Design& Analysis of Algorithms

Session -23

MAXIMUM FLOW

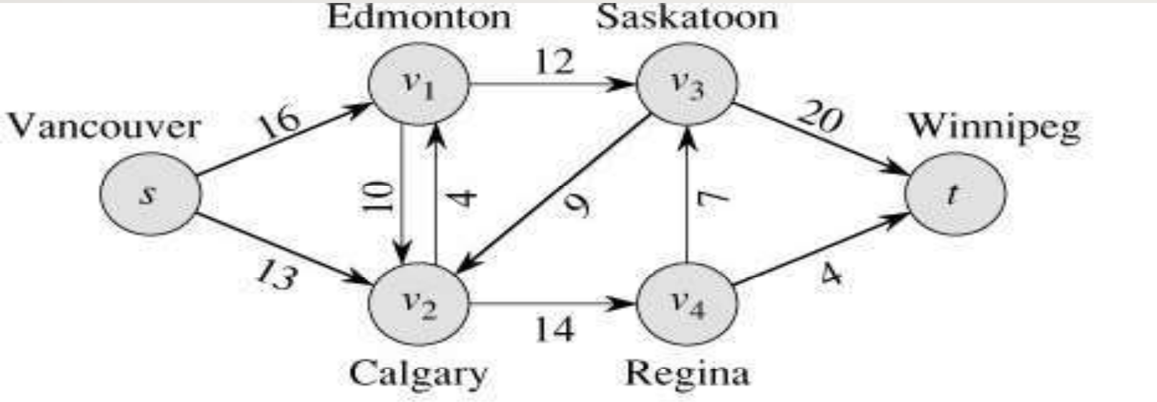
IT IS DEFINED AS THE MAXIMUM AMOUNT OF FLOW THAT THE NETWORK WOULD ALLOW TO FLOW FROM SOURCE TO SINK.

MULTIPLE ALGORITHMS EXIST IN SOLVING THE MAXIMUM FLOW PROBLEM.

TWO MAJOR ALGORITHMS TO SOLVE THESE KIND OF PROBLEMS ARE FORD-FULKERSON ALGORITHM AND DINIC'S ALGORITHM.

WHAT IS NETWORK FLOW ?

- In graph theory, a flow network is defined as a directed graph involving a source(S) and a sink(T) and several other nodes connected with edges.
- Each edge has an individual capacity c(u,v)>=0 which is the maximum limit of flow that edge could allow.
- If (u,v) is not in E assume c(u,v)=0.
- Flow in a network is an integer-valued function f defined on the edges of G satisfying $0 \le f(u,v) \le c(u,v)$, for every edge (u,v) in E.
- Assume that every vertex v in V is on some path from s to t.
- Following is an illustration of a network flow:



Initially,
c(s,v1)=16
c(v1,s)=0
c(v2,s)=0 …

CONDITIONS FOR NETWORK FLOW

For each edge (u,v) in E, the flow f(u,v) is a real valued function that must satisfy following 3 conditions :

- Capacity Constraint : $\forall\ u,v \in V,\ f(u,v) \le c(u,v)$
- Skew Symmetry : $\forall\ u,v \in V,\ f(u,v) = -f(v,u)$
- Flow Conservation: $\forall\ u \in V - \{s,t\}\ \sum_{v \in V} f(s,v) = 0$

Skew symmetry condition implies that f(u,u)=0.

THE VALUE OF A FLOW.

The value of a flow is given by :

$$|f| = \sum_{v \in V} f(s,v) = \sum_{v \in V} f(v,t)$$

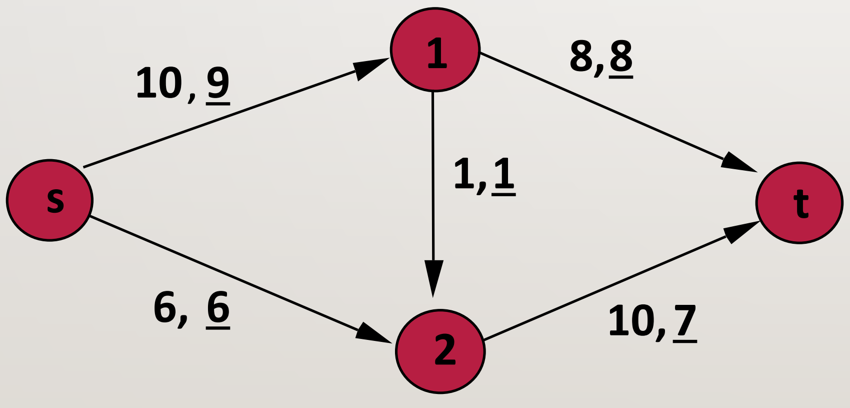
The flow into the node is same as flow going out from the node and thus the flow is conserved. Also the total amount of flow from source s = total amount of flow into the sink t.

THE FORD-FULKERSON METHOD

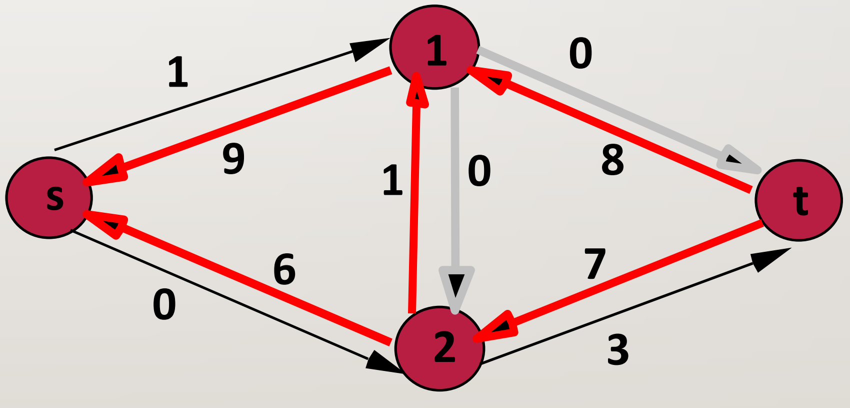
- It was developed by L. R. Ford, Jr. and D. R. Fulkerson in 1956.
- A residual network graph indicates how much more flow is allowed in each edge in the network graph.
- The residual capacity of the network with a flow f is given by:

The residual capacity (rc) of an edge (i,j) equals **c(i,j) – f(i,j)** when (i,j) is a forward edge, and equals **f(i,j)** when (i,j) is a backward edge. Moreover the residual capacity of an edge is always non-negative

$$c_f(u,v) = c(u,v) - f(u,v)$$



Original Network



Residual Network

AUGMENTING PATHS

- An augmenting path is a simple path from source to sink which do not include any cycles and that pass only through positive weighted edges.
- If there are no augmenting paths possible from S to T, then the flow is maximum.
- The result i.e. the maximum flow will be the total flow out of source node which is also equal to total flow in to the sink node.

Implementation

- An augmenting path in residual graph can be found using DFS or BFS.
- Updating residual graph includes following steps: (refer the diagrams for better understanding)
  - For every edge in the augmenting path, a value of minimum capacity in the path is subtracted from all the edges of that path.
  - An edge of equal amount is added to edges in reverse direction for every successive nodes in the augmenting path.

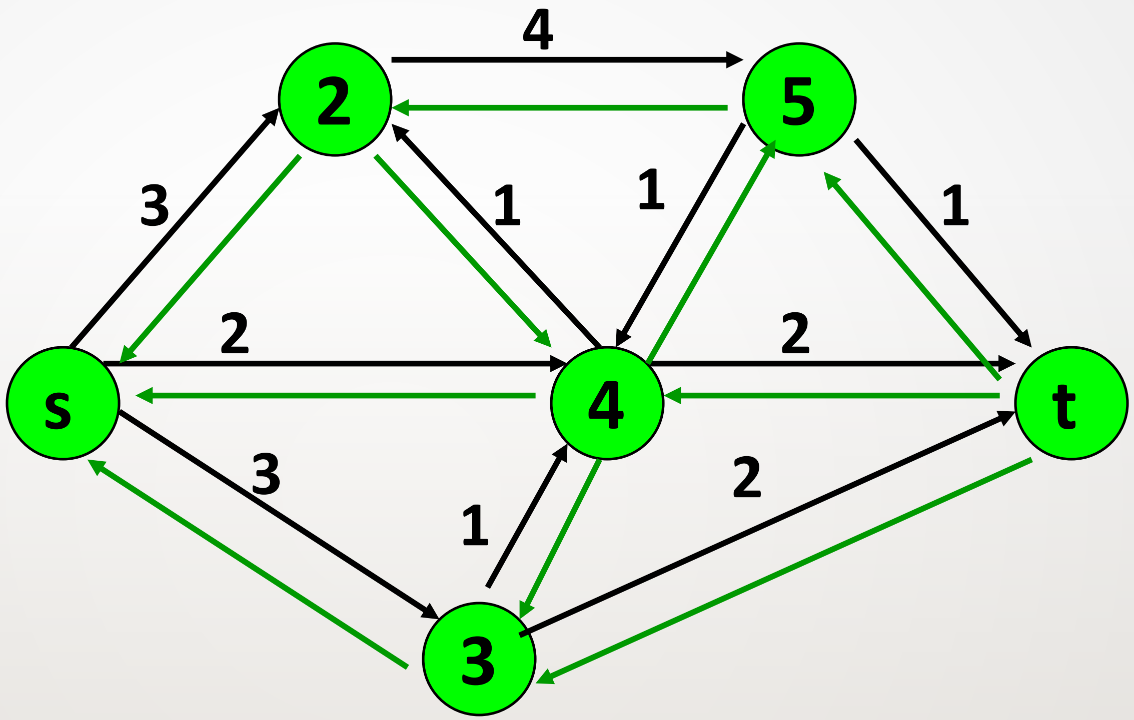
THE FORD-FULKERSON'S ALGORITHM

```
FORDFULKERSON(G,E,s,t)
FOREACH e ∈ E
    f(e) ← 0
G_f ← residual graph
WHILE (there exists augmenting path P)
    f ← augment(f, P)
    update G_f
ENDWHILE
RETURN f
```

```
AUGMENT(f,P)
b ← bottleneck(P)
FOREACH e ∈ P
    IF (e ∈ E)
        // backwards arc
        f(e) ← f(e) + b
    ELSE
        // forward arc
        f(e^R) ← f(e) - b
RETURN f
```

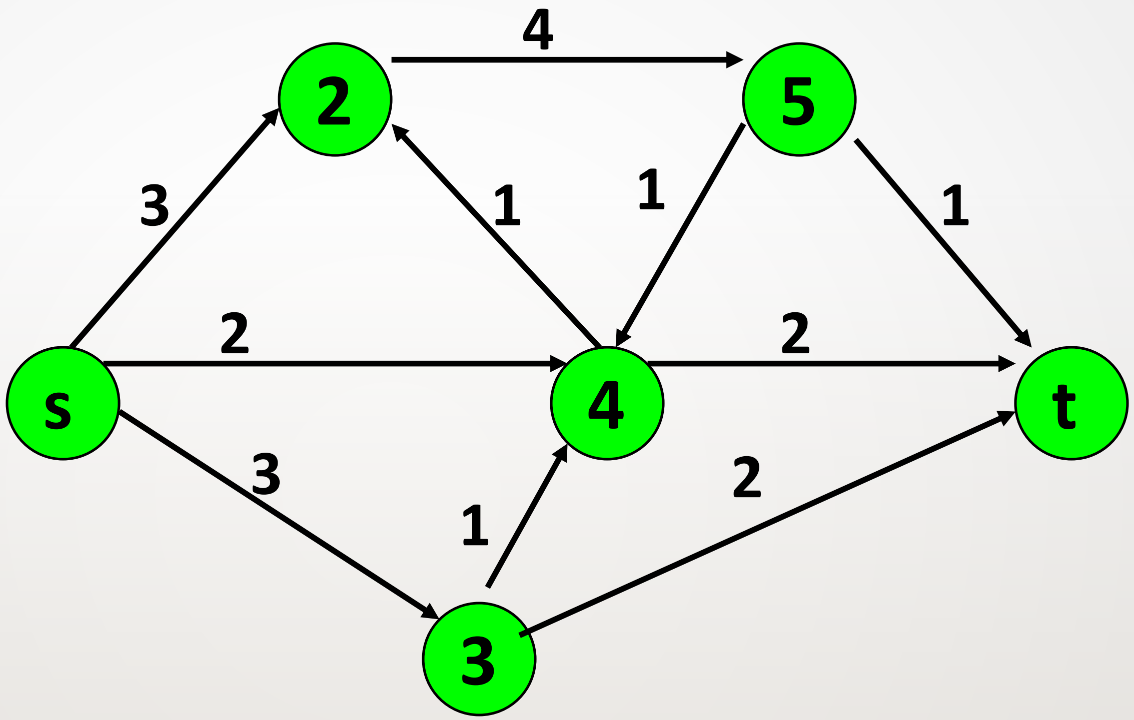
Click To See Ford Fulkerson's Algorithm In Action (Animation)

FORD-FULKERSON MAX FLOW



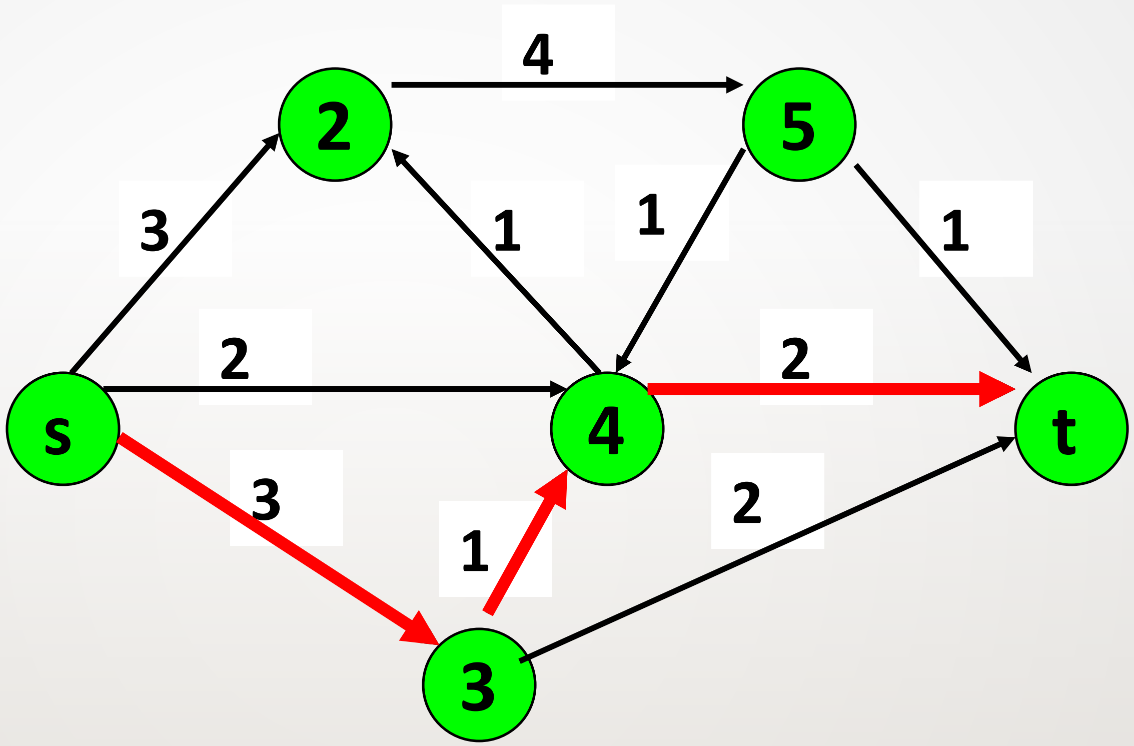
This is the original network, plus reversals of the arcs.

FORD-FULKERSON MAX FLOW



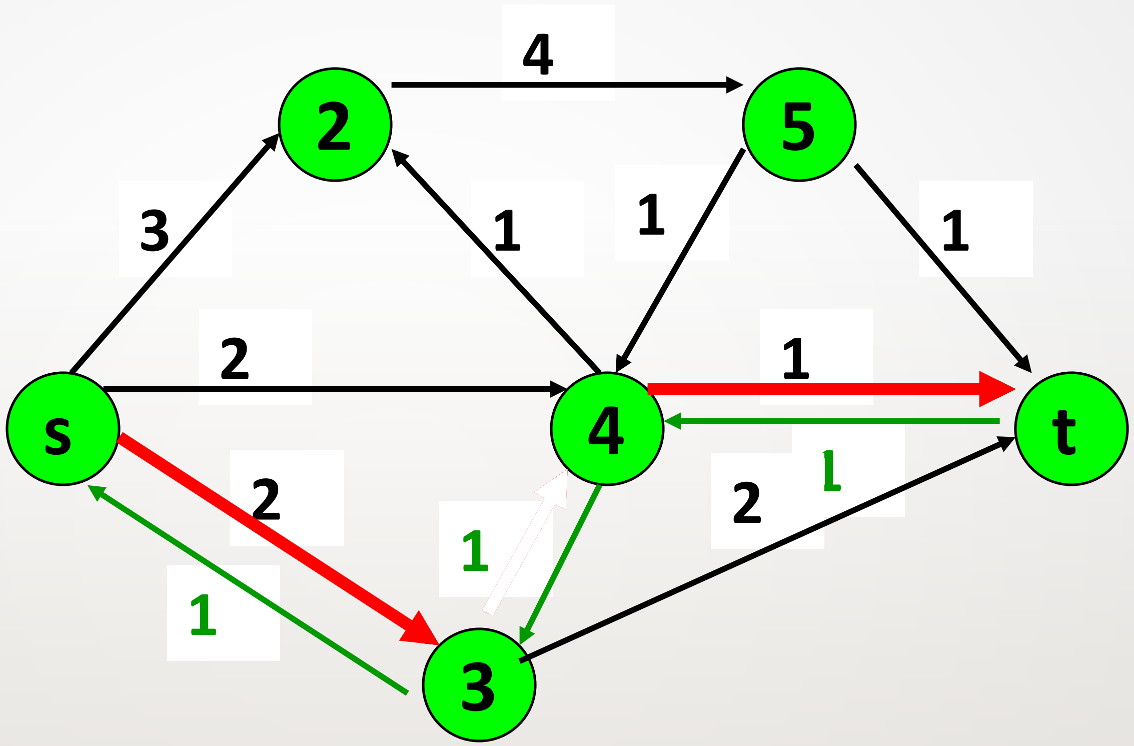
This is the original network, and the original residual network.

FORD-FULKERSON MAX FLOW



Find any s-t path in G(x)

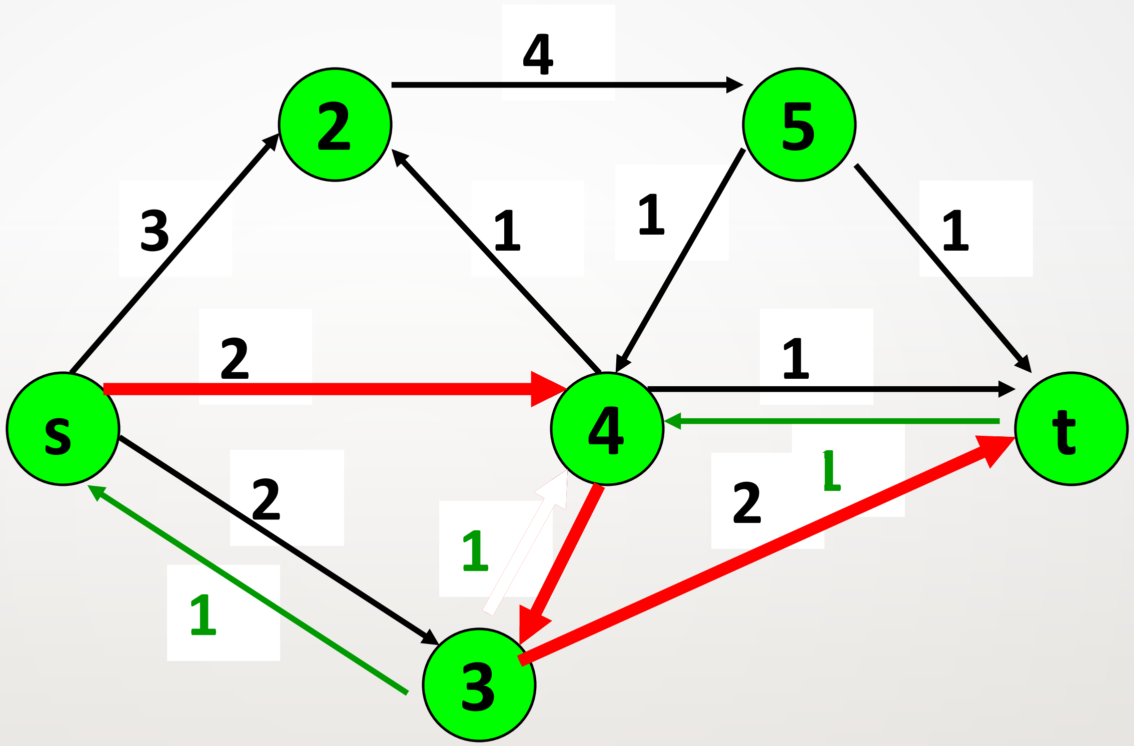
FORD-FULKERSON MAX FLOW



Determine the capacity Δ of the path.

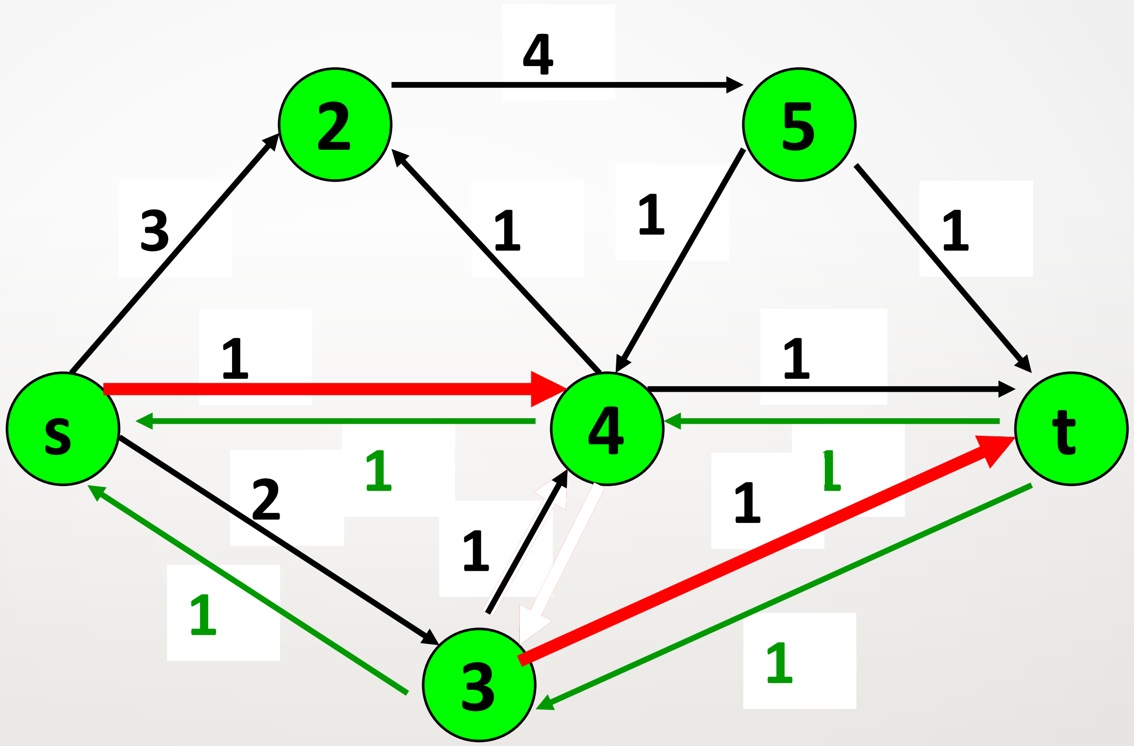
Send Δ units of flow in the path.
Update residual capacities.

FORD-FULKERSON MAX FLOW



Find any s-t path

FORD-FULKERSON MAX FLOW



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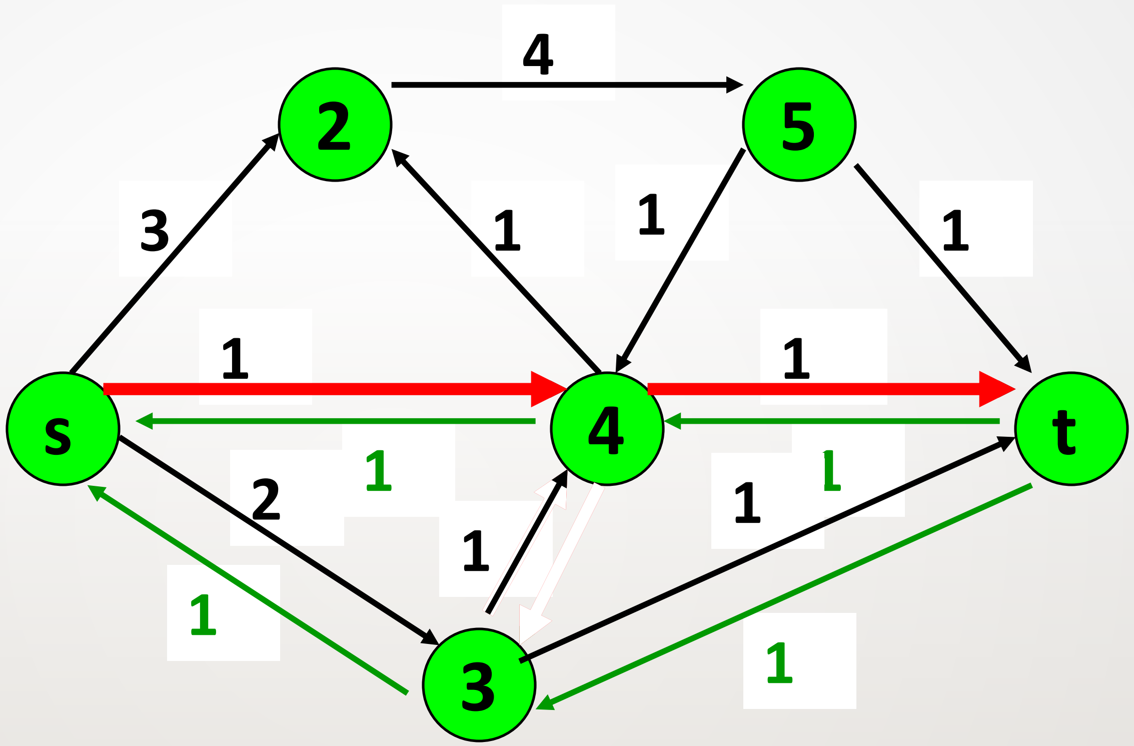
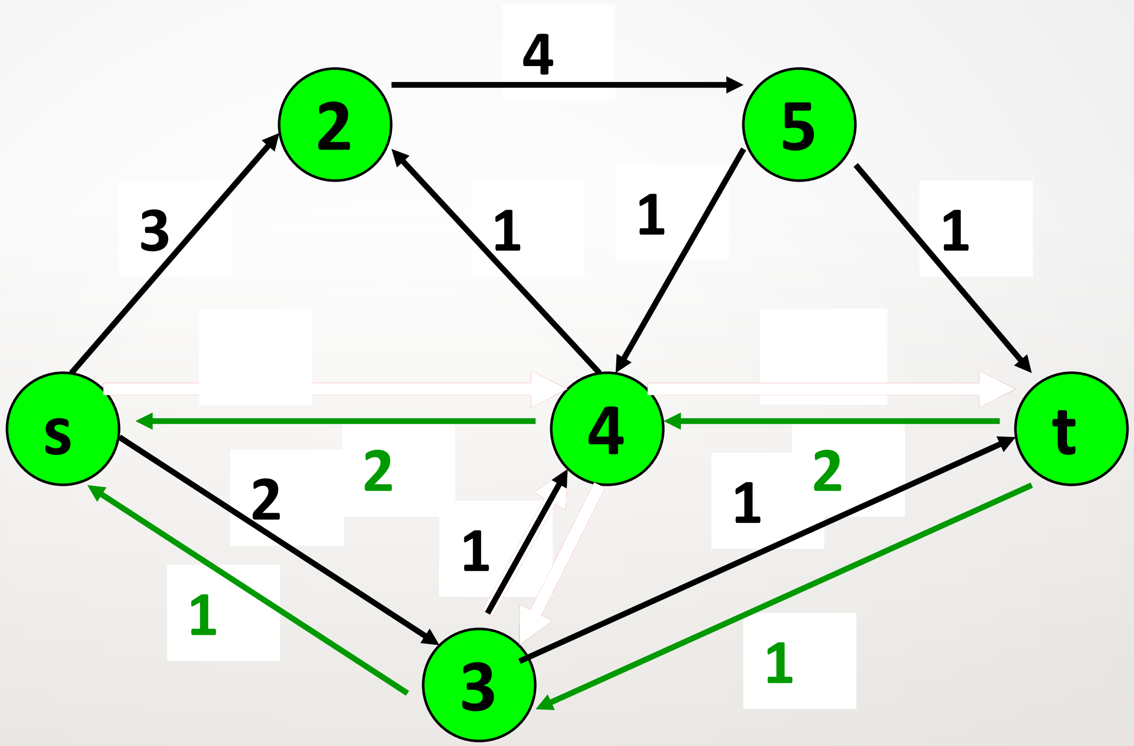


Find any s-t path

FORD-FULKERSON MAX FLOW



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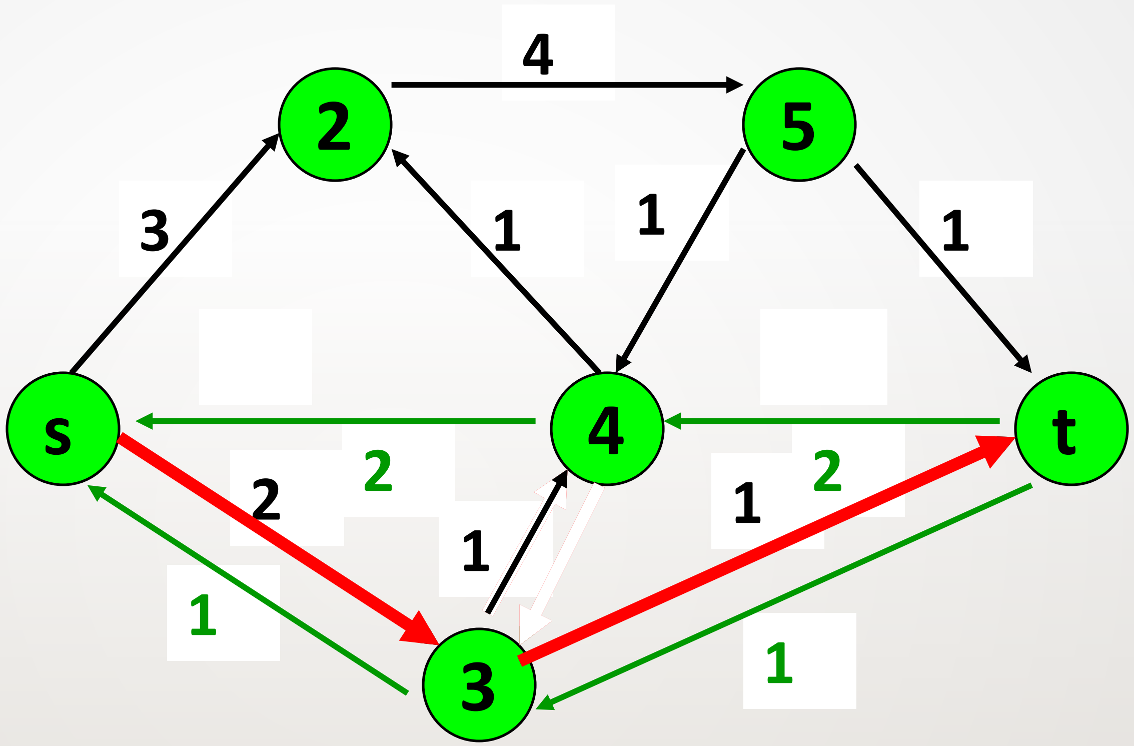
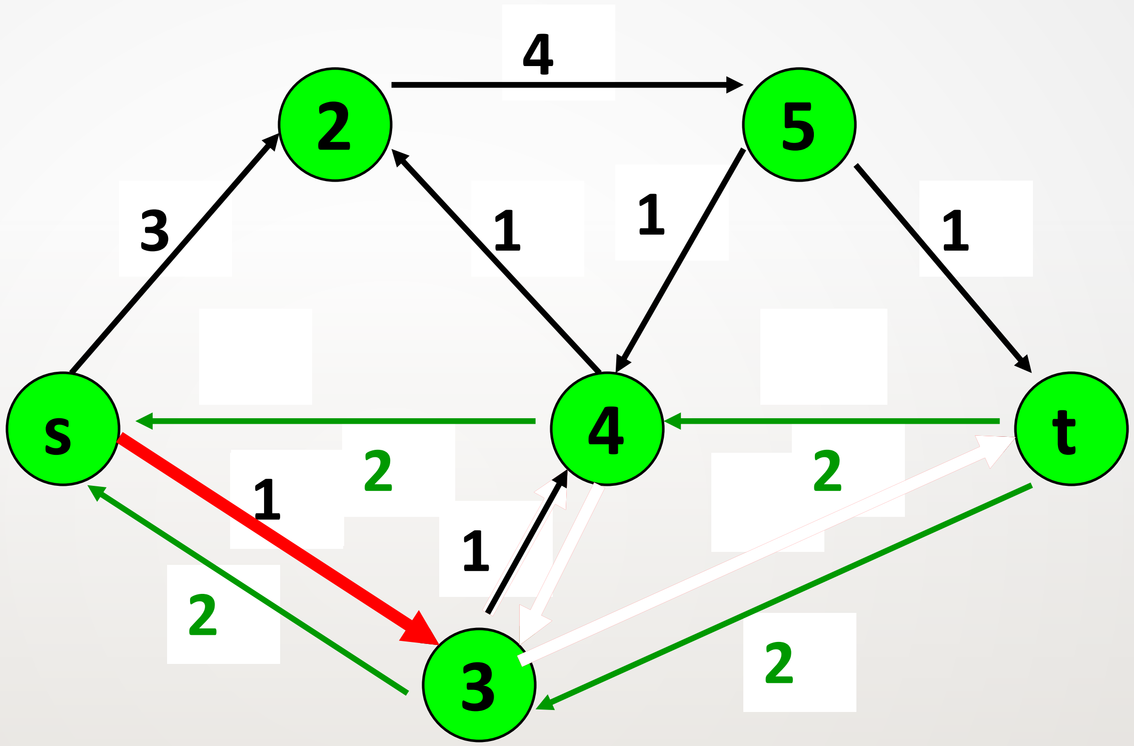
FORD-FULKERSON MAX FLOW



Find any s-t path

FORD-FULKERSON MAX FLOW



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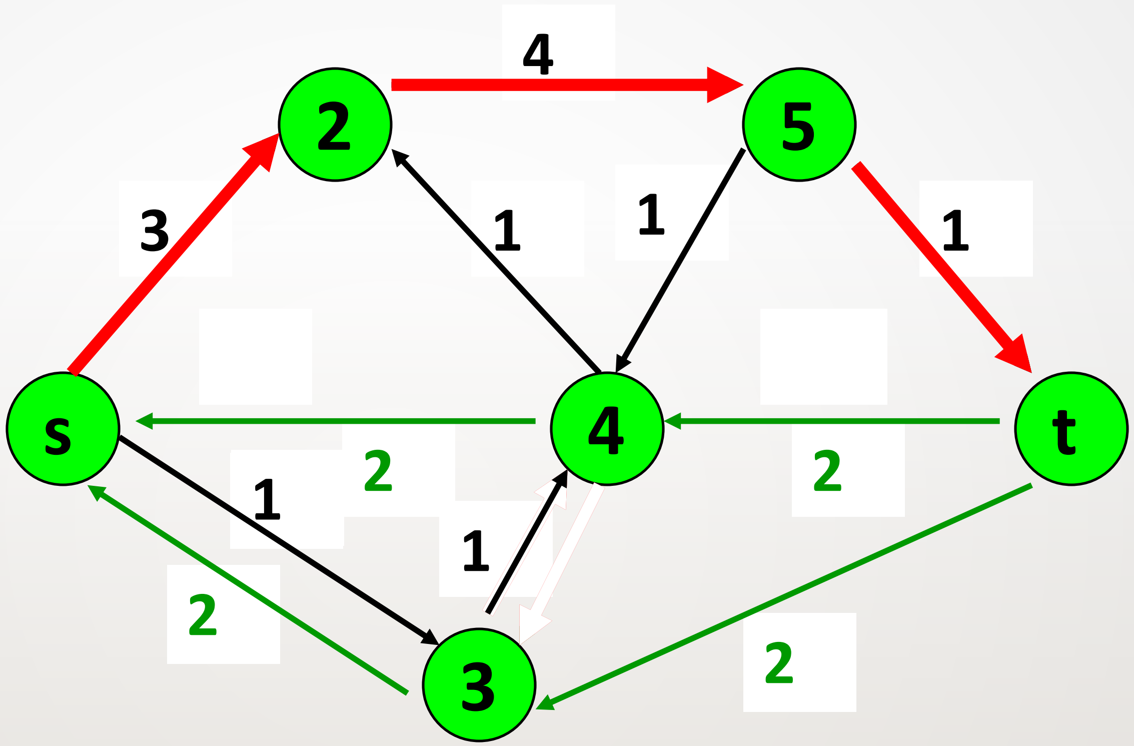
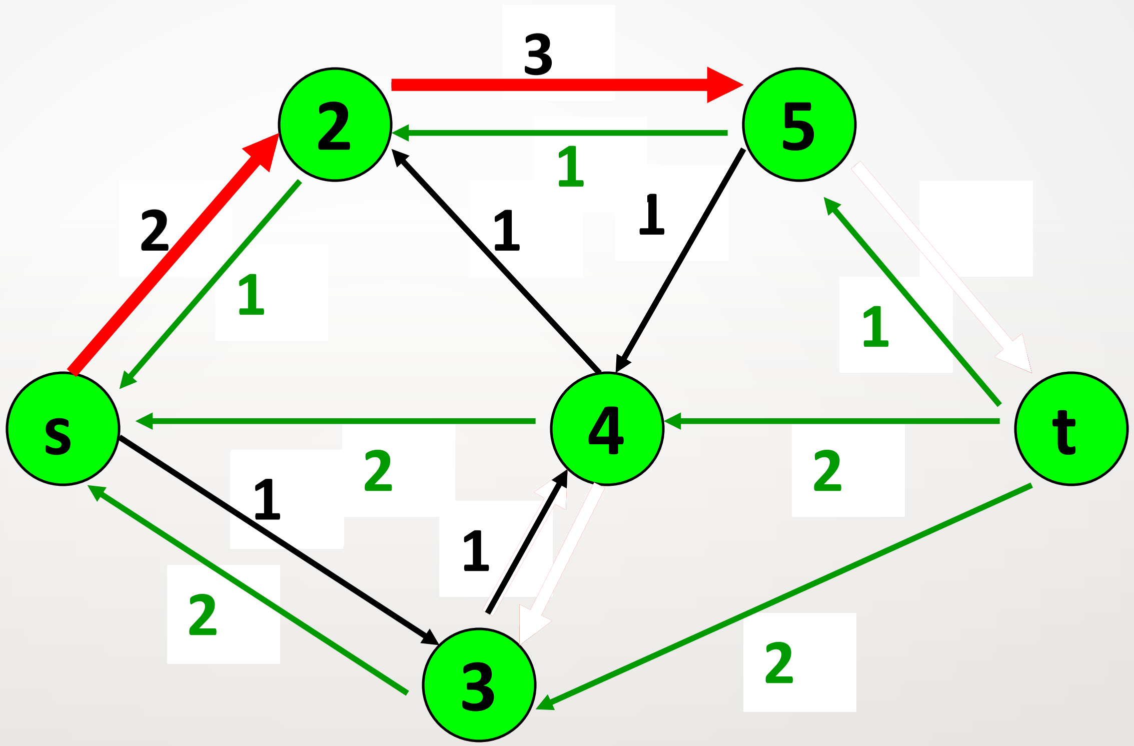
FORD-FULKERSON MAX FLOW



Find any s-t path

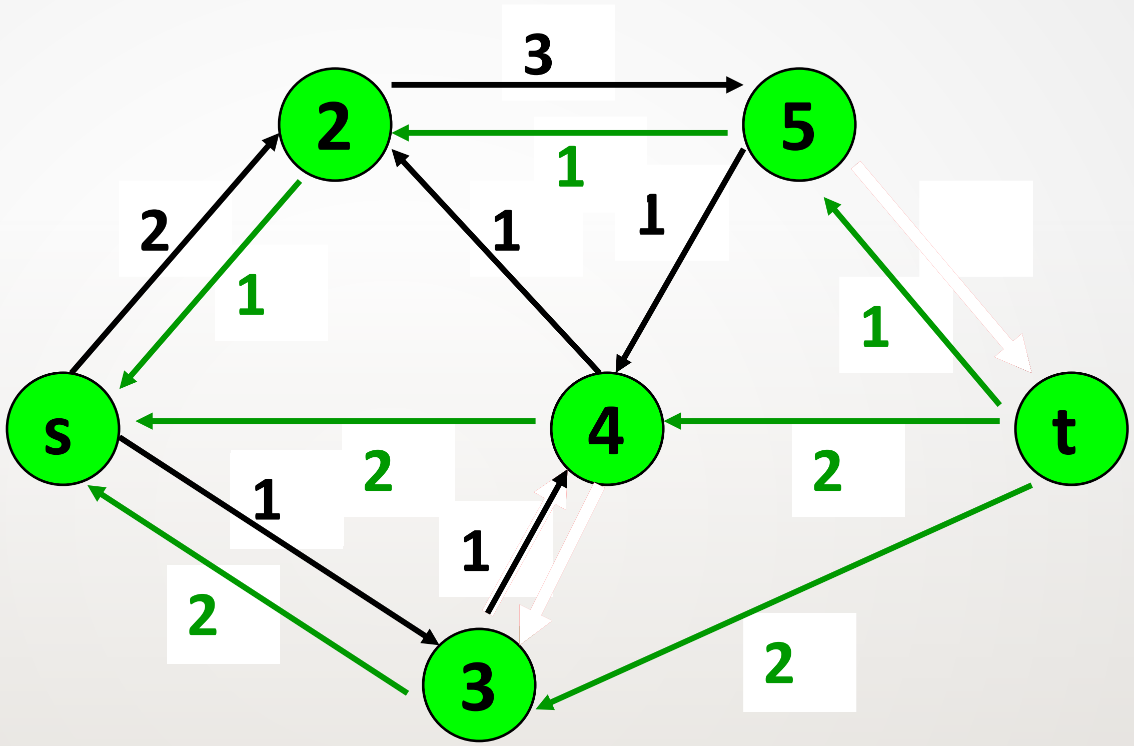
FORD-FULKERSON MAX FLOW



Determine the capacity Δ of the path.

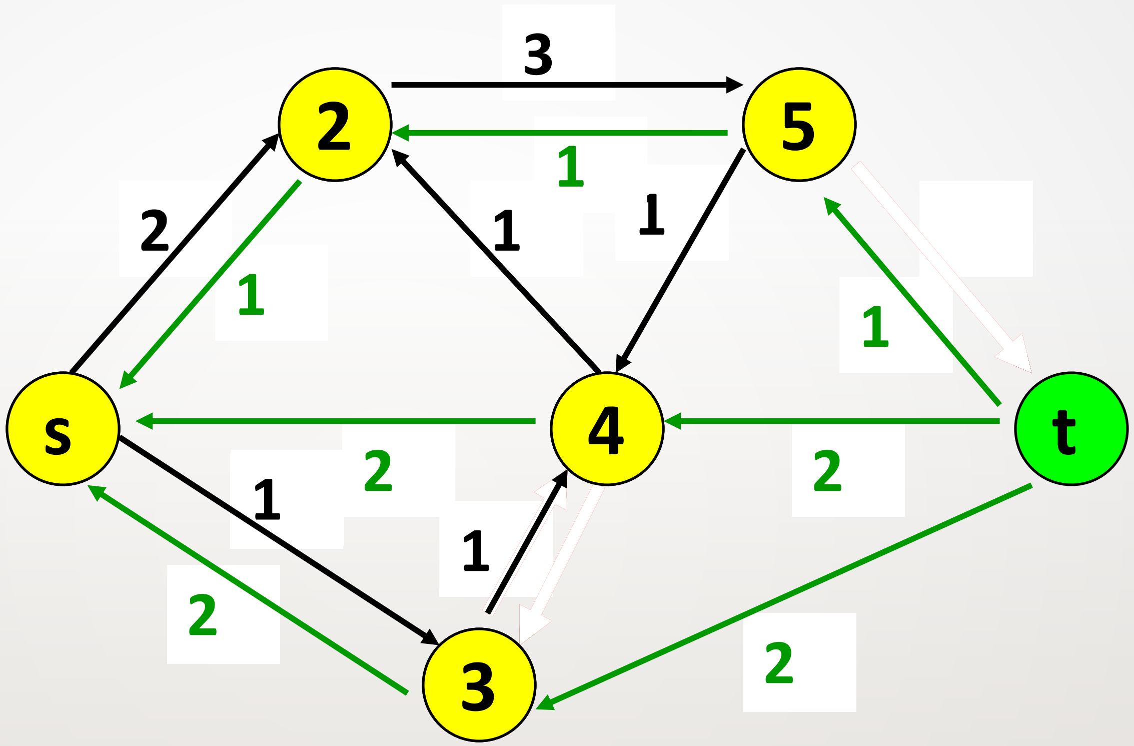
Send Δ units of flow in the path.
Update residual capacities.

FORD-FULKERSON MAX FLOW



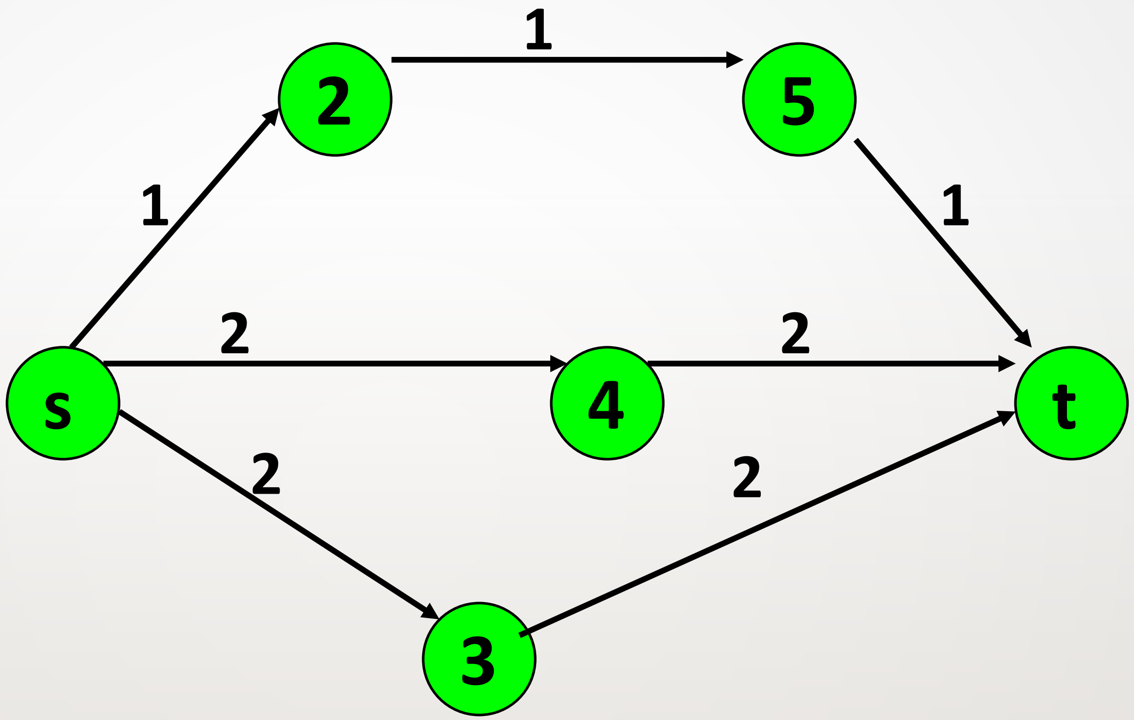
There is no s-t path in the residual network.
This flow is optimal

FORD-FULKERSON MAX FLOW



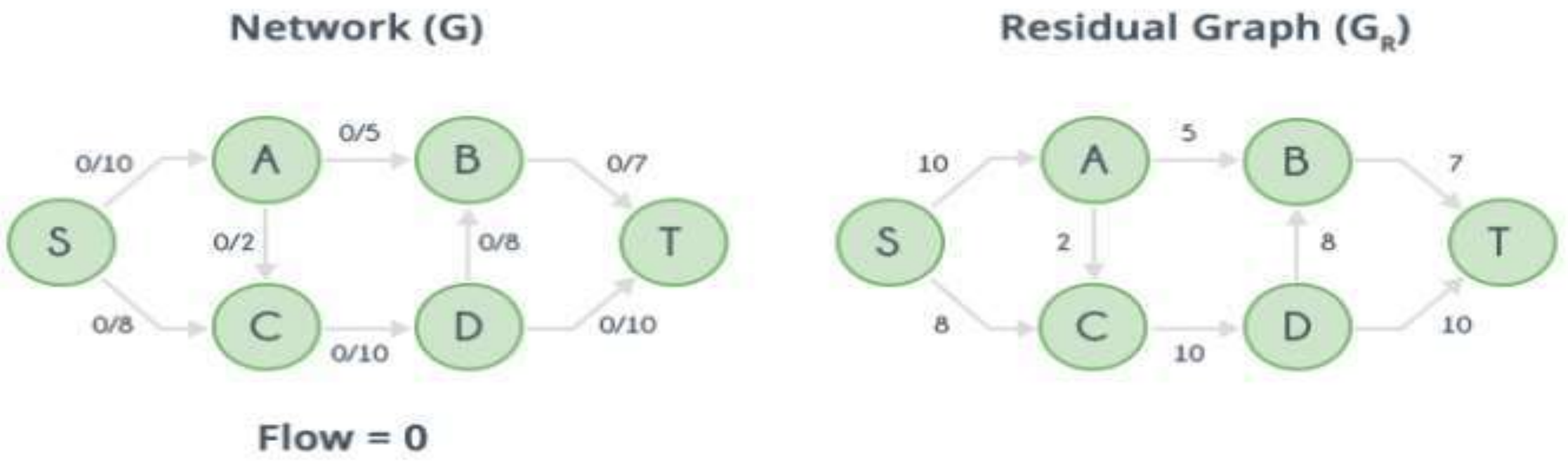
These are the nodes that are reachable from node s.

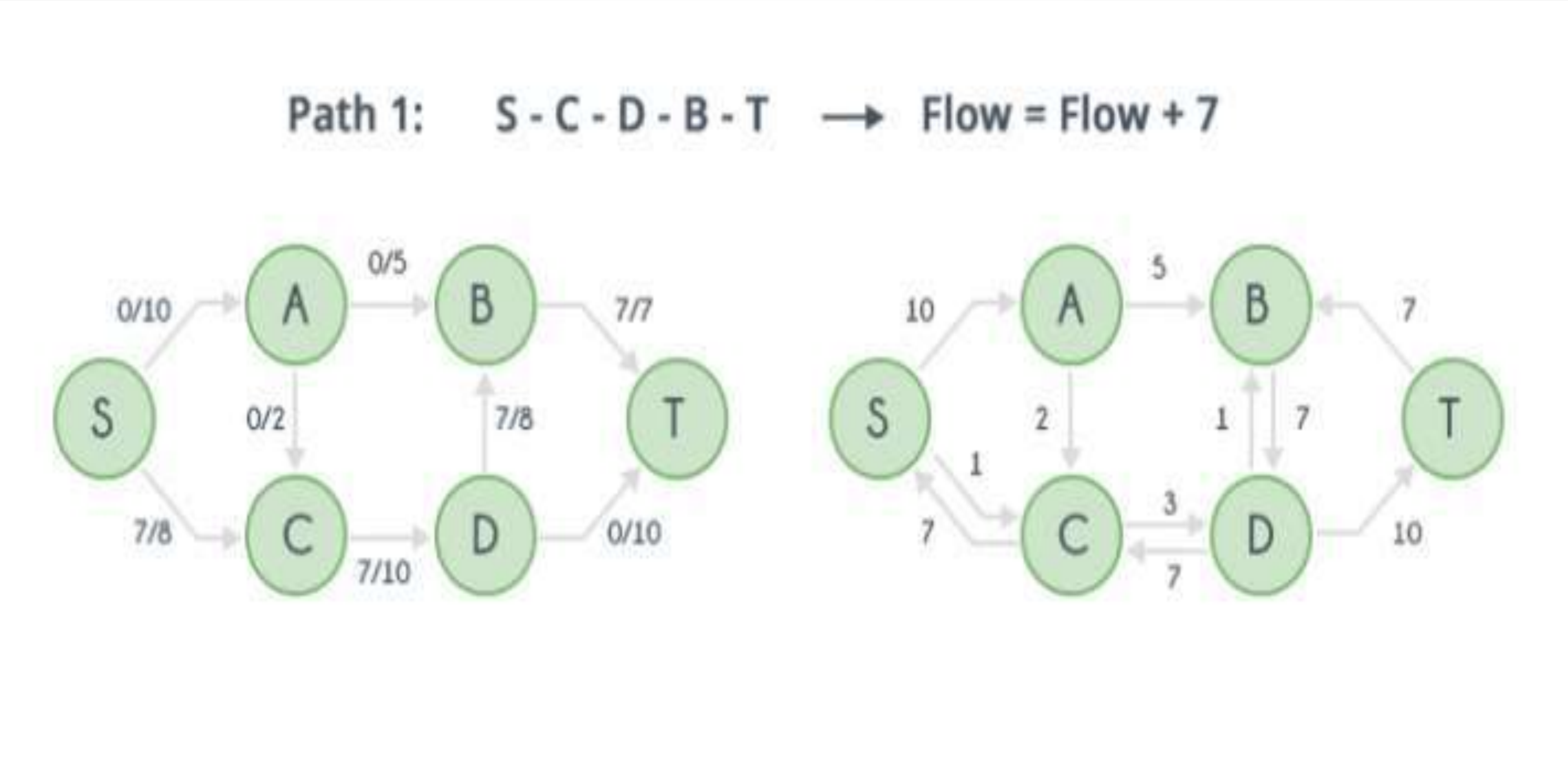
FORD-FULKERSON MAX FLOW

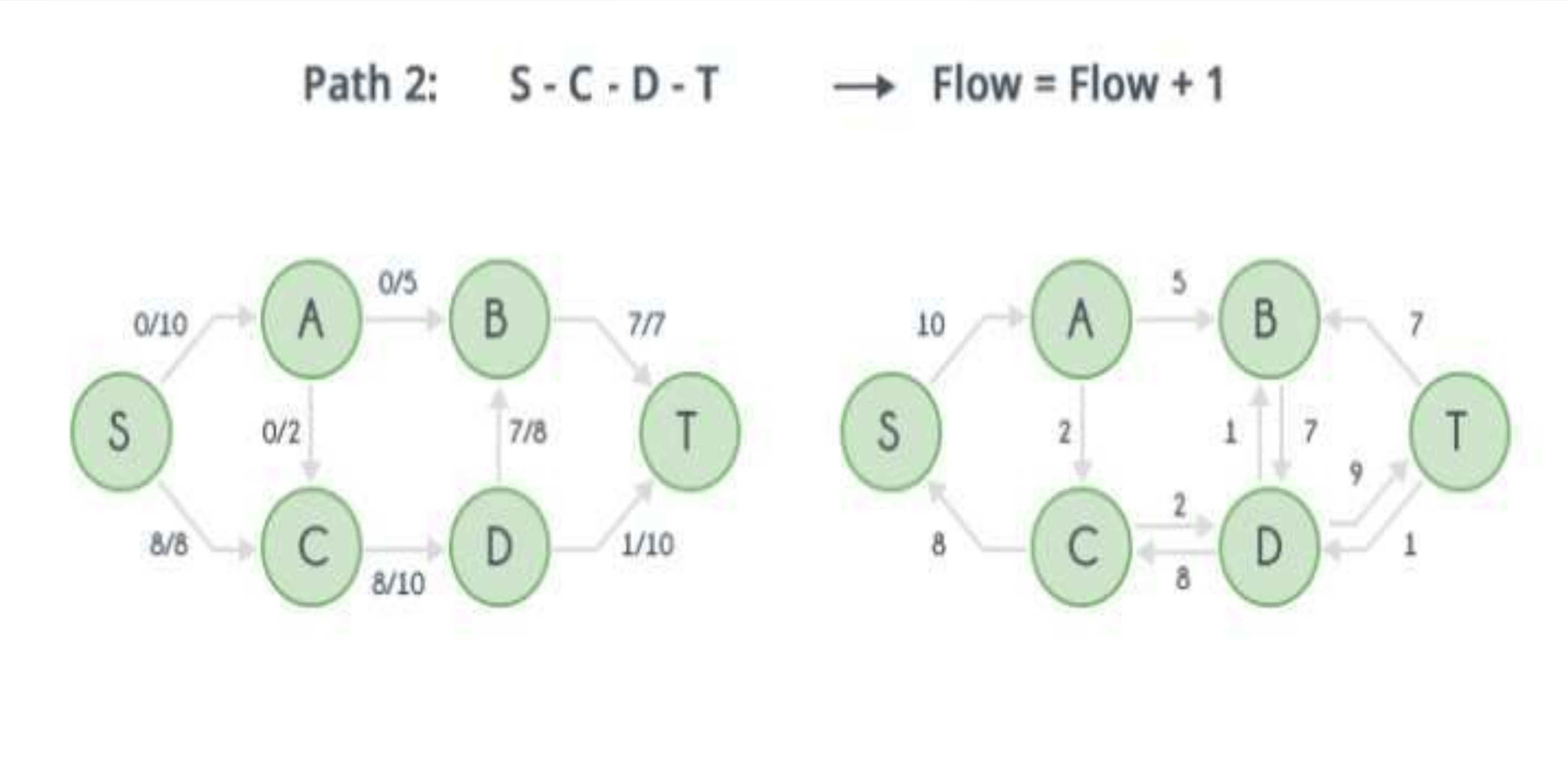


Here is the optimal flow=1+2+2 = 5

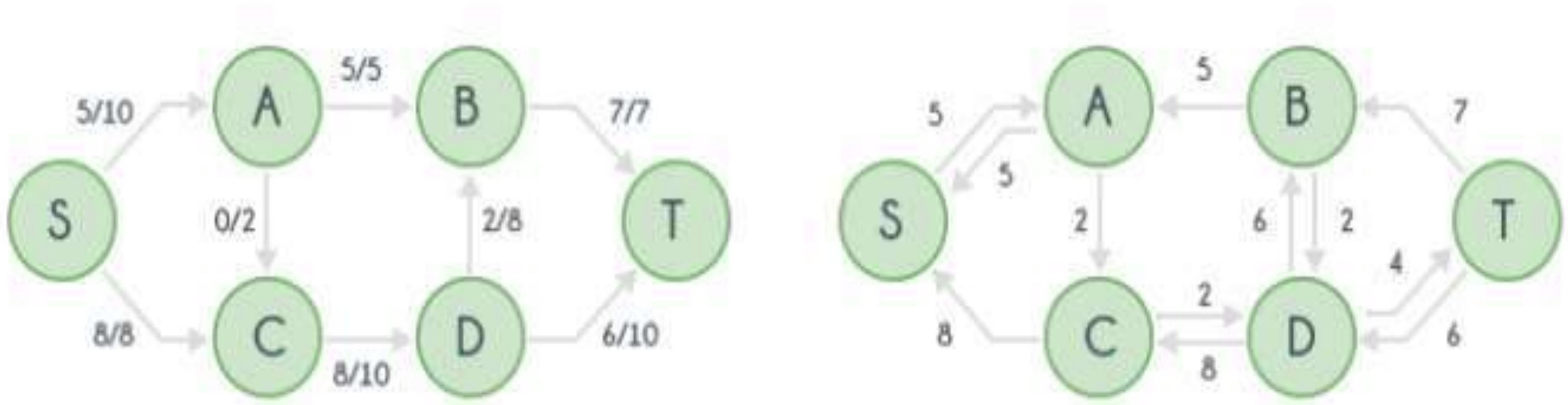
A demonstration of working of Ford-Fulkerson algorithm is shown below with the help of diagrams.




Path 1: S - C - D - B - T → Flow = Flow + 7
0/10
0/5
7/7
0/2
7/8
7/8
7/10
0/10
S
A
B
C
D
T
10
5
7
2
1
7
1
3
7
7
10
S
A
B
C
D
T


Path 2: S - C - D - T → Flow = Flow + 1
0/10
0/5
7/7
0/2
7/8
8/8
8/10
1/10
S
A
B
C
D
T
10
5
7
2
1
7
9
2
8
8
1

Path 3: S - A - B - T → Flow = Flow + 5



Path 4: S - A - C - D - T → Flow = Flow + 2

No More Paths Left

Max Flow = 15

SAMPLE QUESTIONS

- How does the Ford-Fulkerson algorithm handle multiple sources and sinks in the network
- What is Maximum flow in Ford-Fulkerson algorithm
- What is the residual network, and how is it used in the Ford-Fulkerson algorithm
- Discuss the basic steps involved in the Ford-Fulkerson algorithm