Design and Analysis of Algorithms

Session 31 & 32

Travelling Salesman Problem

TRAVELLING SALESMAN PROBLEM

- **Let G = (V, E) be a directed graph.**
- **The problem is to find a tour of minimum cost and we assume that every tour starts and ends at vertex 1.**
- **So, the solution space S is given by S = {(1, r, 1)| where r is a permutation of (2,3, ... ,n)}.**
- **Therefore S= (n — 1)!**

$$c(A) = \begin{cases} \text{length of tour defined by the path from the root to } A, \text{ if } A \text{ is a leaf} \\ \text{cost of a minimum-cost leaf in the subtree } A, \text{ if } A \text{ is not a leaf} \end{cases}$$

- For the given matrix first obtain reduced cost matrix.
- A matrix is reduced iff every row and column is reduced.
- A row (column) is said to be reduced iff it contains at least one zero and all remaining entries are non-negative.
- Let A be the reduced cost matrix for node R.
- Let S be a child of R such that the tree edge (R, S) corresponds to including edge (i, j) in the tour.
- If S is not a leaf, then the reduced cost matrix for S may be obtained as follows:

1. Change all entries in row i and column j of A to oo.
2. Set A(j,1) to oo.
3. Reduce all rows and columns in the resulting matrix except for rows and columns containing only oo. Let the resulting matrix be B.
4. C(S)=C(R)+A(i,j)+ r where r is the total amount subtracted in step (3)

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

(a) Cost matrix

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

(b) Reduced cost matrix

L = 25

Figure 8.11 An example

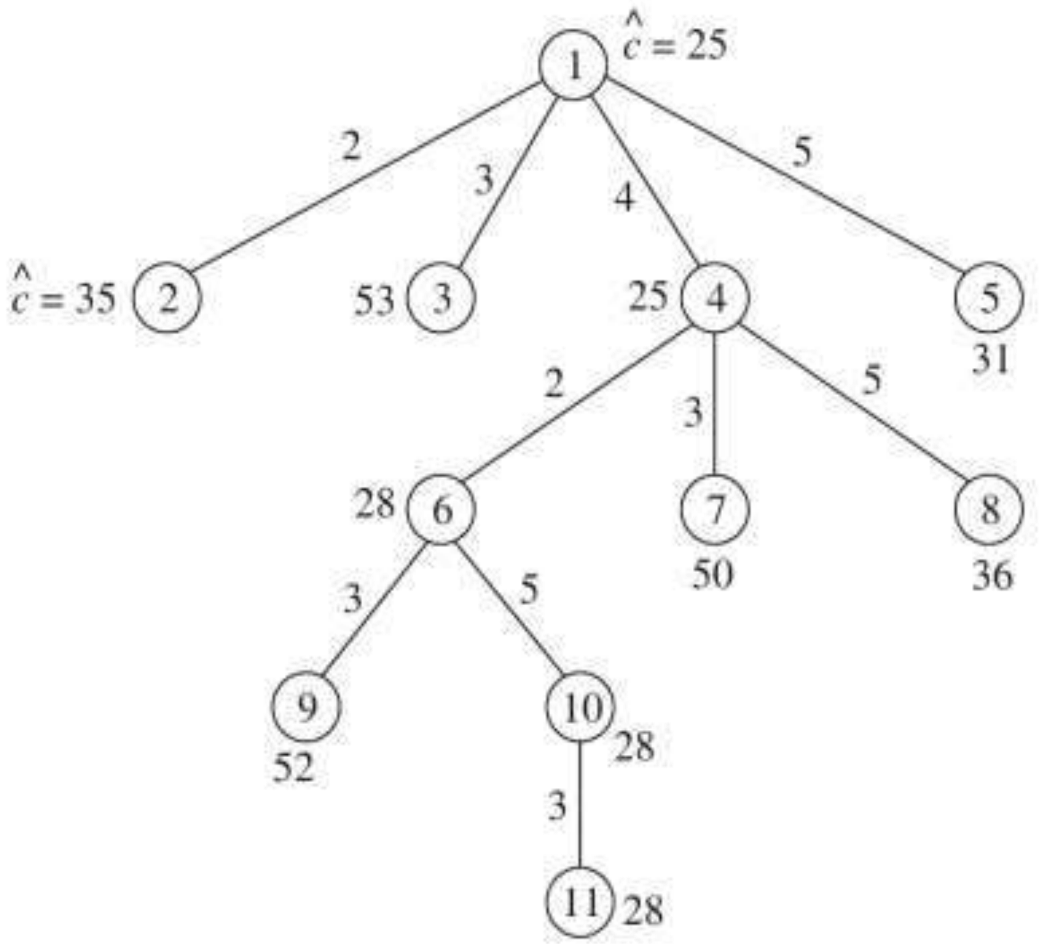

Figure 9.14 State space tree generated by LCBB procedure.

To compute $\hat{c}$ at node 2, we change the entries of first row and second column to ∞, and $A(2, 1)$ is also set to ∞ [Figure 9.15(a)]. Since the matrix after changing the entries are found to be already reduced, no reduction is required. Therefore, $\hat{c}(2)$ can be computed as

$$\hat{c}(1) + A(1, 2)$$

i.e.

$$25 + 10 = 35$$

Similarly, $\hat{c}$ at node 3 can be computed by changing the entries of 1st row, 3rd column and $A(3, 1)$ to ∞ as shown in Figure 9.15(b). The matrix is reduced by subtracting 11 from the 1st column. Hence, $\hat{c}(3)$ is computed as

$$\hat{c}(1) + A(1, 3) + r$$

i.e.,

$$25 + 17 + 11 = 53$$

Similarly, $\hat{c}(4)$ and $\hat{c}(5)$ are computed to 25 and 31 respectively. The matrix and its corresponding reduced matrix are shown in Figure 9.15(c) and 9.15(d) respectively. Since $\hat{c}(4)$ is minimum, node 4 is considered as the next E-node as u remains unchanged.

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix}$$

(a) Matrix at node 2, no reduction required

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 15 & 3 & \infty & \infty & 0 \\ 11 & 0 & \infty & 12 & \infty \end{bmatrix} \Rightarrow \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix}$$

(b) Matrix at node 3 after changing the entries

Reduced matrix at node 3, $r = 11$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

(c) Matrix at node 4

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & 2 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 15 & 3 & 12 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix} \Rightarrow \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$

(d) Matrix at node 5

Reduced matrix at node 5, $r = 5$

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

(e) Matrix at node 6

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & \infty & \infty & 0 \\ \infty & 3 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & 0 & \infty & \infty & \infty \end{bmatrix}$$

(f) Matrix at node 7

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$

Reduced matrix at node 7, $r = 13$

Node 4 is expanded to nodes 6, 7 and 8 that correspond to the inclusion of edges (4, 2), (4, 3) and (4, 5) respectively. The matrix at node 6 can be obtained by changing the entries of row 1, column 4, row 4, column 2 and $A(2, 1)$ to ∞. Since, the matrix is already reduced, no further reduction is required. Hence, $\hat{c}(6)$ can be computed as

$$\hat{c}(4) + A(4, 2)$$

i.e.,
$$25 + 3 = 28$$

The matrix at node 6 is shown in Figure 9.15(e).

Similarly, to compute $\hat{c}(7)$ the entries at row 1, column 4, row 4, column 3 and $A(3, 1)$ are changed to ∞. Then, the matrix is reduced by subtracting 2 from row 3 and 11 from column 1. Hence, $\hat{c}(7)$ is computed to be

$$\hat{c}(4) + A(4, 3) + r = 25 + 12 + 13 = 50$$

Similarly, $\hat{c}(8)$ is computed to 36. The matrix at node 7 and 8 are shown in Figure 9.15(f) and 9.15(g) respectively. Currently, the live nodes are 2, 3, 5, 6, 7 and 8. Since node 6 has the minimum $\hat{c}$ value, it is considered as the next E-node. Node 6 is expanded to nodes 9 and 10 that correspond to edges (2, 3) and (2, 5) respectively.

$\hat{c}(9)$ is computed by changing the entries of row 1, column 4, row 4, column 2, row 2, column 3 and $A(3, 1)$ to ∞. The matrix is reduced by subtracting 2 from row 3 and 11 from column 1. Hence, $\hat{c}(9)$ is computed as

$$\hat{c}(6) + A(2, 3) + 2 = 28 + 11 + 13 = 52$$

Similarly, $\hat{c}(10)$ is computed to 28. The matrices at nodes 9 and 10 are shown in Figure 9.15(h) and 9.15(i) respectively.

Now, the live nodes are 2, 3, 5, 7, 8, 9 and 10. Since $\hat{c}(10)$ is the minimum node, 10 is considered as the next *E*-node. The only node generated from node 10 is node 11 that corresponds to edge (5, 3). The cost matrix at node 11 is shown in Figure 9.15(j), where each entry is ∞. $\hat{c}(11)$ is computed as

$$\hat{c}(10) + A(5, 3) = 28 + 0 = 28$$

Node 11 is the solution node. Hence, U is updated to 28. The node whose $\hat{c}$ is greater than U must be killed. Therefore, all the live nodes are killed, and node 11 is considered as the answer node. Hence, LCBB terminates with 1, 4, 2, 5, 3, 1 as the shortest length tour.

Sample Questions:

1. Explain steps involved in reduced cost matrix.
2. Adjacency matrix is given, Solve the following Travelling Salesman problem and find minimum cost of the tour.

	C0	C1	C2	C3	C4
C0	INF	20	30	10	11
C1	15	INF	16	4	2
C2	3	5	INF	2	4
C3	19	6	18	INF	3
C4	16	4	7	16	INF