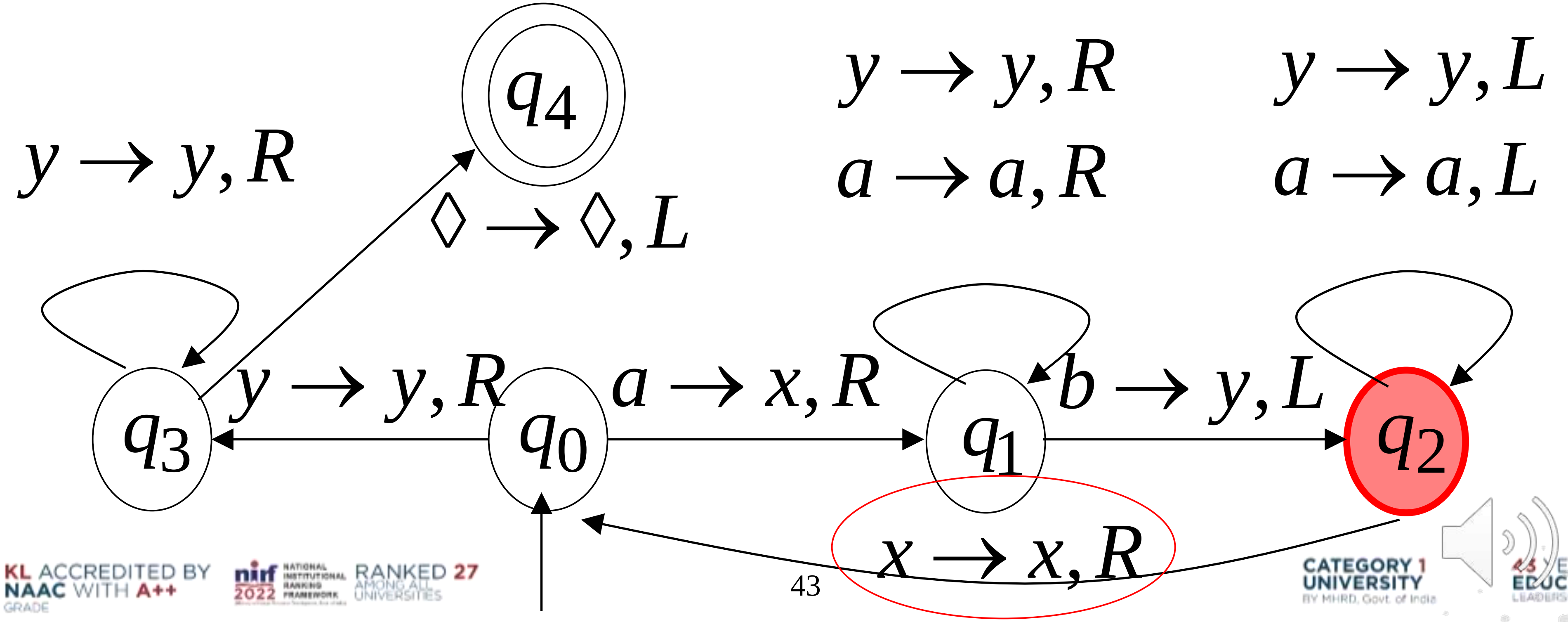

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 4



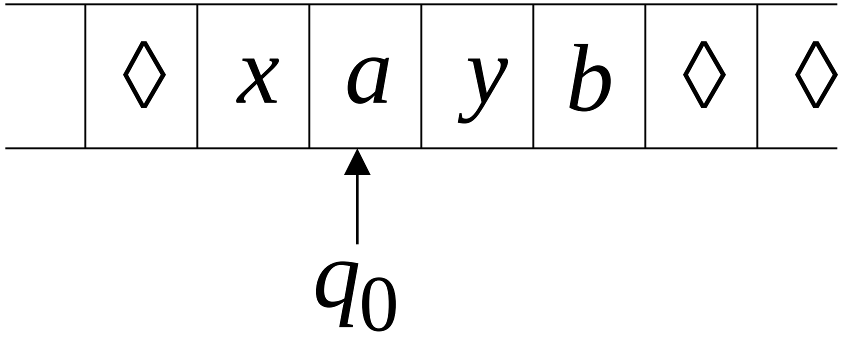
	◊	x	a	y	b	◊	◊

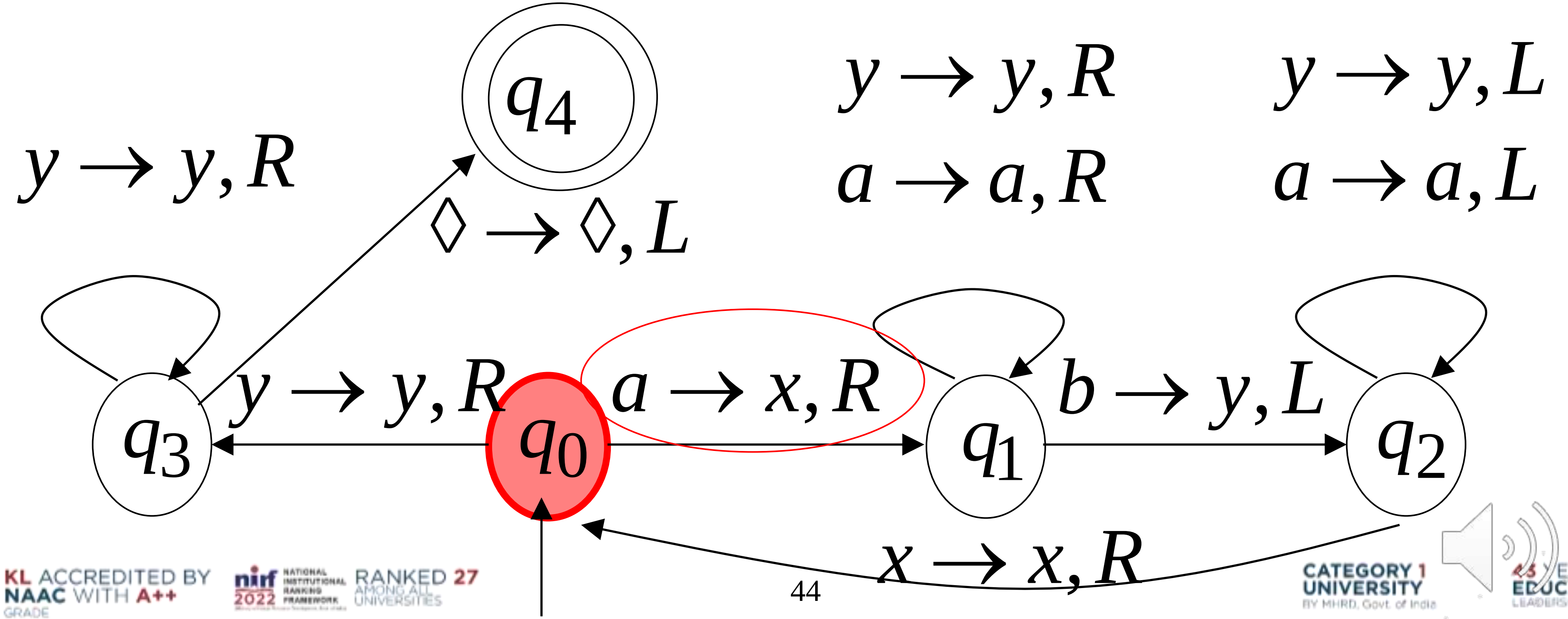
q_2



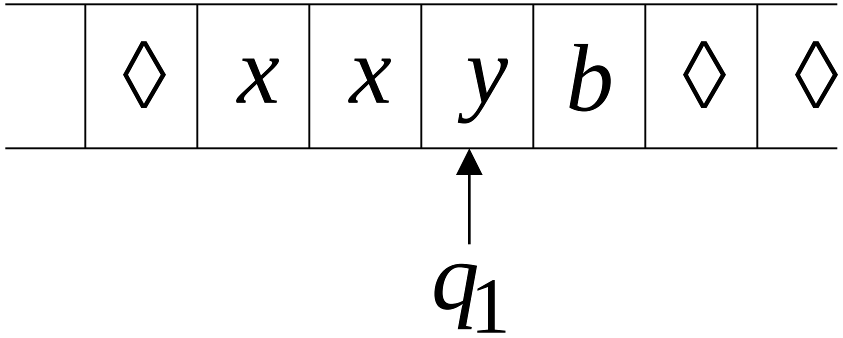
q_4
$y \rightarrow y, R$
◊ $\rightarrow$ ◊$, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 5



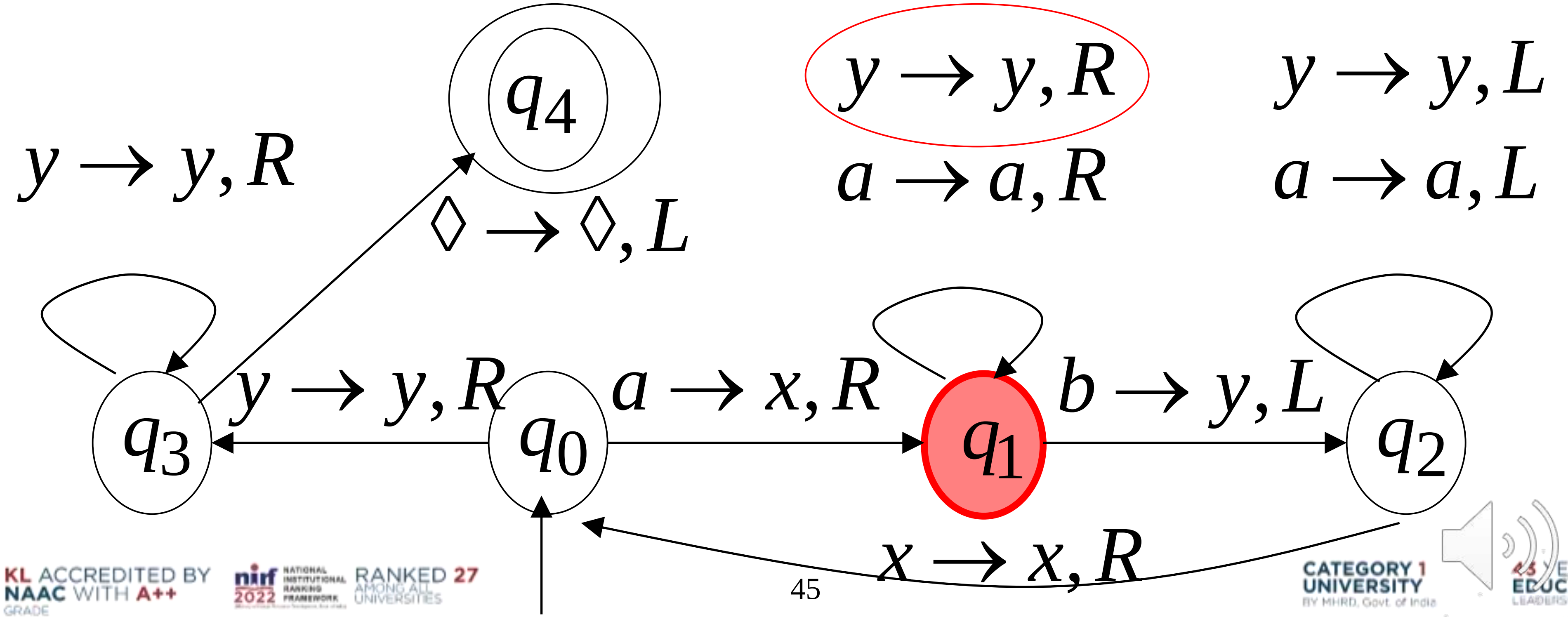


Time 6



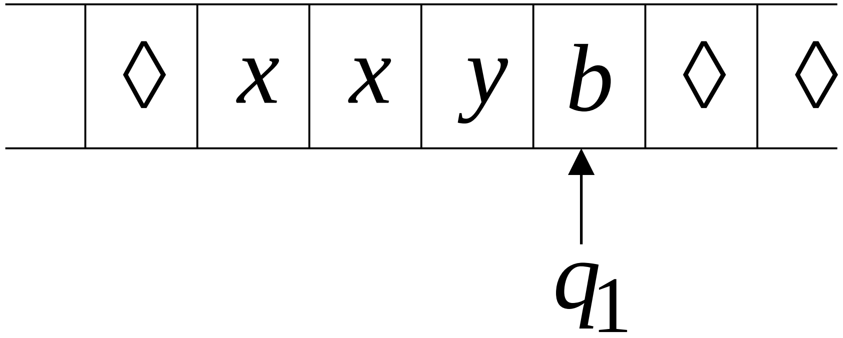
◊	x	x	y	b	◊	◊

q_1

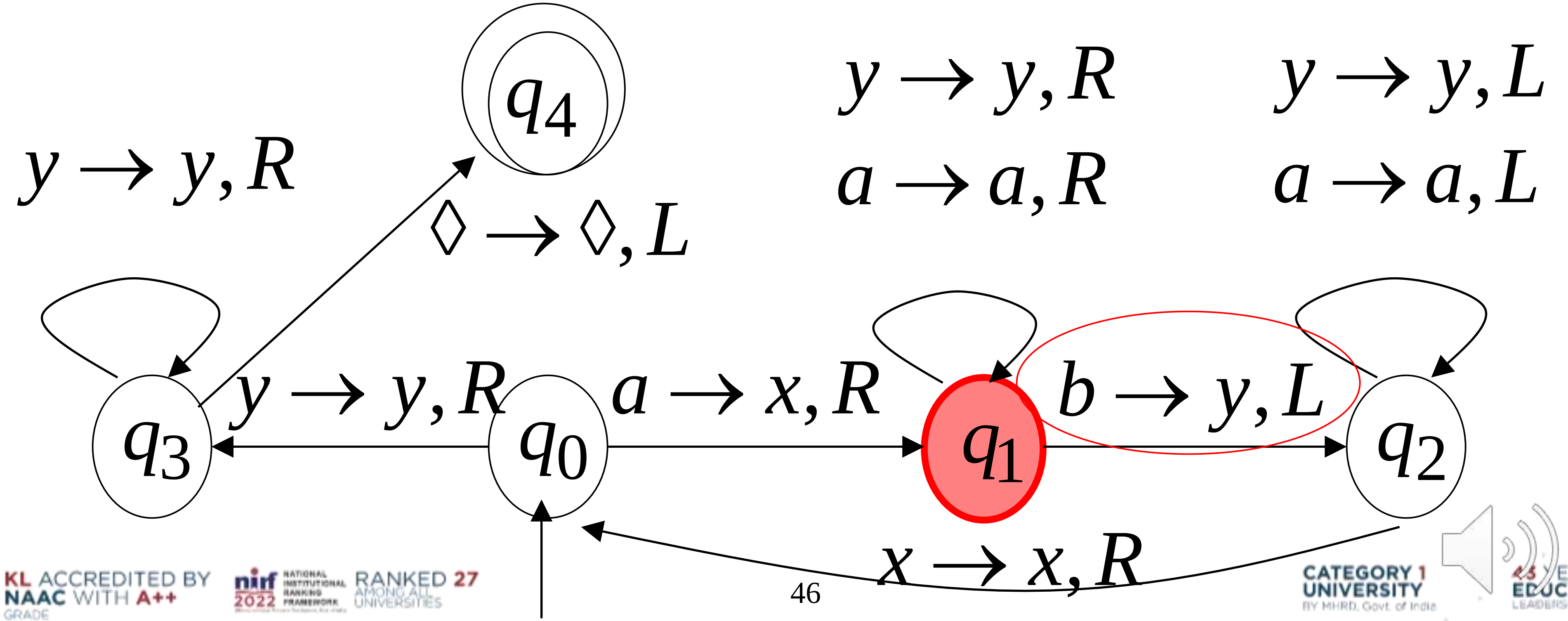


q_4
$y \rightarrow y, R$
◊ → ◊, L
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 7

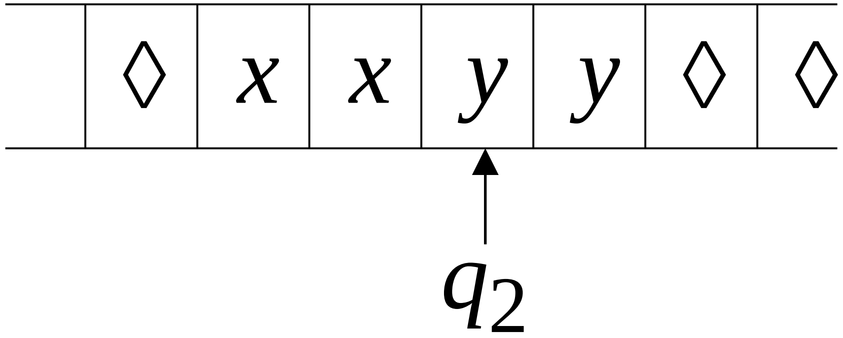


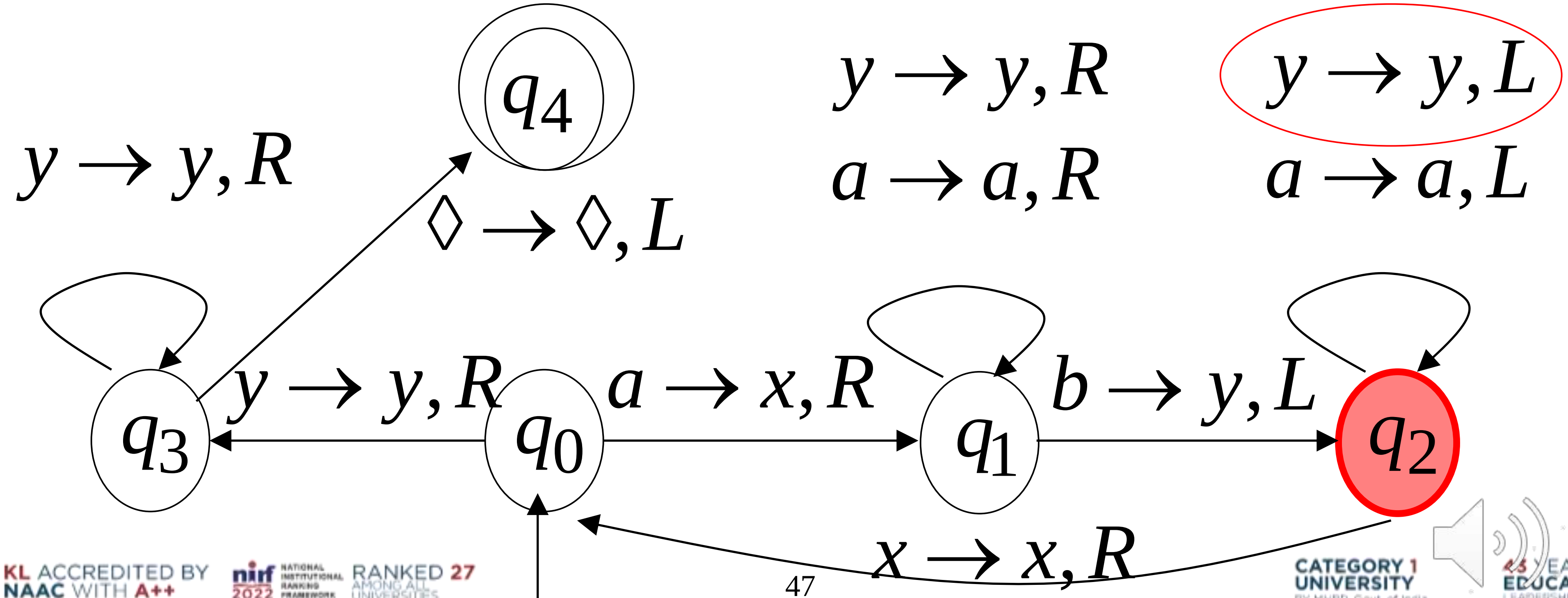
◊
x
x
y
b
◊
◊
q_1



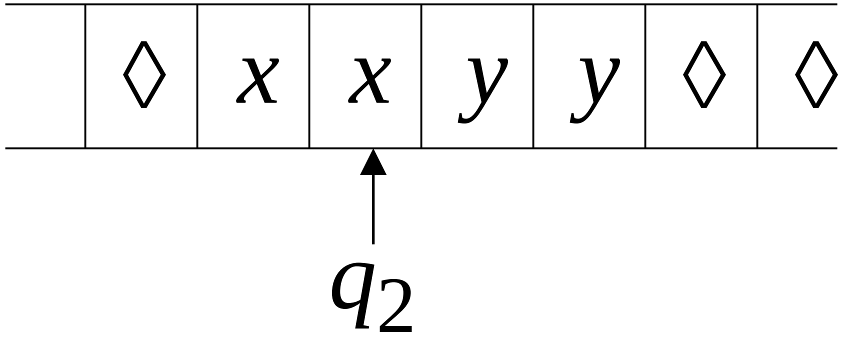
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

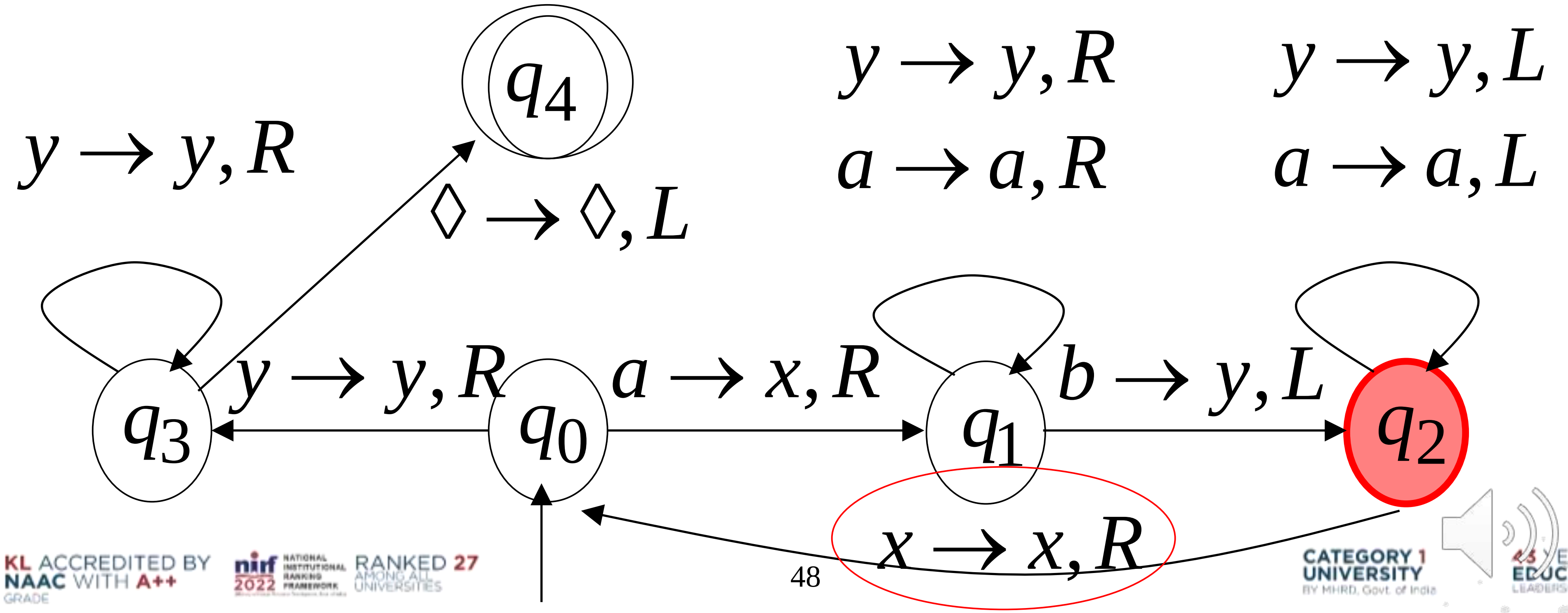
Time 8


◊
x
x
y
y
◊
◊
q_2

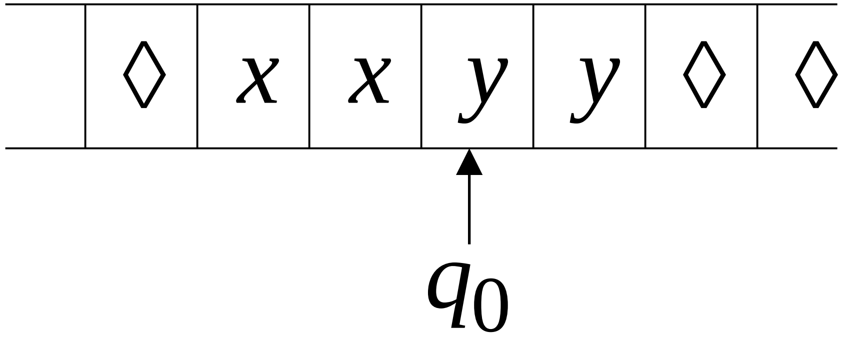

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

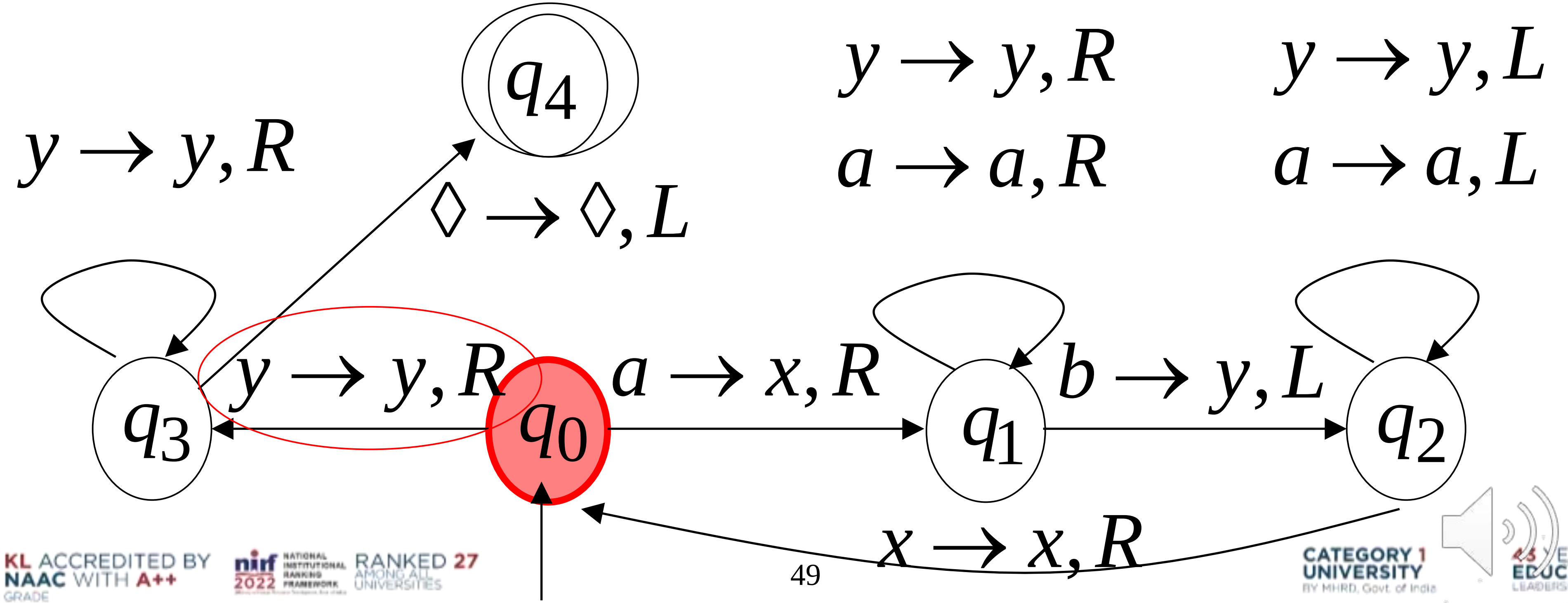
Time 9


◊
x
x
y
y
◊
◊
q_2

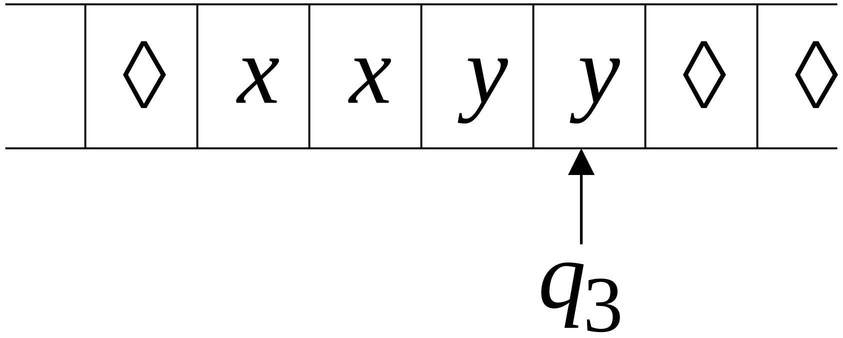

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 10

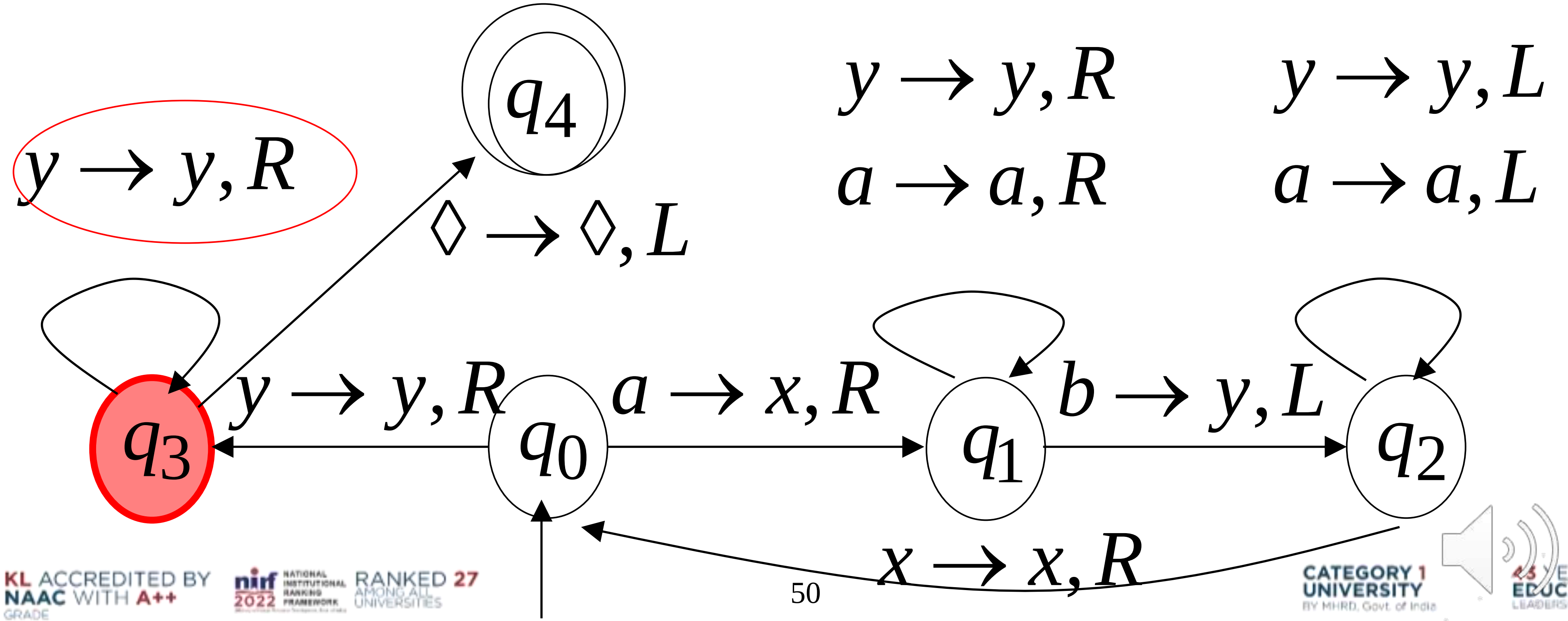

◊
x
x
y
y
◊
◊
q_0


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 11

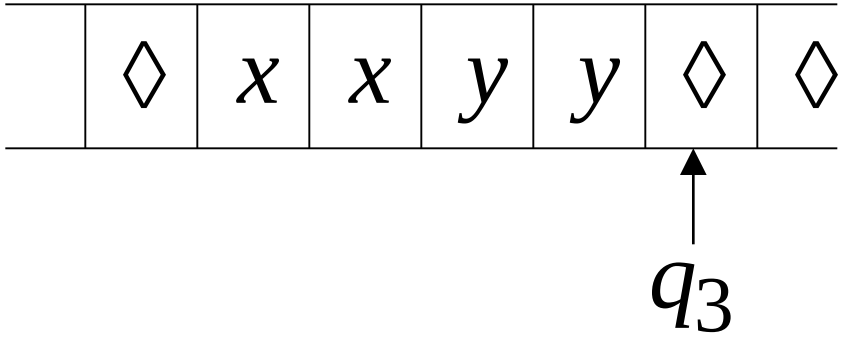


◊ x x y y ◊ ◊
q_3

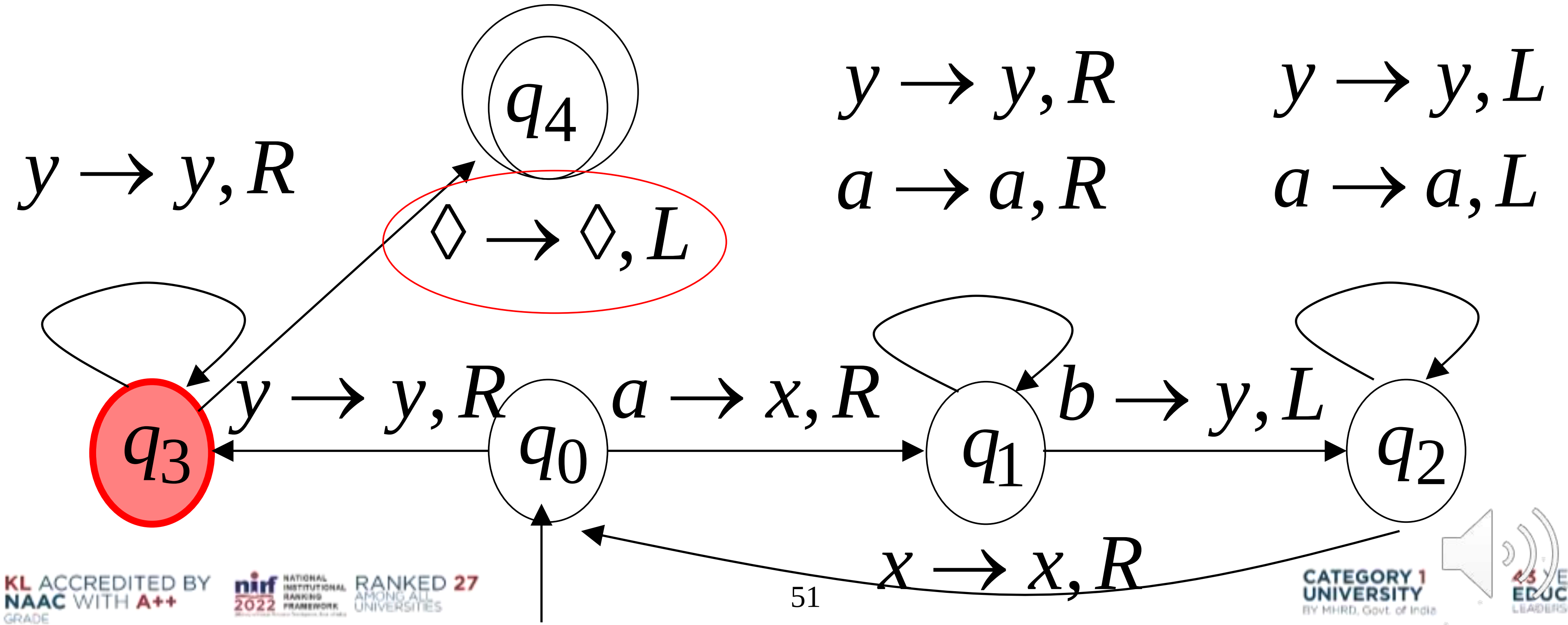


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 12



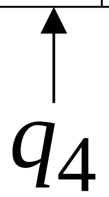
◊
x
x
y
y
◊
◊
q_3



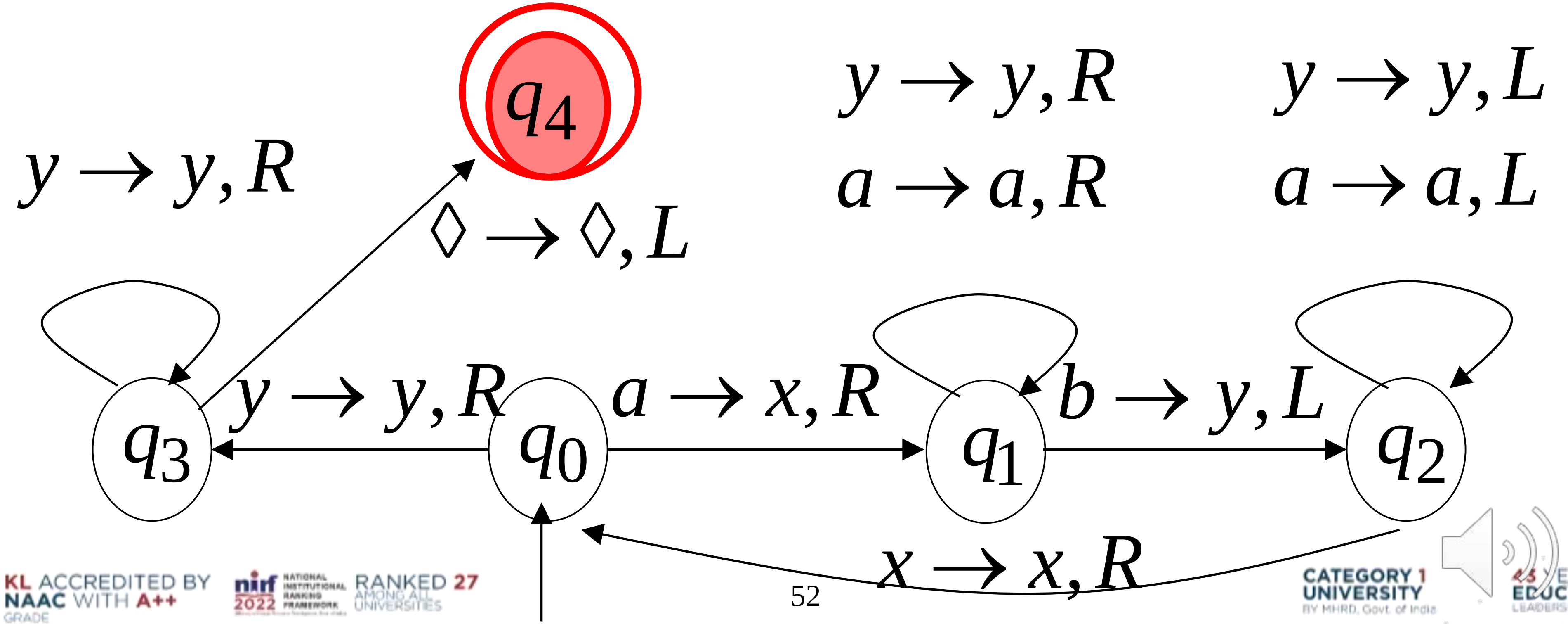
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 13

◊	x	x	y	y	◊	◊



Halt & Accept



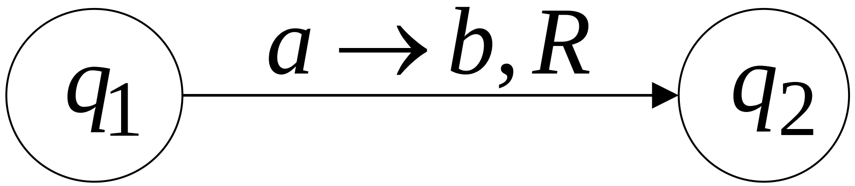
Observation:

If we modify the machine for the language $\{a^n b^n\}$

we can easily construct a machine for the language $\{a^n b^n c^n\}$

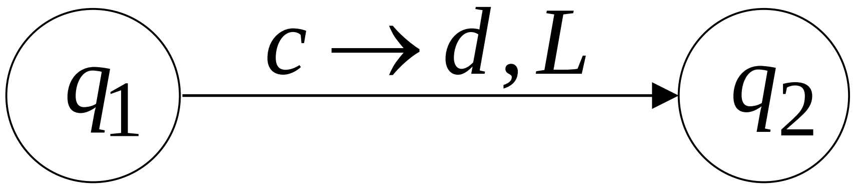
Formal Definitions for Turing Machines

Transition Function



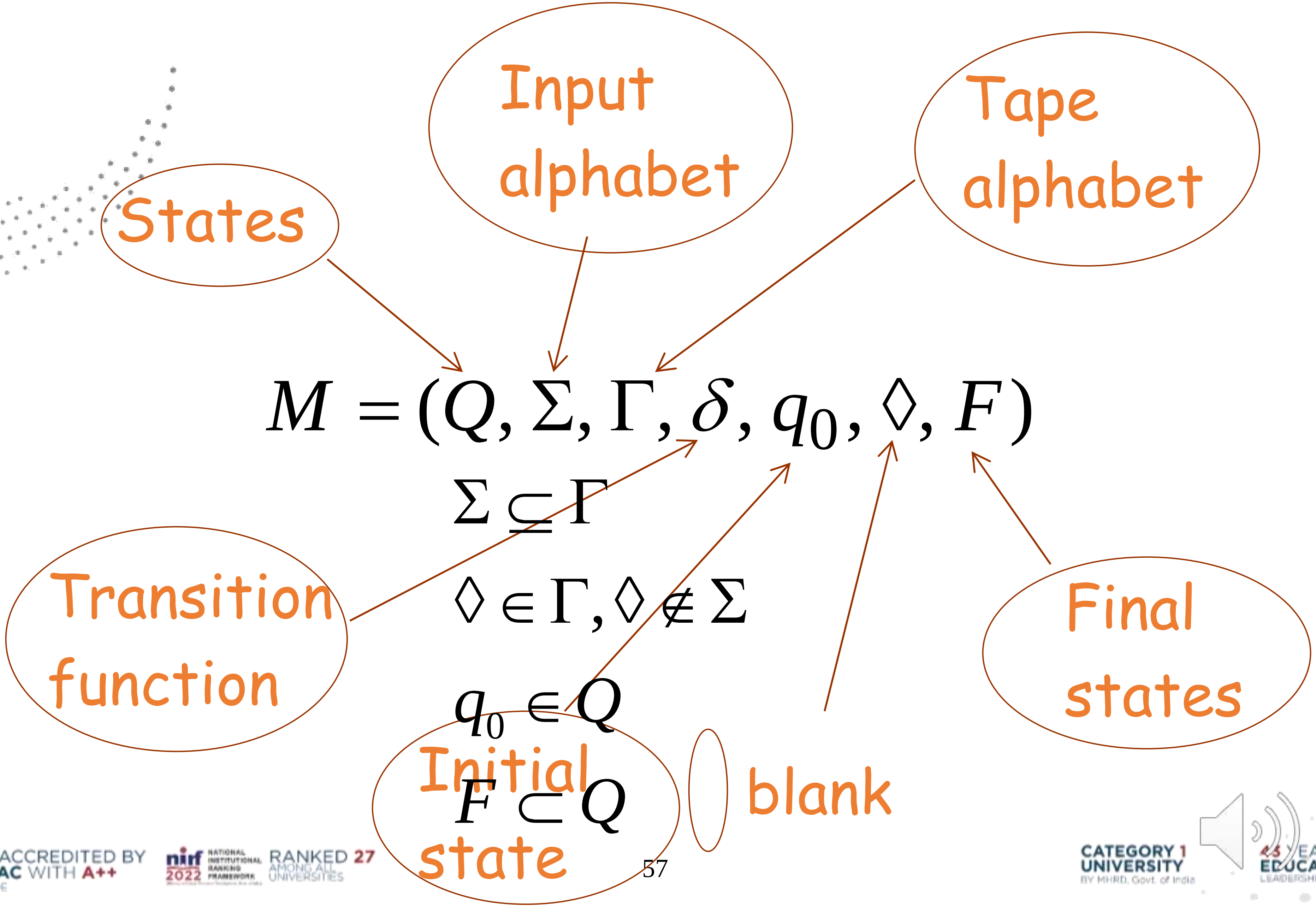
$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:



$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

$$\Sigma \subseteq \Gamma$$

$$\Diamond \in \Gamma, \Diamond \notin \Sigma$$

$$q_0 \in Q$$

$$F \subset Q$$

States

Input alphabet

Tape alphabet

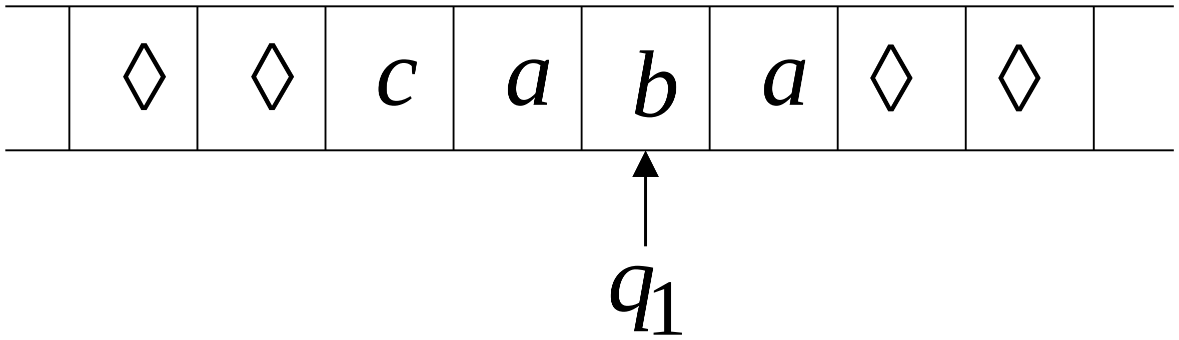
Transition function

Initial state

$\Diamond$ blank

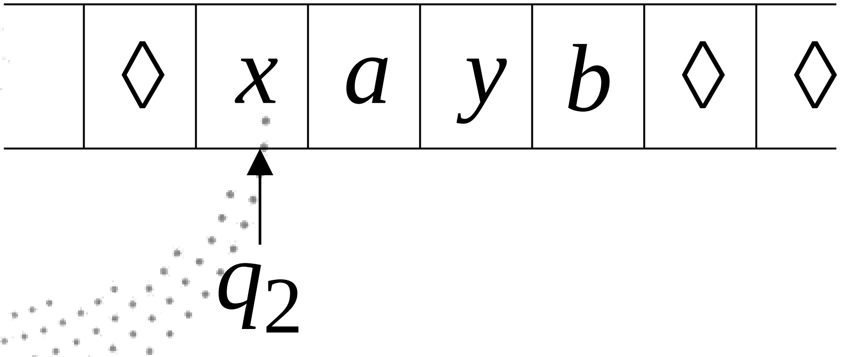
Final states

Configuration

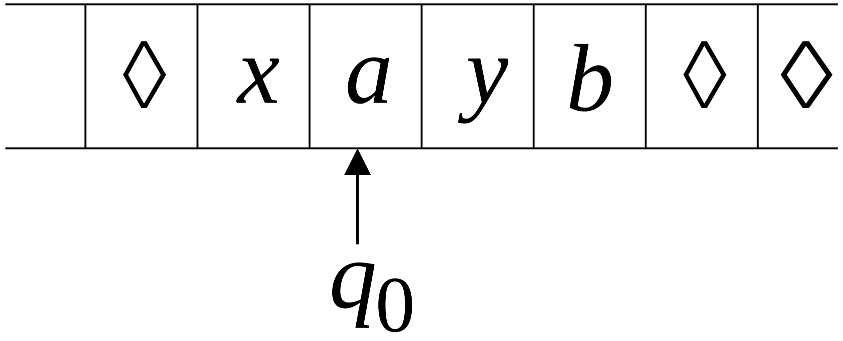


Instantaneous description: $ca\ q_1\ ba$

Time 4

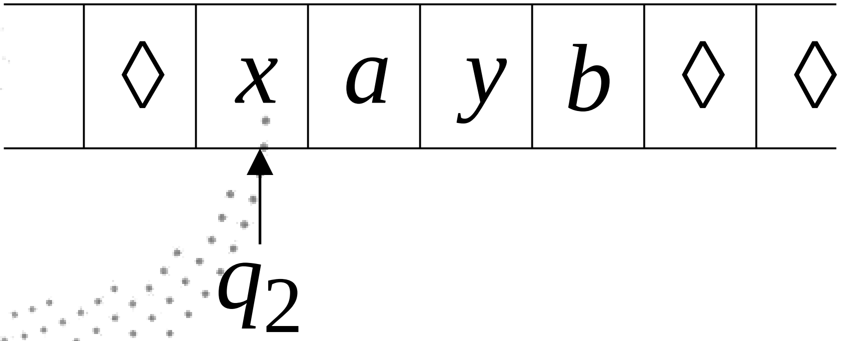


Time 5

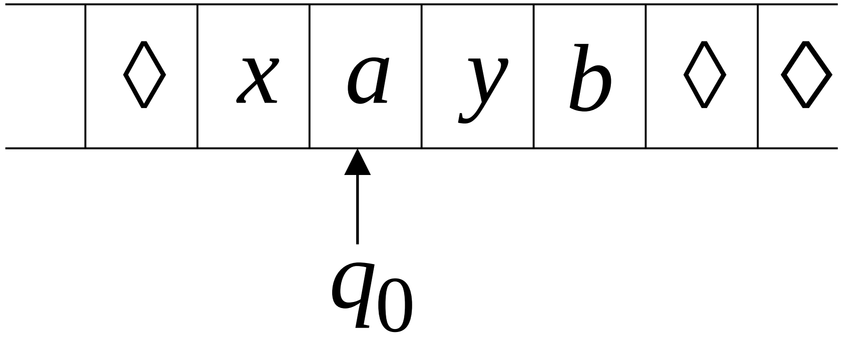


A Move: $q_2\ xayb \succ x\ q_0\ ayb$

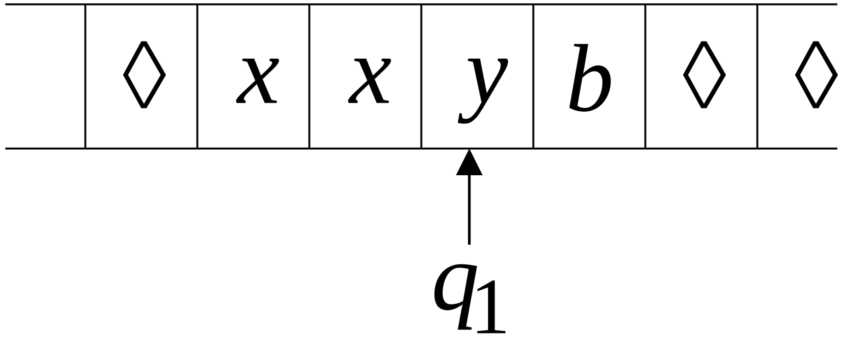
Time 4



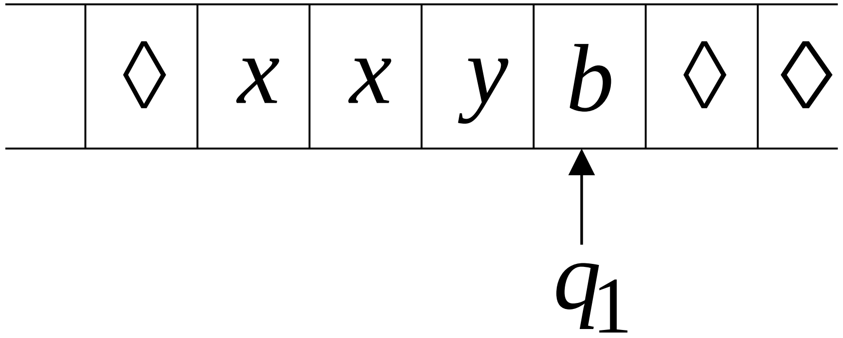
Time 5



Time 6



Time 7

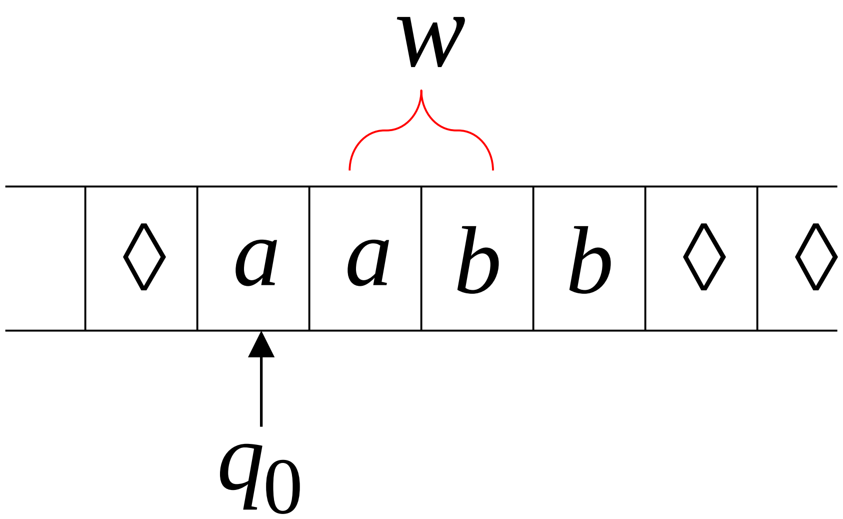


$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\ xayb\ \succ\ x\,q_0\ ayb\ \succ\ xx\,q_1\ yb\ \succ\ xxy\,q_1\ b$$

Equivalent notation: $$q_2\ xayb\ \overset{*}{\succ}\ xxy\,q_1\ b$$

Initial configuration: $q_0\ w$



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

Final state

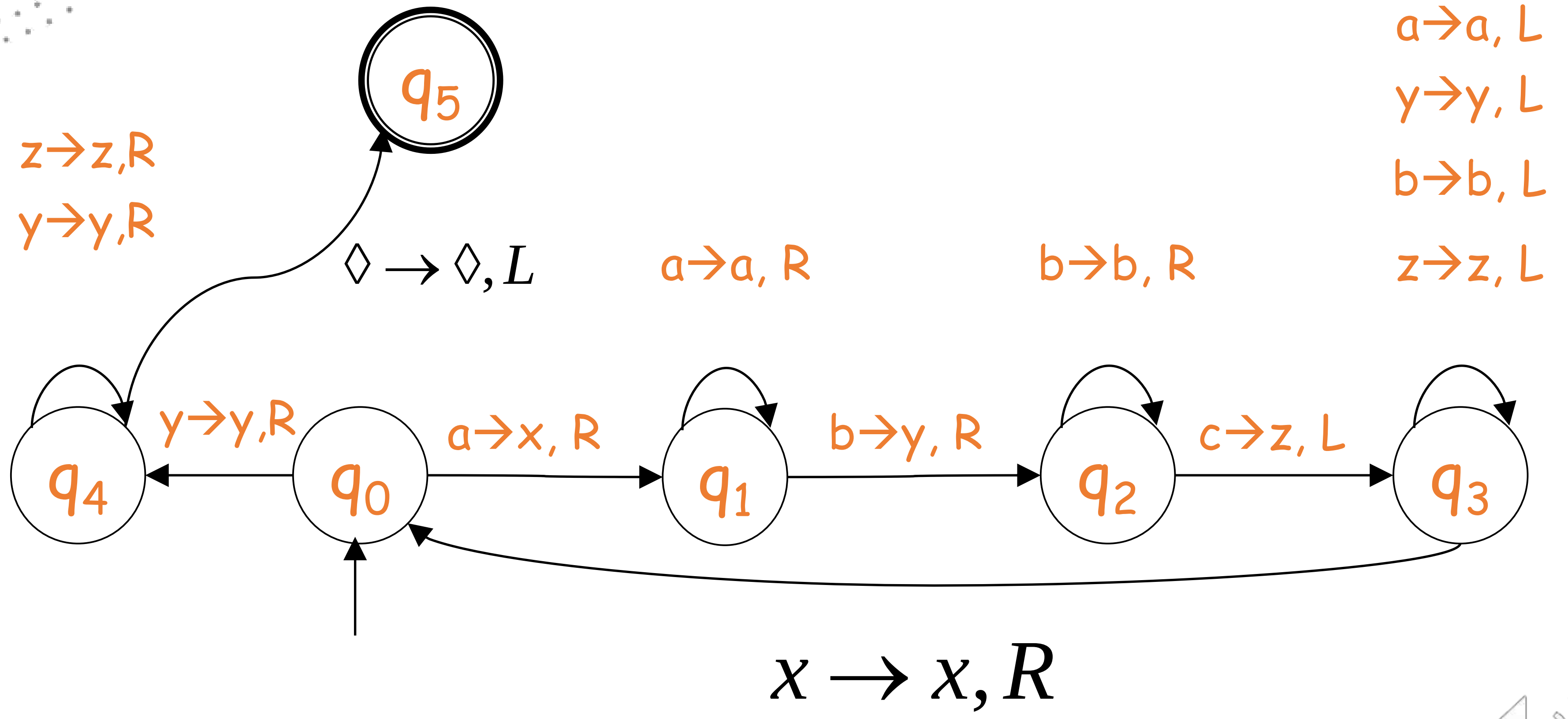
Standard Turing Machine

The machine we described is the standard:

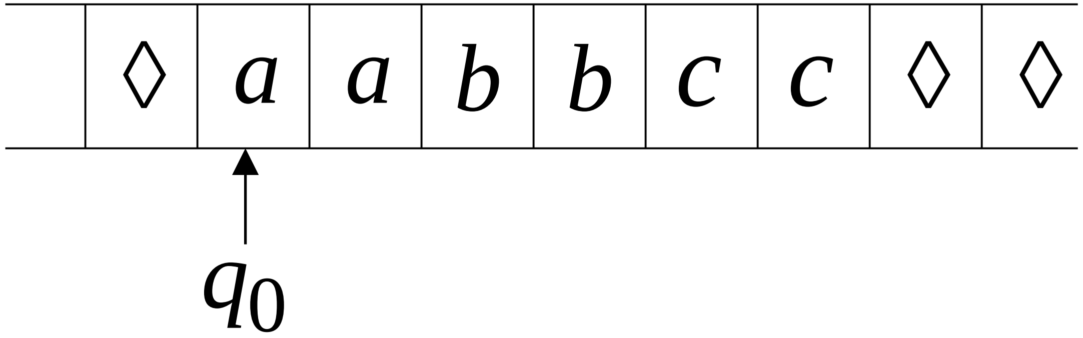
- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

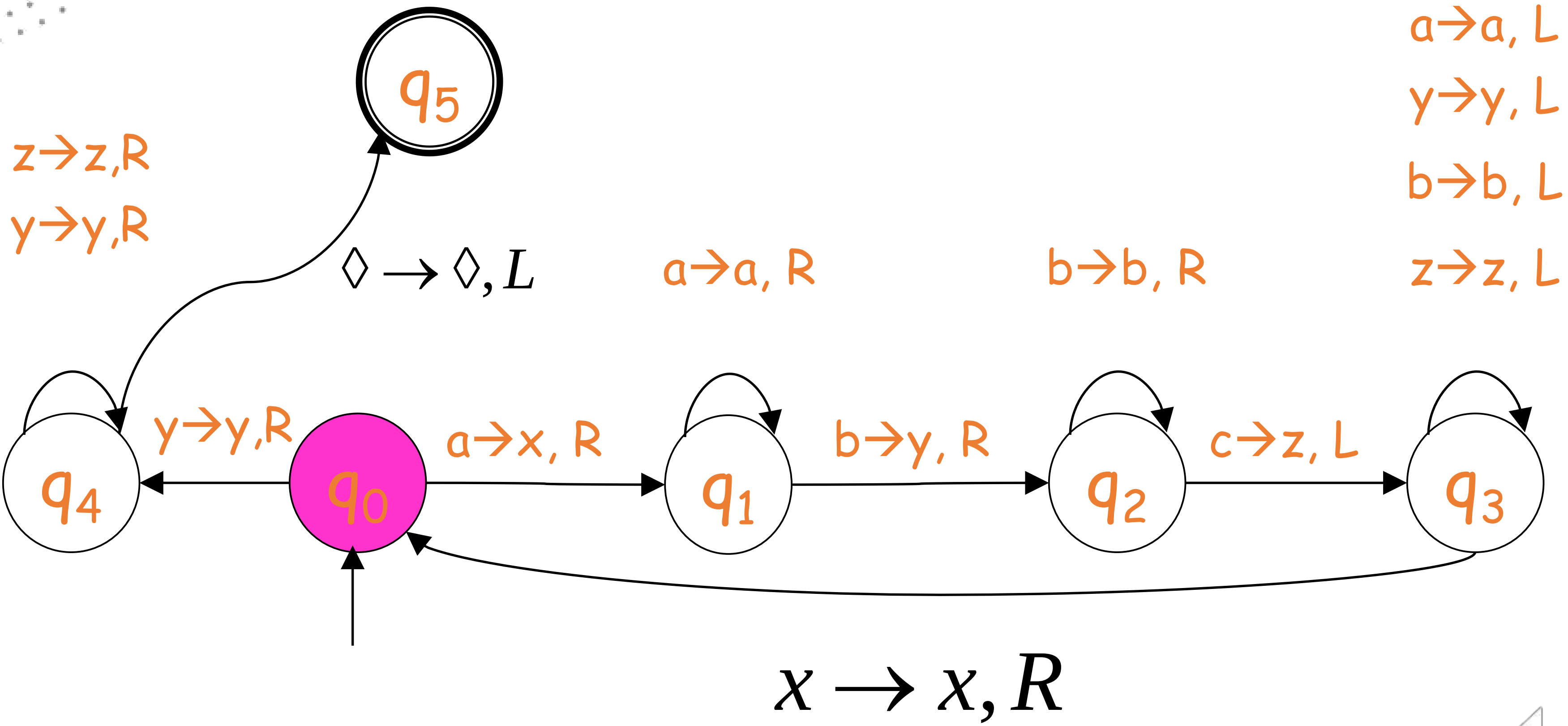
Another Turing Machine Example

Turing machine for the language $\{a^n b^n c^n\}$

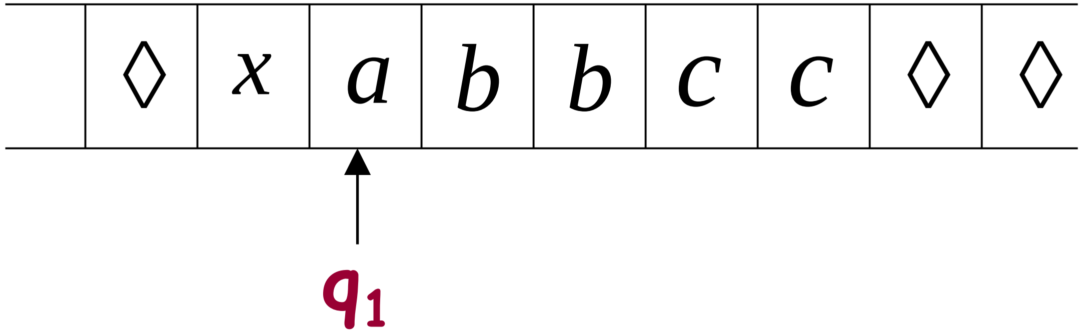


Time 0

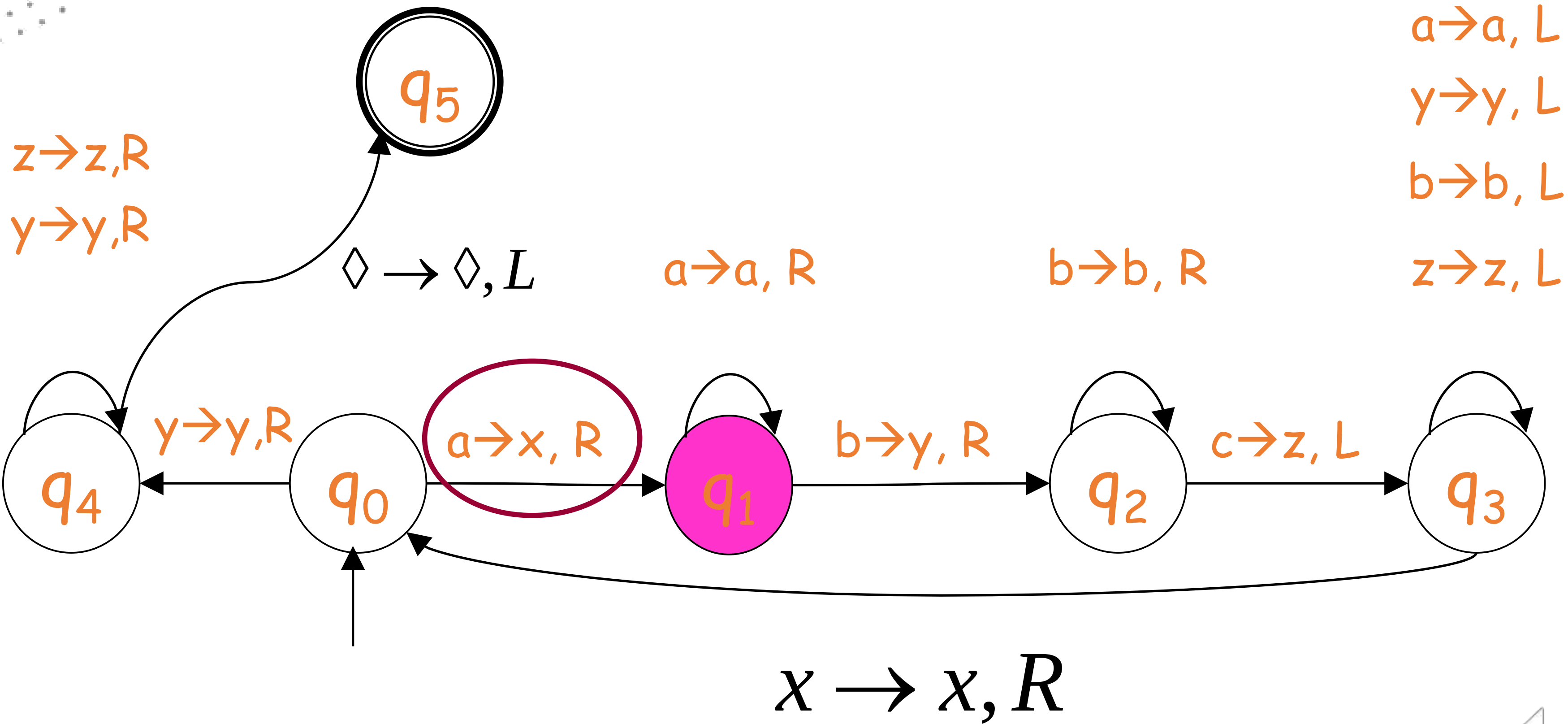

◊
a
a
b
b
c
c
◊
◊
q_0


q_5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
q_4
y→y,R
q_0
a→x, R
q_1
b→y, R
q_2
c→z, L
q_3
$x \rightarrow x, R$

Time 1

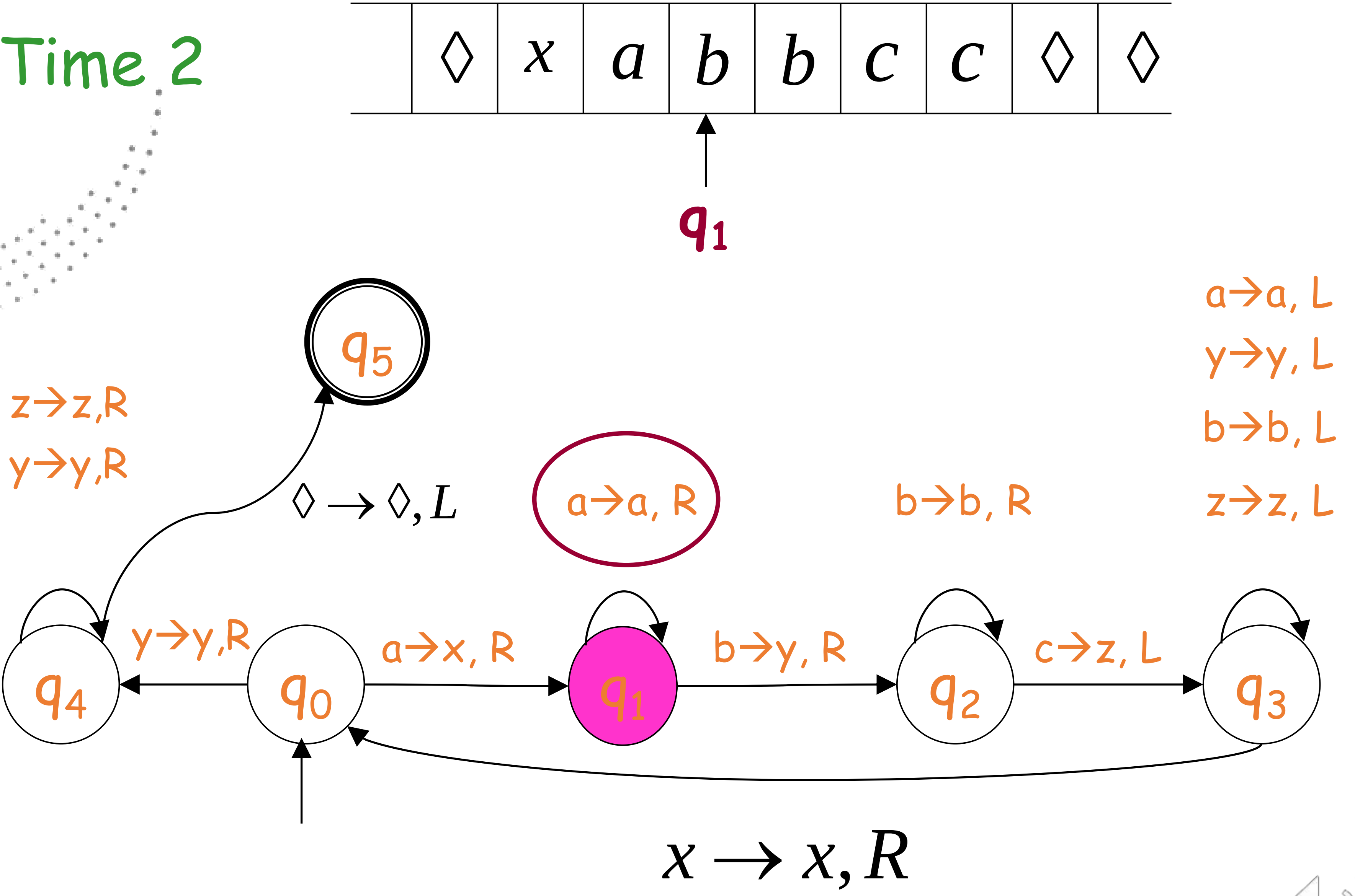


◊
x
a
b
b
c
c
◊
◊
q₁

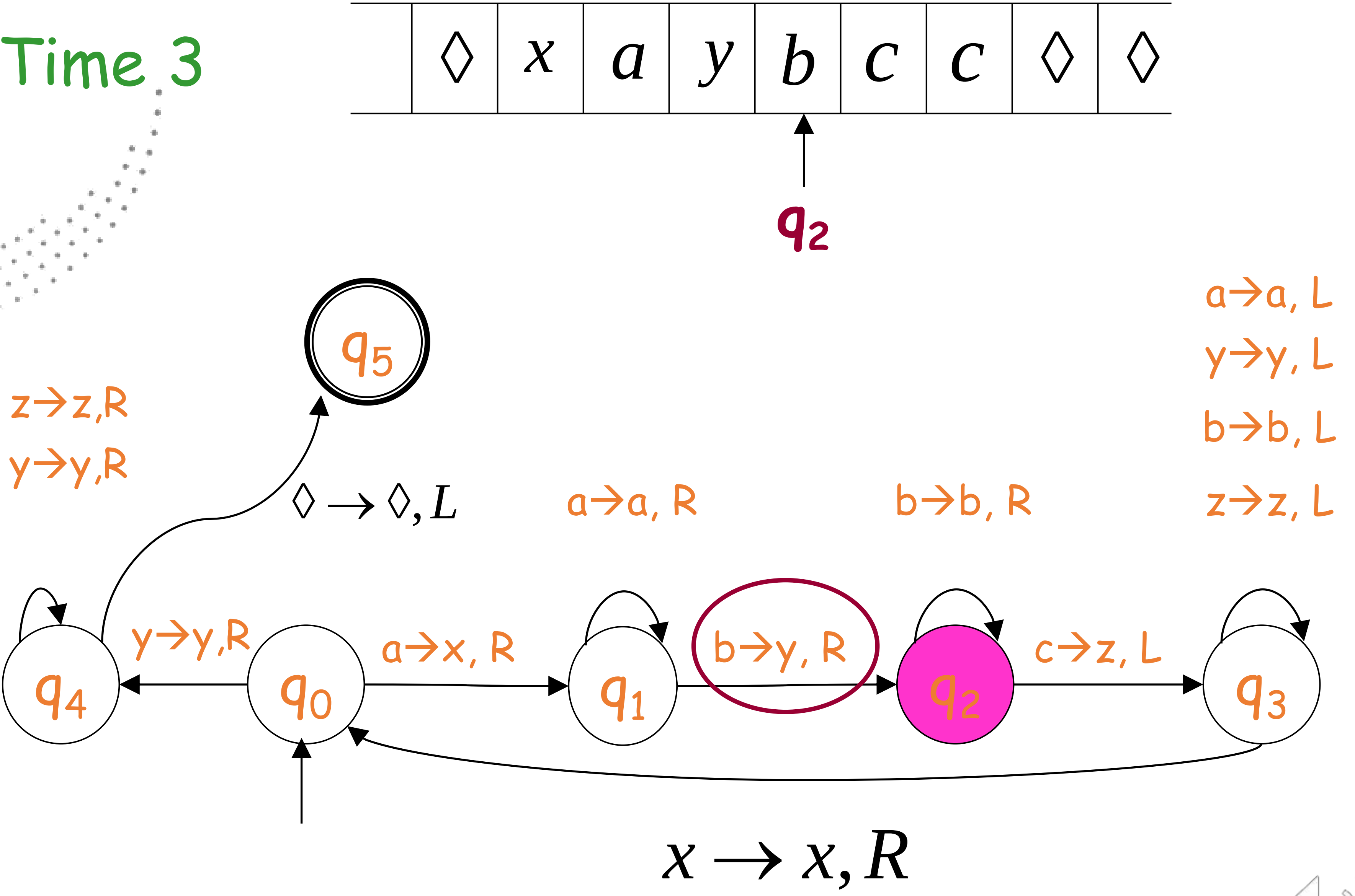


q₅
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q₄
q₀
q₁
q₂
q₃
x → x, R

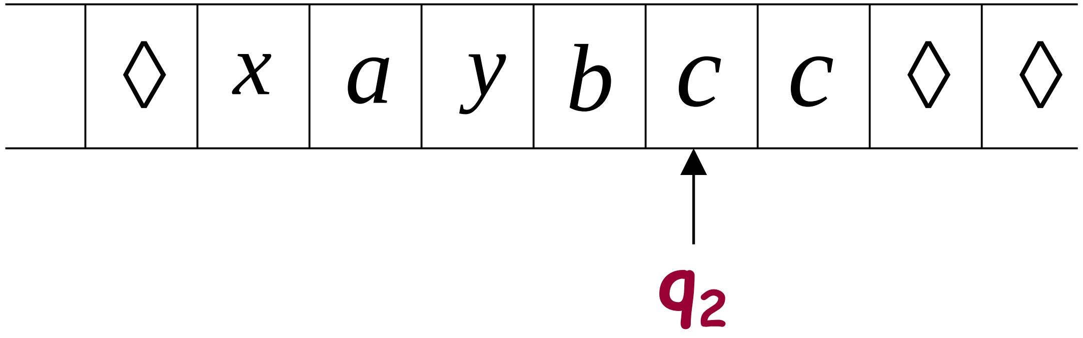
Time 2

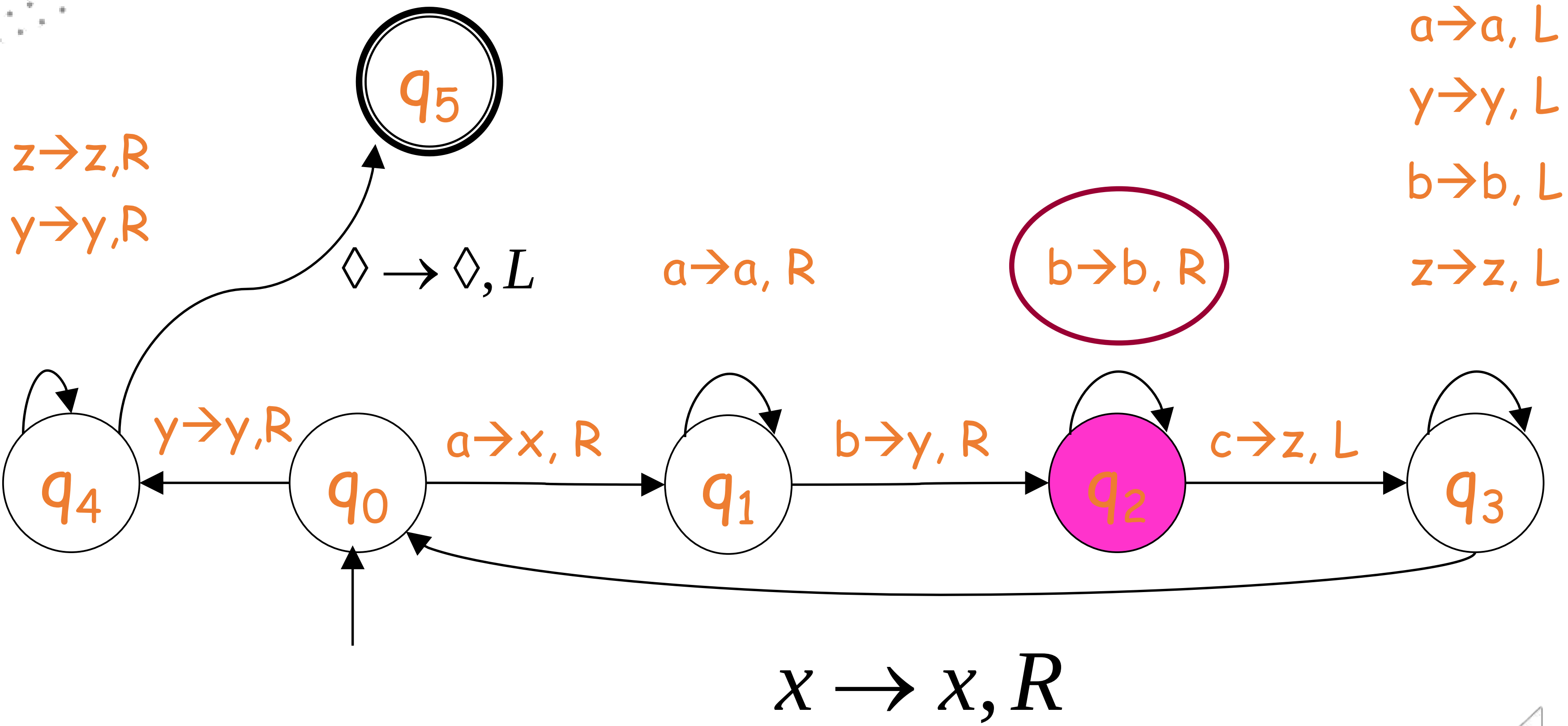

◊
x
a
b
b
c
c
◊
◊
q1
q5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

Time 3

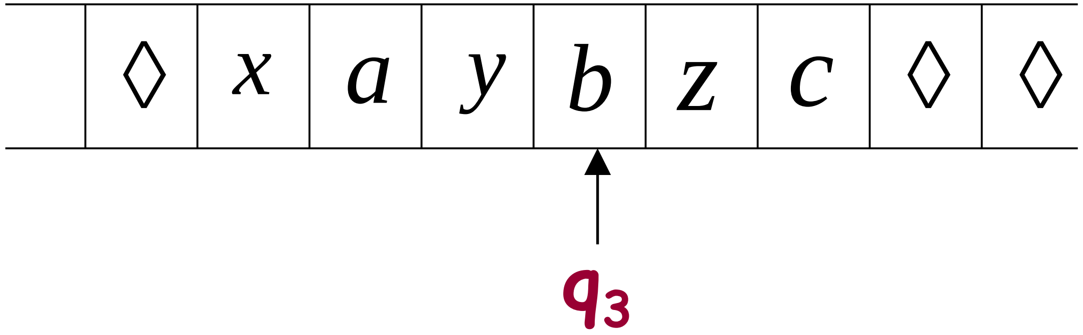

◊
x
a
y
b
c
c
◊
◊
q2
q5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

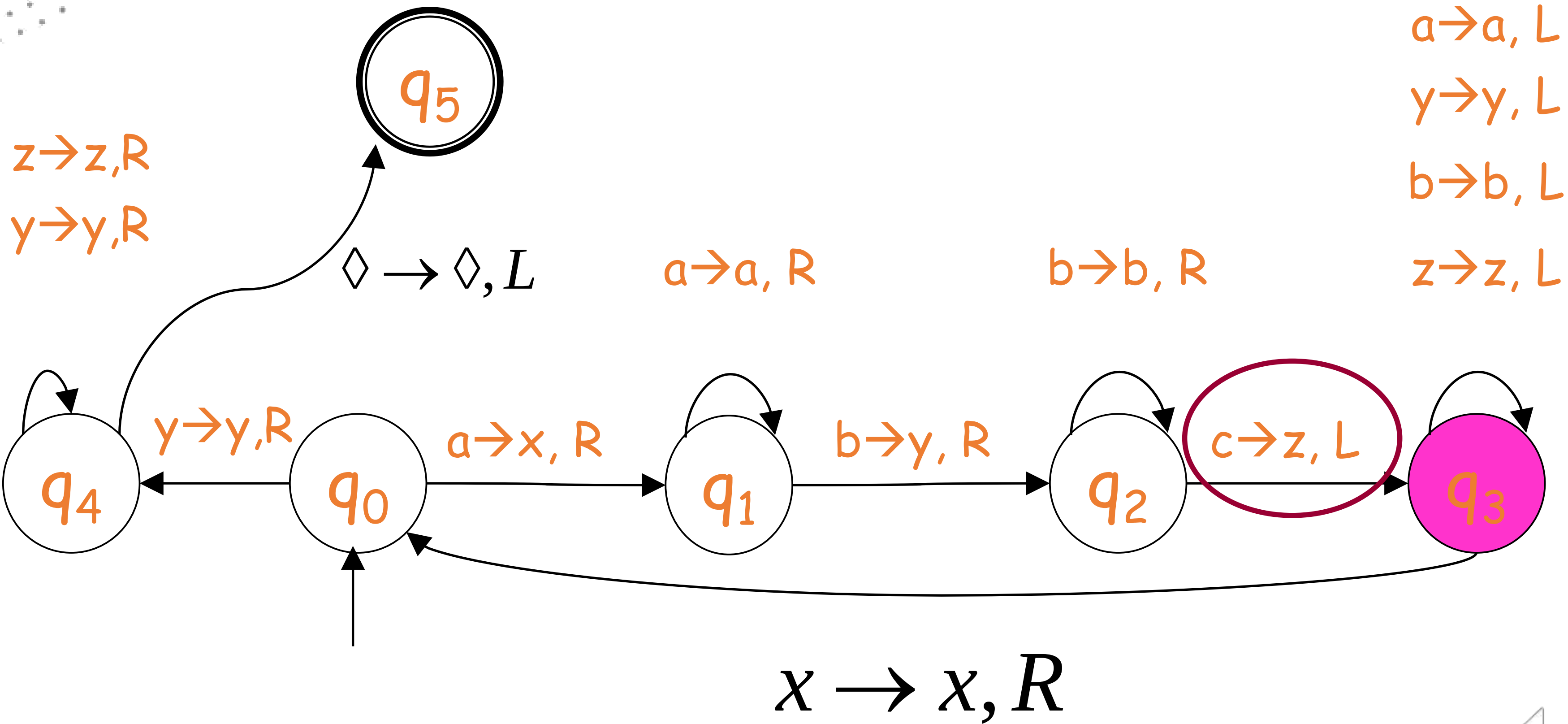
Time 4


◊
x
a
y
b
c
c
◊
◊
q2

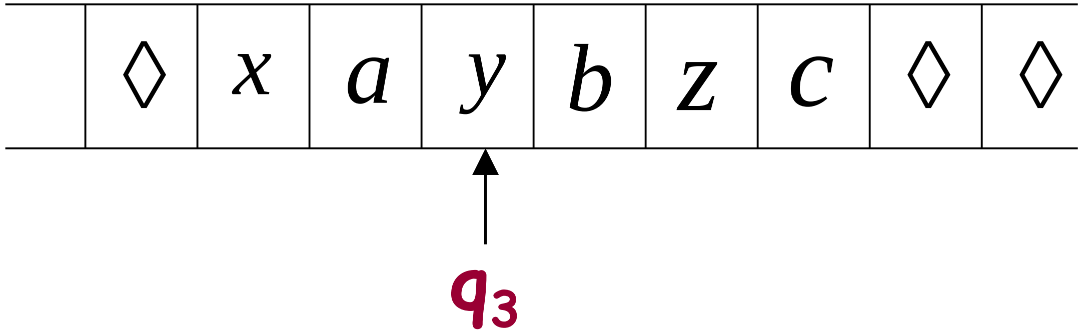

q5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
q4
y→y,R
q0
a→x, R
q1
b→y, R
q2
c→z, L
q3
x → x, R

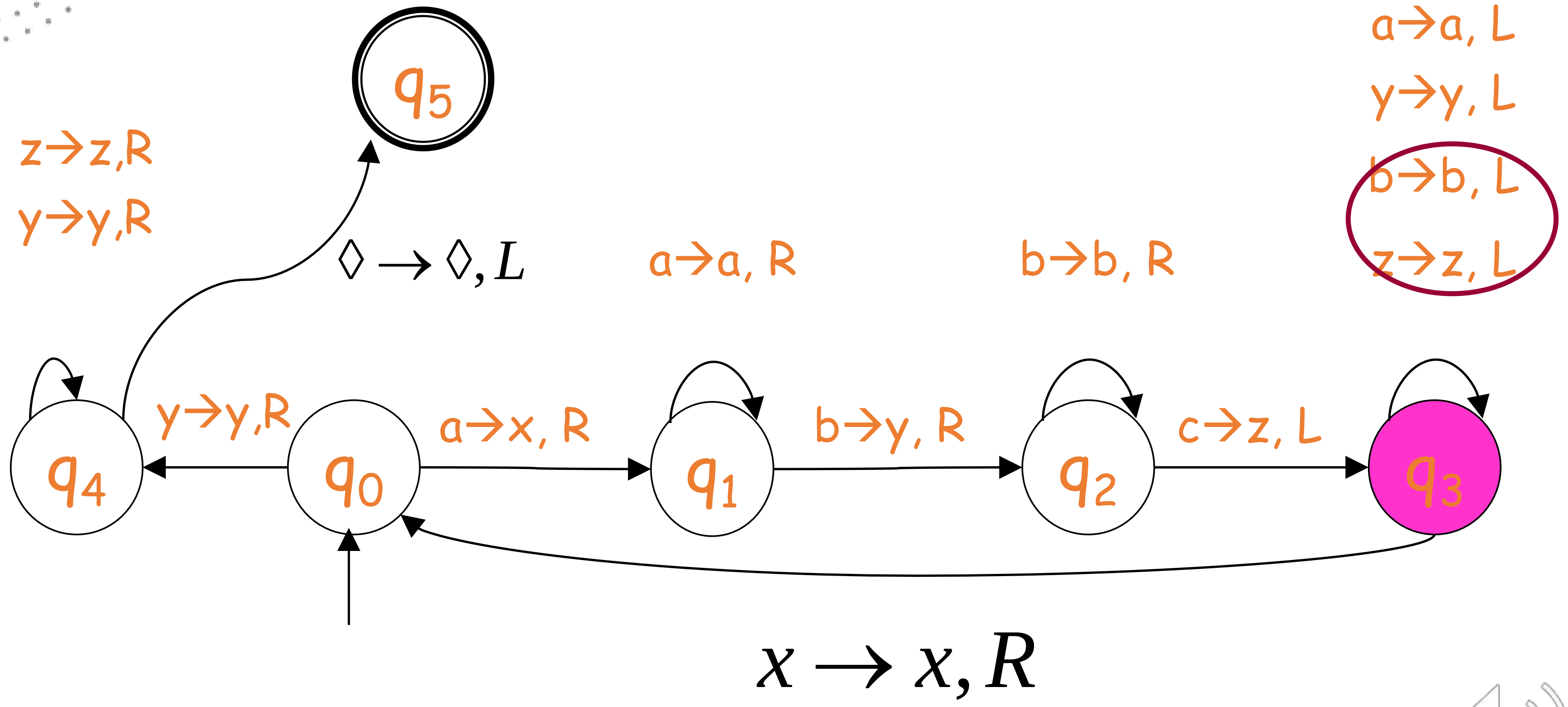
Time 5


◊
x
a
y
b
z
c
◊
◊
q3

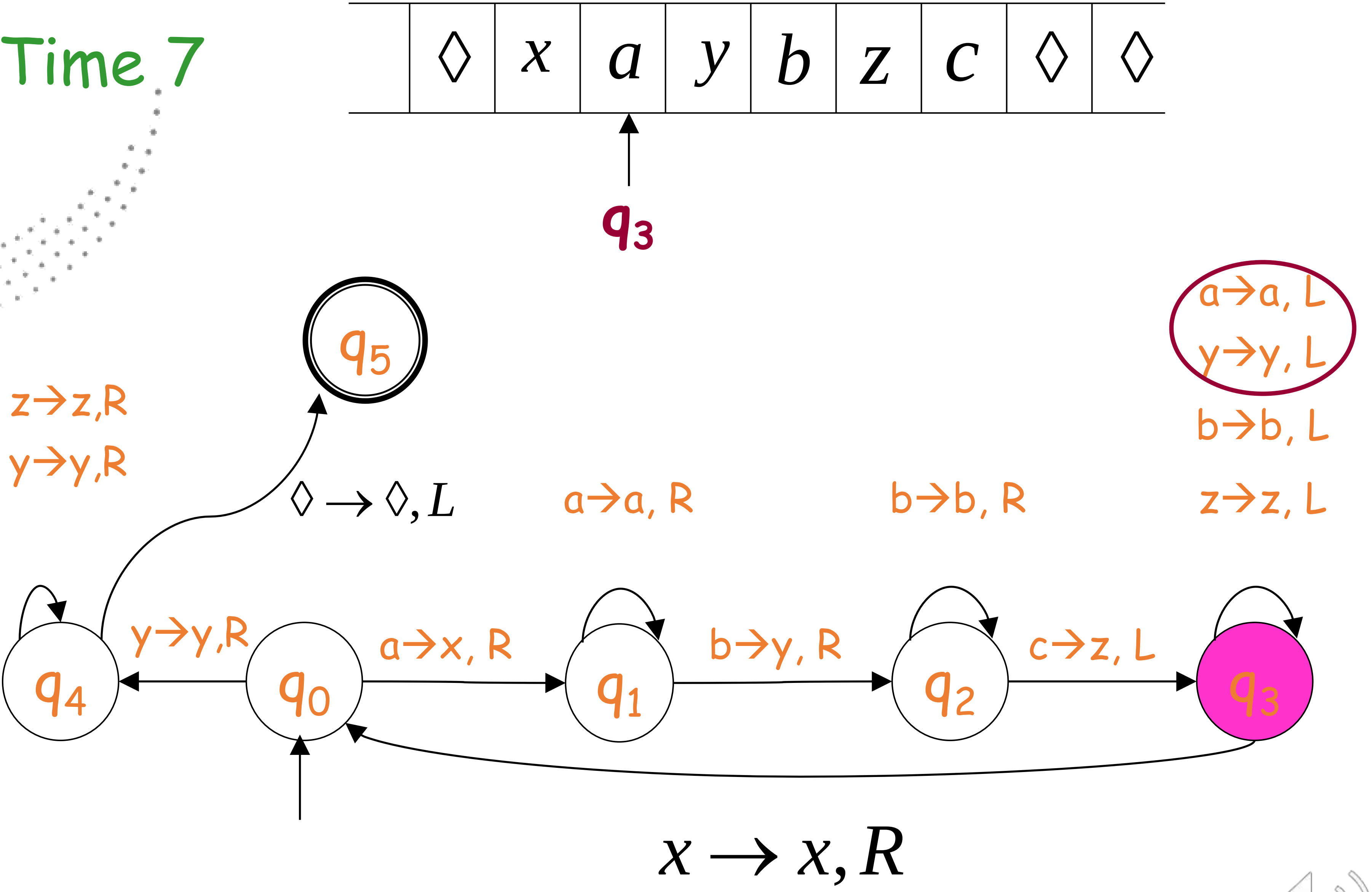

a→a, L
y→y, L
b→b, L
z→z, L
q5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

Time 6


◊
x
a
y
b
z
c
◊
◊
q₃

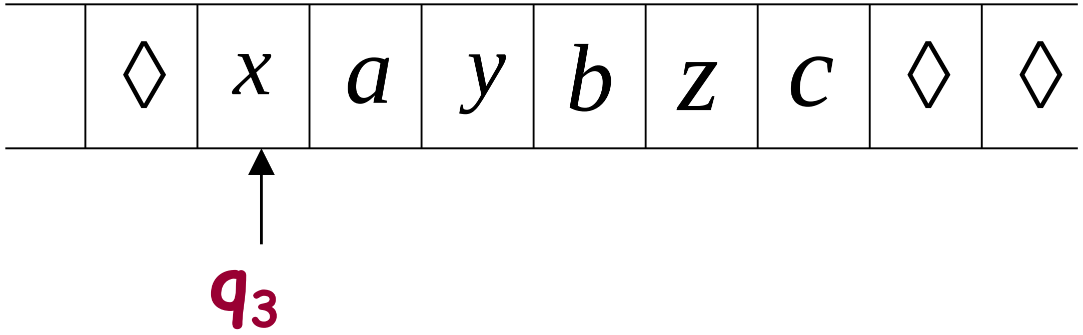

q₅
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q₄
q₀
q₁
q₂
q₃
x → x, R

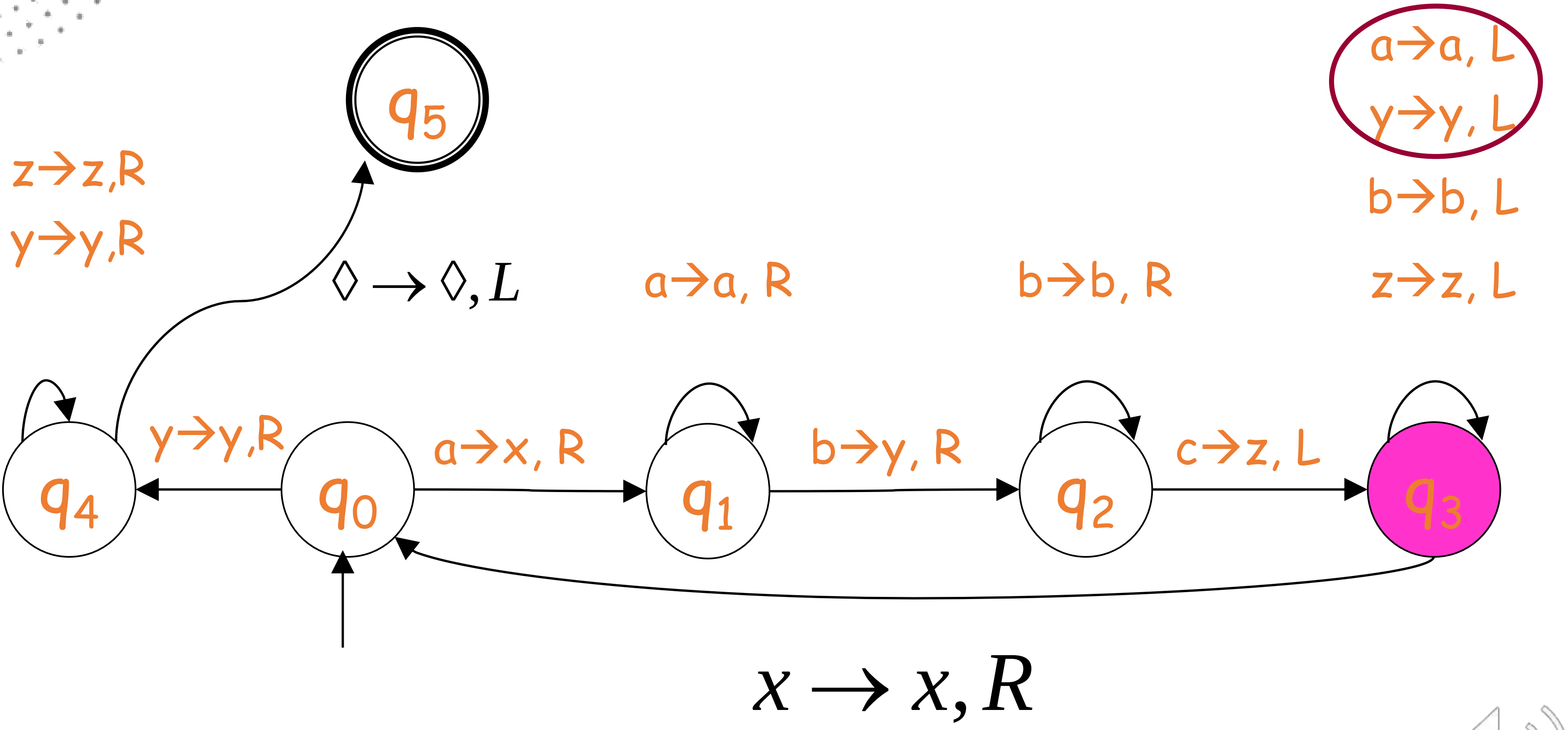
Time 7



	◊	x	a	y	b	z	c	◊	◊

Time 8

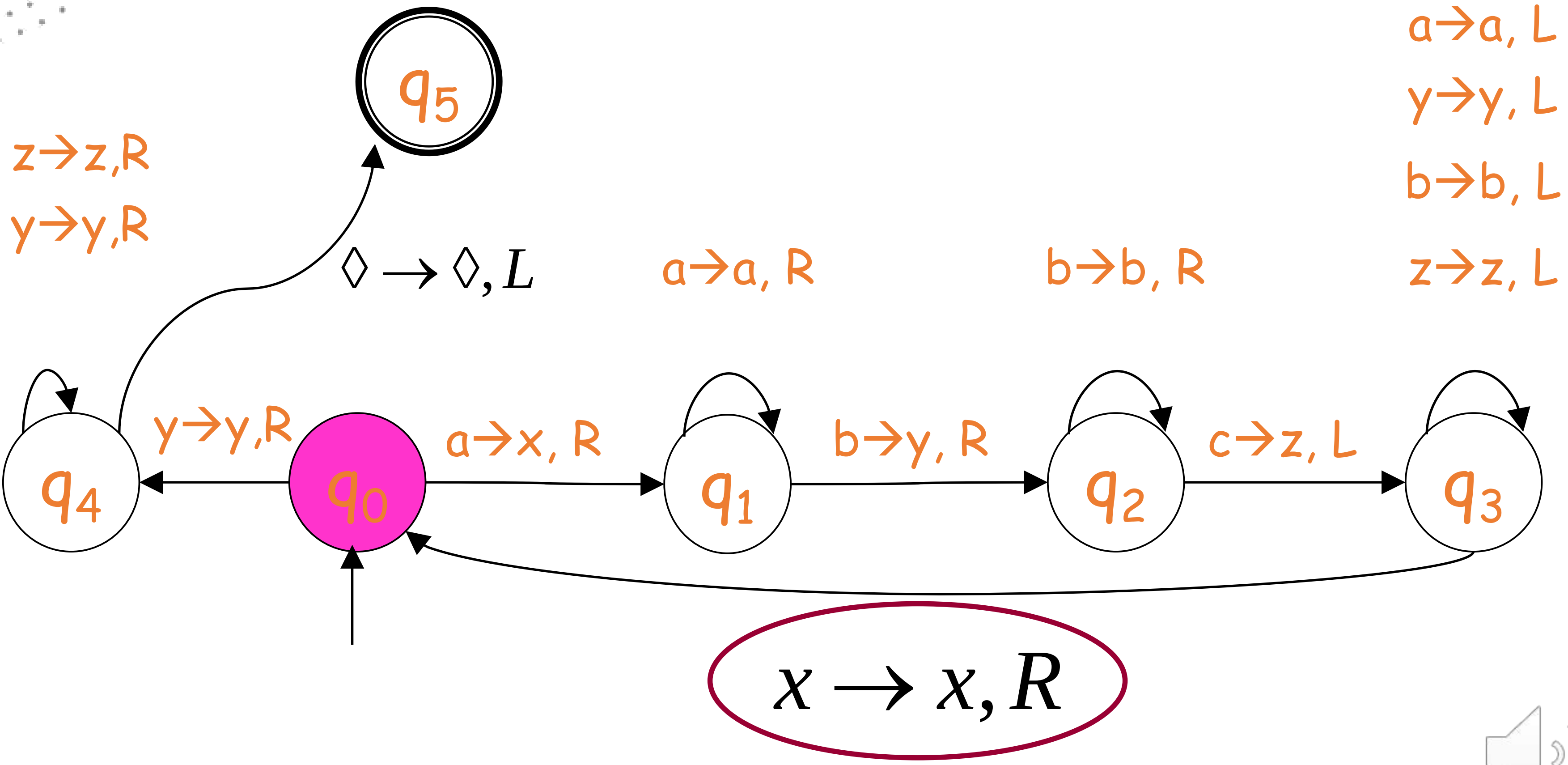

◊
x
a
y
b
z
c
◊
◊
q3


q5
z→z,R
y→y,R
◊ → ◊, L
a→a, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

Time 9

	◊	x	a	y	b	z	c	◊	◊

q_0

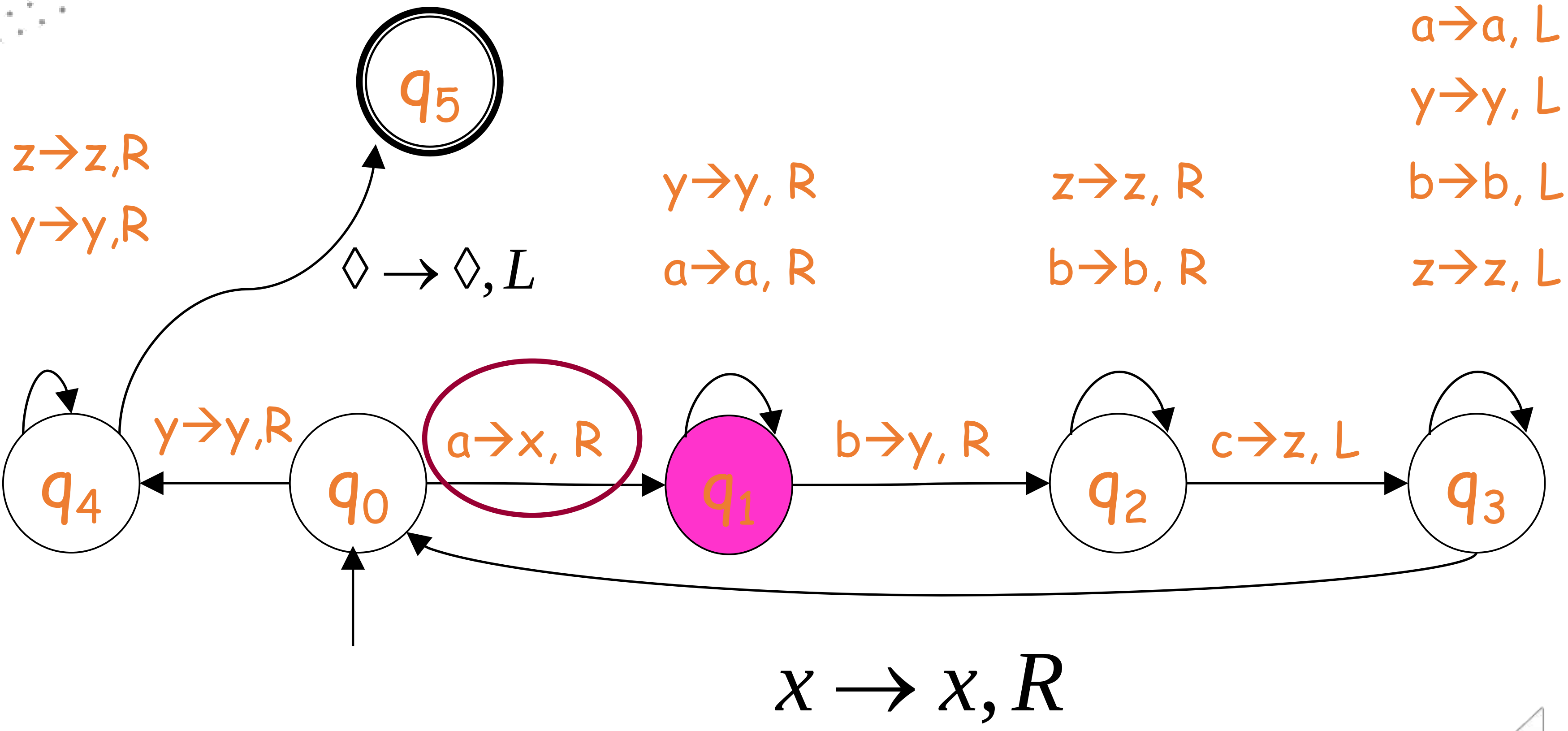


Time 10

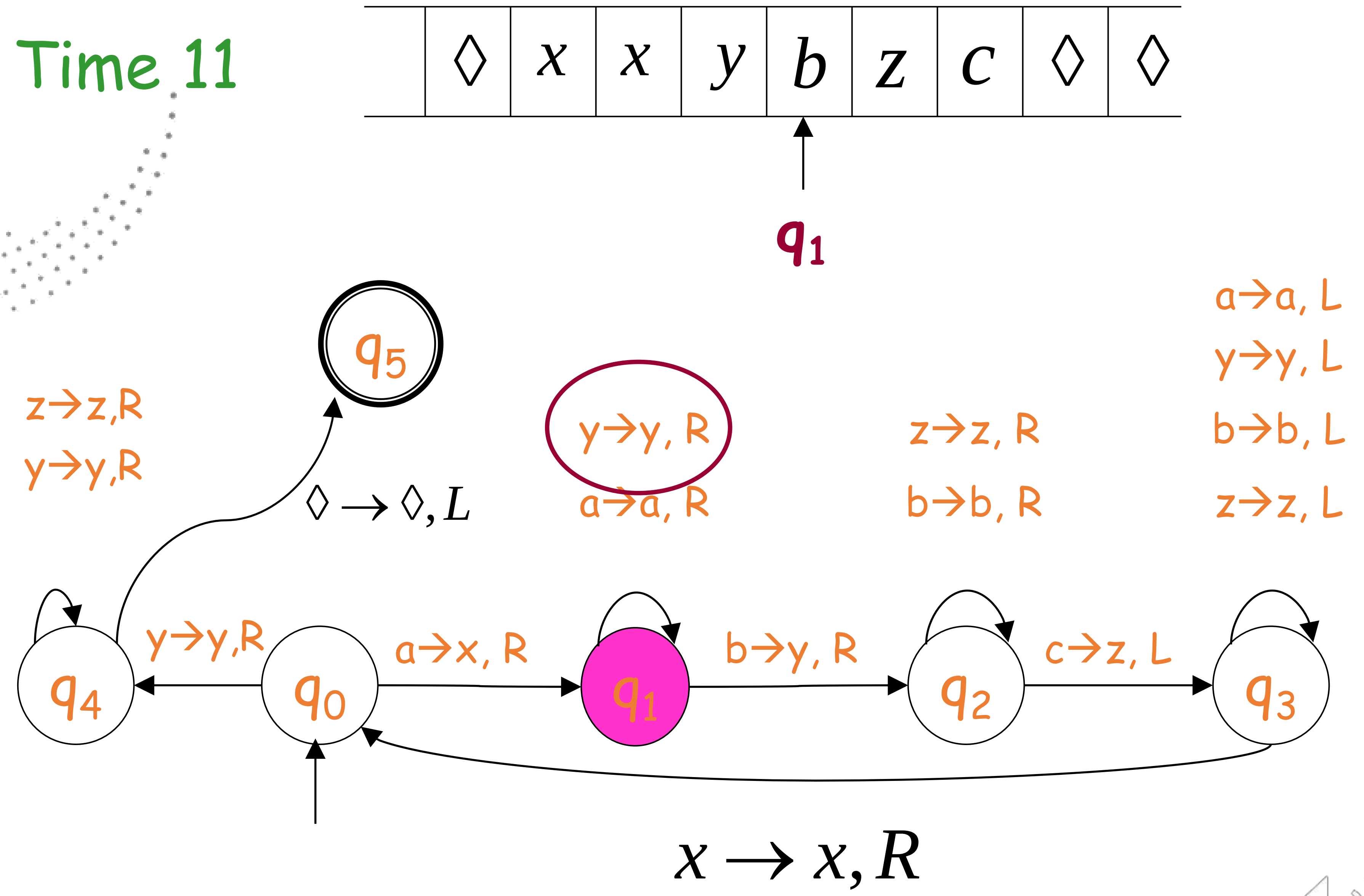
	◊	x	x	y	b	z	c	◊	◊

↑

q_1



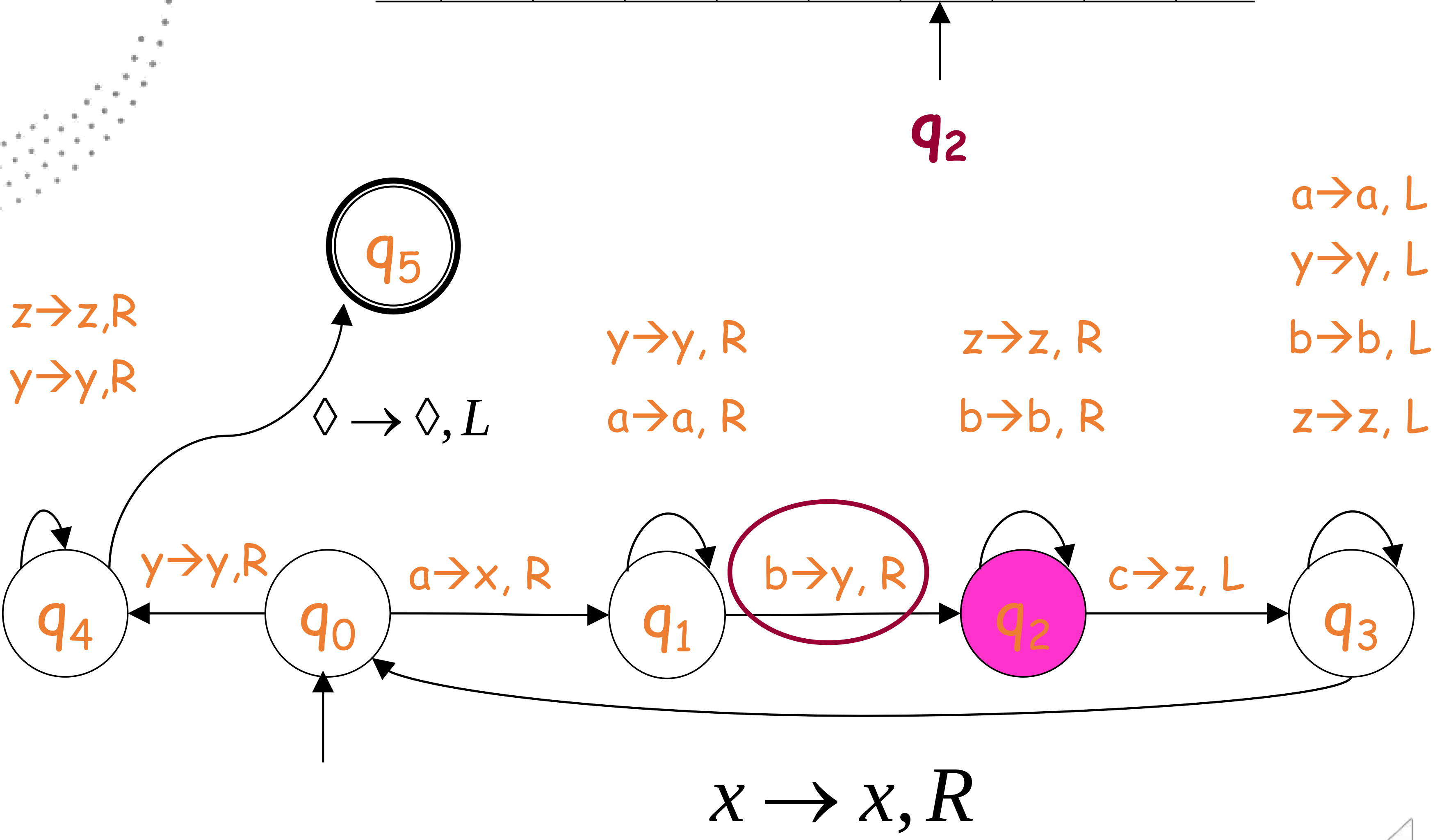
Time 11



◊
x
x
y
b
z
c
◊
◊
q1
q5
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

Time 12

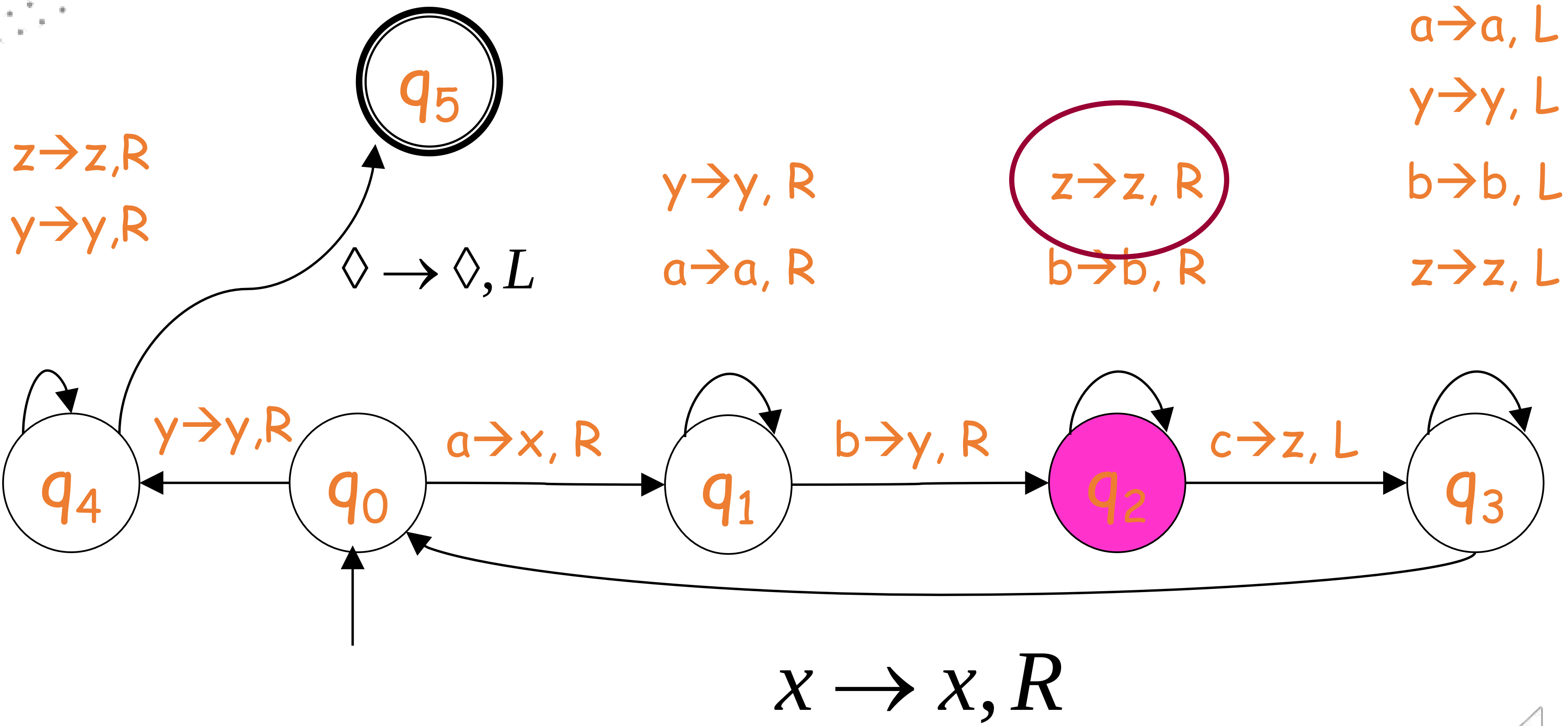
	◊	x	x	y	y	z	c	◊	◊



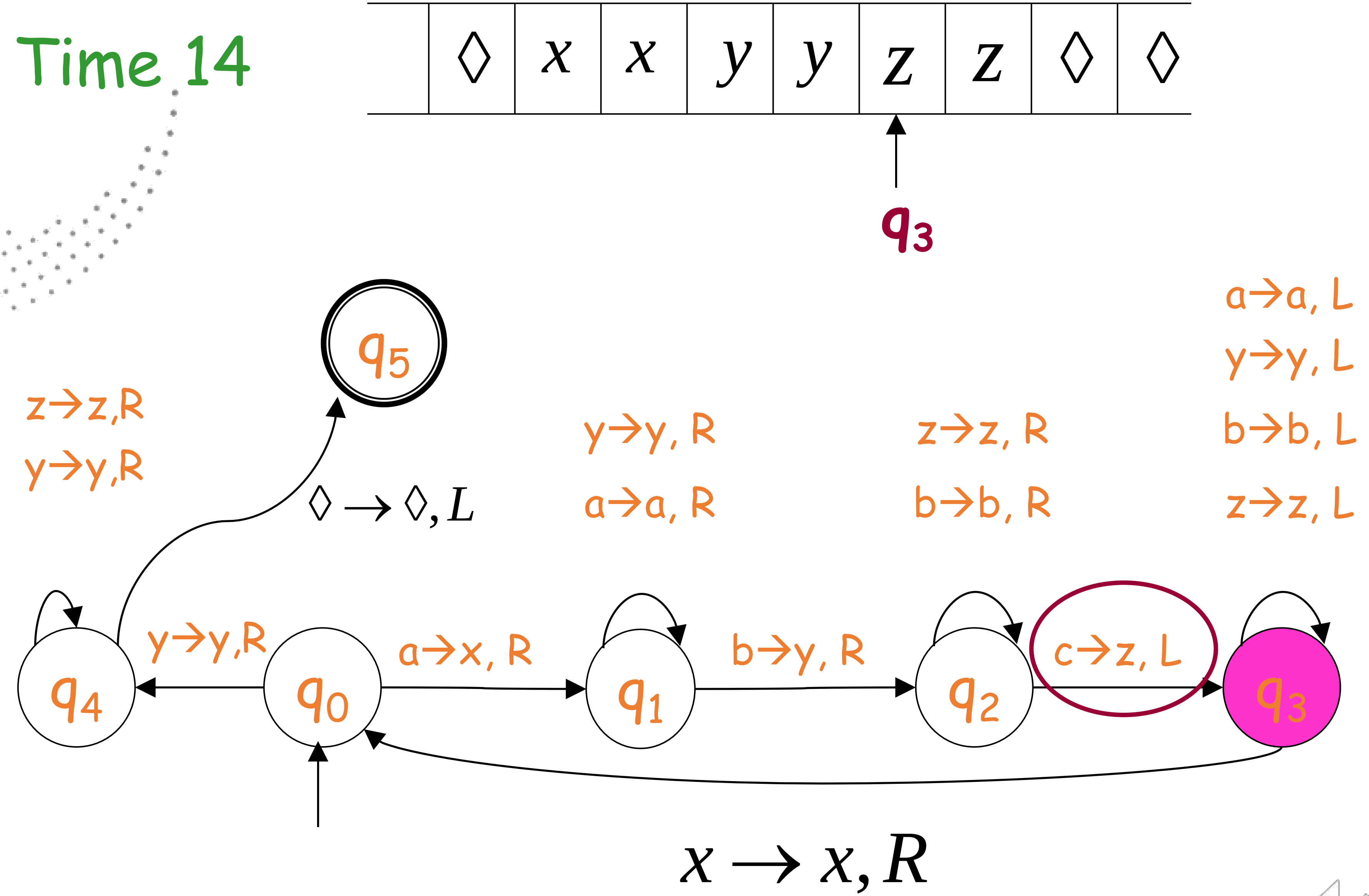
Time 13

	◊	x	x	y	y	z	c	◊	◊

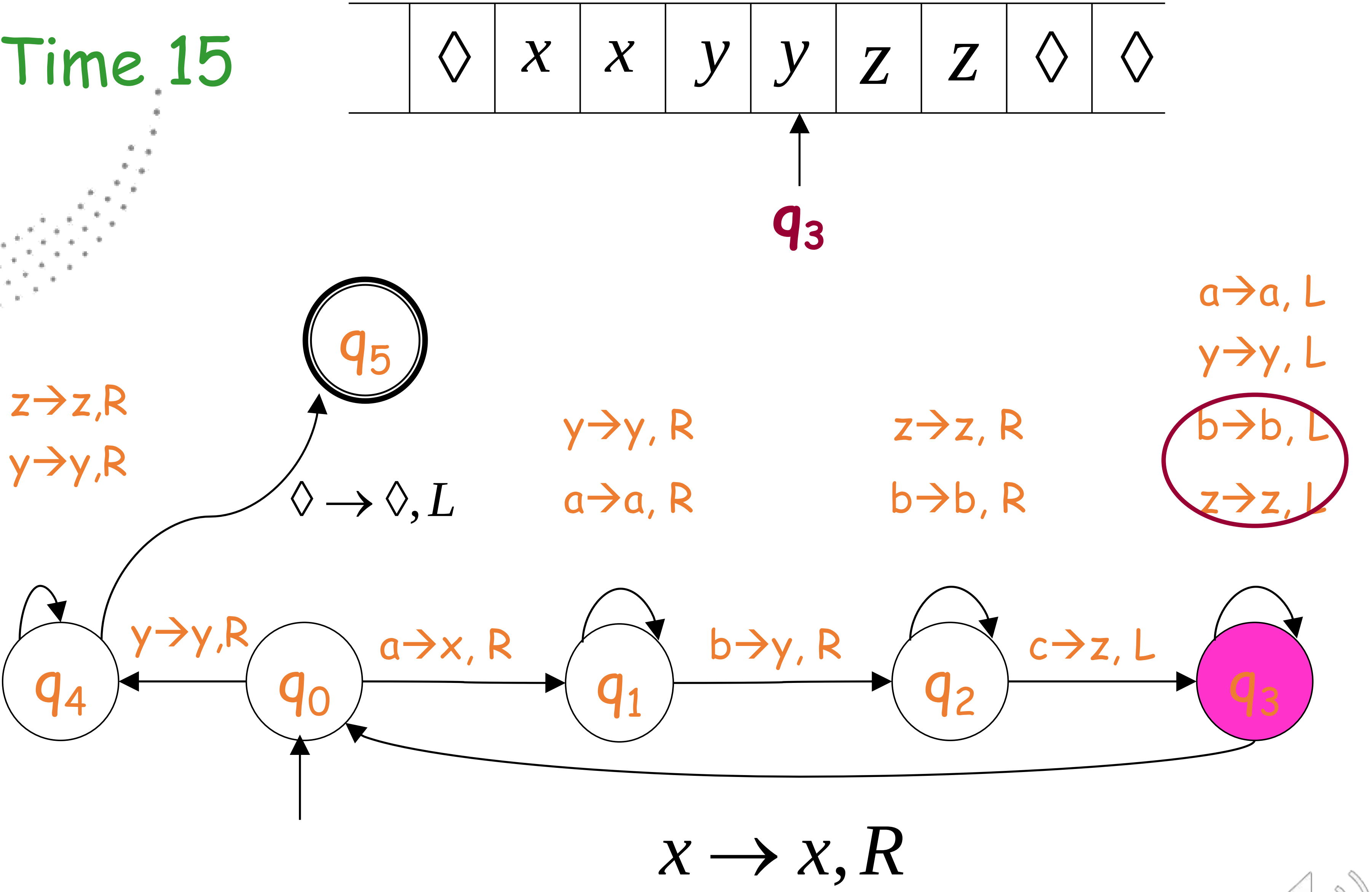
q_2



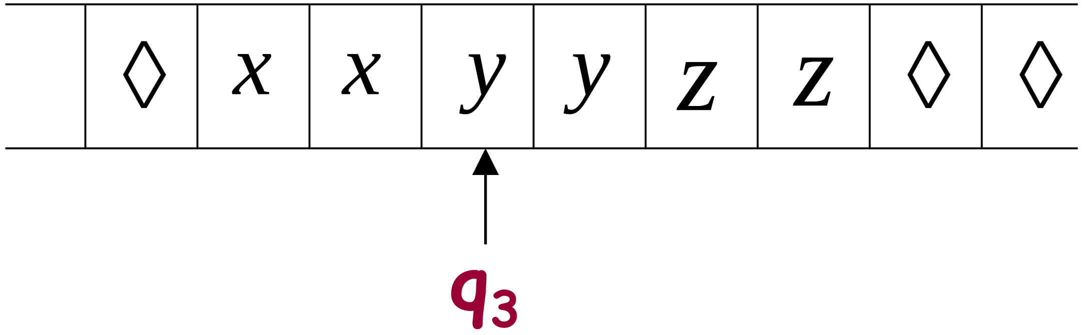
Time 14

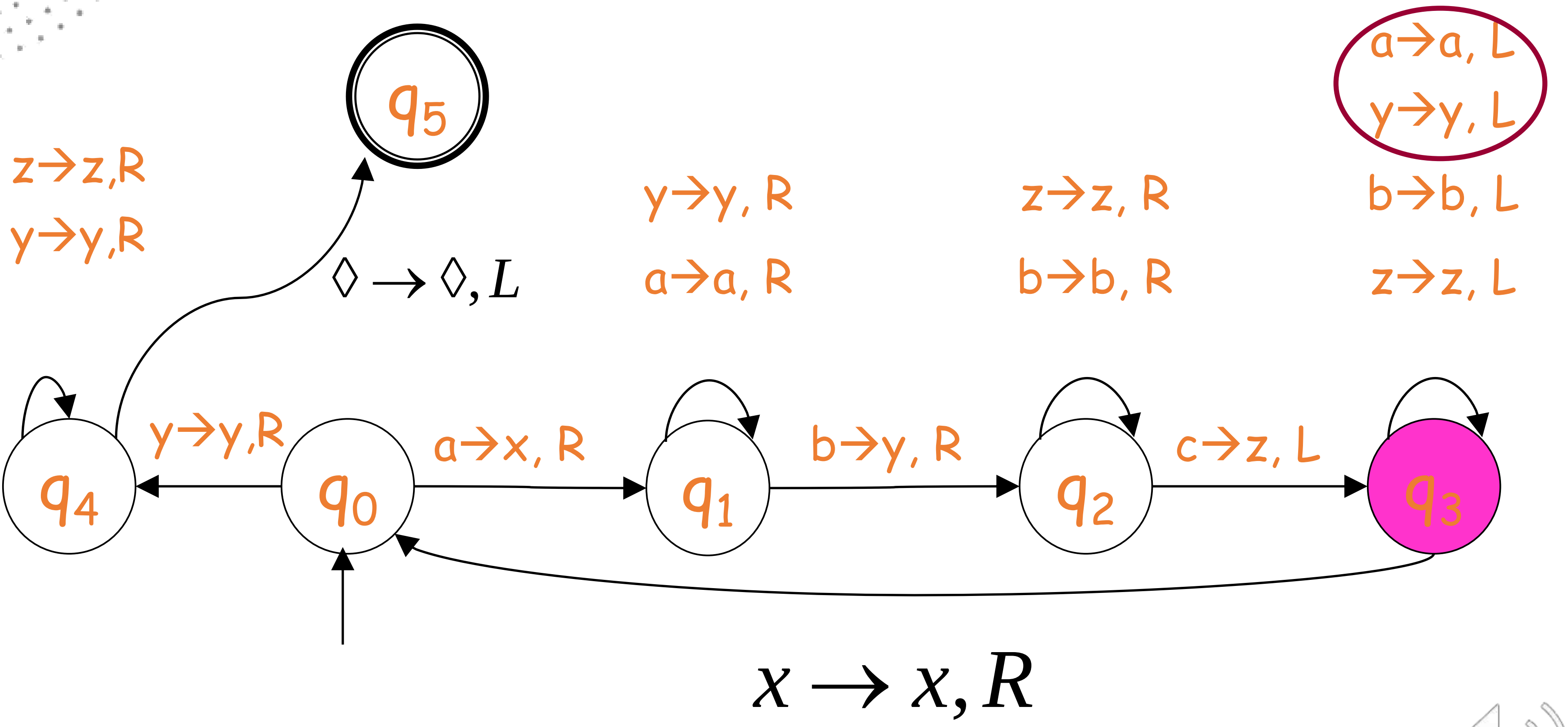

◊
x
x
y
y
z
z
◊
◊
q₃
q₅
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q₄
q₀
q₁
q₂
q₃
x → x, R

Time 15

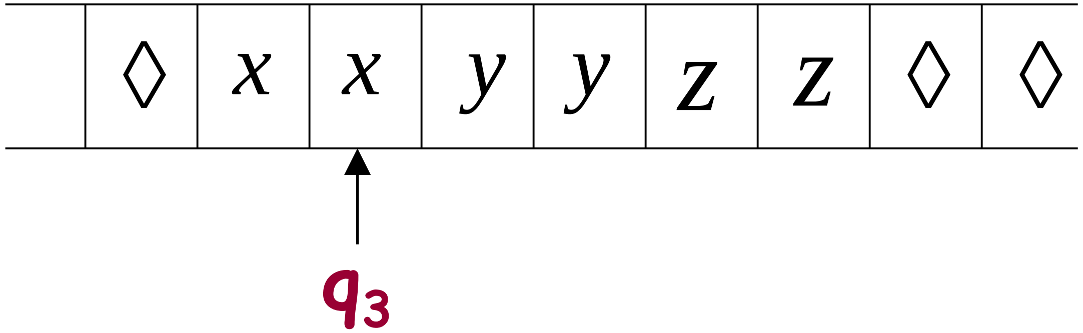

◊
x
x
y
y
z
z
◊
◊
q_3
q_5
z→z,R
y→y,R
$\Diamond \rightarrow \Diamond, L$
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q_4
q_0
q_1
q_2
q_3
$x \rightarrow x, R$

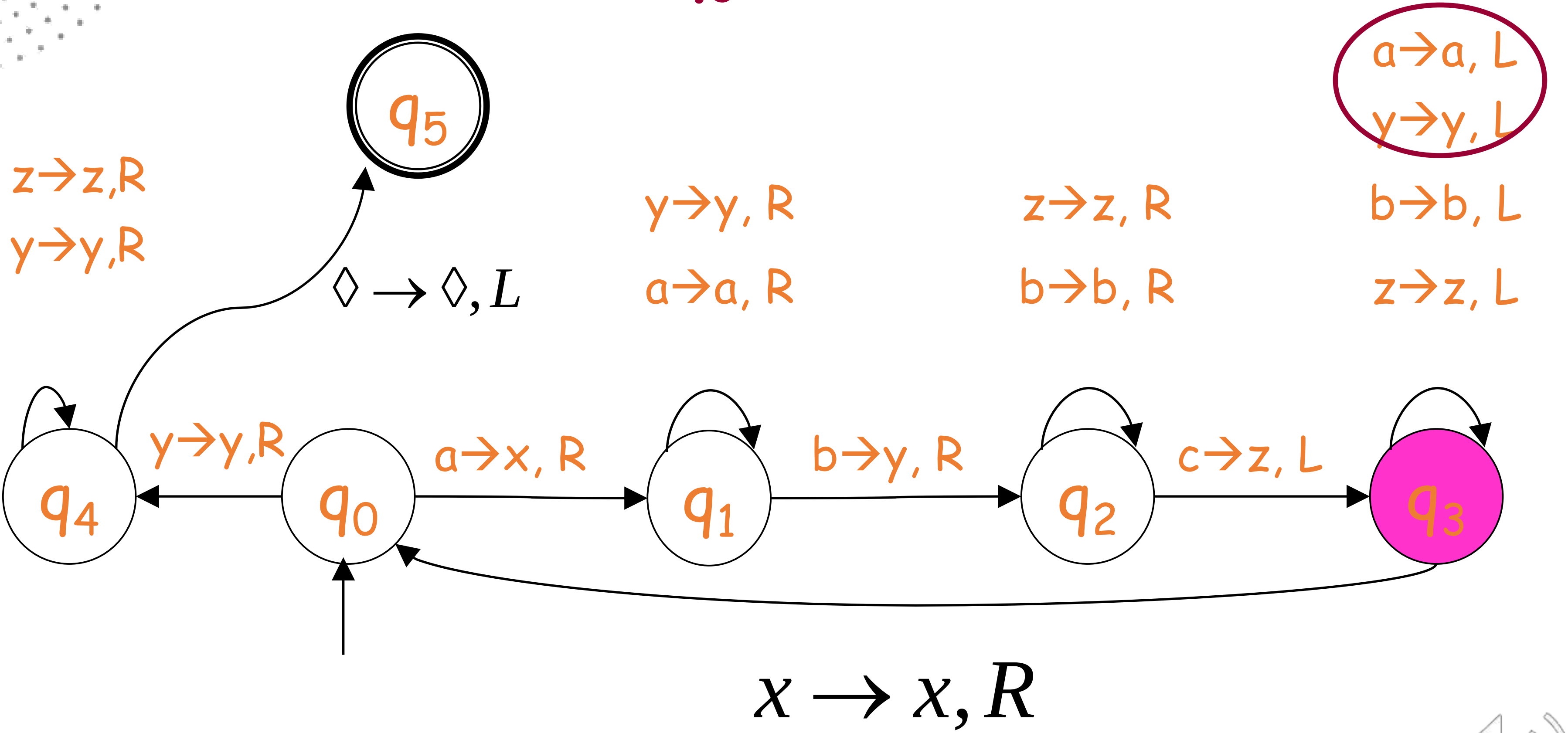
Time 16


◊
x
x
y
y
z
z
◊
◊
q_3

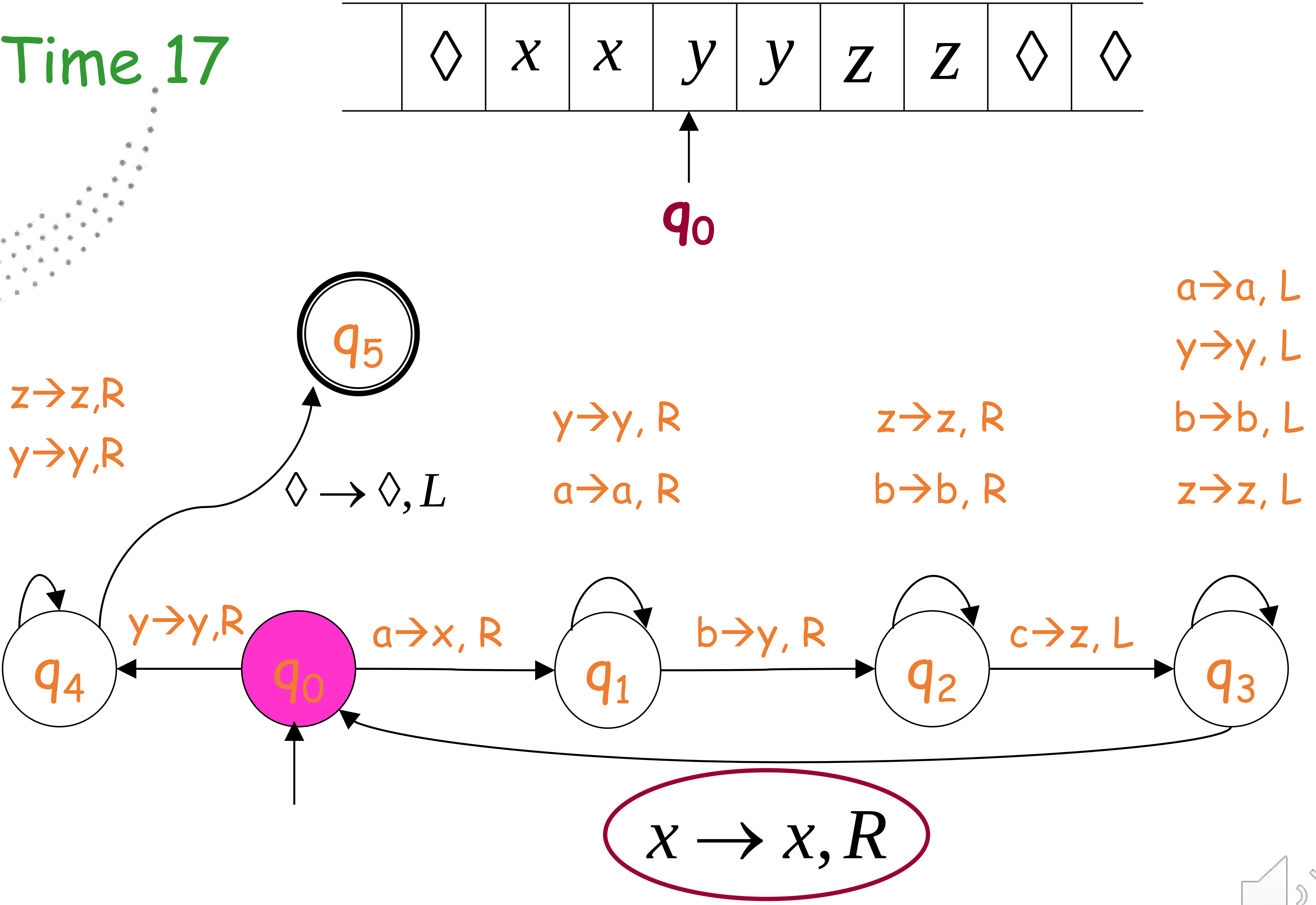

q_5
z→z,R
y→y,R
$\lozenge \to \lozenge, L$
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
q_4
y→y,R
q_0
a→x, R
q_1
b→y, R
q_2
c→z, L
q_3
$x \to x, R$

Time 16


◊
x
x
y
y
z
z
◊
◊
q3

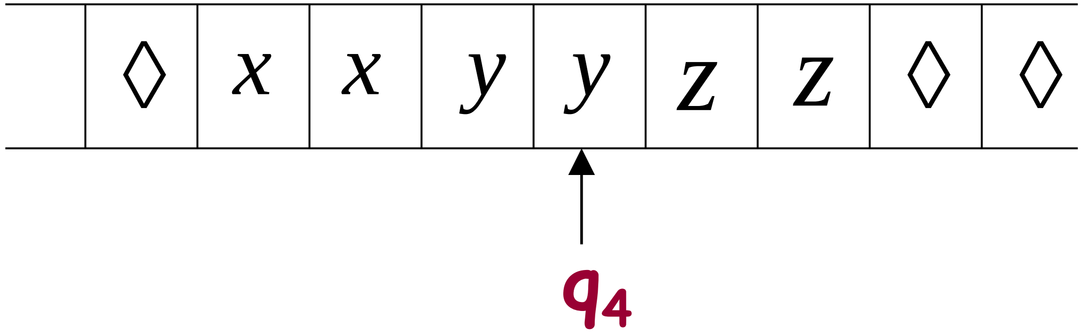

q5
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

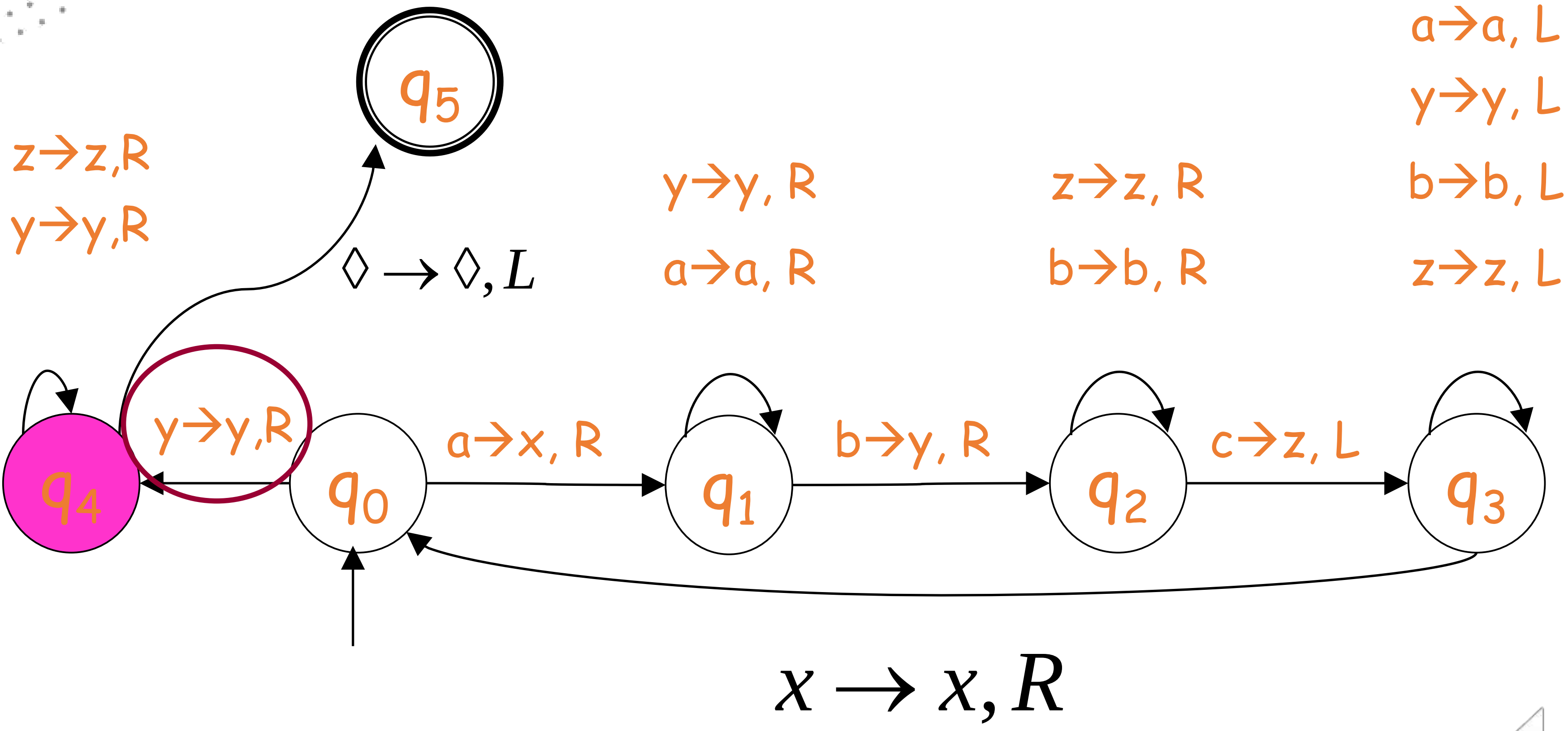
Time 17



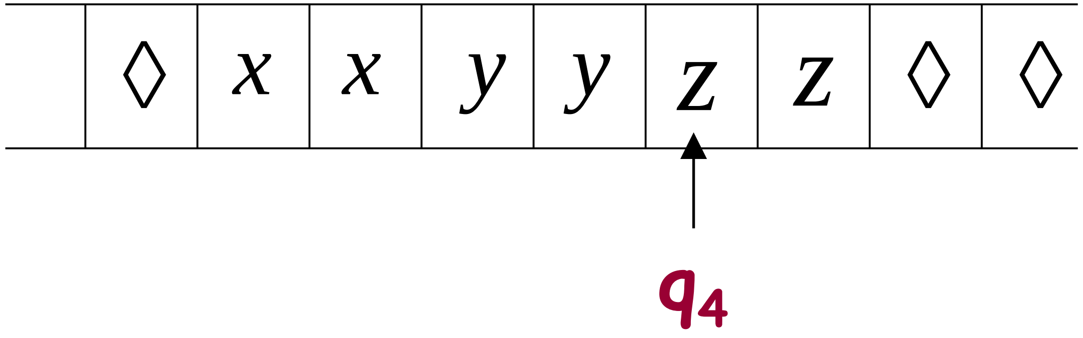
◊ x x y y z z ◊ ◊
q₀
q₅
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q₄
q₀
q₁
q₂
q₃
x → x, R

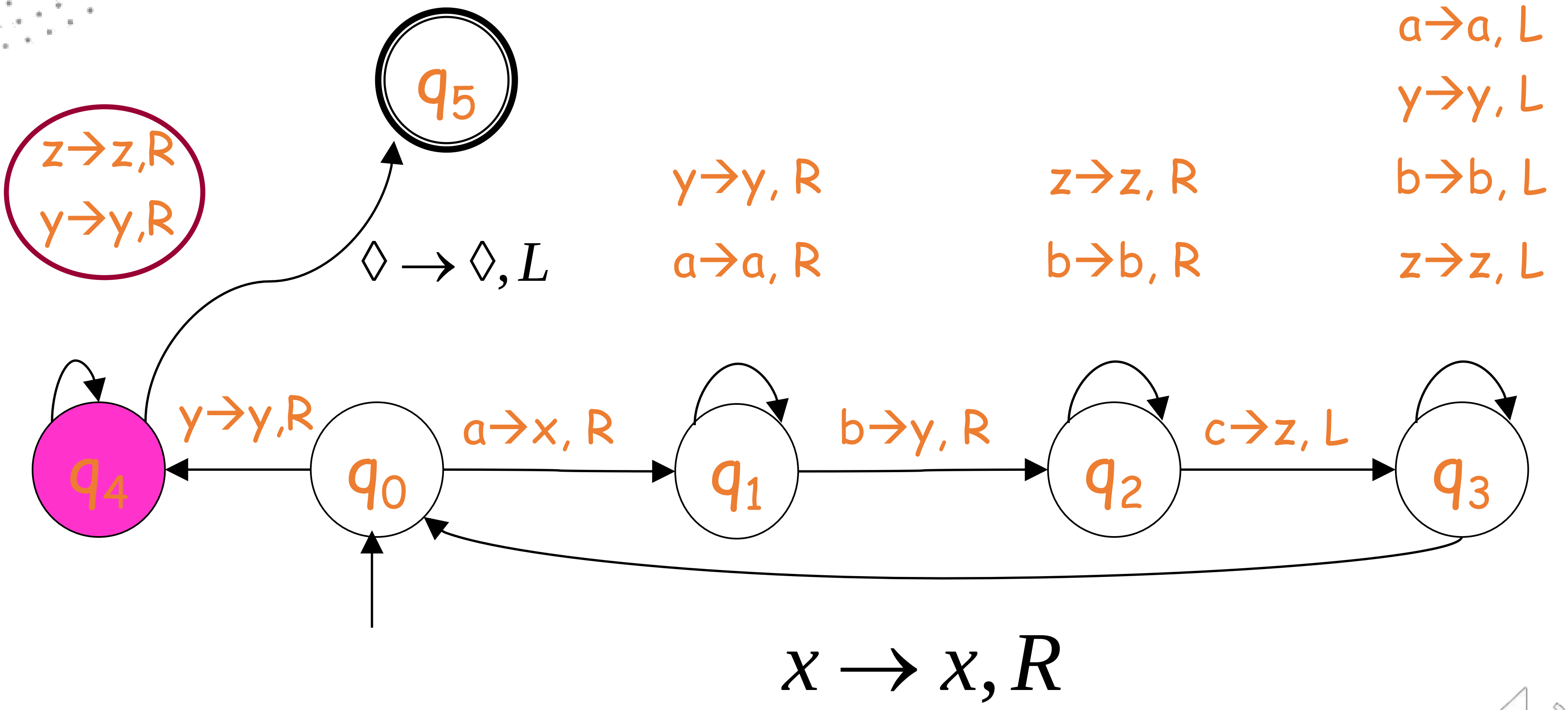
Time 18


◊
x
x
y
y
z
z
◊
◊
q4


q5
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
y→y,R
a→x, R
b→y, R
c→z, L
q4
q0
q1
q2
q3
x → x, R

Time 19

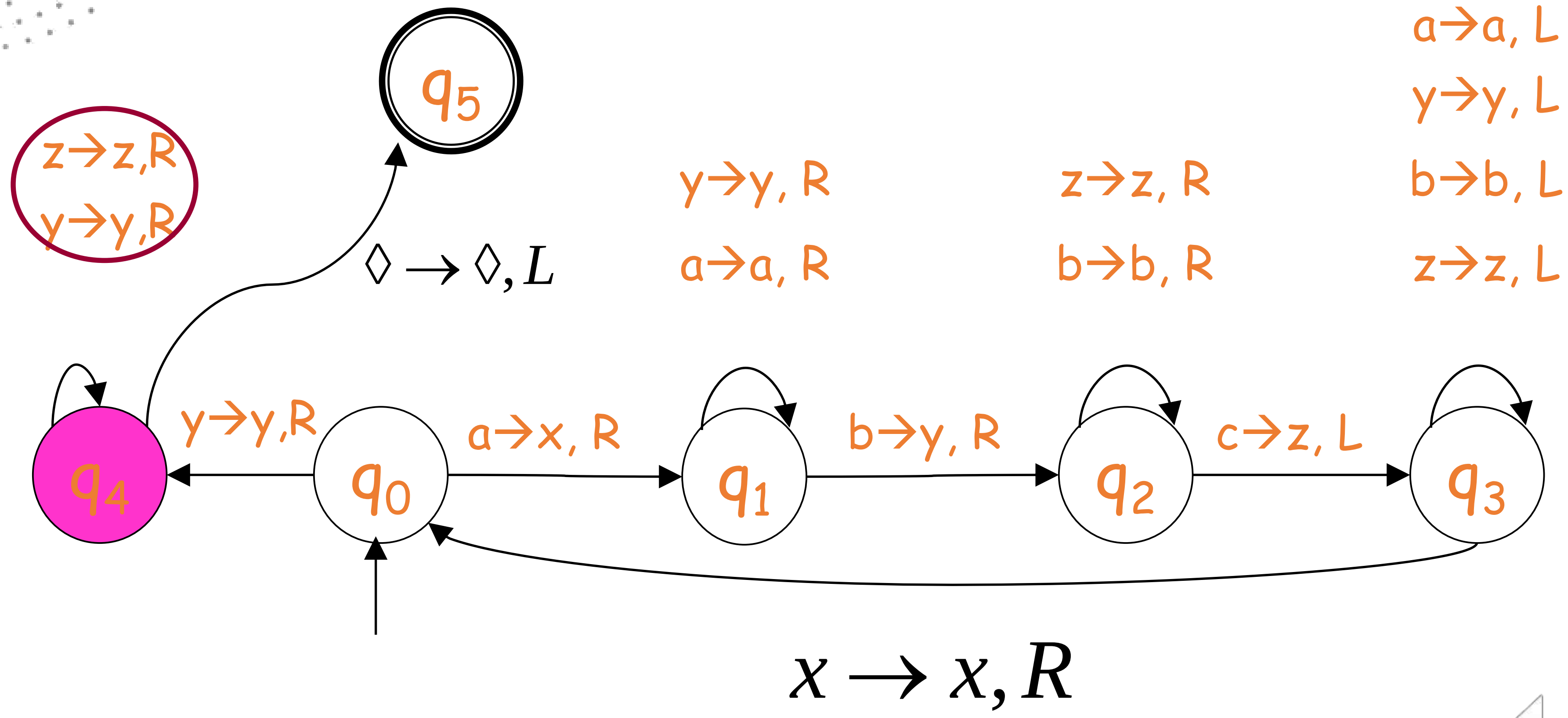

◊
x
x
y
y
z
z
◊
◊
q_4


q_5
z→z,R
y→y,R
$\lozenge \rightarrow \lozenge, L$
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
q_4
y→y,R
q_0
a→x, R
q_1
b→y, R
q_2
c→z, L
q_3
$x \rightarrow x, R$

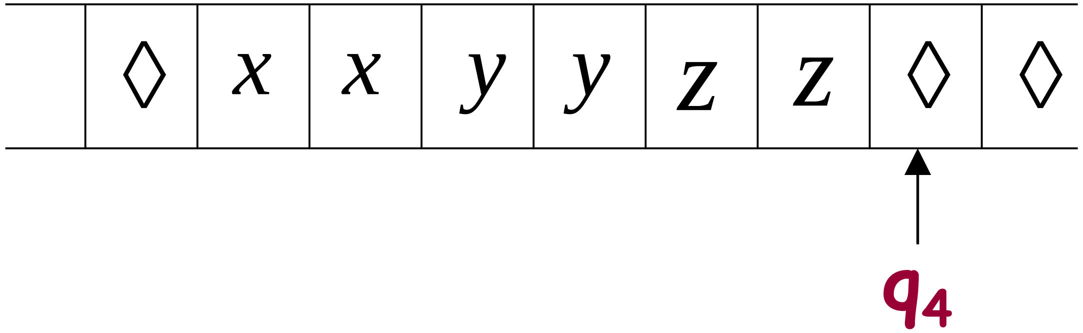
Time 20

	◊	x	x	y	y	z	z	◊	◊

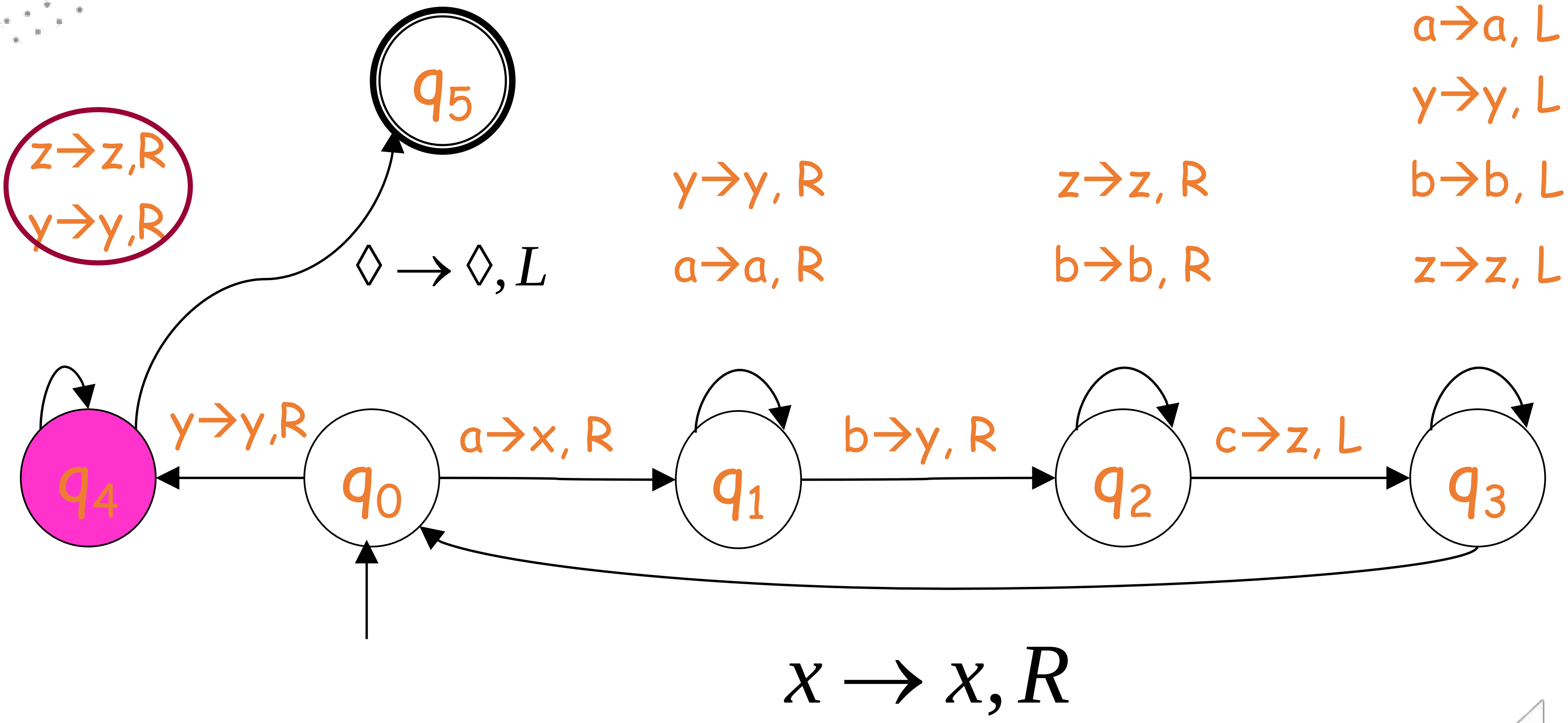
q4



Time 21

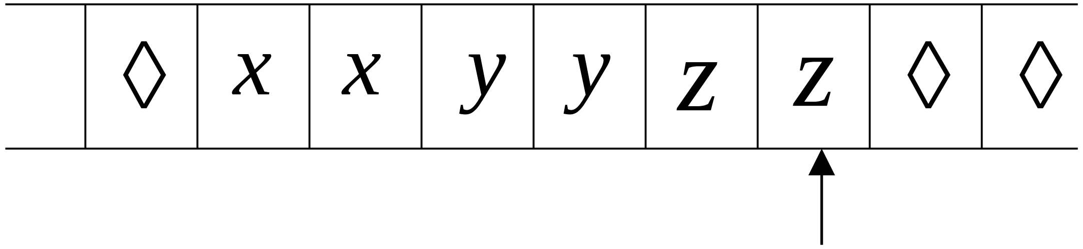


◊
x
x
y
y
z
z
◊
◊
q_4

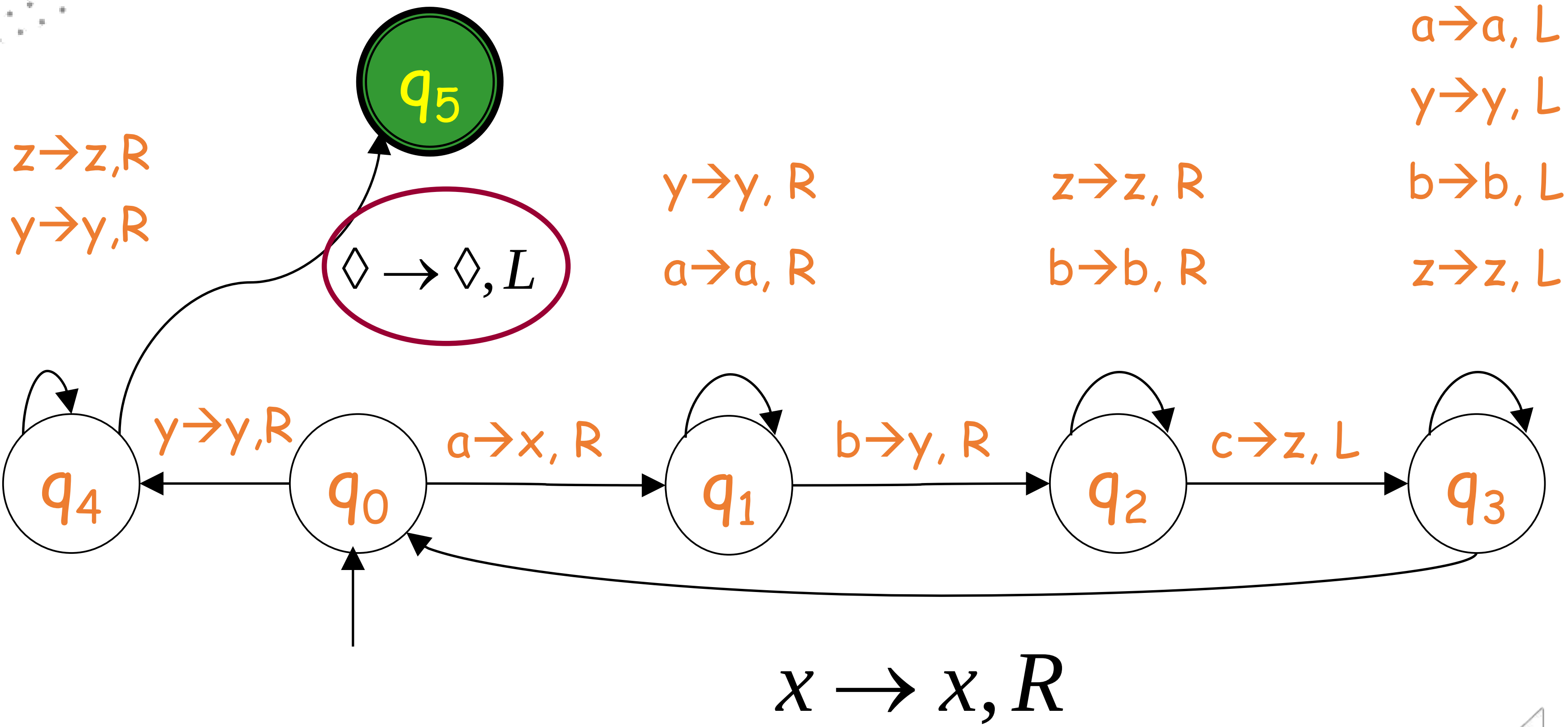


q_5
z→z,R
y→y,R
◊ → ◊, L
y→y, R
a→a, R
z→z, R
b→b, R
a→a, L
y→y, L
b→b, L
z→z, L
q_4
y→y,R
q_0
a→x, R
q_1
b→y, R
q_2
c→z, L
q_3
x → x, R

Time 21



"Accepted"



MCQ

1. What is a Turing Machine?
a) A type of computer hardware
b) A mathematical model of computation
c) A programming language
d) A networking protocol
2. Who introduced the concept of Turing Machines?
a) Alan Turing
b) Charles Babbage
c) Ada Lovelace
d) John von Neumann
3. What are the components of a Turing Machine?
a) Input tape, output tape, and control unit
b) Input tape, output tape, and memory
c) Input tape, work tape, and control unit
d) Input tape, work tape, and memory

TERMINAL QUESTIONS

1. Question: Explain the concept of universality in relation to Turing Machines. How does it enable Turing Machines to solve any computational problem?
2. Question: Discuss the significance of the halting problem in the context of Turing Machines. Why is it impossible to design a Turing Machine that can determine whether another Turing Machine halts or not?
3. Question: How can a Turing Machine simulate the behavior of a computer program? Discuss the necessary steps and components involved in this simulation process.
4. Question: Explain the concept of time complexity in relation to Turing Machines. How is it measured, and why is it important for analyzing the efficiency of algorithms?
5. Question: Discuss the concept of undecidability in relation to Turing Machines. Provide an example of an undecidable problem and explain why it cannot be solved by any Turing Machine.
6. Question: Can a Turing Machine compute irrational numbers such as π or √2? Explain why or why not, considering the limitations of the Turing Machine model.

THANK YOU

Team – TOC