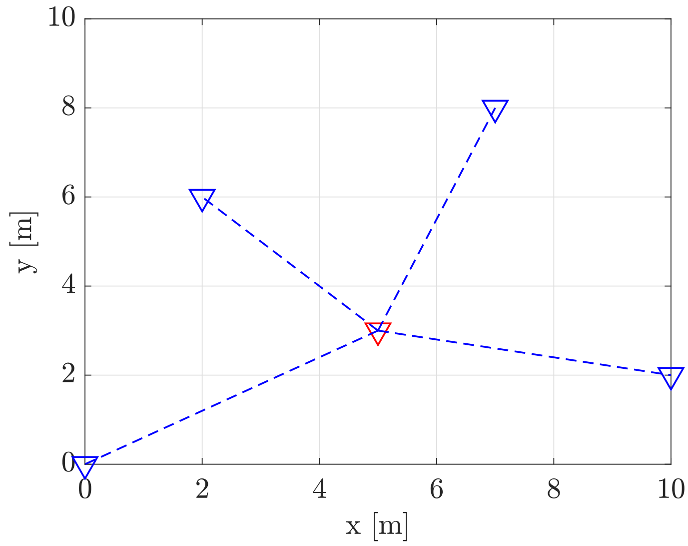

Figure 1: Illustration of the problem. In the figure, the blue triangles illustrate the BS locations, the red triangle depicts position of the car and the dashed lines illustrate the range measurements.

Problem 1 – Let us consider a 2D scenario illustrated in Fig. 1 in which an autonomous car localizes itself by measuring the range to nearby base stations (BSs). The position of the car is $\mathbf{x} = [x_{\text{car}},\ y_{\text{car}}]^\top$ and location of the jth BS is $\mathbf{s}_j = [x_j,\ y_j]^\top$. The range between the car and jth BS is given by

$$\mathbf{g}_j(\mathbf{x}) = \sqrt{(x_j - x_{\text{car}})^2 + (y_j - y_{\text{car}})^2}. \tag{1}$$

Assuming, the car can measure the range to M BSs simultaneously, the noisy measurements $\mathbf{y} = [y_1, \dots, y_M]^\top$ can be written as

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{r}, \quad \text{where} \tag{2}$$

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \sqrt{(x_1 - x_{\text{car}})^2 + (y_1 - y_{\text{car}})^2} \\ \vdots \\ \sqrt{(x_M - x_{\text{car}})^2 + (y_M - y_{\text{car}})^2} \end{bmatrix},$$

and $\mathbf{r}$ denotes the measurement noise which is assumed zero-mean Gaussian with covariance $\mathbf{R}$, that is, $\mathbf{r} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$.

Your task is to develop an estimation algorithm for the nonlinear model given (2) *which minimizes the weighted least squares cost function, given by*

$$J_{WLS}(\mathbf{x}) = (\mathbf{y} - \mathbf{g}(\mathbf{x}))^\top \mathbf{R}^{-1} (\mathbf{y} - \mathbf{g}(\mathbf{x})), \tag{3}$$

with respect to $\mathbf{x}$.

Problem 2 – Let us assume the true location of the car is $\mathbf{x} = [5,\ 3]^\top$, there are four BSs ($M = 4$) for which the positions are:

$$\begin{aligned}\mathbf{S} &= [\mathbf{s}_1,\ \mathbf{s}_2,\ \mathbf{s}_3,\ \mathbf{s}_4], \\ &= \begin{bmatrix} 2 & 0 & 10 & 7 \\ 6 & 0 & 2 & 8 \end{bmatrix}\end{aligned}$$

and the covariance matrix is:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

Implement your estimation algorithm from Problem 1 and using the parameters above[1]*. In your answer, visualize the problem and solution of the estimator.*

Problem 3 – The Cramér-Rao lower bound (CRLB) provides a lower bound on the variance of an unbiased estimator, defined by the inverse of the Fisher information matrix (FIM). Therefore, the unbiased estimator $\hat{\mathbf{x}}$ satisfies $\mathrm{Cov}(\hat{\mathbf{x}}) = \mathrm{E}[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^\top] \leq \mathbf{J}(\mathbf{x})^{-1}$, where $\mathbf{J}(\mathbf{x}) \in \mathbb{R}^2$ denotes the FIM. From the FIM, the position error bound (PEB) can be computed as

$$\mathrm{PEB} = \sqrt{[\mathbf{J}(\mathbf{x})^{-1}]_{1,1} + [\mathbf{J}(\mathbf{x})^{-1}]_{2,2}}. \tag{4}$$

Derive the CRLB for the considered problem and illustrate the PEB in the range $x_{car} \in [0, 10]$ *and* $y_{car} \in [0, 10]$. *Does your estimator achieve the derived PEB?*

[1] Use the coding language of your choice! Matlab or Python are preferred but other languages are also possible.