MATHEMATICAL PROGRAMMING

CO3

NON-LINEAR PROGRAMMING : QUADRATIC PROGRAMS

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LINEAR PROGRAMMING VERSUS NONLINEAR PROGRAMMING

LINEAR PROGRAMMING	NONLINEAR PROGRAMMING
A method to achieve the best outcome in a mathematical model whose requirements are represented by linear relationships	A process of solving an optimization problem where the constraints or the objective functions are nonlinear
Helps to find the best solution to a problem using constraints that are linear	Helps to find the best solution to a problem using constraints that are nonlinear

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NLPP

Non Linear Programming Problem (NLPP)

An optimisation problem in which objective function z or some/all constraints are non linear [Higher power of x_1, x_2 ~~far~~ than one] is called NLPP.

Types :

i) with no constraints

ii) with Equality constraints → Lagrange's method

iii) with inequality constraints. → Kuhn-Tucker conditions.

Types of Nonlinear Programming problems

- Unconstrained optimization

  $$\textit{min} \text{ or } \textit{max} \; f(x_1,\ldots,x_n)$$

  No functional constraints.
- Linearly constrained optimization
  - Objective function nonlinear
  - Functional constraints linear

  Extensions of simplex method can be applied.
- Quadratic programming

  Special case of linearly constrained optimization when the objective function is quadratic.

Quadratic Programming

The quadratic programming problem differs from the linear programming problem only in that the objective function also includes x_i^2 and x_j^2 $x_i x_j$ $(i \neq j)$ terms.

The matrix form of a quadratic programming problem is

$$\text{Maximize} \quad f(\mathbf{x}) = \mathbf{cx} - \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx},$$

subject to

$$\mathbf{Ax} \le \mathbf{b} \quad \text{and} \quad \mathbf{x} \ge \mathbf{0},$$

where **c** is a row vector, **x** and **b** are column vectors, **Q** and **A** are matrices, and the superscript T denotes the transpose (see Appendix 4). The q_{ij} (elements of Q) are given constants such that $q_{ij} = q_{ji}$, say, Q is a symmetrical matrix.

The algebraic form of the objective function of this quadratic programming problem is

$$f(\mathbf{x}) = \mathbf{cx} - \tfrac{1}{2}\mathbf{x}^T\mathbf{Qx} = \sum_{j=1}^{n} c_j x_j - \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j.$$

If $i = j$ in this double summation, then $x_i x_j = x_j^2$.

Example:

To illustrate the notation, consider the following example of a quadratic programming problem.

$$\text{Maximize} \quad f(x_1, x_2) = 15x_1 + 30x_2 + 4x_1x_2 - 2x_1^2 - 4x_2^2,$$
subject to
$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 0, \qquad x_2 \geq 0.$$

$$\text{Maximize} \quad f(\mathbf{x}) = \mathbf{cx} - \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x},$$
subject to
$$\mathbf{Ax} \leq \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0},$$

In this case, $f(x_1, x_2)$ can be rewritten as

$$f(x_1, x_2) = 15x_1 + 30x_2 - \tfrac{1}{2}(4x_1^2 - 4x_2x_1 - 4x_1x_2 + 8x_2^2)$$

$$\mathbf{c} = [15 \quad 30], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix}, \quad \mathbf{A} = [1 \quad 2], \quad \mathbf{b} = [30].$$

Note that
$$\begin{aligned} \mathbf{x}^T\mathbf{Q}\mathbf{x} &= [x_1 \quad x_2]\begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= [(4x_1 - 4x_2) \quad (-4x_1 + 8x_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 4x_1^2 - 4x_2x_1 - 4x_1x_2 + 8x_2^2 \\ &= q_{11}x_1^2 + q_{21}x_2x_1 + q_{12}x_1x_2 + q_{22}x_2^2. \end{aligned}$$

EXAMPLE

⇒ Let objective function $f(x) = 3x_1^2 + 4x_2^2 + 2x_1x_2 - 2x_1 - 3x_2$

⇒ Constraint:

- $3x_1 + 2x_2 \leq 6$
- $x_1 + x_2 \leq 2$
- $x_1, x_2 \geq 0$

⇒ Problem representation

$$min\{f(x)\} = [x_1 \; x_2] \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [-2 \; -3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

⇒ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, Q = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, c = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

Positive semi-definite and symmetric

$\Rightarrow$ $Q = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \rightarrow [q_{ij}]_{2\times 2}$, if $q_{ij} = q_{ji} \rightarrow$ Symmetric

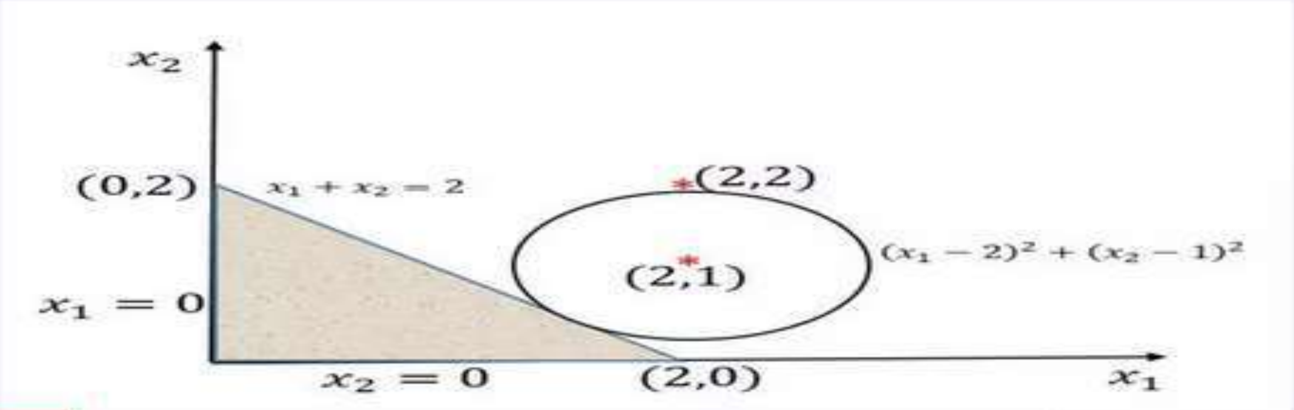
$\Rightarrow$ If $det|Q| \geq 0 \rightarrow$ Positive semi-definite

Solution by graphical method:

$\Rightarrow$ Let objective function $f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$

$\Rightarrow$ Constraint:

- $x_1 + x_2 \leq 2$
- $x_1, x_2 \geq 0$



QUADRATIC PROGRAMMING

- An NLP problem with non-linear objective function and linear constraints. Such an NLP problem is called quadratic programming problem.
- The general mathematical model of quadratic programming problem is as follows:

$$\text{Optimize (Max or Min) } Z = \left\{ \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k \right\}$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

and $\quad x_j \ge 0$ for all i and j

CONT...

- In matrix notations, QP problem is written as:

$$\text{Optimize (Max or Min) } Z = \mathbf{cx} + \frac{1}{2}\mathbf{x}^T\,\mathbf{Dx}$$

subject to the constraints

$$\mathbf{Ax} \le \mathbf{b}$$

and $\mathbf{x} \ge 0$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$; $\mathbf{c} = (c_1, c_2, \ldots, c_n)$; $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$

$\mathbf{D} = [d_{jk}]$ is an $n \times n$ symmetric matrix, i.e. $d_{jk} = d_{kj}$; $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix

CONT…

1. If the objective function in QP problem is of minimization, then matrix D is symmetric and positive definite.
2. if the objective function is of maximization, then matrix D is symmetric and negative-definite.
3. If matrix D is null, then the QP problem reduces to the standard LP problem.

CONT…

1. **Non-Linear Programming** :
   1. **Quadratic programs** – Constrained quadratic programming problems,
      - **Beale's method,**
      - **Wolfe method,**
      - **Karush-Kuhn Tucker (KKT) Conditions.**

KUHN-TUCKER CONDITIONS

1. The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the maximization QP problem subject to linear constraints.

KUHN-TUCKER CONDITIONS

For ONE inequality Constraint: **Kuhn–Tucker Conditions**

If, f – *objective function;* g – *consraint;* λ – *Lagrangian Multiplier*

$$Let,\ K = f - \lambda \cdot g$$

Find x_1, x_2 *and* λ *by solving the eqautions,* $\dfrac{\partial K}{\partial x_1} = 0, \quad \dfrac{\partial K}{\partial x_2} = 0, \quad \dfrac{\partial K}{\partial \lambda} = 0$

KUHN-TUCKER CONDITIONS

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \leq 0$
5. $\lambda \geq 0$ $(if\ Z\ is\ max)$
6. $x_1, x_2 \geq 0$

NOTE:

$For,\quad Maximize: consider\ \lambda \geq 0$

$For,\quad Minimize: consider\ \lambda \leq 0$

$In\ equation\ (3), either\ \lambda = 0\ or\ g = 0.$

$From\ solving\ these\ different\ cases, \&\ verify\ whether\ all\ above\ conditions\ are\ satisfied.$

Solve the following NLPP using the Kuhn-Tucker method:

Maximize: $z = 2x_1^2 - 7x_2^2 + 12x_1x_2 \leftarrow f$

sub. to, $\quad 2x_1 + 5x_2 \le 98 \leftarrow g$

$x_1, x_2 \ge 0$

$g \equiv 2x_1 + 5x_2 - 98$

Let, $K = f - \lambda \cdot g = 2x_1^2 - 7x_2^2 + 12x_1x_2 - \lambda(2x_1 + 5x_2 - 98)$

$$\frac{\partial K}{\partial x_1} = 4x_1 + 12x_2 - 2\lambda = 0 \Rightarrow 2x_1 + 6x_2 - \lambda = 0 \ldots\ldots(1)$$

$$\frac{\partial K}{\partial x_2} = -14x_2 + 12x_1 - 5\lambda = 0 \Rightarrow 12x_1 - 14x_2 - 5\lambda = 0 \ldots\ldots(2)$$

$$\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0 \ldots\ldots(3)$$

$$g \le 0 \Rightarrow (2x_1 + 5x_2 - 98) \le 0 \ldots\ldots(4)$$

$$\lambda \ge 0 \ldots\ldots(5)$$

$$x_1, x_2 \ge 0 \ldots\ldots(6)$$

Now, $\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0$

For ONE inequality Constraint:

f – objective function;
g – consraint;
λ – Lagrangian Multiplier

Let, $K = f - \lambda \cdot g$

Find x_1, x_2 and λ by solving the eqautions,

$$\frac{\partial K}{\partial x_1} = 0, \frac{\partial K}{\partial x_2} = 0, \frac{\partial K}{\partial \lambda} = 0$$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \le 0$
5. $\lambda \ge 0$ (*if Z is max*)
6. $x_1, x_2 \ge 0$

In equation (3), either $\lambda = 0$ or $g = 0$.

From solving these different cases, & verify whether all above conditions are satisfied

Now, $\lambda \cdot g = \lambda \cdot (2x_1 + 5x_2 - 98) = 0$

Case 1: $If\ \boxed{\lambda = 0,}$

$$2x_1 + 6x_2 - \cancel{\lambda} \overset{0}{=} 0 \dots\dots (1)$$

$$12x_1 - 14x_2 - 5\cancel{\lambda} \overset{0}{=} 0 \dots\dots (2)$$

$$\left.\begin{aligned} x_1 + 3x_2 &= 0 \\ 6x_1 - 7x_2 &= 0 \end{aligned}\right\}$$

$$\Longrightarrow x_1 = x_2 = 0$$

$$Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$$

$$Z = 0$$

Hence, this case, does not created feasible solutio

Therefore, assumption of $\lambda = 0$ is not correct.

Therefore, we need to REJECT, these values at λ

Case 2: $If\ 2x_1 + 5x_2 - 98 = 0,$

$$2x_1 + 6x_2 - \lambda = 0 \ \times (5)$$

$$\begin{aligned} &10x_1 + 30x_2 - \cancel{5\lambda} = 0 \\ -\ &12x_1 - 14x_2 - \cancel{5\lambda} = 0 \\ &\quad\ \ -\qquad\ +\qquad\ + \\ \hline &-2x_1 + 44x_2 = 0 \end{aligned}$$

$\&\ \ 2x_1 + 5x_2 = 98$

$$\Rightarrow\ x_1 = 44\ and\ \ x_2 = 2$$

$$2(44) + 6(2) - \lambda = 0 \Rightarrow \lambda = 100 \geq 0$$

These, values satisfies all the necessary conditions

The optimal solution is:

$$x_1 = 44\ ,\ \ x_2 = 2$$

$$Z_{max} = 2(44)^2 - 7(2)^2 + 12(44 \times 2) = 4900$$

Solve the following NLPP using the Kuhn-Tucker method:

$$Minimize: z = x_1^3 - 4x_1 - 2x_2 \rightarrow f$$

$$sub.to, \quad x_1 + x_2 \le 1 \rightarrow g$$

$$x_1, x_2 \ge 0$$

$$Let, K = f - \lambda \cdot g = x_1^3 - 4x_1 - 2x_2 - \lambda(x_1 + x_2 - 1)$$

$$\frac{\partial K}{\partial x_1} = 3x_1^2 - 4 - \lambda = 0 \dots\dots (1)$$

$$\frac{\partial K}{\partial x_2} = -2 - \lambda = 0 \Rightarrow \boxed{\lambda = -2} \dots\dots (2)$$

$$\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0 \dots\dots (3)$$

$$g \le 0 \Rightarrow (x_1 + x_2 - 1) \le 0 \Rightarrow x_1 + x_2 \le 1 \dots\dots (4)$$

$$\lambda \le 0 \dots\dots (5)$$

$$x_1, x_2 \ge 0 \dots\dots (6)$$

Now, $\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0$

For ONE inequality Constraint:

f – *objective function;*
g – *consraint;*
λ – *Lagrangian Multiplier*

$$Let, K = f - \lambda \cdot g$$

Find x_1, x_2 *and* λ *by solving the eqautions,*

$$\frac{\partial K}{\partial x_1} = 0, \frac{\partial K}{\partial x_2} = 0, \frac{\partial K}{\partial \lambda} = 0$$

Kuhn–Tucker conditions are:

1. $K_{x_1} = f_{x_1} - \lambda \cdot g_{x_1} = 0$
2. $K_{x_2} = f_{x_2} - \lambda \cdot g_{x_2} = 0$
3. $\lambda \cdot g = 0$
4. $g \le 0$
5. $\lambda \le 0$ (*if Z is min*)
6. $x_1, x_2 \ge 0$

In equation (3), *either* $\lambda = 0$ *or* $g = 0$.

From solving these different cases, & verify whether all above conditions are satisfied

KUHN-TUCKER CONDITIONS

Now, $\lambda \cdot g = \lambda \cdot (x_1 + x_2 - 1) = 0$ *But as,* $\lambda = -2 \leq 0$

$0 \neq$ --- $\neq$

$$\rightarrow x_1 + x_2 - 1 = 0 \Rightarrow x_1 + x_2 = 1$$

from (1), $3x_1^2 - 4 - \lambda = 0 \Rightarrow 3x_1^2 - 4 - (-2) = 0$

$$3x_1^2 = 2 \quad \Rightarrow x_1 = \sqrt{\frac{2}{3}} = 0.8165$$

Since, $x_1 + x_2 = 1 \Rightarrow x_2 = 1 - \sqrt{\frac{2}{3}} = (1 - 0.8165)$

$$\Rightarrow \quad x_1 = \sqrt{2/3} \; and \; x_2 = (1 - 0.8165)$$

These, values satisfies all the necessary conditions

The optimal solution is:

$$x_1 = 0.8165 \; and \; x_2 = (1 - 0.8165)$$

$$Z_{min} = x_1^3 - 4x_1 - 2x_2 = (0.8165)^3 - 4(0.8165) - 2(1 - 0.8165) = -3.0887$$