Unit 3(ii)

Negative Binomial and Geometric Distributions

Geometric Distributions (textbook)

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the first success occurs, is

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots .$$

When Geometric Distribution is Used

Geometric distribution is applicable to find the probability where we perform an experiment until a success occurs.

Examples:

1) Tossing a coin repeatedly until the first head appears.
2) Shot the target until it hits.
3) Give the test until he will pass it.
4) Throwing a die repeatedly until first time a six appears.

The number of unknown parameter in Geometric distribution is only p.

Example

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

Solution: Using the geometric distribution with $x = 5$ and $p = 0.01$, we have

$$g(5; 0.01) = (0.01)(0.99)^4 = 0.0096.$$

Mean and Variance

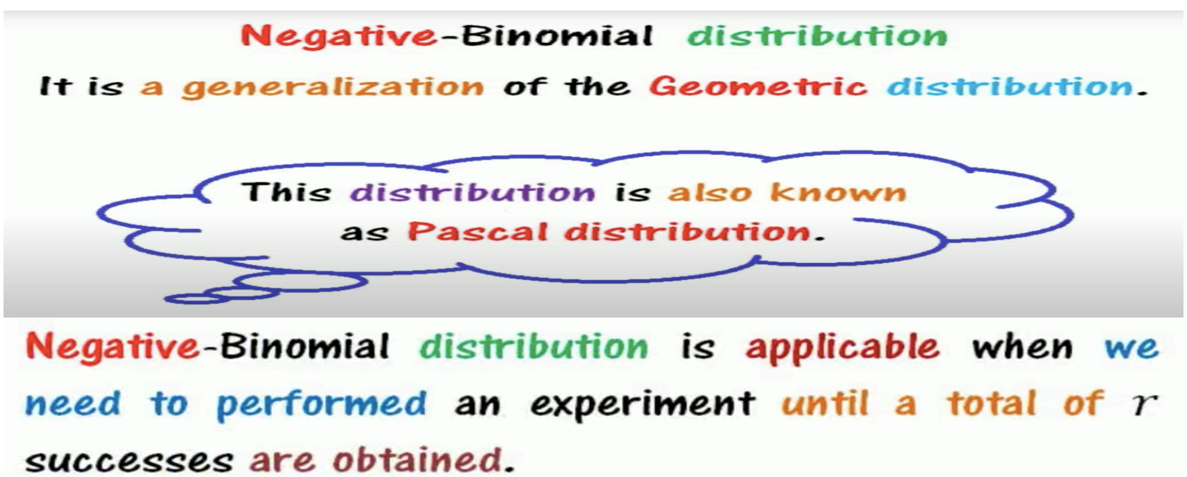
The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}.$$

Negative Binomial

If repeated independent trials can result in a success with probability p and a failure with probability q = 1 − p , then
the probability distribution of the random variable X , the number of the trial on which the **kth** success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, k+2, \ldots .$$



MEAN $\mu = k/p$ Variance = kq/p^2

Example

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B .

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

(c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

(a) $b^*(6; 4, 0.55) = \binom{5}{3} 0.55^4 (1 - 0.55)^{6-4} = 0.1853$

(b) $P(\text{team } A \text{ wins the championship series})$ is

$$\begin{aligned} & b^*(4; 4, 0.55) + b^*(5; 4, 0.55) + b^*(6; 4, 0.55) + b^*(7; 4, 0.55) \\ & \qquad = 0.0915 + 0.1647 + 0.1853 + 0.1668 = 0.6083. \end{aligned}$$

(c) $P(\text{team } A \text{ wins the playoff})$ is

$$\begin{aligned} & b^*(3; 3, 0.55) + b^*(4; 3, 0.55) + b^*(5; 3, 0.55) \\ & \qquad = 0.1664 + 0.2246 + 0.2021 = 0.5931. \end{aligned}$$

Example

Find the probability that a person flipping a coin gets

(a) the third head on the seventh flip;

(b) the first head on the fourth flip.

(a) Using the negative binomial distribution, we get

$$b^*(7; 3, 1/2) = \binom{6}{2} (1/2)^7 = 0.1172.$$

(b) From the geometric distribution, we have $g(4; 1/2) = (1/2)(1/2)^3 = 1/16$.

More examples from reference book

Example 1: If the probability is 0.40 that a child exposed to a certain disease will contain it, what is the probability that the tenth child exposed to the disease will be the third to catch it?

Solution: Required probability $= P(X = 10)$

$$= ({}^{9}C_2 p^2 q^7)p$$

$$= {}^{9}C_2 p^3 q^7$$

$$= \frac{9 \times 8}{2 \times 1}(0.4)^3(0.6)^7$$

$$= 0.0645$$

Example 2: Suppose that the probability for an applicant for a driver's licence to pass the road test on any given attempt is $\frac{2}{3}$. What is the probability that the applicant will pass the road test on the third attempt?

Solution: Required probability $= P(X = 3)$

$$= q^2 p$$

$$= \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)$$

$$= \frac{2}{27}$$

Example 3: In a company, 5% defective components are produced. What is the probability that at least 5 components are to be examined in order to get 3 defective?

Solution: Required probability $= P(X \geq 5)$

$$= 1 - P(X < 5)$$

$$= 1 - P(X = 3) - P(X = 4)$$

$$= 1 - {}^{2}C_2 p^2 q^0 \times p - {}^{3}C_2 p^2 q^1 \times p$$

$$= 1 - (0.05)^2 \times 0.05 - 3(0.05)^2(0.95)^1 \times (0.05)$$

$$= 0.9995$$

Example 4: If the probability that a child exposed to a certain viral fever will be infected is 0.3, find the probability that the eight child exposed to the disease will be the fourth to the infected.

Solution: Required probability = $P(X = 8)$

$$= {}^{7}C_3 p^3 q^4 \times p$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}(0.3)^3(0.7)^4 \times (0.3)$$

$$= 0.0681$$

Example 5: If the probability that a person will believe a rumour is 0.6, find the probability that the tenth person to hear the rumour will be the third person to believe.

Solution: Required probability = $P(X = 10)$

$$= {}^{9}C_2 p^2 q^7 \times p$$

$$= \frac{9 \times 8}{2 \times 1}(0.6)^2(0.4)^7 \times 0.6$$

$$= 0.0127$$

Example 6: A marksman is firing bullets at a target and the probability of hitting the target at any trial is 0.7. Find the probability that his seventh shot is his fourth hit?

Solution: Required probability = $P(X = 7)$

$$= {}^{6}C_3 p^3 q^3 \times p$$

$$= \frac{6 \times 5 \times 4}{3 \times 2 \times 1}(0.7)^3(0.3)^3 \times (0.7)$$

$$= 0.1297$$

Example: Suppose that a trainee solder shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7, what is the

i) Probability that the target would be hit on 10^{th} attempt?

Solution: Required probability

$$= P(X = 10)$$
$$= q^9 p$$
$$= (0.3)^9 (0.7)$$
$$= 1.3777 \times 10^{-5}$$

ii) Probability that it takes him less than 4 shots?

Solution: Required probability

$$= P(X < 4)$$
$$= P(X = 1) + P(X = 2) + P(X = 3)$$
$$= p + qp + q^2 p$$
$$= 0.7 + (0.3)(0.7) + (0.3)^2(0.7)$$
$$= 0.9730$$

iii) Probability that it takes him an even number of shots?

Solution: Required probability

$$= P(X = even)$$
$$= P(X = 2) + P(X = 4) + P(X = 6) + \cdots$$
$$= qp + q^3 p + q^5 p + \cdots$$
$$= qp[1 + q^2 + q^4 + \cdots]$$
$$= qp \times \left(\frac{1}{1 - q^2}\right)$$

Geometric Prog $= \frac{a}{1-r}$

Example: Suppose that a trainee solder shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7, what is the

iv) Average number of shots needed to hit the target?

Solution: Average number $= \frac{1}{p}$

$$= \frac{1}{0.7}$$
$$= 1.4286$$