MP Home Assignment -2 CO-2

1. Discrete Optimization using Cutting Plane method Solve the integer programming problem Maximize: Z=3x1+x2+3x3 Subject to -x1+2x2+x3≤4 2x2-3/2 x3≤1 x1-3x2+2x3≤3 Where x1,x2,x3≥0 and integer. Get the optimal solution as an integer value using Gomory's cutting plane method.

Solution:
Problem is

$$\text{Max } Z = 3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 \le 4$$

$$2x_2 - \frac{3}{2}x_3 \le 1$$

$$x_1 - 3x_2 + 2x_3 \le 3$$

and $x_1, x_2, x_3 \ge 0$; x_1, x_2, x_3 non-negative integers

After introducing slack variables

$$\text{Max } Z = 3x_1 + x_2 + 3x_3 + 0S_1 + 0S_2 + 0S_3$$

subject to

$$-x_1 + 2x_2 + x_3 + S_1 = 4$$

$$2x_2 - \frac{3}{2}x_3 + S_2 = 1$$

$$x_1 - 3x_2 + 2x_3 + S_3 = 3$$

and $x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

Iteration-1		C_j	3	1	3	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_1}$
S_1	0	4	-1	2	1	1	0	0	---
S_2	0	1	0	2	$-\frac{3}{2}$	0	1	0	---
S_3	0	3	(1)	-3	2	0	0	1	$\frac{3}{1} = 3 \rightarrow$
$Z = 0$		Z_j	0	0	0	0	0	0	
		$C_j - Z_j$	$3 \uparrow$	1	3	0	0	0	

$\therefore$ The pivot element is 1.

Entering $= x_1$, Departing $= S_3$, Key Element $= 1$

- $R_3(\text{new}) = R_3(\text{old})$

$R_3(\text{old}) =$	3	1	-3	2	0	0	1
$R_3(\text{new}) = R_3(\text{old})$	3	1	-3	2	0	0	1

- $R_1(\text{new}) = R_1(\text{old}) + R_3(\text{new})$

$R_1(\text{old}) =$	4	-1	2	1	1	0	0
$R_3(\text{new}) =$	3	1	-3	2	0	0	1
$R_1(\text{new}) = R_1(\text{old}) + R_3(\text{new})$	7	0	-1	3	1	0	1

- $R_2(\text{new}) = R_2(\text{old})$

$R_2(\text{old}) =$	1	0	2	$-\frac{3}{2}$	0	1	0
$R_2(\text{new}) = R_2(\text{old})$	1	0	2	$-\frac{3}{2}$	0	1	0

Iteration-2		C_j	3	1	3	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	**MinRatio** $\frac{X_B}{x_2}$
S_1	0	7	0	-1	3	1	0	1	---
S_2	0	1	0	(2)	$-\frac{3}{2}$	0	1	0	$\frac{1}{2} = 0.5 \rightarrow$
x_1	3	3	1	-3	2	0	0	1	---
$Z = 9$		Z_j	3	-9	6	0	0	3	
		$C_j - Z_j$	0	$10 \uparrow$	-3	0	0	-3	

∴ The pivot element is 2.

Entering $= x_2$, Departing $= S_2$, Key Element $= 2$

$R_2(\text{new}) = R_2(\text{old}) \div 2$

$R_2(\text{old}) =$	1	0	2	$-\frac{3}{2}$	0	1	0
$R_2(\text{new}) = R_2(\text{old}) \div 2$	$\frac{1}{2}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{2}$	0

$R_1(\text{new}) = R_1(\text{old}) + R_2(\text{new})$

$R_1(\text{old}) =$	7	0	-1	3	1	0	1
$R_2(\text{new}) =$	$\frac{1}{2}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{2}$	0
$R_1(\text{new}) = R_1(\text{old}) + R_2(\text{new})$	$\frac{15}{2}$	0	0	$\frac{9}{4}$	1	$\frac{1}{2}$	1

$R_3(\text{new}) = R_3(\text{old}) + 3R_2(\text{new})$

$R_3(\text{old}) =$	3	1	-3	2	0	0	1
$R_2(\text{new}) =$	$\frac{1}{2}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{2}$	0
$3 \times R_2(\text{new}) =$	$\frac{3}{2}$	0	3	$-\frac{9}{4}$	0	$\frac{3}{2}$	0
$R_3(\text{new}) = R_3(\text{old}) + 3R_2(\text{new})$	$\frac{9}{2}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	1

Iteration-3		C_j	3	1	3	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	MinRatio $\frac{X_B}{x_3}$
S_1	0	$\frac{15}{2}$	0	0	$\left(\frac{9}{4}\right)$	1	$\frac{1}{2}$	1	$\frac{\frac{15}{2}}{\frac{9}{4}} = \frac{10}{3} = 3.3333 \rightarrow$
x_2	1	$\frac{1}{2}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{2}$	0	---
x_1	3	$\frac{9}{2}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	1	---
$Z = 14$		Z_j	3	1	$-\frac{3}{2}$	0	5	3	
		$C_j - Z_j$	0	0	$\frac{9}{2} \uparrow$	0	-5	-3	

$\therefore$ The pivot element is $\frac{9}{4}$.

Entering $= x_3$, Departing $= S_1$, Key Element $= \frac{9}{4}$

$R_1(\text{new}) = R_1(\text{old}) \times \frac{4}{9}$

$R_1(\text{old}) =$	$\frac{15}{2}$	0	0	$\frac{9}{4}$	1	$\frac{1}{2}$	1
$R_1(\text{new}) = R_1(\text{old}) \times \frac{4}{9}$	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$

$R_2(\text{new}) = R_2(\text{old}) + \frac{3}{4} R_1(\text{new})$

$R_2(\text{old}) =$	$\frac{1}{2}$	0	1	$-\frac{3}{4}$	0	$\frac{1}{2}$	0
$R_1(\text{new}) =$	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$
$\frac{3}{4} \times R_1(\text{new}) =$	$\frac{5}{2}$	0	0	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$
$R_2(\text{new}) = R_2(\text{old}) + \frac{3}{4} R_1(\text{new})$	3	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

$R_3(\text{new}) = R_3(\text{old}) + \frac{1}{4} R_1(\text{new})$

$R_3(\text{old}) =$	$\frac{9}{2}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	1
$R_1(\text{new}) =$	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$
$\frac{1}{4} \times R_1(\text{new}) =$	$\frac{5}{6}$	0	0	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{9}$
$R_3(\text{new}) = R_3(\text{old}) + \frac{1}{4} R_1(\text{new})$	$\frac{16}{3}$	1	0	0	$\frac{1}{9}$	$\frac{14}{9}$	$\frac{10}{9}$

Iteration-4		C_j	3	1	3	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	MinRatio
x_3	3	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	
x_2	1	3	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	
x_1	3	$\frac{16}{3}$	1	0	0	$\frac{1}{9}$	$\frac{14}{9}$	$\frac{10}{9}$	
$Z = 29$		Z_j	**3**	**1**	**3**	**2**	**6**	**5**	
		$C_j - Z_j$	0	0	0	-2	-6	-5	

Since all $C_j - Z_j \leq 0$

Hence, non-integer optimal solution is arrived with value of variables as :

$x_1 = \frac{16}{3}, x_2 = 3, x_3 = \frac{10}{3}$

Max $Z = 29$

To obtain the integer valued solution, we proceed to construct Gomory's fractional cut, with the help of x_3-row as follows:

$$\frac{10}{3} = 1x_3 + \frac{4}{9}S_1 + \frac{2}{9}S_2 + \frac{4}{9}S_3$$

$$\left(3 + \frac{1}{3}\right) = (1 + 0)x_3 + \left(0 + \frac{4}{9}\right)S_1 + \left(0 + \frac{2}{9}\right)S_2 + \left(0 + \frac{4}{9}\right)S_3$$

The fractional cut will become

$-\frac{1}{3} = Sg1 - \frac{4}{9}S_1 - \frac{2}{9}S_2 - \frac{4}{9}S_3 \rightarrow$ (Cut-1)

Iteration-1		C_j	3	1	3	0	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	$Sg1$
x_3	3	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	0
x_2	1	3	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
x_1	3	$\frac{16}{3}$	1	0	0	$\frac{1}{9}$	$\frac{14}{9}$	$\frac{10}{9}$	0
$Sg1$	0	$-\frac{1}{3}$	0	0	0	$\left(-\frac{4}{9}\right)$	$-\frac{2}{9}$	$-\frac{4}{9}$	1
$Z = 29$		Z_j	**3**	**1**	**3**	**2**	**6**	**5**	**0**
		$C_j - Z_j$	0	0	0	-2	-6	-5	0
		Ratio $= \frac{C_j - Z_j}{Sg1,j}$ and $Sg1,j < 0$	---	---	---	4.5 ↑	27	11.25	---

$\therefore$ The pivot element is $-\frac{4}{9}$.

Entering $= S_1$, Departing $= Sg1$, Key Element $= -\frac{4}{9}$

$R_4(\text{new}) = R_4(\text{old}) \times \left(-\frac{9}{4}\right)$

$R_4(\text{old}) =$	$-\frac{1}{3}$	0	0	0	$-\frac{4}{9}$	$-\frac{2}{9}$	$-\frac{4}{9}$	1
$R_4(\text{new}) = R_4(\text{old}) \times \left(-\frac{9}{4}\right)$	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$

$R_1(\text{new}) = R_1(\text{old}) - \frac{4}{9}R_4(\text{new})$

$R_1(\text{old}) =$	$\frac{10}{3}$	0	0	1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	0
$R_4(\text{new}) =$	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$
$\frac{4}{9} \times R_4(\text{new}) =$	$\frac{1}{3}$	0	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	-1
$R_1(\text{new}) = R_1(\text{old}) - \frac{4}{9}R_4(\text{new})$	3	0	0	1	0	0	0	1

$R_2(\text{new}) = R_2(\text{old}) - \frac{1}{3} R_4(\text{new})$

$R_2(\text{old}) =$	3	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0
$R_4(\text{new}) =$	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$
$\frac{1}{3} \times R_4(\text{new}) =$	$\frac{1}{4}$	0	0	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{3}{4}$
$R_2(\text{new}) = R_2(\text{old}) - \frac{1}{3} R_4(\text{new})$	$\frac{11}{4}$	0	1	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$

$R_3(\text{new}) = R_3(\text{old}) - \frac{1}{9} R_4(\text{new})$

$R_3(\text{old}) =$	$\frac{16}{3}$	1	0	0	$\frac{1}{9}$	$\frac{14}{9}$	$\frac{10}{9}$	0
$R_4(\text{new}) =$	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$
$\frac{1}{9} \times R_4(\text{new}) =$	$\frac{1}{12}$	0	0	0	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{9}$	$-\frac{1}{4}$
$R_3(\text{new}) = R_3(\text{old}) - \frac{1}{9} R_4(\text{new})$	$\frac{21}{4}$	1	0	0	0	$\frac{3}{2}$	1	$\frac{1}{4}$

Iteration-2		C_j	3	1	3	0	0	0	0
B	**C_B**	**X_B**	**x_1**	**x_2**	**x_3**	**S_1**	**S_2**	**S_3**	**$Sg1$**
x_3	3	3	0	0	1	0	0	0	1
x_2	1	$\frac{11}{4}$	0	1	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$
x_1	3	$\frac{21}{4}$	1	0	0	0	$\frac{3}{2}$	1	$\frac{1}{4}$
S_1	0	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$
$Z = \frac{55}{2}$		**Z_j**	**3**	**1**	**3**	**0**	**5**	**3**	**$\frac{9}{2}$**
		$C_j - Z_j$	0	0	0	0	-5	-3	$-\frac{9}{2}$
		Ratio	---	---	---	---	---	---	---

Since all $C_j - Z_j \le 0$

Hence, non-integer optimal solution is arrived with value of variables as :

$x_1 = \frac{21}{4}, x_2 = \frac{11}{4}, x_3 = 3$

Max $Z = \frac{55}{2}$

To obtain the integer valued solution, we proceed to construct Gomory's fractional cut, with the help of x_2-row as follows:

$$\frac{11}{4} = 1x_2 + \frac{1}{2}S_2 + \frac{3}{4}Sg1$$

$$\left(2 + \frac{3}{4}\right) = (1 + 0)x_2 + \left(0 + \frac{1}{2}\right)S_2 + \left(0 + \frac{3}{4}\right)Sg1$$

The fractional cut will become

$$-\frac{3}{4} = Sg2 - \frac{1}{2}S_2 - \frac{3}{4}Sg1 \rightarrow \text{(Cut-2)}$$

Iteration-1		C_j	3	1	3	0	0	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	$Sg1$	$Sg2$
x_3	3	3	0	0	1	0	0	0	1	0
x_2	1	$\frac{11}{4}$	0	1	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$	0
x_1	3	$\frac{21}{4}$	1	0	0	0	$\frac{3}{2}$	1	$\frac{1}{4}$	0
S_1	0	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$	0
$Sg2$	0	$-\frac{3}{4}$	0	0	0	0	$-\frac{1}{2}$	0	$\left(-\frac{3}{4}\right)$	1
$Z = \frac{55}{2}$		Z_j	**3**	**1**	**3**	**0**	**5**	**3**	$\frac{9}{2}$	**0**
		$C_j - Z_j$	0	0	0	0	-5	-3	$-\frac{9}{2}$	0
		Ratio $= \frac{C_j - Z_j}{Sg2,j}$ and $Sg2,j < 0$	---	---	---	---	10	---	6 ↑	---

∴ The pivot element is $-\frac{3}{4}$.

Entering $= Sg1$, Departing $= Sg2$, Key Element $= -\frac{3}{4}$

$R_5(\text{new}) = R_5(\text{old}) \times \left(-\frac{4}{3}\right)$

$R_5(\text{old}) =$	$-\frac{3}{4}$	0	0	0	0	$-\frac{1}{2}$	0	$-\frac{3}{4}$	1
$R_5(\text{new}) = R_5(\text{old}) \times \left(-\frac{4}{3}\right)$	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$

$R_1(\text{new}) = R_1(\text{old}) - R_5(\text{new})$

$R_1(\text{old}) =$	3	0	0	1	0	0	0	1	0
$R_5(\text{new}) =$	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$
$R_1(\text{new}) = R_1(\text{old}) - R_5(\text{new})$	2	0	0	1	0	$-\frac{2}{3}$	0	0	$\frac{4}{3}$

$R_4(\text{new}) = R_4(\text{old}) + \frac{9}{4}R_5(\text{new})$

$R_4(\text{old}) =$	$\frac{3}{4}$	0	0	0	1	$\frac{1}{2}$	1	$-\frac{9}{4}$	0
$R_5(\text{new}) =$	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$
$\frac{9}{4} \times R_5(\text{new}) =$	$\frac{9}{4}$	0	0	0	0	$\frac{3}{2}$	0	$\frac{9}{4}$	-3
$R_4(\text{new}) = R_4(\text{old}) + \frac{9}{4}R_5(\text{new})$	3	0	0	0	1	2	1	0	-3

$R_2(\text{new}) = R_2(\text{old}) - \frac{3}{4} R_5(\text{new})$

$R_2(\text{old}) =$	$\frac{11}{4}$	0	1	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$	0
$R_5(\text{new}) =$	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$
$\frac{3}{4} \times R_5(\text{new}) =$	$\frac{3}{4}$	0	0	0	0	$\frac{1}{2}$	0	$\frac{3}{4}$	-1
$R_2(\text{new}) = R_2(\text{old}) - \frac{3}{4} R_5(\text{new})$	2	0	1	0	0	0	0	0	1

$R_3(\text{new}) = R_3(\text{old}) - \frac{1}{4} R_5(\text{new})$

$R_3(\text{old}) =$	$\frac{21}{4}$	1	0	0	0	$\frac{3}{2}$	1	$\frac{1}{4}$	0
$R_5(\text{new}) =$	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$
$\frac{1}{4} \times R_5(\text{new}) =$	$\frac{1}{4}$	0	0	0	0	$\frac{1}{6}$	0	$\frac{1}{4}$	$-\frac{1}{3}$
$R_3(\text{new}) = R_3(\text{old}) - \frac{1}{4} R_5(\text{new})$	5	1	0	0	0	$\frac{4}{3}$	1	0	$\frac{1}{3}$

Iteration-2		C_j	3	1	3	0	0	0	0	0
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	$Sg1$	$Sg2$
x_3	3	2	0	0	1	0	$-\frac{2}{3}$	0	0	$\frac{4}{3}$
x_2	1	2	0	1	0	0	0	0	0	1
x_1	3	5	1	0	0	0	$\frac{4}{3}$	1	0	$\frac{1}{3}$
S_1	0	3	0	0	0	1	2	1	0	-3
$Sg1$	0	1	0	0	0	0	$\frac{2}{3}$	0	1	$-\frac{4}{3}$
$Z = 23$		Z_j	**3**	**1**	**3**	**0**	**2**	**3**	**0**	**6**
		$C_j - Z_j$	0	0	0	0	-2	-3	0	-6
		Ratio	---	---	---	---	---	---	---	---

Since all $C_j - Z_j \le 0$

Hence, integer optimal solution is arrived with value of variables as :
$x_1 = 5, x_2 = 2, x_3 = 2$

Max $Z = 23$

The integer optimal solution found after 2-cuts.

2. given Value, weight 10 5 40 4 30 6 50 3; W=10 Solve the above 0/1 knapsack problem is solved using dynamic programming

Profit	Weight
10	5
40	4
30	6
50	3

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

Resultant Profit

90

Resultant Solution

0101