Expected Value Problems

Theory:

- Let W be the sample set.
- Expected Value of random variable X on W.

  = Sum (p[w_i] * X[w_i]).
- Here w_i is any specific event from W.

Linearity of Expectation

- E [X+Y] = E[X] + E[Y].
- X,Y can be dependent or independent.

If X and Y are independent:

- E[XY] = E[X] * E[Y]

Problem 1:

Suppose you have bought 10$ lottery ticket. Probability of winning $100 is 5% and $200 is 1%. What is the expected value you can win?

Sol:

EV = 5/100 * 100 + 1/100*200 + 94/100*0

= 7$.

Problem 2:

Two fair dice are rolled twice. If S is the sum of the numbers and P is the product of the numbers on them, What is the expected value of S+P?

Sol:

E[S+P] = E[S] + E[P]

= 3.5*2 + 3.5^2 = 19.25

Problem 3

Caroline is going to flip 10 fair coins. If she flips 'n' heads she will be paid $n. What is the expected value of her payout?

Sol:

EV = 10 * EV (X[1]) = 10 * 0.5 = 5$.

Problem 4

Sammy is lost and starts to wander aimlessly. Each minute he walks 1m forward with 1/2, stays where he is with probability 1/3 and 1 m backward with probability 1/6. What is the expected value Sammy moves in 1hr?

Sol:

EV (1 min) = 1 * ½ + 0 * ⅓ - 1 * ⅙ = ⅓

Expected value in 1 hr = 20m.

Problem 5

The digits 1, 2, 3, and 4 are randomly arranged to form two digit numbers AB and CD. What is the expected value of AB * CD?

Sol:

E[AB * CD] != E[AB] * E[CD], AB and CD are dependent.

AB * CD = (10A + B) * (10C + D) = 100AC + 10BC + 10AD + BD.

By symmetry, E[AC] = E[BC] = E[AD] = E[BD].

E[AB * CD] = 121 E [AC] = ⅙ [1*2 + 1*3 + 1*4 + 2*3 + 2*4 + 3*4]

= 705.83

Problem 6 (Using Indicator Variables)

A box contains yellow, orange, green, blue balls. Billy randomly selects 4 balls from the box with replacement. What is the expected value for the number of distinct balls Billy selects?

Sol:

One way (which is clumsy) is to calculate the probabilities of getting 1,2,3,4 distinct values and finding out the expected value.

Alternately using Linearity of Expectation:

Indicator: X1 = {
1 : If a yellow ball is present.
0 : Absent
}

Similarly define X2, X3, X4.

E (distinct balls) = E (X1 + X2 + X3 + X4) = 4 * E(X1)
= 4 * (1 - (3/4)^4) = 175/64.

Problem 7

25 independent, fair coins are tossed in a row. What is the expected number of consecutive HH pairs?

Sol:

E (sum of consecutive HH pairs) = E(X1 + X2 + X3 + ….)

X1 : {
1 : first pair is HH,
0 : -
}

$E = 24 * ½ * ½ = 6$

Problem 8 (Using states)

Allison has an unfair coin which lands on heads with probability p. What is the expected value for the number of times she will have to flip till she gets a head? [Bernouli's trial]

Sol:

state 0 : she flipped tails so far.
state 1 : she flipped a head.

E (state : 0) : EV if she gets a tail.
E (state : 1) : EV if she gets a head.

$E(x1) = 0$
$E(x0) = 1 + (1-p) * E(x0) + p * E(x1)$
$E(x0) = 1/p$

Alternative way:

$x = (1 - p) * (1 + x) + p * 1$
$x = 1 + x - p - px + p$
$x = 1 / p$

Problem 9

With each purchase of SlurpeeShack you receive a random piece of the puzzle. Once you collect all 12 pieces you get a free Slurpee. What is the expected value of the number of purchases you will need in order to collect all 12 pieces?

Sol:

Define X_i : number of purchases we need to make to get ith distinct coupon.

EV = Sum of X_i's, $1 <= i <= n$.
p[getting a distinct ith candy after (i-1) distinct candies] = $[12 - (i-1)] / 12$
Let EV $[X_i] = x$.

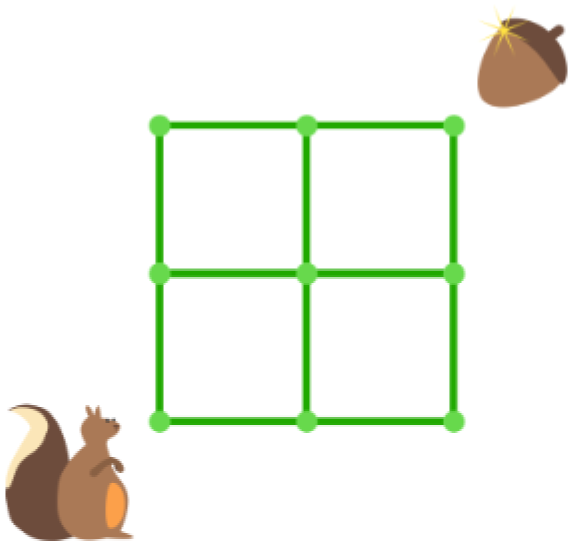
$x = p * 1 + (1-p) * (1 + x)$ [Bernoulli distribution]
$x = p + 1 - p + x - px$
$px = 1$
$x = 1/p$.

$EV[X_i] = 12 / [12 - (i-1)]$.
EV = Sum $12 / [12 - (i-1)]$
$= 12 (1/12 + 1/11 + \ldots + 1/1)$
~ 37.

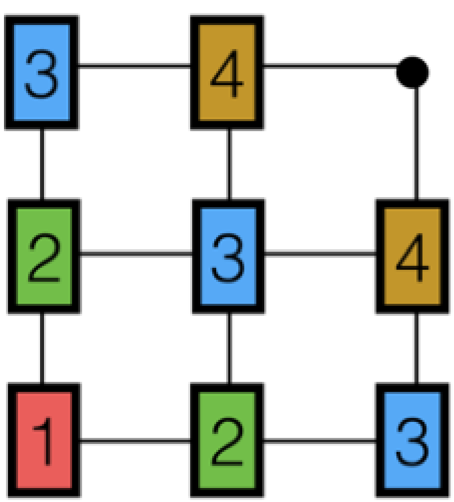
This is a famous problem often referred to as the coupon collector's problem.
In general for n coupons the expected value of purchases to get n distinct item is nH_n where H_n is the harmonic number.

For large n, $H_n \sim \ln n$.

Problem 10



Sarah the squirrel is trying to find her acorn but she can't remember where she left it. She starts at the lower left corner of the grid and randomly steps on to one of the adjacent vertices. What is the expected value for the number of steps for her to find her acorn?



1st claim:

- EV (center vertex) = E_3.
- Proof:
  - $E_C = \frac{1}{4} * (1+E_4) * 2 + \frac{1}{4} * (1+E_2) * 2$
    $= \frac{1}{2} * E_4 + \frac{1}{2} * E_2$
    $= E_3$

Transition states:

$E_1 = x$, $E_2 = y$, $E_3 = z$, $E_4 = w$.

$x = y + 1$
$y = \frac{1}{3} * (1+x) + \frac{2}{3} * (1+z)$
$z = \frac{1}{2} *(1+ y) + \frac{1}{2} * (1+w)$
$w = \frac{1}{3} * 1 + \frac{2}{3} * (1+z)$

Solve:

$y = x - 1$
$3(x - 1) = 3 + x + 2z$
$2x = 2z + 6$
$2x - 6 = 2 + y + w = 2 + x - 1 + w$
$w = x - 7$
$3w = 3 + 2z$
$3x - 21 = 3 + 2x - 6$
$x = 21 - 3 = 18$

Problem 11

Jorge has an N sided fair die and wonders how many times he would need to roll until he has rolled all the numbers from 1 to N.

$n * H_n = N$ (solve for n).

Problem 12

An infinite line of stepping stones stretches out into an infinitely large lake. A frog starts on the second stone from the shore. Every jump he has a 60% chance of jumping one stone closer to the shore and 40% away. What is the expected value of the number of jumps he will take before reaching the first stone?

Sol:

_1_ _2_ _3_ _4_ _4_ _5_

Let x_i be the expected number of jumps to reach stone i from stone_i.

$p = 60\%$

$x_1 = 0$
$x_2 = p * (1+x_1) + (1-p) * (1+x_3)$
and so on.
In general : $x_i = p(1+x_{(i-1)}) + (1-p)(1+x_{(i+1)})$
Solving this is not possible as it stretches to infinite.

Observation:

$x_3 = 2*x_2$ [because $x_{32} = x_2$]
$x_2 = x$.
$x= p + (1-p) * (1+2x)$
$x= p + 1 - p + 2x - 2xp \Rightarrow x = 1 / (2p - 1) = 5$.

Alternate way:

After 1 jumps, the frog closes p*-1 + (1-p) gap from the 1st stone.
= 1 - 2p from the first stone.
After n jumps: n (1 - 2p) = 1.
n = 1/(1-2p).

Problem 13 (Applications of linearity of expectations)

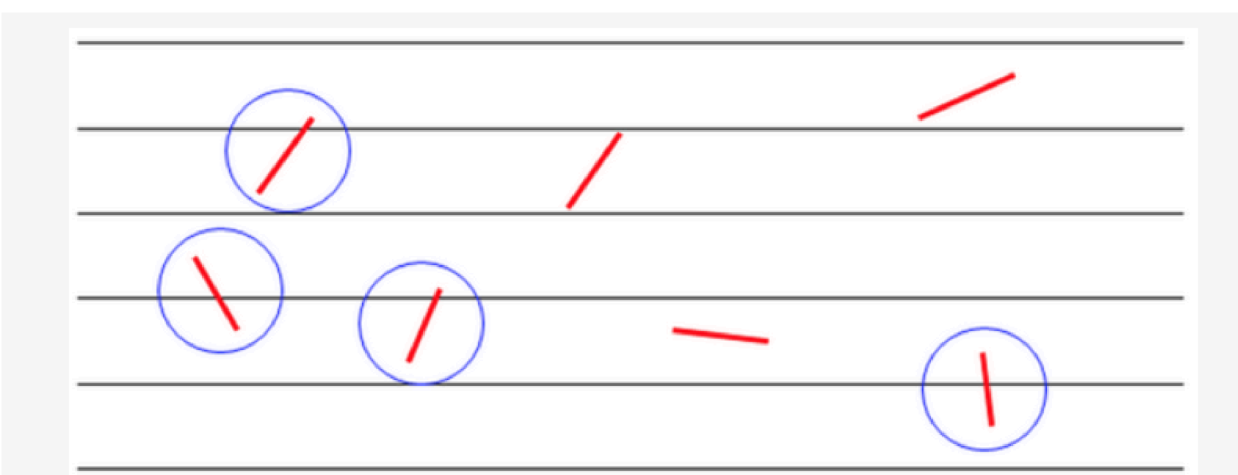
There is a lottery contest in which there are m possible tickets and n participants. Out of 'm' tickets, only 1 can win a prize. What is the probability that someone wins among n participants?

Sol:

Straightforward way:

- 1 - P (none wins)
- P (none win) = (1 - 1/m)^n

Problem 14 (Buffon's needle problem - Barbier's proof)



Suppose you drop a needle of length 1 onto a floor with strips of wood 1 unit apart; What is the probability it lands across two strips of wood?

Consider a needle piece of length 1. Let the expected number of crossing be E_1.
$E_1 = p_1 * 1 \Rightarrow p_1 = E_1$.

Now consider a strip (can be bent or polygon), consisting of x and y units.

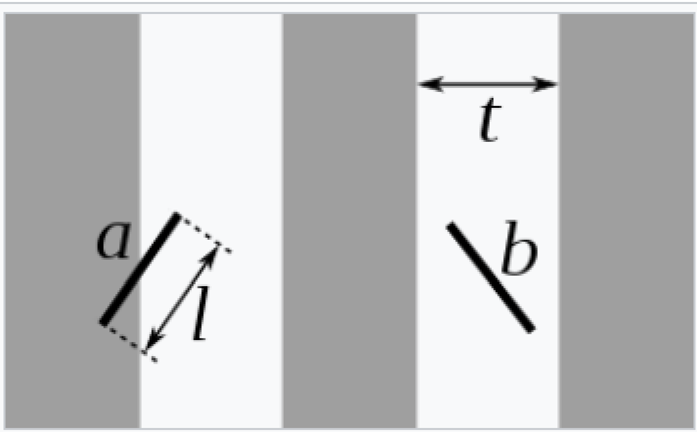
$E(x+y) = E(x) + E(y)$
$E(L) = L * E_1$ or $E_L = L/w * E_w$, w is some small length.
$E_1 = 1/w*E_w$.

A circle of diameter 1 will always intersect at two points.

$\pi * E_1 = 2$
$E_1 = 2/\pi \sim 64$ %.

Probability of landing across two strips ~ 36%.

General case:



The a needle lies across a line, while the b needle does not.

circumference = $\pi * t$
$\pi * t * E_1 = 2$
$E_1 = 2/(\pi * t)$
For needle of length l, $E_l = 2l / (\pi * t)$.

E_l = l / dx * E[dx]
pi * t / dx * E[dx] = 2
E_l = 2l / (pi * t).

Problem 14

There are a group of ‘n’ potential friends and each pair of people becomes friends with probability ½. What is the expected number of friend triplets? Mutually all are friends within a triplet?

Sol:

- x_i : indicator variable on ith triplet being a friend.
- Ans = Sum (x_i) = nC3 * ⅛.

Problem 15 (Birthday Paradox)

If n people are in a room what is the expected number of distinct birthdays represented? Assume 365 days in a year.

Sol:

x_i (indicator variable): {
1 : if ith day is someone’s birthday
0 : otherwise.
}
No of distinct birthdays = x_1 + x_2 + … + x_365
EV (x_i) = 1 * (1 - 364/365^n) + 0*p
EV = Sum (X_i) = 365 * EV(x_i)

Problem 16 (Coin Toss)

Kwu is bored. He flips a coin continuously until he gets a head followed by a tail. What is the expected number of times he will flip the coin?

Sol:

- Let x : expected number of times he will flip the coin given he is on heads.
  Let y : …. he is on tails.

  $x = ½ * 1 + ½ * (1 + x)$
  $x = 2$

  $y = ½ * (1 + y) + ½ * (1 + x)$
  $2y = x + y + 2$
  $y = x + 2 = 4$

  Expected number of flips = $1 + ½ (x+y) = 4$

Expected number of flips for HH = 6
… HT = 4

Problem 17

Kwu is bored. He flips a coin continuously until he gets heads immediately followed by tails immediately followed by heads e.g. HTH. What is the expected number of times he will flip the coin?

Sol:

Define the expected values as follows :

- x1 : expected number of flips when the last two are HH.
- x2 : … HT
- x3 : … TH
- x4 : … TT

Transitions:

$x1 = \frac{1}{2} * (1+x2) + \frac{1}{2} * (1+x1)$
$x2 = \frac{1}{2} * 1 + \frac{1}{2} (1+x4)$
$x3 = \frac{1}{2} * (1+x2) + \frac{1}{2} * (1+x1)$
$x4 = \frac{1}{2} * (1+x3) + \frac{1}{2} * (1+x4)$

Solve for x1, x2, x3, x4
$ans = 2 + \frac{1}{4} * (x1 + x2 + x3 + x4) = 10.$

Problem 18

There are 2 cans, one is red and the other is blue. We start with 3 marbles in the red can. We play a game where we choose one marble <u>at random</u> out of 3 and move it to the opposite can. What is the expected number of moves needed for all marbles to be in the blue can?

Sol:

x1 : blue [...] red []
x2 : blue [..] red [.]
x3 : blue [.] red [..]
x4 : blue [] red [...]

$x1 = 0$
$x2 = \frac{1}{2} * (1+x3) + \frac{1}{2} * 1$
$x3 = \frac{1}{2} * (1+x2) + \frac{1}{2} * (1+x4)$
$x4 = 1+x3$

The above transition is wrong because the probability of picking up a ball from [..] is not as from [.].

Correct Transition:

$x1 = 0$
$x2 = \frac{2}{3} * (1+x3) + \frac{1}{3} * 1$
$x3 = \frac{1}{3} * (1+x4) + \frac{2}{3} * (1+x2)$
$x4 = 1 + x3$

$3x2 = 2 + 2x3 + 1$
$3x3 = x4 + 1 + 2 + 2x2$
$3(x4-1) = x4 + 3 + 2x2$
$3x2 = 2 + 2x4-2 + 1$
$3x2 = 2x4 + 1$
$2x4 = 6 + 2x2$
$3x2 - 1 = 6 + 2x2$
$x2 = 7$
$x4 = 10$ (Ans)

Problem 19 https://na-cho.github.io/files/states.pdf

There are 11 lily pads in a pond. A frog jumps from pad 1. Pad 0 has a snake and Pad 10 it will exit. Frog jumps right 1 pad with probability 1-i/10 and left with probability i/10. What is the probability it leaves the pond?

0 1 2 3 4 5 6 7 8 9 10

Consider pad 1 and 9. Probability of leaving the pond from 9 is same as getting eaten by the snake in pad 1. Symmetry.

Same is 2,8, 3,7 4,6
Consider pad 5: p(getting out) = $\frac{1}{2} * (1 - p6) + \frac{1}{2} * p6 = \frac{1}{2}$ (Computation by symmetry)

$x1 = p1 * x2$
$x2 = p2 * (1+x1) + (1-p2)*(1+x3)$

$x3 = p3 * (1+x2) + (1-p3)*(1+x4)$
$x4 = p4 * (1+x3) + (1-p4)*½$
Solving: Ans = 63/146

Problem 20

What is the expected number of coin tosses to get 5 consecutive heads?

Sol:

x1: H
x2: HH
x3: HHH
x4: HHHH
x5: T

$x1 = ½ (1+x5) + ½ * (1+x2)$
$x2 = ½ (1+x5) + ½ * (1+x3)$
$x3 = ½ (1+x5) + ½ * (1+x4)$
$x4 = ½ (1+x5) + ½ * 1$
$x5 = ½ (1+x1) + ½ * (1+x5)$

$2x1 = x2 + x5 + 2$
$2x2 = x3 + x5 + 2$
$2x3 = x4 + x5 + 2$
$2x4 = x5 + 2$
$2x5 = x1 + x5 + 2$

$x1 = 60$
$x5 = 62$

Ans : ½ * (1+x1) + ½ * (1+x5) = **62**

Alternately,
let x be the expected value
$x = ½ (x+1) + ¼ (2+x) + ⅛ (3+x) + 1/16 (4+x) + 1/32 (5+x)$

Problem 21 (Errichto's blog I)

In a deck of 52 cards, there are 4 aces. 10 cards are drawn at random.

a. **What is the probability that the 7th card is an ace?**
b. **What is the EV of the number of aces in 10 cards?**

Sol:

a.

Pr (7th card is ace) = 4/52 = 1/13

Pr (any card is an ace | w/o knowing anything about the other cards) = 4/52.

b.

EV (#aces in 10 cards) = 10 * E (any card is ace) = 10/13.

(a) can be counter-intuitive. Think about the whole deck of 52 cards, the probability of an ace in any card is obviously 4/52. Now sample 10/20 cards from this does not inherently change the probability of that card, it's just moving the cards around. So probability is still 4/52.

Inductively: Let expected value of ace in any cards in (i-1) cards be equal E.
E[i] = E * n - E * (i-1) / (n-(i-1)) = E. (Linearity of expectations).

Problem 22

A deck of n playing cards, which contains three aces, is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is (n+1)/2.

Sol:

Method 1:

Second ace can be in position 2 … n-1.
Calculate E(k).

* * (A) * * A * * (A) * *
Total choices = C(n, 3)
Number of combinations where second ace is in kth = (k-1)*(n-k).
E(k) = k * (k-1) * (n-k)
EV (number of cards to be turned up)
= **Sum (k=1..N)** k * (k-1) * (n-k) / C(n, 3) = n+1 / 2.

Calculating the above sum is tedious but the above process is straightforward.

Method 2: (Symmetry)

For every arrangement where 2nd ace is in the kth position, there is an equally probable arrangement if you take cards from backwards for the ace to lie in (n-k+1).

½ * (k + n - k+1)/2
= (n+1) / 2.

Method 3:

3 Aces partition the arrangement into 4 sections say x, y, z, w.
E[x+y+z+w] = n-3.
Now any permutation (x,y,z,w) can be permuted equally likely.
E[x] = E[y] = E[z] = E[w].
4 E[x] = n-3.
E[x] = (n-3)/4.
E[position of second ace] = E[x+y+2] = 2 + E[x]*2 = 2 + (n-3)/2 = (n+1) / 2.

Problem 23 (AIME, 96)

A permutation a1, a2 … a10 of integers {1…10} form the sum, |a1-a2| + |a3-a4| + |a5-a6| + |a7-a8| + |a9-a10|. Find the expected value of the sum across all permutations.

Sol:

$x1 = |a1-a2|$

$x2 = |a3-a4|$ … and so on!

$EV = E[x1+x2+x3+x4+x5] = 5 E [x1]$

Consider - - (p)- - a - - (q) - - b - - (r) - -

$E[p+q+r] = n - 2$
$E[p] = E[q] = E[r]$
$E[q] = (n-2)/3$
$E[x1] = 1 + E[q] = (n+1) / 3$
$EV = 5/3 * (n+1) = 55/3$

Problem 24 (Noodles' problem)

You have 100 noodles. Being blindfolded you take two ends of some noodles in your bowl and connect them. You continue unless there are no free ends. Calculate the expected number of circles.

Sol:

let E[n] be the expected number of circles using 'n' noodles.

In the first try you either pick up two same ends of some noodle with prob p or pick different noodles with prob (1-p).

$p = n / C(2n, 2) = n * 2 / 2n * (2n-1) = 1 / (2n - 1)$

$E[n] = p * (E[n-1] + 1) + (1-p) * E[n-1]$
$= E[n-1] + 1 / (2n-1)$
$E[1] = 1.$
$E[100] = 1 + 1/3 + 1/5 + 1/7 + 1/9 + .. 1/199$

Problem 25

Suppose we draw cards out of a deck without replacement. How many cards do we expect to draw out before we get an ace?

Sol:

Method : 1

There are 4 aces.

* * * A * * * A * * * A * * * A * * *

4 aces partition the array into 5 segments, say p, q, r, s, t.

E[p+q+r+s+t] = 48.

E[p] = 48/5

E[ans] = 48/5 + 1 = 53/5.

Method : 2

Indicator X_i : ith non-ace cards appears before 1st ace.

E[ans] = 1 + E[x1 + x2 … + x48]

= 1 + 48 * E [x_i]

E[x_i] = (T/4!/5!) / T [permute: x_i a1, a2, a3, a4]

= ⅕

E[ans] = 53/5.

Problem 26 (Errichto's)

The price of a TV is 1000$. Each day the prices go up by 5$ or 10$ with probability 50%. Final EV of the final prices.

EV = 1000 + N * 7.5%.

Problem 27

You roll a 6 sided die twice. Find EV of the bigger of two scores.

Sol :

Method 1:

E[ans] = 1/36 * (1*1 + 2*3 + 3*5 + 4*7 + 5*9 + 6*11) = 161 / 36.

Method 2:

E[i] : (i/6)^2 - (i-1/6)^2 = E(bigger score is <= i) - E(bigger score is <= i-1).

Problem 28

You roll a 6 sided die n times. Find EV of the biggest scores.

Sol :

let E[i] denote the expected value of getting i as the biggest score.

E[i] = (⅙)^n * (i^n - (i-1)^n) = (i/6)^n - (i-1/6)^n.
E[ans] = Sum (i=1…6) { **i * E[i]** }.

Dynamic Programming Approach:
Define dp[i][j]: expected value of getting j as the biggest score after i turns

dp[i+1][3] ← dp[i+1][3] * ⅙ (flip 1)
dp[i+1][3] ← dp[i+1][3] * ⅙ (flip 2)
dp[i+1][3] ← dp[i+1][3] * ⅙ (flip 3)
dp[i+1][4] ← dp[i+1][3] * ⅙ (flip 4)
dp[i+1][5] ← dp[i+1][3] * ⅙ (flip 5)
dp[i+1][6] ← dp[i+1][3] * ⅙ (flip 6)

For large n, you may not calculate dp states and directly report 6 with precision.

Problem 29

12 teams including Poland play in a volleyball tournament. Teams are divided into 4 groups with 3 teams. In each group every team plays against every other team (in case of tie, random teams qualify) and the best two teams advance to quarterfinals, semifinals and finals. In every match a random of two teams wins.

a. **Find p-bility that Poland wins the tournament.**
b. **Find EV of number of matches won by Poland**
c. **Find EV of the number of matches won by Poland assuming they won the tournament.**

Sol:

a. Each team is symmetrical.
   Pr (Poland wins the tournament) = 1/12.
   Alternately:
   $= \frac{2}{3} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = 1/12.$
b. EV of number of matches won by Poland = E (Group+Quarter+Semi+Final)
   E(Group) = 1 (12 matches played in group stage, 12 winners slots)
   $= \frac{1}{2} * 1 + \frac{1}{2} * 1 = 1$
   E(Quarter) = 4/12 (4 winner slots, each has 1/12 chance of Poland in it).
   $= \frac{2}{3} * \frac{1}{2} = \frac{1}{3}$
   E(Semis) = 2/12 (2 winner slots, …) = $\frac{1}{6}$
   $= \frac{2}{3} * \frac{1}{2} * \frac{1}{2} = \frac{1}{6}$.
   E(Finals) = 1/12.
   = E(Semis) * $\frac{1}{2}$ = 1/12
   E = 1 + $\frac{1}{3}$ + $\frac{1}{6}$ + 1/12 = 19/12.
   Alternately,
   No of winner slots = 12 + 4 + 2 + 1 = 19
   Each has 1/12 probability for Poland to win.
   Hence 19/12.

c. EV of the number of matches won by Poland assuming they won the tournament = E

E = E (group|winner + quarter|winner + semi|winner + finals|winner)
= E (group|winner + 1 + 1 + 1)
= 3 + E (group|winner)

E(group|winner) = 1 * P(1|adv) + 2 * P (2|adv)
P(2|adv) = P(2 and advance) / P (advanced)

By Bayes theorem,
P(2|adv) * P(adv) = P(adv|2) * P(2)
P(2|adv) * ⅔ = 1 * ¼
P(2|adv) = ⅜

P(1|adv) = 1 - ⅜ = ⅝.
EV = 35/8.

Contribution Technique

Hills : Given a sequence of length N < 1e5. Count the number of triples i<j<k that a[i] < a[j] > a[k].
Bonus: Count zig zags of length 10. a[i_1] < a[i_2] > a[i_3] < a[i_4] …

Sol:

For each i, calculate the number of elements < i which are less than a[i] and greater than i which are less than a[i] say x and y.

answer = Sum … x[i] * y[i].

Bonus : use dp
dp[i][length][greater] = Sum … (dp[1…i-1][length-1][smaller] if smaller)
use prefix sum.

Paths in a Tree: Given a tree of length N (N<10^5). Find the sum of lengths of all paths. Find the sum of squares of lengths of all paths.

Sol:

- Each path has contribution from x * (n-x). x is the size of a sub-tree.
- Sum … all edges.

Sum over subsets : There are N competitions, with prize a[i]. You win each with a probability of 50%. Find EV of total prizes.

Sol:

Method 1: a[1] * 0.5 + a[2] * 0.5 + … + a[N] * 0.5 = S/2.

Method 2:

Possible results:

0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
…

= 1/(2^n) * (0 + a[1] * 2^n-1 + a[2] * 2^n-1 + …)
= S/2.

Math Encoder: You are given a sequence of ‘n’ numbers. Find the expected value of difference of maximum and minimum elements across all non-empty subsets. What is the expected value if we choose a random interval?

Sol:

Method 1:

E (Maximum - Minimum) : Sum … E[a[j] - a[i]].

Method 2:

E[Max - Min] = E[Max] - E[Min].
E[Max] = Sum … a[i] * 2^n-i.
E[Min] = Sum … a[i] * 2^i-1.
E[Ans] = Sum … a[i] * (2^n-i - 2^i-1).

For random intervals: use stack, NGE, PGE.

Eating Ends

You're given a sequence of length N (N ≤ 2000 or N ≤ 1e5). (N - 1) times we will remove the first or the last element, each with p-bility 50%. Find EV of the last remaining number.

Sol:

Pr. first element remains = 1 / (2^n-1) * C (n-1, 0)
Pr. second element remains = 1 / (2^n-1) * C (n-1, 1)
…
Pr. ith element remains = 1 / (2^n-1) * C (n-1, i-1)
Expected Ans = Sum … Pr(i) * a[i].

Part II, Power Technique

Theory:

Sum of squares [a, b, c]

Pr of pairs = Pab * f(a,b) + Pac * f(a,c) …
= P_pairs * f(_pairs)

Brute Force Approach:

```
for i in 1…N
    for j in 1…N
        s += p[i,j]*f(i,j)
```

```
for i = 0 … n-1
        for j = 0 … n-1
                answer += p(i,j)
```

p(i,j) = p(i) * p(j), i!=j
p(i,i) = p(i)

answer = Sum … p(i) * (S - p(i)) + p(i) over all i

Cubes:

```
for i
        for j
                for k
                        p[i]*p[j]*p[k], i != j != k
                        p[i]*p[k], i=j!=k
                        p[i], i=j=k
```

Assume i!=j.

p[i] * p[j] (S - p[i] - p[j]) + p[i] * p[j]

Problems:

Square of wins:

You bought N tickets. The ith of them is winning with p-bilitiy p_i. The events are independent. Find EV of the square of the number of winning tickets.

x_i = 1 if you win ith ticket;
0 otherwise.

EV = E [(x1 + x2 + x3 + … + xN) * (x1 + x2 + x3 + … + xN)]
= E [x1 * x1 + x1 * x2 + …]
= E [p_1 * (S - p_1) + p_1+ …]
= Sum … p_i * (S - p_i) + p_i.

Cube of wins

(x1 + x2 + x3 + … + xN) * (x1 + x2 + x3 + … + xN) * (x1 + x2 + x3 + … + xN)
for i!=j,

p[i] * p[j] * (S - p[i] - [j]) + 2 * p[i]*p[j]

Codeforces problems to practice:
https://codeforces.com/problemset/problem/1453/D