Chapter 5
Two-Dimensional Random Variables

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Basic Concepts

- Often, a single random variable cannot adequately provide all of the information needed about the outcome of an experiment.
- In these situations, two or more random variables are considered jointly and the description of their joint behavior is our concern.
- For example, tomorrow's weather is really best described by an array of random variables that includes wind speed, wind direction, atmospheric pressure, relative humidity and temperature.
- It would not be either easy or desirable to attempt to combine all of this information into a single measurement.
- We would like to extend the notion of a random variable to deal with an experiment that result in several observations each time the experiment is run.

Basic Concept...

- For example, let T be a random variable representing tomorrow's maximum temperature and let R be a random variable representing tomorrow's total rainfall. It would be reasonable to ask for the probability that tomorrow's temperature is greater than 70 degree Celsius and tomorrow's total rainfall is less than 0.1 cubic inch. In other words, we wish to determine the probability of the event $A = T > 70; R < 0.1$
- Another question that we might like to have answered is, "What is the probability that the temperature will be greater than 70 degree Celsius regardless of the rainfall?" To answer this question, we would need to compute the probability of the event $B = T > 70$
- In this lesson, we will build on our probability model and extend our definition of a random variable to permit such calculations.
- In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a **joint probability distribution**.

Joint Probability Distributions

- If X and Y are discrete random variables, the joint probability distribution of X and Y is a description of the set of points (x, y) in the range of (X, Y) along with the probability of each point.
- The joint probability distribution of two random variables is sometimes referred to as the bivariate probability distribution or bivariate distribution of the random variables.
- One way to describe the joint probability distribution of two discrete random variables is through a joint probability mass function.
- Also, P(X= x and Y=y) is usually written as $f_{XY}(x, y) = P(X = x, Y = y)$.

Joint Probability Distributions...

- The joint probability mass function of the discrete random variables X and Y denoted by $f_{XY}(x, y)$, satisfies
  1. $f_{XY}(x, y) \geq 0$
  2. $\sum_x \sum_y f_{XY}(x, y) = 1$
  3. $f_{XY}(x, y) = P(X = x, Y = y)$
- **Example**: Two unbiased coins are tossed. Let X= 1 if the first coin shows head and X=0 if it shows tail, and let Y denote the number of heads thrown. Write down the joint probability mass function of (X, Y).

Joint Probability Distributions...

Solution

- The sample space is HH, TH, HT, TT since the coins are given to be unbiased, each simple even is to be assigned the probability $\frac{1}{4}$. The possible values of (X,Y) or the range of variation of (X,Y) is $R = (x, y)|x = 0, 1, y = 0, 1, 2$. Let $P_{xy}(x_i, y_i)$ be the joint probability mass function. Now,
$P_{xy}(0,0) = P(X = 0, Y = 0) = P(TT) = \frac{1}{4}$
$P_{xy}(0,1) = P(X = 0, Y = 1) = P(TH) = \frac{1}{4}$
$P_{xy}(0,2) = P(X = 0, Y = 2) = 0(as\ X = 0, Y = 2$ is an impossible event)
$P_{xy}(1,0) = P(X = 1, Y = 0) = 0(as\ X = 1, Y = 0$ is an impossible event)
$P_{xy}(1,1) = P(X = 1, Y = 1) = P(HT) = \frac{1}{4}$
$P_{xy}(1,2) = P(X = 1, Y = 2) = P(HH) = \frac{1}{4}$
- The values of P(x,y) may be given in a tabular form as follow

Y \ X	0	1
0	0.25	0
1	0.25	0.25
2	0	0.25

Marginal Probability Distributions

- If more than one random variable is defined in a random experiment, it is important to distinguish between the joint probability distribution of X and Y and the probability distribution of each variable individually.
- The individual probability distribution of a random variable is referred to as its marginal probability distribution.
- It shows the distribution of one of the variables without respect to the other variable.
- To determine $P(X{=}x)$, we sum $P(X{=}\ x, Y{=}y)$ over all points in the range of (X, Y) for which $X{=}x$.

Marginal Probability Distributions...

- If X and Y are discrete random variables with joint probability mass function $f_{XY}(x, y)$, then the **marginal probability mass functions** of X and Y are

$$f_X(x) = P(X = x) = \sum_{R_x} f_{XY}(x, y)$$

and

$$f_Y(y) = P(Y = y) = \sum_{R_y} f_{XY}(x, y)$$

where R_x denotes the set of all points in the range space of (X,Y) for which X=x and R_y denotes the set of all points in the range space of (X,Y) for which Y=y.

Marginal Probability Distributions...

Example

Using the above example about the two unbiased coins tossed, find the marginal probability distribution of X and Y.

Y \ X	0	1	$f_Y(y)$
0	0.25	0	0.25
1	0.25	0.25	0.5
2	0	0.25	0.25
$f_X(x)$	0.5	0.5	1

Conditional Probability Distributions

- Suppose that two random variables are defined in a random experiment.
- If the knowledge of one can change the probabilities that we associate with the values of the other, consequently the random variables X and Y can be considered to be dependent.
- Knowledge of the value obtained for X changes the probabilities associated with the values of Y.
- Given discrete random variables X and Y with joint probability mass function $f_{XY}(x, y)$, the conditional probability mass functions of Y given X=x and X given Y=y are

$$f_{Y/X}(y) = \frac{f_{XY}(x, y)}{f_X(x)} \;\; for \;\; f_X(x) > 0$$

$$f_{X/Y}(X) = \frac{f_{XY}(x, y)}{f_Y(y)} \;\; for \;\; f_Y(y) > 0$$

Conditional Probability Distributions...

The conditional probability mass function must satisfy the following conditions

1. $f_{Y/X}(y) \geq 0$ and $f_{X/Y}(X) \geq 0$
2. $\sum_{R_x} f_{Y/X}(y) = 1$ and $\sum_{R_y} f_{X/Y}(x) = 1$
3. $P(Y = y/X = x) = f_{Y/X}(y)$ and $P(X = x/Y = y) = f_{X/y}(x)$

Example

A company produces two types of compressors, grade A and grade B. Let X denote the number of grade A compressors produced on a given day. Let Y denote the number of grade B compressors produced on the same day. Suppose that the joint probability mass function $f_{XY}(x, y) = P(X = x; Y = y)$ is given by the following table. Given that no grade B compressors were produced on a given day, what is the probability that two grade A compressors were produced?

$f_{XY}(x;y)$		Y = 0	Y = 1	$f_X(x)$
X	0	0.1	0.3	0.4
	1	0.2	0.1	0.3
	2	0.2	0.1	0.3
$f_Y(y)$		0.5	0.5	1

Independence

- In some random experiments, knowledge of the values of X does not change any of the probabilities associated with the values for Y.
- Two random variables are said to be independent whenever $f_{XY}(x, y) = f_X(x)f_Y(y$ for all x and y.
- If we find one pair of x and y in which the equality fails, X and Y are not independent
- For discrete random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are **independent**.

(1) $f_{XY}(x, y) = f_X(x)f_Y(y)$ for all x and y
(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$
(3) $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$
(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively.

Independence...

- **Example**
  Consider the compressor example again

$f_{XY}(x,y)$		Y		$f_X(x)$
		0	1	
X	0	0.1	0.3	0.4
	1	0.2	0.1	0.3
	2	0.2	0.1	0.3
$f_Y(y)$		0.5	0.5	1

  Check whether X and Y are independent.

  Solution

  The random variables X and Y are not independent, because

$$f_{XY}(0,0) = 0.1 \neq f_X(0)f_Y(0) = (0.4)(0.5) = 0.2$$

Joint Probability Distributions

- The joint probability distribution of two continuous random variables X and Y can be specified by providing a method for calculating the probability that X and Y assume a value in any region R of two-dimensional space.
- The double integral of over a region R provides the probability that assumes a value in R.
- This integral can be interpreted as the volume under the surface over the region R.

A **joint probability density function** for the continuous random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies the following properties:

(1) $f_{XY}(x, y) \geq 0$ for all x, y

(2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy = 1$

(3) For any region R of two-dimensional space

$$P([X, Y] \in R) = \iint_R f_{XY}(x, y)\, dx\, dy$$

Marginal Probability Distributions

- Similar to joint discrete random variables, we can find the marginal probability distributions of X and Y from the joint probability distribution.

If the joint probability density function of continuous random variables X and Y is $f_{XY}(x, y)$, the **marginal probability density functions** of X and Y are

$$f_X(x) = \int_{R_x} f_{XY}(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_{R_y} f_{XY}(x, y)\, dx$$

where R_x denotes the set of all points in the range of (X, Y) for which $X = x$ and R_y denotes the set of all points in the range of (X, Y) for which $Y = y$

- A probability involving only one random variable can be found from the marginal probability distribution of the random variable or from the joint distribution.

$$P(a < X < b) = \int_a^b \int_{R_x} f_{XY}(x,y)dydx = \int_a^b \left(\int_{R_x} f_{XY}(x,y)dy \right) dx$$

$$= \int_a^b f_X(x)dx$$

Conditional Probability Distributions

- Analogous to discrete random variables, we can define the conditional probability density function of X=x.

Given continuous random variables X and Y with joint probability density function $f_{XY}(x, y)$, the **conditional probability density function** of Y given $X = x$ is

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0$$

Because the conditional probability density function $f_{Y|x}(y)$ is a probability density function for all y in R_x, the following properties are satisfied:

(1) $f_{Y|x}(y) \geq 0$

(2) $\int_{R_x} f_{Y|x}(y)\, dy = 1$

(3) $P(Y \in B | X = x) = \int_B f_{Y|x}(y)\, dy$ for any set B in the range of Y

Independence

- The definition of independence for a continuous random variables is similar to the definition for discrete random variable.

For continuous random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are said to be **independent.**

(1) $f_{XY}(x, y) = f_X(x) f_Y(y)$ for all x and y

(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

(3) $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$

(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively.

- **Example**

Let X and Y are two continuous random variables. Assume the joint probability density function for X and Y is

$$f_{XY}(x, y) = \frac{1}{4}xy \text{ for } o \leq x \leq 2 \text{ , } o \leq y \leq 2$$

a. Verify that $f_{XY}(x, y)$ is a joint density function.
b. Determine the probability that $0 < x < 1$ and $1 < y < 2$
c. find the ,marginal distributions of X and Y.
d. Find the conditional distributions of Y/X=x and X/Y=y.