Course Title:

Mathematical programming (2-2-0-0)

Topic :

Formulation of Linear Programming Problem

CO-1

Session - 1

AIM

To familiarize students with the concept of formulation of linear programming problem

INSTRUCTIONAL OBJECTIVES

This session is designed to:

1. Describe develop and understanding of linear programming problem.
2. Demonstrate how to formulate linear programming problem
3. Illustrate in detail of objective function.

LEARNING OUTCOMES

At the end of this unit, you should be able to:.

1. Describe the ways to construct the mathematical formulation of linear programming problem
2. Solve the formulation of linear programming problem.

MATHEMATICAL PROGRAMMING- COS

CO1: Linear Programming: Formulation of LP Problem (LPP), Graphical method, Simplex method, Transportation problem, Duality concept in LPP, Feasibility of solution using Farka's Lemma, Ellipsoid method, Karmarkar's Algorithm .

CO2: Combinatorial Optimization , Integer Programming, Branch & bound algorithms, valid inequalities & cuts. Fractional Programming. Combinatorial Optimization: Approximation Algorithms, Submodular functions, Matroids, Continuous approximation algorithms. Dynamic programming: Knapsack problem, Travelling salesman problem

CO3: Non-Linear Programming : Quadratic programs – Constrained quadratic programming problems, Beale's method, Wolfe method, Karush-Kuhn Tucker (KKT) Conditions. Geometric Programming: Problems with one-degree of difficulty with positive coefficients, Geometric programming with constraints, Problems with positive and negative coefficients

CO4: Infinite Dimensional Optimization : Heuristic and Meta heuristics, Single solution vs. population-based, Parallel meta heuristics, Evolutionary algorithms, Nature-inspired metaheuristics, Genetic Algorithm, Ant-colony optimization, Particle swarm optimization, Simulated annealing

MATHEMATICAL PROGRAMMING COURSE OBJECTIVES

- CO1 (Linear Programming)
  - Formulation of linear programming
  - Finding optimal solution to LPP by
    - Graphical Method
    - Simplex Method
  - Duality Principle
  - Farka's Lemma
  - Karmakar algorithm → fast computation of optimal solutions
  - Transportation problem formulation and minimum cost computations
  - Network flow models
- CO3 (Non-Linear Optimization Problems)
  - Introduction to convex sets, convex function and convex optimization
  - Semi-definite programming, conic programming, second order conic programming
  - Quadratic programming
    - KKT conditions
    - Beale's method
    - Wolf method
  - Geometric programming
- CO2 (Integer Programming)
  - Formulation of integer programming problems(IPP)
  - Finding optimal solution to IPP
    - Branch and Bound
    - Gomory's cut method
  - Matroids
  - Approximation algorithms
- CO4 (optimization based search algorithms)
  - Finding the global optimal solution
    - Hill climbing methods
    - Heuristics and meta heuristics
    - Simulated annealing
    - Ant colony optimization
    - Evolutionary Genetic algorithms
    - Particle Swarm Optimization

OBJECTIVE

- To develop and understanding of linear programming problem
- To formulate linear programming problem

EXAMPLE PROBLEM

A Company manufactures and sells 2 types of Products A and B; The cost of production of each unit of A and B is $ 200 and $500 respectively each unit of A produce a profit of $20 and each unit of b produce a profit of $ 15 on selling. Company estimates the monthely demand of A and B to beat a maximum of 500 units in all. The production budget for month is set at $ 50,000. How many units should the company manufactures in order to earn maximum profit from its monthely sales of A and B?

FORMULATION OF LPP

A company manufactures and sells 2 types of Products: 'A' and 'B'.
The cost of production of each unit of 'A' and 'B' is $200 and $ 150 respectively.
Each unit of 'A' yields a profit of $ 20 and each unit of 'B' yields a profit of $ 15 on selling.
Company estimates the monthly demand of 'A' and 'B' to be at a maximum of 500 units in all.
The Production budget for the month is set at $ 50,000. How many units should the company manufacture in order to earn maximum profit from its monthly sales of 'A' and 'B'?

Products	Cost of production per unit	profit per unit	Total demand
A B	$ 200 $ 150	$ 20 $ 15	500 units
Production budget	$ 50,000	-	-

- **Objective function:** The direct function of form Z = ax + by, where a and b are constant, which is reduced or enlarged is called the objective function. For example, if Z = 13x + 21y. The variables x and y are called the decision variable.
- **Constraints:** The restrictions that are applied to a linear inequality are called constraints.

  Non-negative constraints: $x > 0, y > 0$ etc.

  General constraints: $x + y > 30, 3x + 4y \geq 20$ etc.
- **Optimization problem:** A problem that seeks to maximization or minimization of variables of linear inequality problem is called optimization problems.
- **Optimal (most feasible) solution:** Any point in the emerging region that provides the right amount (maximum or minimum) of the objective function is called the optimal solution.

FORMULATION OF LPP

Products	Cost of production per unit	profit per unit	Total demand
A B	\$ 200 \$ 150	\$ 20 \$ 15	500 units
Production budget	\$ 50,000	-	-

Number of units of A ⟶ x

Number of units of B ⟶ y

Budget constraint :

$200x + 150y \leq 50{,}000$

Total demand constraint :

$x + y \leq 500$

z = Total profit

$z = 20\,x + 15\,y$

CASE STUDY

A small cooperative craft workshop makes two types of table: a standard rectangular table and a de luxe circular table. The market can absorb as many of either type of table as the workshop can produce, and so we can assume unlimited demand. Each type of table is made from the same wood and, once the wood has been cut, each table has to go through three processes: joinery, pre finishing and final finishing (in that order). Sufficient cut wood is always available.

Each rectangular table takes 2 hours for joinery, 40 minutes for pre finishing and 5 hours 20 minutes for final finishing. Each circular table requires 3 hours for joinery, 2 hours for pre-finishing and 4 hours for final finishing. The workshop employs five joiners, two sanders and eight polishers. The joiners each work a fixed six-hour day, while the sanders and polishers each work a fixed eight-hour day on the pre finishing and final finishing respectively. No overtime is worked, and full six-hour or eight-hour days are worked by each employee irrespective of whether there is work for that employee to do. All running costs, including wages, are fixed. The cooperative sells each rectangular table for £120 and each circular table for £150. How many of each type of table should it produce each day in order to maximize its profit?

FORMULATING OBJECTIVE FUNCTION

- The quantity to be optimized in a mathematical programming problem is known as the objective function, and the objective is to optimize this function.
- **The first step in formulating a linear programming model is to identify the objective and it's objective function.**
- In this example, we want to maximize the daily profit, so we could take maximizing the daily product as the objective and the daily profit as the objective function.
- However, the daily profit is equal to the daily income from selling tables less the fixed daily running costs (including the cost of wood and wages). So, since the running costs are fixed, maximizing daily profit is equivalent to maximizing daily income.

OBJECTIVE FUNCTION (CONT..)

- As omitting the fixed running costs will simplify the model, we shall therefore take the objective to be to maximize daily product and the objective function to the daily income.
- The second step is to identify the variables, other than the objective function, and to specify their units of measurement.
- In this example, the only variables are the numbers of tables of each type made per day.
- The third step is to identify the constraints and parameters. In this example the only constraints are the numbers of person-hours available for joinery, pre finishing and final finishing each day. The parameters are the time it takes for each process for each type of table, the hours available for each process each day, and the selling prices of the tables

DECISION VARIABLES

- The fourth step is to assign algebraic symbols to the objective function, the variables and, if necessary, the parameters. It is usual in linear programming to use z for the objective function and x1, x2, ... for the variables. So, in this case, we have:
- z the daily income, in pounds
- x_1 the number of rectangular tables made per day
- x_2 the number of circular tables made per day.

- We shall therefore adopt this approach here, so that, usually, instead of assigning symbols to the parameters, we shall state their numerical values in a table that relates them to the variables, the objective function and the constraints, as in the following Table .

TABULATING DATA FOR FORMULATION

	Rectangle Table	Circular Table	upper limit(Per Day)
income (£)	120	150	-
joinery (hours)	2	3	30
Pre finishing (hours)	2/3	2	16
final finishing (hours)	5 1/3	4	64

- The next step is to derive algebraic relationships for the objective function and the constraints.
- Before deriving these relationships, it is important to remember that they must be linear.
- In this case, there is a clear linear relationship between the daily income and the numbers of tables sold, given by

$$z = 120x_1 + 150x_2$$

FORMULATION OF CONSTRAINTS

- This is the objective function and need to maximize it
$$maximize\ z = 120x_1 + 150x_2$$
- The numbers of hours spent daily on each of joinery, pre finishing and final finishing are given, using Table , by the simple linear expressions
- $2x_1 + 3x_2 (Joinery)$
- $\frac{2}{3}x_1 + 2x_2 (Pre\ finishing)$
- $5\frac{1}{3}x_1 + 4x_2 (final\ finishing)$
- The upper limits on the numbers of hours available for each of these processes each day can then be combined with these expressions to give the following linear constraints
- $2x_1 + 3x_2 \leq 30$
- $\frac{2}{3}x_1 + 2x_2 \leq 16$
- $5\frac{1}{3}x_1 + 4x_2 \leq 64$

FINAL FORMULATION BY ADDING NON-NEGATIVE CONSTRAINTS

- Finally we must, as so often in mathematics, state the obvious: the cooperative cannot make a negative number of tables. So we must also include the constraints:
  - $x_1 \geq 0$ *and* $x_2 \geq 0$
  - We can write the objective and constraints succinctly as follows.
  - $maximize\ z = 120x_1 + 150x_2$
  - Subject to

$$2x_1 + 3x_2 \leq 30$$

$$\frac{2}{3}x_1 + 2x_2 \leq 16$$

$$5\frac{1}{3}x_1 + 4x_2 \leq 64$$

- $x_1 \geq 0$ *and* $x_2 \geq 0$

QUESTIONS

- What is the significance of objective function in a linear programming problem
- Give examples of a minimization problem and maximization problem
- What are decision variables
- What are constraints
- What are non-negative constrains

CLASS WORK: LPP FORMULATION PROBLEM

A food manufacturer buys edible oils in their raw state, and refines and blends them to produce margarine. The raw oils can be vegetable or non-vegetable. The vegetable oils are ground-nut, soya-bean and palm. The non-vegetable oils are lard and fish. The margarine sells at £2400 a tonne, and the manufacturer can sell as much margarine as can be made. The prices of the raw oils are shown in Table . Sufficient quantities of all five oils can be assumed to be readily available.

oil	ground-nut	soya-bean	palm	lard	fish
cost (£ per tonne)	960	1600	1600	2200	1900
hardness	1.2	3.4	8.0	10.8	8.3

PROBLEM CONTINUED…

- To avoid contamination, vegetable and non-vegetable oils are refined separately, using different equipment. It is possible to refine up to 1000 tones of vegetable oil and up to 800 tones of non-vegetable oil in a month. When refined, some or all of the oils are blended to produce the margarine.
- The quantities of oil lost during the refining and blending processes are negligible.
- The hardnesses of the refined oils, given in Table , are assumed to combine linearly to give the hardness of the margarine. To ensure that the margarine spreads easily but is not runny, the hardness of the margarine must be between 5.6 and 7.4, measured in the same units as the hardnesses of the oils in Table . All running costs can be taken to be fixed. The manufacturer wants to know how much of each type of oil to use in order to maximize profit while maintaining the quality of the margarine.

FORMULATION

We follow the steps of Procedure

(a) The purpose of the model is to maximize profit, subject to constraints on resources, while maintaining quality. Since the refining capacities are given per month, it is sensible to work on a monthly basis. We could therefore take maximizing monthly profit as the objective. However, since the running costs are fixed, maximizing monthly profit is equivalent to maximizing monthly income. Therefore, as omitting the fixed running cost will simplify the model, we shall take the objective as maximizing monthly income. This monthly income could be measured in pounds. However, looking at the quantities involved, it would seem more sensible to work in units of £10 000.

(b) The variables are the quantities of each oil used per month in blending the margarine. Also, as it will be useful to know the total quantity of margarine manufactured per month, we shall regard this as a variable.

(c)identify the parameters .

(d) We now assign algebraic symbols to the objective function and the variables, giving precise definitions, including units of measurement:

- z the monthly profit, in tens of thousands of pounds;
- x_1 the quantity, in hundreds of tones, of ground-nut oil used per month;
- x_2 the quantity, in hundreds of tones, of soya-bean oil used per month;
- x_3 the quantity, in hundreds of tones, of palm oil used per month;
- x_4 the quantity, in hundreds of tones, of lard used per month;
- x_5 the quantity, in hundreds of tones, of fish oil used per month;
- x_6 the quantity, in hundreds of tones, of margarine manufactured per month.

(e) We next identify the linear relationships for the objective function and the constraints. We can do this by extending Table , and making suitable adjustments to the numerical values to take into account the units of measurement decided on in (a) and (b), and listed in (d). This results in the following Table :

	ground-nut oil	soya-bean oil	palm oil	lard	fish oil	upper limit	lower limit
cost (£10000s per 100 tonnes)	9.6	16	16	22	19	—	—
refining of vegetable oils (100s of tonnes)	✓	✓	✓	—	—	10	—
refining of non-vegetable oils (100s of tonnes)	—	—	—	✓	✓	8	—
hardness (hardness units)	1.2	3.4	8.0	10.8	8.3	7.4	5.6

- (f) Determine the objective function and objective of the problem in terms of x1, . . . , x6
- (g) We now want to express the constraints algebraically in terms of the variables.
- The hardness constraints require some thought.
- Remember that the hardnesses of the components combine linearly to give the hardness of the margarine, which must be at least 5.6.
- Using Table this means that we must have

$$1.2x_1 + 3.4x_2 + 8.0x_3 + 10.8x_4 + 8.3x_5 \geqslant 5.6x_6.$$

- If, for consistency, we keep all variables to the left-hand side of inequality or equality signs, this becomes

$$1.2x_1 + 3.4x_2 + 8.0x_3 + 10.8x_4 + 8.3x_5 - 5.6x_6 \geqslant 0.$$

- We must also remember the quantity constraint, which can be written as

$$x_1 + x_2 + x_3 + x_4 + x_5 = x_6$$

- or, in the format of the other constraints, as

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = 0.$$

(h) Finally, as the manufacturer cannot use negative quantities of oil, we have the trivial constraints

$$x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0.$$

The linear programming model for Example is thus:

$$\text{maximize } z = -9.6x_1 - 16x_2 - 16x_3 - 22x_4 - 19x_5 + 24x_6$$

subject to

$$\begin{array}{rcll}
x_1 + x_2 + x_3 & \leqslant & 10 & \text{(vegetable refining capacity)} \\
x_4 + x_5 & \leqslant & 8 & \text{(non-vegetable refining capacity)} \\
1.2x_1 + 3.4x_2 + 8.0x_3 + 10.8x_4 + 8.3x_5 - 5.6x_6 & \geqslant & 0 & \text{(lower hardness)} \\
1.2x_1 + 3.4x_2 + 8.0x_3 + 10.8x_4 + 8.3x_5 - 7.4x_6 & \leqslant & 0 & \text{(upper hardness)} \\
x_1 + x_2 + x_3 + x_4 + x_5 - x_6 & = & 0 & \text{(quantity)} \\
x_1, x_2, x_3, x_4, x_5, x_6 \geqslant 0 & & & \blacksquare
\end{array}$$

Food Type	Yeild per unit			Cost per Unit
	Protiens	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming problem for the given problem.

Solution:

- Let x_1, x_2, x_3 and x_4 denote the number of units of food type 1,2,3 and 4 respectively. The objective is to minimize the cost.

Minimize $Z=45x_1+40x_2+85x_3+65x_4$

subject to the constraints

$$3x_1+4x_2+8x_3+6x_4>=800$$

$$2x_1+2x_2+7x_3+5x_4>=200$$

$$6x_1+4x_2+7x_3+4x_4>=700$$

where x_1, x_2, x_3 and $x_4>=0$.

This is called an Linear Programming Problem(LPP).

Machines	Time per unit (minutes)			Machine capacity
	Product-1	Product-2	Product-3	
M1	2	3	2	440
M2	4	--	3	470
M3	2	5	--	430

Solution:

Let the amounts of products 1,2 and 3 manufactured daily be x1,x2 and x3 units respectively. Clearly x1,x2 and x3 are clearly >=0

In this problem, The objective function is

Maximize $Z=4x_1+3x_2+6x_3$

Since, the constraints are on machine capacities and can be mathematically expressed as

$$2x_1+3x_2+2x_3<=440$$

$$4x_1+0x_2+3x_3<=470$$

$$2x_1+5x_2+0x_3<=430$$

where x_1,x_2,x_3 are >=0.

Hence, The given problem can be formulated as an LP problem by

Maximize $Z = 4x_1+3x_2+6x_3$

Subject to the constraints

$2x_1+3x_2+2x_3<=440$

$4x_1+0x_2+3x_3<=470$

$2x_1+5x_2+0x_3<=430$

where x_1,x_2,x_3 are $>=0$

REFERENCES

Textbook:

1. *Introduction to Mathematical Programming* by Russell C. Walker, published by Pearson Custom Publishing, 2006

2. S. P. Bradley, A. C. Hax, and T. L. Magnanti, *Applied Mathematical Programming*, Addison-Wesley Publishing Company, 1977

3. Lenstra, Rinnooy Kan, & Schrijver (eds.), *History of Mathematical Programming: A Collection of Personal Reminiscences*, Elsevier, 1991.

Web Resources

1. https://nptel.ac.in/courses/110105096
2. https://www.coursera.org/learn/operations-research-modeling
3. https://www.semanticscholar.org/paper/Applied-Mathematical-Programming-Bradley-Hax/8a4ee083b23505df221410e6a2b41fc56fa250a6

THANK YOU

(DEEMED TO BE UNIVERSITY)

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