Contents

1 Matrices and Systems of Linear Equations

1.1 Definition of a Matrix and Basic Operations

Definition 1.1 A matrix is any rectangular array of real numbers or variables of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad (1)$$

The numbers or the variables in the matrix are called ***entries*** or ***elements*** of the matrix. If a matrix has *m* rows and *n* columns then we way that its size is *m* by *n* (m×n) matrix. An n×n matrix is called a **square matrix** or a **matrix of order n.** A 1×1 matrix is simply a real number. Matrices will be denoted by capital bold-faced letters **A**, **B**, etc, or by (a_{ij}) or (b_{ij}). For instance if

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 8 & 5 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 5 & 1 & 0 \\ \frac{1}{3} & -2 & 6 \\ \pi & e & \sqrt{3} \end{pmatrix} \qquad (2)$$

then **A** is a 2×3 matrix while **B** is a 3×3 square matrix or a matrix of order 3. The entry in the *i*th row and *j*th column of an *m*×*n* matrix **A** is written a_{ij}. For an *n*×*n* square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the **main diagonal** elements. The main diagonal entries for the matrix **B** in (2) are 5, –2, $\sqrt{3}$.

Definition 1.2 Column and Row Vectors

An n×1 matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called a column vector. A 1× n matrix, $(a_1, a_2, \ldots, a_n)$ is called a row vector.

Special Matrices

In matrix theory there are many special kinds of matrices that are important because they posses certain properties. The following is a list of some of these matrices.

- A matrix that consists of all zero entries is called a **zero matrix** and is denoted by **0**.

  For example $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

- An n × n matrix **A** is said to be a **triangular matrix** if all its entries below the main diagonal are zeros or if all its entries above the main diagonal are zero, [in other words a square matrix **A** is triangular if $a_{ij} = 0$ for $i > j$ or $a_{ij} = 0$ $i < j$.] More specially, in the first case the matrix is called **upper triangular** and in the second case the matrix is called lower triangular. The following are triangular matrices.

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 7 \end{pmatrix} \qquad \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 \\ 8 & 9 & 3 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 15 & 2 & 3 & 4 & 2 \end{pmatrix}$$

Upper triangular matrix Lower triangular matrix

- An n × n matrix **A** is said to be a **diagonal matrix** if all its entries not on the main diagonal are zeros. In terms of the symbolism $\mathbf{D} = (d_{ij})_{n\times n}$, **D** is a diagonal matrix if $d_{ij} = 0 \ \ for \ i \neq j$. The matrix **D** thus is given by $\mathbf{D} = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ (3)
- If in (3) if all the diagonal elements are equal, it is referred to as a ***scalar matrix*** $\mathbf{S}_n$, and if these elements are equal to 1, we have a ***unit*** or an ***identity*** matrix I_n of order n.

Thus $\mathbf{S}_n = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$ $\qquad \mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

are respectively a scalar and an identity matrix.

Operations on Matrices

Definition 1.3 Equality of Matrices

Two n × n matrices **A** and **B** are equal if $a_{ij} = b_{ij}$ for each *i* and *j*.

In other words, two matrices are equal if and only if they have the same size and their corresponding entries are equal.

Matrix Addition

When two matrices **A** and **B** are of the same size we can add them by adding their corresponding entries.

Definition 1.4 If A and B are m × n matrices, then their sum is

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m\times n}.$$

Example 1: Addition of Two Matrices.

a) Let $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -1+7 & 3+(-8) \\ 0+9 & 4+3 & 6+5 \\ -6+1 & 10+(-1) & -5+2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{pmatrix}.$$

b) The sum of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is not defined, since **A** and **B** are of different sizes.

Definition 1.5 Scalar Multiplication of a Matrix.

If k is a real number, then the scalar multiple of a matrix A is

$$k\mathbf{A} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix} = (ka_{ij})_{m\times n}$$

In other words, to compute k**A,** we simply multiply each entry of **A** by k. For instance, from definition 2.5, $5\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 5\cdot 2 & 5\cdot(-3) \\ 5\cdot 4 & 5\cdot(-1) \end{pmatrix} = \begin{pmatrix} 10 & -15 \\ 20 & -5 \end{pmatrix}$.

The difference of two m × n matrices defined in the usual manner **A –B =A+(–B**) where –1**B = –B**.

Properties of Matrix Addition and Scalar Multiplication

Suppose **A, B,** and **C** are m × n matrices and α and β are scalars. Then

i) **A + B = B + A** (Commutative law of addition)

ii) **(A + B) + C = A + (B + C)**(Associative law of addition)

iii) α(**A + B**) = α**A** + α**B**

iv) **(**α + β)**A =** α**A** + β**A**

v) **(**αβ)**A** =α(β**A)**

vi) 1**A = A**

Note: Each of the above six properties can be proved by using Definition 2.4 and 2.5.

1.2 Product of Matrices and Some Algebraic Properties, Transpose

Definition 2.6 Let the number of columns in matrix **A** be the same as the number of rows in matrix **B**, then the matrix product **AB** exists and the element in row i and column j of **AB** is obtained by multiplying the corresponding elements of row i of **A** and column j of **B** and adding the product.

In other words if matrix **A** has n column and matrix **B** has n rows then the ith row of **A** is $(a_{i1}, a_{i2}, ..., a_{in})$ and the jth column of **B** $\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$. Thus if **C = AB** then

$$\mathbf{C}_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{1n} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Moreover the number of rows and the number of columns of **C** are equal to the number of rows of **A** and the number of columns of **B,** respectively. Thus

$$\underset{m\times n}{\mathbf{A}} \; \underset{n\times r}{\mathbf{B}} = \underset{m\times r}{\mathbf{C}}$$

Example 1 If

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Then $\mathbf{AB} = \begin{pmatrix} 3+12+2-1 & 6+16+0+1 \\ 1+6+3-1 & 2+8+0+1 \\ 0+3+2-3 & 0+4+0+3 \end{pmatrix} = \begin{pmatrix} 16 & 23 \\ 9 & 11 \\ 2 & 7 \end{pmatrix}.$

We note here that the size of **A** is 3×4 and the size of **B** is 4×2 consequently the size of **AB** is 3×2.

Properties of Matrix Multiplication

In defining the properties of matrix multiplication below, the matrix **A**, **B**, and **C** are assumed to be of compatible dimensions for the operations in which they appear.

***Property* I** Matrix multiplication is, in general, not commutative. That is **AB** ≠ **BA**. Observe that in Example 1 of this section **BA** is not even defined because the first matrix in this case **B** does not have the same number of columns as the number of rows of the second matrix **A.**

***Property* II** From **AB** = **0,** it does not follow that either **A = 0** or **B = 0**. Here **0**'s are null matrices of appropriate order.

Example 2 For the matrix **A** and **B** given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

we have $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

a null matrix even though **A** or **B** is not a null matrix.

***Property* II** The relation **AB = AC** or **BA = CA** does not imply that **B = C**. The cancellation law does not hold in general as in a real numbers.

Example 4 For the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 4 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

we have, by direct multiplication,

$$\mathbf{AB} = \begin{pmatrix} 9 & 10 & 7 \\ 6 & 7 & 6 \\ 9 & 8 & -1 \end{pmatrix} = \mathbf{AC}\text{ , although } \mathbf{B} \neq \mathbf{C}.$$

***Property* IV** Matrix multiplication is associative. That is
A(BC)=(AB)C.

***Property* V** The multiplication of matrices is distributive with respect to addition i.e.

A(B+C)=(AB+AC), **(B+C)A = BA + CA.**

Example 5 If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \quad and \quad \mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

verify that A(BC) = (AB)C and A(B+C) = AB+AC.

Solution:

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

$$(\mathbf{AB})\mathbf{C} = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix}$$

Thus

$$\mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 6 & 5 \\ 16 & 11 \end{pmatrix} = (\mathbf{AB})\mathbf{C}$$

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

$$\mathbf{AB}+\mathbf{AC} = \begin{pmatrix} -4 & 5 \\ -6 & 11 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 11 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 5 & 15 \end{pmatrix}$$

Therefore

A(B + C) = AB + AC.

Notation. Since **A(BC)** = **(AB)C**, one may simply omit the parentheses and write **ABC**. The same is true for a product of four or more matrices. In the case where an n×n matrix is multiplied by itself a number of times, it is convenient to use exponential notation. Thus, if k is a positive integer, then

$$\mathbf{A}^k = \underbrace{\mathbf{AA}\ldots\mathbf{A}}_{k\ times}$$

Example 6 If

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{AAA} = \mathbf{AA}^2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

and in general

$$\mathbf{A}^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}.$$

Example 7 Simplify the following matrix expression

A(A + 2B) + 3B(2A – B) – $\mathbf{A^2}$ + $\mathbf{7B^2}$ – 5AB

Solution: Using the properties of matrix operation we get

A(A + 2B) + 3B(2A – B) – $\mathbf{A^2}$ + $\mathbf{7B^2}$ – 5AB = $\mathbf{A^2}$ + 2AB + 6BA – $\mathbf{3B^2}$ – $\mathbf{A^2}$ + $\mathbf{7B^2}$ – 5AB = – 3AB + 6BA + $\mathbf{4B^2}$.

Transpose of a Matrix

Definition 2.7 The transpose of a matrix **A**, denoted $\mathbf{A}^{\mathbf{T}}$, is the matrix whose columns are the rows of the given matrix **A.**

Symbolically the transpose of an m × n matrix $\mathbf{A} = (a_{ij})_{m\times n}$ is an n × m matrix

$$\mathbf{A}^{\mathbf{T}} = (a_{ij}^{\mathbf{T}})_{n\times m} = (a_{ji})_{n\times m}$$

where $a_{ij}^{\mathbf{T}} = a_{ji}$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$, *then* $\mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$. *If* $\mathbf{B} = (5 \quad 3)$, *then* $\mathbf{B}^{\mathbf{T}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

In the next theorem we give some important properties of the transpose.

Theorem 1.8 Suppose **A** and **B** are matrices and *k* a scalar. Then

i) $(\mathbf{A}^{\mathbf{T}})^{\mathbf{T}} = \mathbf{A}$ ii) $(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$

iii) $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$ iV) $(k\mathbf{A})^{\mathbf{T}} = k\mathbf{A}^{\mathbf{T}}$

Proof: We give here the proof of iii) here the rest is left as Exercise.

Note that

$$\mathbf{A} = (a_{ik})_{m\times n}, \quad \mathbf{B} = (b_{kj})_{n\times r}$$

then

$$\mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} = (b_{ik}^{\mathbf{T}})_{r\times n}(a_{kj}^{\mathbf{T}})_{n\times m} = \left(\sum_{k=1}^{n} b_{ik}^{\mathbf{T}} a_{kj}^{\mathbf{T}}\right)_{r\times m} = \left(\sum_{k=1}^{n} a_{jk} b_{ki}\right)_{r\times m} \qquad (1)$$

and the last step follows from the definition of a transpose. Also,

$$\mathbf{AB} = (a_{ik})_{m\times n}(b_{kj})_{n\times r} = \left(\sum_{k=1}^{n} a_{ik} b_{kj}\right)_{m\times r}$$

which on being transpose (i.e., on interchanging the subscript *i* and *j*) gives

$$(\mathbf{AB})^{\mathbf{T}} = \left(\sum_{k=1}^{n} a_{jk} b_{ki}\right)_{r\times m} \qquad (2)$$

now iii) follows from (1) and (2).

The remaining properties can be proved similarly.

Definition 1.9 An n×n matrix $\mathbf{A} = (a_{ij})$ is said to be:

i) Symmetric if $a_{ij} = a_{ji}$ for all *i* and *j*, that is if $\mathbf{A}^{\mathbf{T}} = \mathbf{A}$.

ii) Skew-symmetric if $a_{ij} = -a_{ji}$ for all *i* and *j*, that is $\mathbf{A}^{\mathbf{T}} = -\mathbf{A}$.

The following are examples of symmetric matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{pmatrix}$$

Class Work 1

1. If $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$ $\quad \mathbf{B} = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & -1 \end{pmatrix}$ $\quad \mathbf{C} = \begin{pmatrix} 2 & -4 & 5 \\ 1 & 0 & 0 \end{pmatrix}$

a) **B+C** b) **B – C** c) **AB** d) **AC** e) $\mathbf{B}^T\mathbf{A}^T$ f) $(\mathbf{AB})^T$ g) Determine the following elements of **D = AB + 2C**, without computing the complete matrix. i) d_{12} ii) d_{23}

2. Let **A** be 3 × 5 matrix, **B** be 5 × 2 matrix, **C** be 3 × 4 matrix, **D** be 4 × 2 matrix, **E** be 4 × 5 matrix, give the size of a) 2(**EB**) + **DA** b) **CD** – 2**(CE)B**.
3. If $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$ compute $\mathbf{A}^4$.
4. Simplify $A(A - 4B) + 2B(A + B) - A^2 + 7B^2 + AB$.
5. Let
$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
compute $\mathbf{A}^2$ and $\mathbf{A}^3$. What will $\mathbf{A}^n$ turn to be?
6. Show that for a square matrix $\mathbf{A} = (a_{ij})$
   i) $A + A^T$ is a symmetric matrix
   ii) $A - A^T$ is skew-symmetric matrix
   iii) AA^T and A^TA, A^2 are symmetric matrices.

1.3 Elementary row operations and echelon form

We use matrices to describe systems of linear equations. There are two important matrices associated with every system of linear equations. The coefficients of the variables form a matrix called the **matrix of coefficients** of the system. The coefficients, together with the constant terms, form a matrix called the **augmented matrix** of the system. For example, the matrix of coefficients and the augmented matrix of the following system of linear equations are as shown.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned} \qquad \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix}}_{matrix\ of\ coefficients} \qquad \underbrace{\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right)}_{augmented\ matrix}$$

Observe that the matrix of coefficients is a submatrix of the augmented matrix. The augmented matrix completely describes the system.

Transformations called elementary transformations can be used to change a system of linear equation into another system of linear equations that has the same solution. These transformations are used to solve systems of linear equations by eliminating variables. In practice it is simpler to work in terms of matrices using equivalent transformations called elementary row operations. These transformations are as follows:

<u>Elementary transformations</u>
1. Interchanging two equations
2. Multiplying both sides of an equation by a nonzero constant
3. Add a multiple of one equation on to another equation.

<u>Elementary row operations</u>
1. Interchanging two rows of a matrix
2. Multiply the elements o row by a nonzero constant
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Systems of equations that are related through elementary transformations, and thus have the same solutions, are called **equivalent systems**. The symbol ≈ is used to indicate equivalent system of equations. The next example compares the elementary transformation with elementary row operations.

Example 1 Solve the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Solution:

Equation Method

Initial system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Eliminate x_1 from the 2nd and 3rd equations

$$\begin{array}{ll} \approx & x_1 + x_2 + x_3 = 2 \\ Eq2 + (-2)Eq1 & x_2 - x_3 = -1 \\ Eq3 + (-1)Eq1 & -2x_2 - 3x_3 = -8 \end{array}$$

Eliminate x_2 from the 1st and 3rd equations

$$\begin{array}{ll} \approx & x_1 + \quad + 2x_3 = 3 \\ Eq1 - Eq2 & x_2 - x_3 = -1 \\ Eq_3 + 2Eq_2 & 5x_3 = -10 \end{array}$$

Make coefficient of x_3 in 3rd Eq 1

$$\begin{array}{ll} \approx & x_1 + \quad + 2x_3 = 3 \\ -\frac{1}{5}Eq3 & x_2 - x_3 = -1 \\ & x_3 = 2 \end{array}$$

Eliminate x_3 from 1st and 2nd equations

$$\begin{array}{ll} \approx & x_1 + \quad\quad = -1 \\ Eq1 - Eq3 & x_2 \quad = 1 \\ Eq2 + Eq3 & x_3 = 2 \end{array}$$

The solution is $x_1 = -1, x_2 = 1, x_3 = 2$

Matrix Method

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right)$$

we refer to the first row as the *pivot row* and the entry 1 circled in the first row as the *pivot*

$$\begin{array}{l} \approx \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array}\right)$$

Create appropriate zeros in column 2

$$\begin{array}{l} \approx \\ R_1 - R_2 \\ R_3 - 2R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array}\right)$$

Make the (3,3) element 1

$$\begin{array}{l} \approx \\ -\frac{1}{5}R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Create zeros in column 3

$$\begin{array}{l} \approx \\ R_1 - 2R_3 \\ R_2 + R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Matrix corresponding to the system

$$\begin{aligned} x_1 + \quad\quad &= -1 \\ x_2 \quad &= 1 \\ x_3 &= 2 \end{aligned}$$

The solution is $x_1 = -1, x_2 = 1, x_3 = 2$

Class work 2

Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

Reduced row echelon form and elementary row operations:

In above motivating example, the key to solve a system of linear equations is to transform the original augmented matrix to some matrix with some properties via a few elementary row operations. As a matter of fact, we can solve **any** system of linear equations by **transforming the associate augmented matrix to a matrix in some form.** The form is referred to as the **reduced row echelon form**.

Definition of a matrix in reduced row echelon form:

A matrix in reduced row echelon form has the following properties:

1. All rows consisting entirely of 0 are at the bottom of the matrix.
2. For each nonzero row, the first entry is 1. The first nonzero entry is called a leading 1.
3. For two successive nonzero rows, the leading 1 in the higher row appears farther to the left than the leading 1 in the lower row.
4. If a column contains a leading 1, then all other entries in that column are 0.

Note: a matrix is in *row echelon form* as the matrix has *the first 3 properties*.

Example:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are the matrices in reduced row echelon form.

The matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is **not** in reduced row echelon form but **in row echelon form** since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not 0. The matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are **not** in row echelon form (also **not** in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not 0.

Definition of elementary row operation:

There are 3 elementary row operations:

1. Interchange two rows
2. Multiply a row by some nonzero constant
3. Add a multiple of a row to another row.

Example:

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}.$$

- Interchange rows 1 and 3 of A

$$\Rightarrow \quad \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

- Multiply the third row of A by $\frac{1}{3}$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

- Multiply the second row of A by -2, then add to the third row of A

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Important result:

- Every nonzero $m \times n$ matrix can be transformed to a unique matrix in reduced row echelon form via elementary row operations.

- If the augmented matrix $[A \vdots b]$ can be transformed to the matrix in reduced row echelon form $[C \vdots d]$ via elementary row operations, then the solutions for the linear system corresponding to $[C \vdots d]$ is exactly the same as the one corresponding to $[A \vdots b]$.

C lass work 2

Reduce the following matrices to row echelon and reduced row echelon forms.

1. $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ **2.** $\begin{pmatrix} 2 & 2 & 4 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ **3.** $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 3 & 9 & 3 \end{pmatrix}$

1.4 Inverse of a matrix and its properties

We motivate the idea of the inverse of a matrix by looking at the multiplicative inverse of a real number. If number b is the inverse of a, then

$$ab = 1 \qquad \text{and} \qquad ba = 1$$

for example, ½ is the inverse of 2 and we have

$$2\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)2 = 1.$$

These are the ideas we extend to matrices.

Definition 2.13 An n×n matrix **A** is said to be **nonsingular** or **invertible** if there exists a matrix **B** such that $\mathbf{AB}=\mathbf{BA}=\mathbf{I}_n$. The matrix **B** is said to be the multiplicative inverse of **A**.

Note: If B and C are both multiplicative inverses of A, then

$\mathbf{B = BI_n = B(AC) = (BA)C = I_nC = C}$.

Thus an invertible matrix has a unique inverse.

Example 1 Show that the matrix B is the inverse of matrix A if

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$$

Solution: Observe that

$$\mathbf{AB} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

and

$$\mathbf{BA} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

Thus $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_3$, which shows matrix B is the inverse of A.

Caution. I) Inverse of a matrix is only defined for square matrices.
II) A matrix may not be invertible even if it is square matrix.

For example, Let $\mathbf{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ then if a is invertible then there exists a matrix say $\mathbf{B}=\begin{pmatrix}x&x'\\y&y'\end{pmatrix}$ such that $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}x&x'\\y&y'\end{pmatrix}=\begin{pmatrix}y&y'\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Implying 0=1, which is absurd. Thus **A** is not invertible.

Definition 1.14 An $n\times n$ matrix is said to be singular if it does not have a multiplicative inverse.

Notation: Let **A** be an invertible matrix. We denote its inverse by $\mathbf{A}^{-1}$.

Gauss-Jordan Elimination for finding the inverse of a matrix

Let **A** be an $n\times n$ matrix.

1. Adjoin the identity $n\times n$ matrix $\mathbf{I}_n$ to **A** to form the augmented matrix $(\mathbf{A}: \mathbf{I}_n)$
2. Compute the reduced echelon form of $(\mathbf{A}: \mathbf{I}_n)$. If the reduced echelon form is of the type $(\mathbf{I}_n: \mathbf{B})$, then B is the inverse of **A**. If the reduced echelon form is not of the type $(\mathbf{I}_n: \mathbf{B})$, in that the first $n\times n$ submatrix is not $\mathbf{I}_n$, then **A** has no inverse.

Example 2 Determine the inverse of the matrix

$$\mathbf{A}=\begin{pmatrix}1&-1&-2\\2&-3&-5\\-1&3&5\end{pmatrix}$$

Solution: Applying the method of Gauss-Jordan Elimination, we get

$$(\mathbf{A}:\mathbf{I}_3)=\begin{pmatrix}1&-1&-2&1&0&0\\2&-3&-5&0&1&0\\-1&3&5&0&0&1\end{pmatrix}\begin{matrix}\approx\\R_2-2R_1\\R_3+R_1\end{matrix}\begin{pmatrix}1&-1&-2&1&0&0\\0&-1&-1&-2&1&0\\0&2&3&1&0&1\end{pmatrix}$$

$$\begin{matrix}\approx\\-R_2\\ \\ \end{matrix}\begin{pmatrix}1&-1&-2&1&0&0\\0&1&1&2&-1&0\\0&2&3&1&0&1\end{pmatrix}$$

$$\begin{matrix}\approx\\R_1+R_2\\R_3-2R_2\end{matrix}\begin{pmatrix}1&0&-1&3&-1&0\\0&1&1&2&-1&0\\0&0&1&-3&2&1\end{pmatrix}$$

$$\begin{matrix}\approx\\R_1+R_3\\R_2-R_3\end{matrix}\begin{pmatrix}1&0&0&0&1&1\\0&1&0&5&-3&-1\\0&0&1&-3&2&1\end{pmatrix}$$

Thus

$$\mathbf{A}^{-1}=\begin{pmatrix}0&1&1\\5&-3&-1\\-3&2&1\end{pmatrix}$$

The following example illustrates the application of the method for a matrix that does not have an inverse. Letter on in this chapter we devise more effective method to decide whether a matrix invertible.

Example 3 Determine the inverse of the matrix below, if it exists.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{pmatrix}$$

Solution: Applying the method of Gauss-Jordan Elimination we get

$$(\mathbf{A} : \mathbf{I}_3) = \begin{pmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \approx \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix} \begin{pmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \approx \\ R_1 - R_2 \\ R_3 + 3R_2 \end{matrix} \begin{pmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{pmatrix}$$

There is no need to proceed further. The reduced echelon from cannot have a one in the (3,3) location. That is the reduced echelon form cannot be of the form ($\mathbf{I}_n$: **B**). Thus A^{-1} does not exist.

Class work 4 Find the inverse of the matrix below if it exists.

a) $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{pmatrix}$

Properties of Inverse Matrices

Let **A** and **B** be invertible matrices and c a nonzero scalar. Then

1. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
2. $(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$
3. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
4. $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$
5. $(\mathbf{A}^t)^{-1} = (\mathbf{A}^{-1})^t$

we verify the 1st and 3rd results to illustrate the techniques involved leaving for the reader the remaining results to verify.

$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ this result follows directly from the definition of inverse of a matrix. Since $\mathbf{A}^{-1}$ is the inverse of **A**, we have

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

This statement also tells us that **A** is the inverse of $\mathbf{A}^{-1}$. Thus $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ we want to show that the matrix $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of the matrix **AB.** We get, using the properties of matrices,

$$\begin{aligned} \mathbf{AB}\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) &= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} \\ &= \mathbf{AI}_n\mathbf{A}^{-1} \\ &= \mathbf{AA}^{-1} = \mathbf{I}_n \end{aligned}$$

Similarly, it can be shown that $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}_n$. Thus $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of the matrix **AB**.

Example: If $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$, then it can be shown that $\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$. Use this information to compute $(\mathbf{A}^t)^{-1}$.

Solution: Result 5 above tells us that if we know the inverse of a matrix we also know that inverse of its transpose. We get

$$(\mathbf{A}^t)^{-1} = (\mathbf{A}^{-1})^t = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

Class work

If $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 9 & 2 \end{pmatrix}$, then $\mathbf{A}^{-1} = \begin{pmatrix} 2 & -1 \\ -9 & 5 \end{pmatrix}$. Use this information to determine

a) $(2\mathbf{A}^t)^{-1}$, b) $\mathbf{A}^{-3}$ c) $(\mathbf{AA}^t)^{-1}$.

1.5 Determinants and its properties

To every square matrix $\mathbf{A} = [a_{ij}]_{n\times n}$ is associated a number or an expression called the determinant of **A** and is denoted by |**A**| or det (**A**).

Determinant of order one

Let $\mathbf{A} = [a_{11}]$ be a square matrix of order one. Then det (**A**) = a_{11}. By definition, if **A** is invertible, then $a_{11} \neq 0$ and so det **A**≠0. Also, conversely if det **A**≠0, then $a_{11} \neq 0$ and so, A is invertible.

Determinant of order two

The *determinant* of a 2×2 matrix is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example 1 Calculate the determinant of the following matrices.

a) b)

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 \\ -1 & 7 \end{pmatrix}$$

Solution:

(*a*)

$$\det\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = 3\cdot 1 - 2\cdot 0 = 3$$

(*b*)

$$\det\begin{pmatrix} -1 & 2 \\ -1 & 7 \end{pmatrix} = (-1)7 - (-1)2 = -5$$

Definition 1.15 Let $\mathbf{A} = [a_{ij}]_{n\times n}$ and $\mathbf{M}_{ij}$ be the (n-1)×(n-1) matrix obtained from **A** by deleting the row i and j column containing a_{ij}. The det($\mathbf{M}_{ij}$) is called the **minor** of a_{ij}. We define the **cofactor** $\mathbf{C}_{ij}$ of a_{ij} by $\mathbf{C}_{ij} = (-1)^{i+j}\det(\mathbf{M}_{ij})$.

Example 2 Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

Solution: Applying the above definitions we get the following.

Minor of a_{11}:

$$\det(\mathbf{M}_{11}) = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = (1\cdot 6) - (4\cdot 2) = -2$$

Cofactor of a_{11}: $\mathbf{C}_{11} = (-1)^{1+1}\det(\mathbf{M}_{11}) = -2$.

Minor of a_{32}: $\det(\mathbf{M}_{32}) = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = (2.2) - (3.4) = -8$

Cofactor of a_{32}: $\mathbf{C}_{32} = (-1)^{3+2}\det(\mathbf{M}_{32}) = 8$.

Definition 1.16 The **determinant of a square matrix** is the sum of the product of the elements of the first row and their cofactors.

If **A** is 3×3, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

If **A** is 4×4, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

$\vdots$

If **A** is n×n, $|\mathbf{A}| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}$

These equations are called cofactor expansions of |**A**|.

Example 3 Evaluate the determinant of the following matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{pmatrix}$$

Solution: Using the elements of the first row and their corresponding cofactors, we get

$$\begin{aligned} |\mathbf{A}| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-1)^2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} + 5(-1)^3\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix} + 4(-1)^4\begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} \\ &= 2(6-8) - 5(18-10) + 4(12-5) = -16 \end{aligned}$$

We have defined the determinant of a matrix in terms of its first row. It can be shown that the determinant can be computed using a different row or one of the columns. For example the cofactor expansion along the second column yields

$$|\mathbf{A}| = -5\begin{vmatrix}3 & 2\\5 & 6\end{vmatrix} + 1\begin{vmatrix}2 & 4\\5 & 6\end{vmatrix} - 4\begin{vmatrix}2 & 4\\3 & 2\end{vmatrix}$$
$$= -5(18-10) + 1(12-20) - 4(4-12)$$
$$= -16$$

As we have seen it is not necessary to limit ourselves to using the first row for the cofactor expansion. We state the following theorem without proof.

Theorem 1.17 If **A** is an n × n matrix with n ≥ 2, then det(**A**) can be expressed as a cofactor expansion using any row or column of **A**.

$$\det(\mathbf{A}) = a_{i1}\mathbf{C}_{i1} + a_{i2}\mathbf{C}_{i2} + \dots + a_{in}\mathbf{C}_{in}$$
$$= a_{1j}\mathbf{C}_{1j} + a_{2j}\mathbf{C}_{2j} + \dots + a_{nj}\mathbf{C}_{nj}$$

for i = 1, . . .,n and j = 1, . . .,n.

The cofactor expansion of a 4 × 4 determinant will involve four 3 × 3 determinants. One can often save work by expanding along the row or column that contains the most zeros.

Note: There is a useful rule that can be used to give the sign part, $(-1)^{i+j}$, of the cofactors in these expansion. The rule is summarized in the following array.

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & & & & \end{pmatrix}$$

If, for example, one expands in terms of the second row, the signs will be – + – etc. The signs alternate as one goes along any row or column.

Example 4 To evaluate the determinant of

$$\begin{pmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

one would expand down the first column. The first three terms will drop out, leaving

$$-2\begin{vmatrix}2 & 3 & 0\\4 & 5 & 0\\1 & 0 & 3\end{vmatrix} = -2\cdot 3\cdot\begin{vmatrix}2 & 3\\4 & 5\end{vmatrix} = 12.$$

Example 5 Calculate

$$\det\begin{pmatrix} 2 & -1 & 4 & 6 \\ 3 & -2 & 7 & 17 \\ 0 & 0 & 0 & 0 \\ -4 & 3 & -6 & 12 \end{pmatrix}$$

Solution **Choose row 3, since it has the most zeros**

$$\det\begin{pmatrix} 2 & -1 & 4 & 6 \\ 3 & -2 & 7 & 17 \\ 0 & 0 & 0 & 0 \\ -4 & 3 & -6 & 12 \end{pmatrix} = 0\cdot\det(\ ) + 0\cdot\det(\ ) + 0\cdot\det(\ ) + 0\cdot\det(\ ) = 0$$

This example illustrates the fact that if a matrix has a row (or column) containing all zeros, the determinant is zero.

Example 6 Calculate

$$\det\begin{pmatrix} 2 & 16 & 17 & 4 \\ 0 & -3 & 22 & -3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution Choose column 1 since it has the most zeros.

$$\det\begin{pmatrix} 2 & 16 & 17 & 4 \\ 0 & -3 & 22 & -3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 2\det\underbrace{\begin{pmatrix} -3 & 22 & -3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Use column 1}} = +2(-3)\det\begin{pmatrix} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= +2(-3)(6) = -36$$

Another look at Example 6 shows us that the determinant of the given matrix was the product of the diagonal elements. Although this does not happen for all matrices, it does if the matrix is upper or lower triangular.

Theorem 2.18 If $\mathbf{A}_{n\times n}$ is upper (or lower) triangular, then. $\det(\mathbf{A}) = a_{11}a_{22}\ldots a_{nn}$

Proof. Let us use the principle of mathematical induction. The proposition P(n) is as follows: An $n \times n$ upper triangular matrix A has determinant. $a_{11}a_{22}\ldots a_{nn}$ First, we check P(2). When $n = 2$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

and by definition $\det(\mathbf{A}) = a_{11}a_{22}$. The proposition is true for $n = 2$. For the induction hypothesis we suppose that P(k) is true. That is, suppose that if $\mathbf{A}_{k\times k}$ is upper triangular, then $\det(\mathbf{A}_{k\times k}) = a_{11}a_{22}\ldots a_{kk}$.

To complete the proof, we must show that $\det(\mathbf{A}_{k+1\times k+1}) = a_{11}a_{22}\ldots a_{k+1k+1}$. . Writing A_{k+1k+1}, we have

$$\begin{pmatrix} a_{11} & a_{12} & & \cdots & & a_{1k} & a_{1,k+1} \\ 0 & a_{22} & & & & a_{2k} & a_{2,k+1} \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & & & & \ddots\ a_{kk} & a_{k,k+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1,k+1} \end{pmatrix}$$

We compute $\det(\mathbf{A}_{k+1\times k+1})$ **by using row k +1 to find**

$$\det A_{k+1\times k+1} = (-1)^{2k+2} a_{k+1,k+1} \det A_{k\times k}$$

$$\underline{\text{By induction hypothesis}} = a_{k+1,k+1}(a_{11}\cdots a_{kk})$$

$$= a_{11}\cdots a_{kk}a_{k+1,k+1}$$

Thus by the principle of mathematical induction, the proposition is true for all n . ☐

***Proof*. [(Alternative) Proof] Let**

$$A = \begin{pmatrix} a_{11} & & & & \\ 0 & a_{22} & & & \\ 0 & 0 & a_{33} & a'_{ij}s & \\ \vdots & 0's & \vdots & \ddots & \\ 0 & \cdots & 0 & \cdots & a_{nn} \end{pmatrix}$$

Use the first column to calculate:

$$\det = A = a_{11}\det \begin{pmatrix} a_{22} & & & & \\ 0 & & a_{33} & a'_{ij}s & \\ \vdots & 0's & & \ddots & \\ 0 & \cdots & 0 & \cdots & a_{nn} \end{pmatrix}$$

Use the first column again:

$$\det A = a_{11}a_{22}\det \begin{pmatrix} a_{33} & & a'_{ij}s \\ & \ddots & \\ 0's & & \\ & & a_{nn} \end{pmatrix}$$

Continuing to always use the first column gives

$$\det A = a_{11}a_{22}\cdots a_{n-2,n-2}\det\begin{pmatrix} a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{nn} \end{pmatrix}$$

$$= a_{11}a_{22}\cdots a_{nn}$$

Both proofs are almost the same for lower triangular matrices. This is left to the problems. ☐

So, if **A** is upper or lower triangular, the determinant is easy to calculate. To use this fact, we can row-reduce a matrix to upper or lower triangular form, calculate the determinant of the resulting matrix, and then relate that determinant to the determinant of the original matrix.

Class work

1. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 & -5 \\ 8 & -1 & 2 & 1 \\ 4 & -3 & -5 & 0 \\ 1 & 4 & 8 & 2 \end{pmatrix}$$

Find the following minors and cofactors of **A**.

(a) M_{12} and C_{12} (b) M_{43} and C_{43}

2. Find the determinants of the following matrix using as little computation as possible.

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 5 & 0 \\ 7 & -3 & 8 & 4 \\ -3 & 0 & 4 & 0 \end{pmatrix}$$

Properties of a Determinant

The following theorem tells us how elementary row operation affect determinants. It also tells us that these operations can be extended to columns.

Theorem 1.19 Let A be an n × n matrix and *c* be a nonzero scalar.

a) If a matrix **B** is obtained from **A** by multiplying the elements of a row (column) by *c* then $|\mathbf{B}| = c|\mathbf{A}|$.
b) If a matrix **B** is obtained from **A** by interchanging two rows (column) then $|\mathbf{B}| = -|\mathbf{A}|$
c) If a matrix **B** is obtained from **A** by adding a multiple of one row (column) to another row(column), then $|\mathbf{B}| = |\mathbf{A}|$.

The proof is left as exercise.

Example 7 Evaluate the determinant

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$$

Solution: We examine the rows and columns of the determinant to see if we can create zeros in a row column the above operations. Note that we can create zeros in the second column by adding twice the third column to it:

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \underset{C_2+2C_3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix}$$

Expand this determinant in terms of the second column to take advantage of the zeros.

$$= (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = (-3)(9-2) = -21.$$

We shall find that matrices that have zero determinant play a significant role in theory of matrices.

Definition 1.20 A square matrix **A** is said to be **singular** if |**A**| = 0. **A** is **nonsingular** if |**A**| ≠ 0.

The following theorem gives information about some of the circumstance under which we can expect a matrix to be singular.

Theorem 1.21 Let **A** be a square matrix. **A** is singular if

a) all the elements of a row (column) are zero
b) two rows (columns) are equal.
c) Two rows(columns) are proportional.

Example 8 Show that the following matrices are singular.

1.

a) $\mathbf{A} = \begin{pmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{pmatrix}$ b) $\mathbf{B} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$

Solution:

a) All the elements in column 2 of **A** are zero. Thus |**A**| = 0.
b) Observe that every element in row 3 of **B** is twice the corresponding element in row 2. We write

(row 3) = 2(row 2)

Row 2 and row 3 are proportional. Thus |**B**| = 0.

The following theorem tells us how determinants interact with various matrix operations. The examples following it demonstrate the theorem in use.

Theorem 1.22 Let A and B be n × n matrices and *c* be a nonzero scalar.

a) Determinant of a scalar multiple : $|c\mathbf{A}| = c^n|\mathbf{A}|$
b) Determinant of a product: $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
c) Determinant of a transpose: $|\mathbf{A}^t| = |\mathbf{A}|$
d) Determinant of an inverse: $\left|\mathbf{A}^{-1}\right| = \dfrac{1}{|\mathbf{A}|}$ (assuming $\mathbf{A}^{-1}$ exists).

Example 9 If A is a 2×2 matrix with |A| = 4, use Theorem 2.22 to compute the following determinants.

a) |3**A**| b) $|\mathbf{A}^2|$ c) $|5\mathbf{A}^t\mathbf{A}^{-1}|$, assuming $\mathbf{A}^{-1}$ exists.

Solution:

a) $|3\mathbf{A}| = (3^2)|\mathbf{A}| = (9).4 = 36$

b) $|\mathbf{A}^2| = |\mathbf{AA}| = |\mathbf{A}||\mathbf{A}| = (4).(4) = 16$

c) $|5\mathbf{A}^t\mathbf{A}^{-1}| = (5^2)|\mathbf{A}^t\mathbf{A}^{-1}| = 25|\mathbf{A}^t||\mathbf{A}^{-1}| = 25|\mathbf{A}|\frac{1}{|\mathbf{A}|} = 25.$

Example 10 Prove that $\left|\mathbf{A}^{-1}\mathbf{A}^t\mathbf{A}\right| = |\mathbf{A}|$

Solution: by the properties of matrices, determinants, and real numbers we get

$$\left|\mathbf{A}^{-1}\mathbf{A}^t\mathbf{A}\right| = \left|(\mathbf{A}^{-1}\mathbf{A}^t)\mathbf{A}\right| = \left|\mathbf{A}^{-1}\mathbf{A}^t\right||\mathbf{A}| = \left|\mathbf{A}^{-1}\right|\left|\mathbf{A}^t\right||\mathbf{A}| = \frac{1}{|\mathbf{A}|}|\mathbf{A}||\mathbf{A}| = |\mathbf{A}|.$$

Class Work

1. Find all the values of x that make the following determinant zero. $\begin{pmatrix} x-1 & -2 \\ x-2 & x-1 \end{pmatrix}$

2. If $\mathbf{A} = \begin{pmatrix} 1 & -1 & -3 \\ 2 & 0 & -4 \\ -1 & 1 & 2 \end{pmatrix}$, then $|\mathbf{A}| = -2$. Use this information, together with the properties of determinants, to compute the determinant of the following matrices. a) $\begin{pmatrix} 1 & -1 & -3 \\ 2 & 0 & -4 \\ -2 & 2 & 4 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 0 & -4 \\ 1 & -1 & -3 \\ -1 & 1 & 2 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -1 & -3 \\ 4 & -2 & -10 \\ -1 & 1 & 2 \end{pmatrix}$

3. If **A** and **B** are 3 × 3 matrices and $|A| = -3$, $|B| = 2$, compute the following determinants. a) $|\mathbf{AB}|$ b) $|\mathbf{AA}^t|$ c) $|(\mathbf{A}^t\mathbf{B}^{-1})^t|$

1.6 Determinant Method of Finding Inverse Matrices

We first introduce tools necessary for developing a formula for the inverse of a nonsingular matrix.

Definition 2.23 Let **A** be an n × n matrix and C_{ij} be the cofactor of a_{ij}. The matrix whose (i, j)th element is C_{ij} is called the matrix of cofactors. The transpose of this matrix is called the adjoint of **A** and is denoted adj(**A**).

$$\underbrace{\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}}_{matrix\ of\ cofactor} \qquad \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^t = \underbrace{\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}}_{adjoint\ matrix}$$

Example 11 Find the matrix of cofactors and the adjoint matrix of the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & -1 & 3 \\ -4 & 5 & 2 \\ -3 & 1 & 4 \end{pmatrix}$$

Solution: The cofactors of A are

$$C_{11} = \begin{vmatrix} 5 & 2 \\ 1 & 4 \end{vmatrix} = 18 \qquad C_{12} = -\begin{vmatrix} -4 & 2 \\ -3 & 4 \end{vmatrix} = 10 \qquad C_{13} = \begin{vmatrix} -4 & 5 \\ -3 & 1 \end{vmatrix} = 11$$

$$C_{21} = -\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} = 7 \qquad C_{22} = \begin{vmatrix} -2 & 3 \\ -3 & 4 \end{vmatrix} = 1 \qquad C_{23} = -\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix} = 5$$

$$C_{31} = \begin{vmatrix} -1 & 3 \\ 5 & 2 \end{vmatrix} = -17 \qquad C_{32} = -\begin{vmatrix} -2 & 3 \\ -4 & 2 \end{vmatrix} = -8 \qquad C_{33} = \begin{vmatrix} -2 & -1 \\ -4 & 5 \end{vmatrix} = -14$$

Thus the matrix of cofactors of **A** is

$$\begin{pmatrix} 18 & 10 & 11 \\ 7 & 1 & 5 \\ -17 & -8 & -14 \end{pmatrix}$$

and the adjoint of A is the transpose of this matrix

$$adj(\mathbf{A}) = \begin{pmatrix} 18 & 7 & -17 \\ 10 & 1 & -8 \\ 11 & 5 & -14 \end{pmatrix}$$

The next lemma is important for the proof of the theorem on inverse of a nonsingular matrix.

Lemma 1.24 Let A be an n × n matrix. If C_{jk} denotes the cofactor of a_{jk} for k=1,…,n then

$$a_{i1}\mathbf{C}_{j1} + a_{i1}\mathbf{C}_{j1} + ... + a_{i1}\mathbf{C}_{j1} = \begin{cases} |\mathbf{A}| & if \ \ i = j \\ 0 & if \ \ i \neq j \end{cases} \qquad (1)$$

Proof: If i = j, (1) is just the cofactor expansion of det(**A**) along the ith row of **A**. If i ≠ j, then it is the expansion of the determinant of a matrix in which the jth row of **A** has been replaced by the ith row of **A**, thus this is matrix having two identical rows consequently its determinant is zero as in (1).

Theorem 2.25 Let **A** be a square matrix with $|\mathbf{A}| \neq 0$. **A** is invertible with

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj(\mathbf{A}).$$

Solution: From Lemma 2.24 we observe that the product **A**adj(**A**) is thus a diagonal matrix with the diagonal elements to get $|\mathbf{A}|\mathbf{I}_n$. Thus

$$\mathbf{A} \ adj(\mathbf{A}) = |\mathbf{A}|\mathbf{I}_n$$

If **A** is nonsingular, det(**A**) is a nonzero scalar and we may write

$$\mathbf{A}\left(\frac{1}{|\mathbf{A}|} adj(\mathbf{A})\right) = \mathbf{I}_n$$

Thus

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj(\mathbf{A}).$$

Example 12 For a 2 × 2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$adj\ \mathbf{A} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

If A is nonsingular, then

$$\mathbf{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Example 13 Use the result of Theorem 2.25 to compute the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & -1 & 3 \\ -4 & 5 & 2 \\ -3 & 1 & 4 \end{pmatrix}$$

Solution: |**A**| is computed and found to be –13. This matrix was discussed in Example 11. There we found that

$$adj(\mathbf{A}) = \begin{pmatrix} 18 & 7 & -17 \\ 10 & 1 & -8 \\ 11 & 5 & -14 \end{pmatrix}$$

The formula for the inverse of a matrix gives

$$\mathbf{A}^{-1} = -\frac{1}{13} adj(\mathbf{A}) = \begin{pmatrix} -\frac{18}{13} & -\frac{7}{13} & \frac{17}{13} \\ -\frac{10}{13} & -\frac{1}{13} & \frac{8}{13} \\ -\frac{11}{13} & -\frac{5}{13} & \frac{14}{13} \end{pmatrix}.$$

Class work

Determine whether the following matrices have inverse. If a matrix has an inverse, find the inverse using the formula for the inverse of a matrix.

a) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 3 \end{pmatrix}$

We now discuss the relationship between the existence and uniqueness of the solution to a system of n linear equations in n variables and the determinant of the matrix of coefficient of the system.

1.7 System of linear equations and characterization of solutions

Theorem 2.26 (**Cramer's Rule**). Let **A** be an n × n nonsingular matrix and let $\mathbf{B} \in \mathbf{R}^n$. Let A_i be the matrix obtained by replacing the ith column of **A** by **B**. If **X** is the unique solution to **AX =B**, then

$$x_i = \frac{|\mathbf{A}_i|}{|\mathbf{A}|} \qquad for\ i = 1, 2, \ldots, n$$

Proof: Since

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{|\mathbf{A}|}(adj\mathbf{A})\mathbf{B}$$

it follows that

$$x_i = \frac{b_1C_{1i} + b_2C_{2i} + ... + b_nC_{ni}}{|\mathbf{A}|}$$

$$= \frac{|\mathbf{A}_i|}{|\mathbf{A}|}.$$

The next theorem characterizes the solution of a system of equations using the determinant of the coefficient matrix and the matrix $\mathbf{A}_i$ defined in Cramer's rule above.

Theorem 1.26 Let **AX = B** be a system of n linear equations in n variables.

i) If $|\mathbf{A}| \neq 0$, then **AX = B** a ***unique solution***. The system has a ***trivial solution*** that is **X=0** if **B = 0**.

ii) If $|\mathbf{A}| = 0$, and at least one of the $\mathbf{A}_i$s is nonzero the system has no solution. For, if $|\mathbf{A}| = 0$ and $|\mathbf{A}_1| \neq 0$, then $X|\mathbf{A}| = |\mathbf{A}_1|$ leads to a contradiction. Such systems are called ***inconsistent***

iii) If $|\mathbf{A}| = 0$ and $|\mathbf{A}_i| = 0,\ i = 1,2,...,n$ the system my have an infinite number of solutions or may not have a solution. A system having an infinite number of solutions is called ***dependent***.

Definition 1.27 If **AX = 0** then the system of equations is said to be ***homogeneous***.

Example 1 Solve

$$3x_1 - x_2 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 6$$
$$x_1 - x_2 - x_3 = -4$$

by using Cramer's rule.

Solution First we calculate $\det A$:

$$\det A = \det\begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \underset{R_2+R_3}{=} \det\begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \end{pmatrix} = 2\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -4$$

Now substitute

$$B = \begin{pmatrix} 4 \\ 6 \\ -4 \end{pmatrix}$$

for column 1 and calculate

$$x_1 = \frac{\det\begin{pmatrix} 4 & -1 & 1 \\ 6 & 1 & 1 \\ -4 & -1 & 1 \end{pmatrix}}{-4} \underset{-C2+C3}{=} \frac{\det\begin{pmatrix} 4 & -1 & 2 \\ 6 & 1 & 0 \\ -4 & -1 & 0 \end{pmatrix}}{-4} = \frac{-4}{-4} = 1$$

Similarly,

$$x_2 = \frac{\det\begin{pmatrix} 3 & 4 & 1 \\ 1 & 6 & 1 \\ 1 & -4 & -1 \end{pmatrix}}{-4} \underset{-4C3+C2}{=} \frac{\det\begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}}{-4} = \frac{-8}{-4} = 2$$

$$x_3 = \frac{\det\begin{pmatrix} 3 & -1 & 4 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{pmatrix}}{-4} \underset{\substack{R2+R3 \\ R1+R2}}{=} \frac{\det\begin{pmatrix} 3 & -1 & 4 \\ 4 & 0 & 10 \\ 2 & 0 & 2 \end{pmatrix}}{-4} = \frac{-12}{-4} = 3$$

The answer checks, by direct substitution.
The following two systems of linear equations, each of which has a singular matrix of coefficients, illustrate that there may be many or no solutions.

$$\underbrace{\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 4x_2 + 5x_3 &= 3 \\ 2x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}}_{\textit{many solution}} \qquad\qquad \underbrace{\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3 \\ 2x_1 + x_2 + 3x_3 &= 3 \\ x_1 + x_2 + 2x_3 &= 0 \end{aligned}}_{\textit{no solution}}$$

$$x_1 = t + 1, x_2 = 2t, x_3 = t$$

Example 2 Determine values of r for which the following system of equations has nontrivial solutions. Find the solutions for each value of r.

$$\begin{aligned} (r+2)x_1 + (r+4)x_2 &= 0 \\ 2x_1 + (r+1)x_2 &= 0 \end{aligned}$$

Solution: This system is a homogeneous system of linear equations. It thus has the trivial solution by Theorem 2.26 (i). The same theorem part (iii) tells us that there is the possibility of other solution only if the determinant of the matrix of coefficients is zero. Equating this determinant to zero, we get

$$\begin{vmatrix} r+2 & r+4 \\ 2 & r+1 \end{vmatrix} = 0$$

$$(r+2)(r+1) - 2(r+1) = 0$$

$$r^2 + r - 6 = 0$$

$$(r-2)(r+3) = 0$$

Thus the determinant is zero if r = –3 or r = 2.

r = –3 results in the system

$$-x_1 + x_2 = 0$$

$$2x_1 - 2x_2 = 0$$

this system has infinitely many solutions, $x_1 = t, x_2 = t.$

$r = 2$ **results in the system**

$$4x_1 + 6x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

This system has many solutions, $x_1 = -3t/2, x_2 = t.$

Cramer's rule gives us a convenient method for writing down the solution to an n × n system of equations in terms of determinants. In this method we can solve for any one of the $x_i s$ with out solving the solution of the entire system. However to compute the solution of the system as a whole, one must evaluate n + 1 determinants of order n. Evaluating even two of these determinants generally involves more computation than solving the system using Gaussian Elimination that we are going to see bellow.

The Gaussian Elimination Method

Let AX = B be a linear system of equations then

1. Write down the augmented matrix of the system of linear equations.
2. Find an echelon form of the augmented matrix using elementary row operations.
3. Write down the system of equations corresponding to the echelon form.
4. Use back substitution to arrive at the solution.

Example 3 Solve the following system of linear equations using the method of Gaussian elimination.

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$

$$-x_1 - 2x_2 - 2x_3 + x_4 = 2$$

$$2x_1 + 4x_2 + 8x_3 + 12x_4 = 4$$

Solution: Solving the augmented matrix, create zeros below the pivot in the first column.

$$\begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{pmatrix} \underset{\substack{R_2+R_1 \\ R_3-2R_1}}{\approx} \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{pmatrix}$$

$$\underset{R_3-2R_2}{\approx} \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{pmatrix}$$

$$\underset{\frac{1}{2}R_3}{\approx} \begin{pmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

We have arrived at the echelon form

The corresponding system of equation is

$$x_1 + 2x_2 + 3x_3 + 2x_4 = -1$$
$$x_3 + 3x_4 = 1$$
$$x_4 = 2$$

The system is now solved by *back substitution* i.e the value of x_4 is substituted into the second equation to give x_3. x_3 and x_4 are then substituted into the first equation to get x_1. We get

$$x_3 + 3(2) = 1$$
$$x_3 = -5$$

Substituting $x_4 = 2$ *and* $x_3 = -5$ into the first equation we have

$$x_1 + 2x_2 + 3(-5) + 2(2) = -1$$
$$x_1 + 2x_2 = 10$$
$$x_1 = -2x_2 + 10$$

Let $x_2 = t$ hence the system has infinitely many solutions.

Example 4 Determine the value of *k* so that the following system of unknown x, y, z has (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions.

$$x + y - z = 1$$
$$2x + 3y + kz = 3$$
$$x + ky + 3y = 2$$

Solution: The augmented matrix determined by the system is

$$\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & -k & 3 \\ 1 & k & 3 & 2 \end{pmatrix} \underset{\substack{R_2-2R_1 \\ R_3-R_1}}{\approx} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{pmatrix}$$

$$\underset{R_3-(k-1)R_2}{\approx} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & -(k-1)(k+2)+4 & 2-k \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & (2-k)(k+3) & 2-k \end{pmatrix}$$

The system has a unique solution if the solution if the coefficient of z in the third equation is not zero; that is, if k ≠ 2 and k ≠ -3. In case k = 2, the third equation reduces to 0 = 0 and the system has infinite equation reduces to 0 = 5 and the system has no solution.
Summarizing (i) k ≠ 2 and k ≠ 3, (ii) k = -3, (iii) k =2.

Class Work

Solve the following systems of equations using

a) Cramer's rule (if possible)
b) Gaussian elimination method.

i) $\begin{aligned} 2x_1+7x_2+3x_3 &= 7 \\ x_1+2x_2+x_3 &= 2 \\ x_1+5x_2+2x_3 &= 5 \end{aligned}$

ii) $\begin{aligned} x_1+6x_2-x_3 &= 3 \\ x_1-2x_2+3x_3 &= 2 \\ 4x_1-2x_2+5x_3 &= 5 \end{aligned}$

MATH 231 WORKSHEET II

1. Given that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ find **A**.

2. For $n \in N$, $let\ \mathbf{A_n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ then show that

   (i) $\mathbf{A_n A_m} = \mathbf{A_{n+m}}$ (ii) $\mathbf{A}_n^{-1} = \mathbf{A}_{-n}$

3. Let $f(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Then show that $f^n(\theta) = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$ for $n \in N$

4. Find the inverse of the following matrices if possible.

   (a) $\begin{pmatrix} 1 & 3 \\ 8 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 2 \\ 9 & 4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 3 & 7 & 5 \\ -1 & 2 & -1 & 3 \\ 2 & 0 & 1 & 4 \\ 1 & -1 & -1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

5. Find the values of x for which the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & x \end{pmatrix}$ is invertible. In that case give $\mathbf{A}^{-1}$.
6. Find the determinants of the matrices in 4 above.
7. What is the determinant of the product matrix below?

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & \frac{1}{3} \\ 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. Given that $\begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5\mathbf{I}_{3,}$ what is det(**A**)?
9. Find the determinant and inverse of the following matrix

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & d \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ (c) $\begin{pmatrix} x+\lambda & x & x \\ x & x\lambda & x \\ x & x & x \end{pmatrix}$

10. For what values of k will the system

$$x_1 + x_2 + kx_3 = 0$$
$$x_1 + kx_2 + x_3 = 0$$
$$kx_1 + x_2 + x_3 = 0$$

have a non-trivial solution? In each case what are these solutions?

11. Solve each of the following system of linear equations by using:
    i) Gaussian elimination method
    ii) Cramer's rule, when ever possible

a) $\begin{aligned} 2x - 5y &= 1 \\ 3x - 2y &= -4 \end{aligned}$ b) $\begin{aligned} 3x_1 + 2x_2 - x_3 &= -1 \\ -x_1 + 2x_2 - 9x_3 &= 9 \end{aligned}$ c) $\begin{aligned} 2x + y + 6z &= 6 \\ 3x + 2y - 2z &= -2 \\ x + y + 2z &= 4 \end{aligned}$

d) $\begin{aligned} 3x + 4y + 7z &= 0 \\ y - 2z &= 3 \\ x + 3y - z &= -5 \end{aligned}$ e) $\begin{aligned} -x_1 + x_2 - x_3 &= -1 \\ -2x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$ d) $\begin{aligned} x + y + z - 2w &= -4 \\ 2y + z + 3w &= 4 \\ 2x + y - z + 2w &= 5 \\ x - y + w &= 4 \end{aligned}$

12. A man refused to tell anyone his age, but he likes to drop hints about it. He then remarks that twice his mother's age add up to 140 and also that his age plus his father's age add up to 105 Furthermore, he says that the sum of his age and his mother's age is 30 more than his father's age. Calculate the man's age or show that his hints contradict one another.

13. Find the eigenvalues and eigenvectors of the matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 5 & -1 & 0 \\ 0 & -5 & 9 \\ 5 & -1 & 0 \end{pmatrix}$$

14. If **D** is a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

   a) What is the characteristic polynomial of **D**?
   b) What are its eigenvalues?

15. Show that if $\theta \in \Re$ and θ is not an integral multiple of π, then the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ does not have a nonzero eigenvector in $\Re^2$.

16. The eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of a nonsingular matrix **A**. Furthermore, the eigenvectors for **A** and $\mathbf{A}^{-1}$ are the same. Verify these facts for the matrices given below

$$\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

2 Vectors and Vector Spaces

2.1 Scalars and Vectors; Located Vectors in R2 and R3

A **scalar** is simply a real number, a complex number or a quantity that has magnitude but no direction. For instance length, temperature, and blood pressure are represented by real numbers hence are scalar quantities. A **vector**, on the other hand, is usually described as a quantity that has both magnitude and direction. Geometrically, a vector is represented by a directed line segment that is an arrow and is written either as a boldface symbol **v** or $\vec{v}$ *or* $\overrightarrow{AB}$ for instance weight, velocity, frictional force are vector quantity.

Notations and Terminologies

A vector whose initial point is A and whose terminal point is B is given by $\overrightarrow{AB}$ and the magnitude (or length) of vector $\overrightarrow{AB}$ is denoted by $\left\|\overrightarrow{AB}\right\|$. Moreover two vectors that have the same magnitude and the same directions are said to be equal. Thus in fig 1 below $\overrightarrow{AB} = \overrightarrow{CD}.$

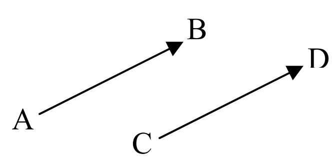


Fig 1

Because of this property of vectors that we can move vectors from one position to another provided its magnitude and direction are maintained, so we say that vectors are **free** by their very nature. The negative of a vector $\overrightarrow{AB}$, written $-\overrightarrow{AB}$, is a vector that has the same magnitude as $\overrightarrow{AB}$ but opposite in direction. If $k \neq 0,$ then k$\overrightarrow{AB}$ is a vector that is $|k|$ as long as $\overrightarrow{AB}$. When k=0 we say 0$\overrightarrow{AB}$ = **0** (zero vector). Two vectors are said to be parallel if and only if they are **nonzero** scalar multiples of each other.

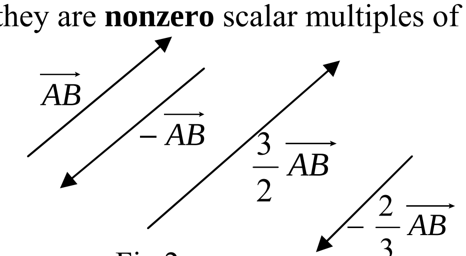
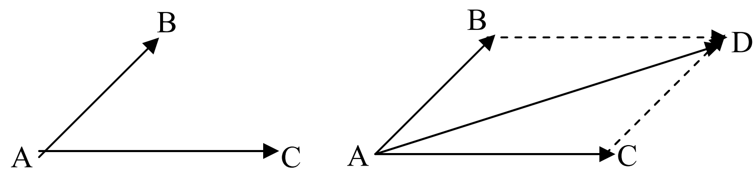


Fig 2

Addition and Subtraction

Two vectors can be considered as having a common initial, such as in fig 3a. Thus, if nonparallel vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the sides of a parallelogram as in fig 3b, we say the vector that is the main diagonal, or $\overrightarrow{AD}$, is the sum of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and we write

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}.$$



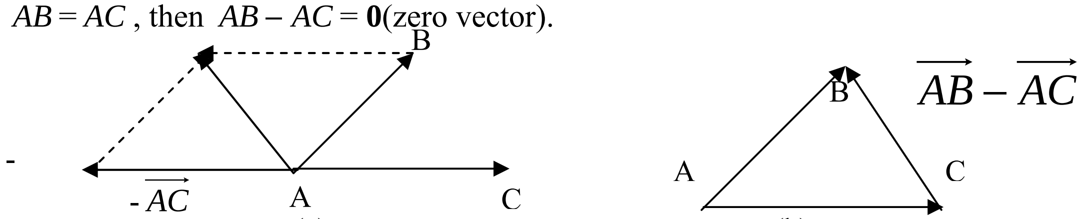
(a) (b)

Fig. 3

The difference of two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is defined by

$$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{AB} + (-\overrightarrow{AC}).$$

As seen in fit 4(a), the difference $\overrightarrow{AB} - \overrightarrow{AC}$ can be interpreted as the main diagonal of a parallelogram with sides $\overrightarrow{AB}$ and $-\overrightarrow{AC}$. However, as shown, in fig 4b, we can also interpret it as the third side of a triangle with sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In this second interpretation, observe that the vector difference $\overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC}$ points toward the terminal point of the vector from which we are subtracting the second vector. If $\overrightarrow{AB} = \overrightarrow{AC}$, then $\overrightarrow{AB} - \overrightarrow{AC} = \mathbf{0}$(zero vector).



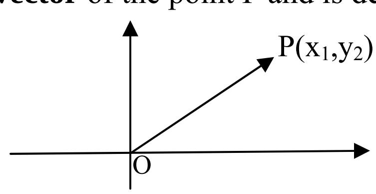
(a) (b)

Fig. 4

Vectors in R^2

To describe a vector analytically, let us consider vectors in two-dimensional coordinate plane. The vector with initial point the origin O and terminal point P(x_1,y_1) in fig 5, is called a **position vector** of the point P and is denoted by $\overrightarrow{OP} = \langle x_1, y_1 \rangle$

Fig. 5



In general, a vector a in R^2 is any ordered pair of real numbers the kind

$$\mathbf{a} = \langle a_1, a_2 \rangle.$$

The numbers a_1 and a_2 are said to be components of the vector a.

As we shall see in the first example, the vector **a** is not necessarily a position vector.

Example 1
The displacement between the point (x,y) and (x+4,y+3) in fig 5a is written <4,3>. As seen in fig. 6b, the position vector of <4,3> is the vector emanating from the origin and terminating at the point P(4,3).

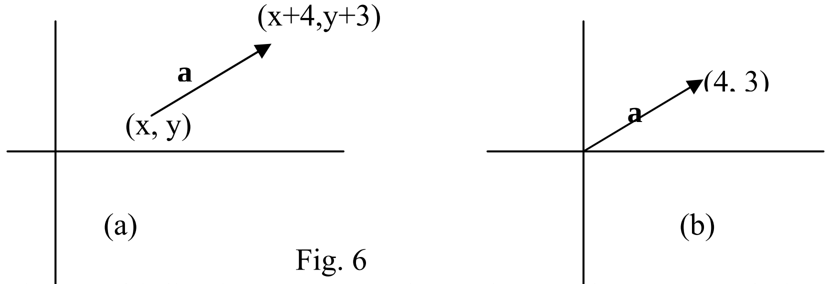


Fig. 6

In $\mathbf{R}^2$ addition, subtraction, multiplication of vectors by scalars, and so on, are defined in terms of their components.

Definition 2.1 Let $\mathbf{a} =< a_1, a_2 >$ and $\mathbf{b} =< b_1, b_2 >$ be vectors in $\mathbf{R^2}$ then

i) Addition: $\mathbf{a}+\mathbf{b} =< a_1 + b_1, a_2 + b_2 >$

ii) Subtraction: $\mathbf{a}-\mathbf{b} =< a_1 - b_1, a_2 - b_2 >$

iii) Equality: $\mathbf{a}=\mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2$

iv) Scalar multiplication: $k\mathbf{a} =< ka_1, ka_2 >$

Example 2 If $\mathbf{a} =< 1,4 >$ and $\mathbf{b} =< -6,3 >$, find $\mathbf{a}+\mathbf{b}, \mathbf{a}-\mathbf{b}$, *and* $2\mathbf{a}+3\mathbf{b}$.

Solution: By definition 1.1

$$\mathbf{a}+\mathbf{b} =< 1+(-6), 4+3 >=< -5,7 >$$
$$\mathbf{a}-\mathbf{b} =< 1-(-6), 4-3 >=< 7,1 >$$
$$2\mathbf{a}+3\mathbf{b} =< 2,8 > + < -18,9 >=< -16,17 >$$

Definition 2.2 The **magnitude**, **length**, or **norm** of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is denoted by $\|a\|$, and defined by

$$\|\mathbf{a}\| = \sqrt{a_1^{\ 2} + a_2^{\ 2}}\ .$$

Example 3 If $\mathbf{a} =< 6,-2 >$, **then** $\|\mathbf{a}\| = \sqrt{6^2 + (-2)^2} = \sqrt{40} = 2\sqrt{10}$.

Clearly, $\|\mathbf{a}\| \ge 0$ for any vector **a,** and $\|\mathbf{a}\| = 0$ *if and only if* $\mathbf{a} = \mathbf{0}$. Especially we define a **unit vector** as a vector with norm unity. We can obtain a unit vector **u** in the direction of **a** by multiplying **a** by $\frac{1}{\|\mathbf{a}\|}$. i.e. **u** = $\frac{1}{\|\mathbf{a}\|}\mathbf{a}$ is a unit vector in the direction of **a**.(why?).

Example 4 Given $\mathbf{a} =< 6,-2 >$, form a unit vector in the direction of **a** and in the opposite direction of **a**.

Solution:- We sow in example 3 that the norm of **a** is $2\sqrt{10}$. Thus the unit vector **u** in the direction of **a** is given by

$$\mathbf{u} = \frac{1}{2\sqrt{10}}\mathbf{a} = \frac{1}{2\sqrt{10}} < 6,-2 >= \left\langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$$

and the vector in the opposite direction of **a** is given by

$$-\mathbf{u} = \left\langle -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle$$

There are two especial unite vectors in $\mathbf{R}^2$ that simplify describing and operating on vectors which are

$$\mathbf{i} =< 1,0 > \textit{ and } \mathbf{j} =< 0,1 >$$

any vector $\mathbf{a} =< a_1, a_2 >$ can be written as a sum:

$$\mathbf{a} =< a_1, a_2 >=< a_1, 0 > + < 0, a_2 >= a_1 < 1,0 > + a_2 < 0,1 >= a_1\mathbf{i} + a_2\mathbf{j}.$$

Example 5 Vector operations using **i** and **j**.

a) <4,7> = 4**i** + 7**j**

b) (2**i** – 5**j**) + (8**i** + 13**j**) = 10**i** + 8**j**

c) $\|\mathbf{i} + \mathbf{j}\| = \sqrt{2}$

d) **a** = 6**i** + 4**j** and **b** = 9**i** + 6**j** are parallel vectors since **b**=3/2**a**

Vectors in $\mathbf{R}^3$

A vector **a** in $\mathbf{R}^3$ is an ordered triple of real numbers $\mathbf{a} =< a_1, a_2, a_3 >$ where a_1, a_2, a_3 are the components of the vector. The set of all vectors in $\mathbf{R}^3$ will be denoted by the symbol $\mathbf{R}^3$. The position vector of a point $P(x_1, y_1, z_1)$ in space is the vector OP = $< x_1, y_1, z_1>$ whose initial point is the origin O and whose terminal point is P.

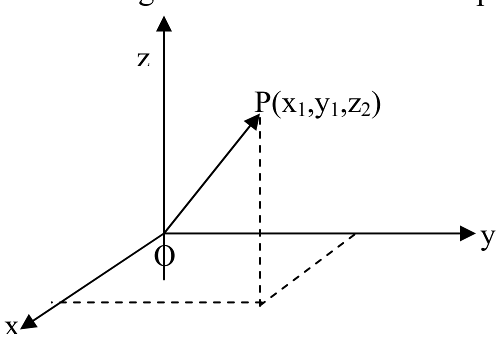


Fig 7

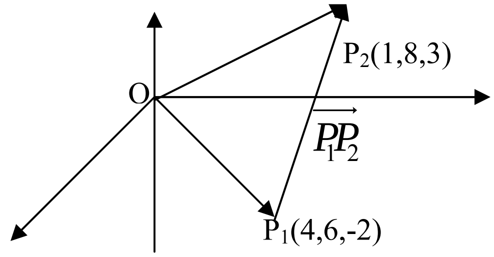
The component definition of addition, subtraction, scalar multiplication and so on are natural generalizations of those given for vectors in $\mathbf{R}^2$.

Definition 2.3 Let $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$ be vectors in $\mathbf{R}^3$. Then

i) Addition: $\mathbf{a} + \mathbf{b} =< a_1 + b_1, a_2 + b_2, a_3 + b_3 >$
ii) Subtraction: $\mathbf{a} - \mathbf{b} =< a_1 - b_1, a_2 - b_2, a_3 - b_3 >$
iii) Scalar multiplication: $k\mathbf{a} =< ka_1, ka_2, ka_3 >$
iv) Equality: $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3$.
v) Zero vector: $\mathbf{0} =< 0,0,0 >$
vi) Magnitude: $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example 6 Find the vector $\overrightarrow{P_1P_2}$ if the points P_1 and P_2 are given by $P_1(4,6,-2)$ and $P_2(1,8,3)$.

Solution: Observe that we may sketch the vectors as in the figure below



Since $\overrightarrow{P_1P_2}$ **=** $\overrightarrow{OP_2} - \overrightarrow{OP_1}$ **b** Fig. 8 **we have**

$\overrightarrow{P_1P_2}$ =<1-4,8-6,3-(-2)>= <-3,2,5>.

Example 8 Find the norm of **a** where $\mathbf{a} = \left\langle \frac{-2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$.

Solution: By definition 1.3vi

$$\|\mathbf{a}\| = \sqrt{\left(\frac{-2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4+9+36}{49}} = 1.$$

Thus **a** is a unit vector.

As we have special unit vectors in $\mathbf{R}^2$ (**i** and **j**) we also have special unit vectors in $\mathbf{R}^3$ defined as

i = <1,0,0> **j** = <0,1,0> **k** = <0,0,1>

so that any vector $\mathbf{a} =< a_1, a_2, a_3 >$ may be written as

$$\mathbf{a} =< a_1, a_2, a_3 >= a_1 < 1,0,0 > + a_2 < 0,1,0 > + a_3 < 0,0,1 >$$
$$= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

For instance $< 7,-5,13 >= 7\mathbf{i} - 5\mathbf{j} + 13\mathbf{k}$.

2.2 Dot (Scalar) Product

In this and the following section, we shall consider two kinds of products between vectors that originat in the study of mechanics, electricity and magnetism. The first of these products, known as the **dot** or **inner** or **scalar product**, yields a scalar.

Definition 2.4 Let $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$ be two vectors. The dot product of **a** and **b** is the number **a.b** defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 .$$

Observe that if $\mathbf{a} =< a_1, a_2, a_3 >$ **the norm of a is given by**

$$\|\mathbf{a}\| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}} = \sqrt{\mathbf{a} \cdot \mathbf{a}} \text{ or}$$

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$$

In particular

$$\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1.$$

Example 1 Find the scalar product of $\mathbf{a} =< 1,-2,4 >$ and $\mathbf{b} =< 3,0,2 >$

Solution: From definition 1.10 we see that

$$\mathbf{a} \cdot \mathbf{b} = (1 \times 3) + (-2 \times 0) + (4 \times 2) = 11.$$

The scalar product satisfies many of the laws that hold for real numbers. For example

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} \qquad\qquad (c\mathbf{a})\cdot\mathbf{b}=c(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(c\mathbf{b})$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \qquad\qquad (\mathbf{a}+\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$$

The following theorem gives us the relation between the dot product of two vectors and the angle between them.

Theorem 2.5 If **a** and **b** are two nonzero vectors in either $\mathbf{R}^2$ or $\mathbf{R}^3$ and θ is the angle between them, then

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta,\ \textit{where}\ 0\le\theta\le\pi.$$

Proof: We will prove the result for $\mathbf{R}^2$ while $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}$. The proof for vectors in $\mathbf{R}^3$ is similar.

The vectors $\mathbf{a},\mathbf{b},\ \textit{and}\ \mathbf{b}-\mathbf{a}$ may be used to form a triangle as in fig below, then by the low of cosines we have

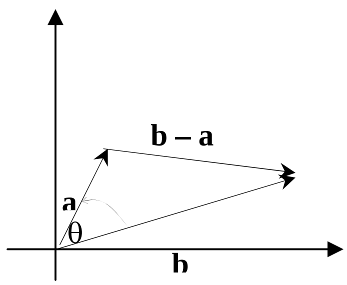


Fig 9

$$\|\mathbf{b}-\mathbf{a}\|^2=\|\mathbf{a}\|^2+\|\mathbf{b}\|^2-2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

where θ is the angle between **a** and **b**. consequently

$$\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta=\frac{1}{2}\left[\|\mathbf{a}\|^2+\|\mathbf{b}\|^2-\|\mathbf{b}-\mathbf{a}\|^2\right]$$

$$=\frac{1}{2}\left[a_1{}^2+a_2{}^2+b_1{}^2+b_2{}^2-(a_1-b_1)^2-(a_2-b_2)^2\right]$$

$$=\frac{1}{2}(2a_1b_1+2a_2b_2)$$

$$=\mathbf{a}\cdot\mathbf{b}$$

We can observe from theorem 1.11 above is that if the two vectors are perpendicular to each other i.e θ =90° then $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta=0$ and conversely. This proves corollary 1.12 below

Corollary 2.6 The nonzero vectors a and b are perpendicular to each other if and only if

$$\mathbf{a}\cdot\mathbf{b}=0.$$

The other important result that we get from theorem 1.11 is that

$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

which intern implies, the angle between **a** and **b** is uniquely determined as $0\le\theta\le\pi$.

Example 2 a) The vector 0 is perpendicular to every vector in $\mathbf{R}^2$

b) The vector <3,2> and <-4,6> are perpendicular in $\mathbf{R}^2$

c) The vector <2,-3,1> and <1,1,1> are orthogonal in $\mathbf{R}^3$

Example 3 Determine the angle between the vectors **u** = **i** and **v = i + k** in $\mathbf{R}^3$.

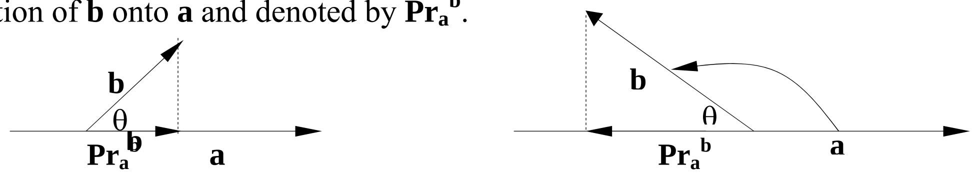
Solution: Since $\mathbf{u}\cdot\mathbf{v}=1,\ \|\mathbf{u}\|=1,\ \|\mathbf{v}\|=\sqrt{2}$ *and if* θ *is the angle between* **u** *and* **v**

we have
$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}=\frac{1}{\sqrt{2}}$$

consequently $\theta = \arccos\dfrac{1}{\sqrt{2}} = 45°$.

The Orthogonal Projection of one Vector onto Another

Suppose that two nonzero vectors **a** and **b** are positioned as Fig (a) and (b) below and that the sun casts a shadow on the line containing a vector parallel to **a** which we call the projection of **b** onto **a** and denoted by $\mathbf{Pr_a^b}$.



(a) (b)

since $\mathbf{Pr_a^b}$ is parallel to **a** or is **0**, it must be a scalar multiple of **a**. The length of $\mathbf{Pr_a^b}$ is evidently $\|\mathbf{b}\|\cos\theta|$, where θ is the angle between **a** and **b** ($0 \le \theta \le \pi$). It follows that

$$\mathbf{Pr_a}^{\mathbf{b}} = \begin{cases} \|\mathbf{b}\| \cos\theta \dfrac{\mathbf{a}}{\|\mathbf{a}\|} & \text{for } 0 \le \theta \le \dfrac{\pi}{2} \\ \|\mathbf{b}\|(-\cos\theta)\dfrac{-\mathbf{a}}{\|\mathbf{a}\|} & \text{for } \dfrac{\pi}{2} < \theta \le \pi \end{cases}$$

hence irrespective of the angle θ we have

$$\mathbf{Pr_a^b} = \|\mathbf{b}\| \cos\theta \frac{\mathbf{a}}{\|\mathbf{a}\|} = \|\mathbf{b}\|\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

Now we are prepared to define the $\mathbf{Pr_a^b}$.

Note: $\left\|\mathbf{Pr_a}^{\mathbf{b}}\right\| = \left\|\dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}\right\| = \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|^2}\|\mathbf{a}\| = \dfrac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}$.

Definition 1.13 Let **a** be a nonzero vector. The projection of vector **b** on to **a** ($\mathbf{Pr_a^b}$) is defined by

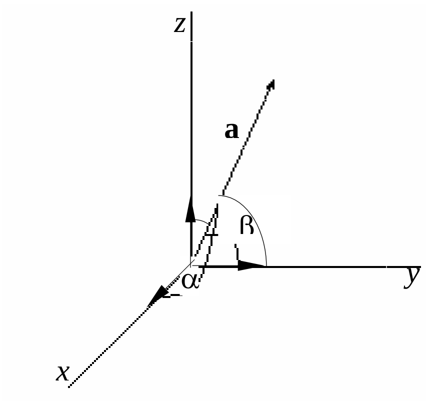
$$\mathbf{Pr_a^b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}.$$

Example 4 Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ *and* $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Find $\mathbf{Pr_a^b}$.

Solution: Observe that $\mathbf{a}\cdot\mathbf{b} = -1 + 2 = 1$ *and* $\|\mathbf{a}\| = \sqrt{2}$, hence

$$\mathbf{Pr_a^b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{1}{(\sqrt{2})^2}(\mathbf{i}+\mathbf{j}) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

Direction Cosines: For a nonzero vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ in $\mathbf{R}^3$, the angle α,β, and γ between **a** and the unit vectors **i**, **j**, and **k**, respectively, are called dirction angles of **a**. See Fig below, then



$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\|\mathbf{i}\|},\ \cos\alpha = \frac{a_1}{\|\mathbf{a}\|}$$
$$\cos\beta = \frac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\|\mathbf{i}\|},\ \cos\beta = \frac{a_2}{\|\mathbf{a}\|}$$
$$\cos\gamma = \frac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\|\mathbf{i}\|},\ \cos\gamma = \frac{a_3}{\|\mathbf{a}\|}$$

We say that $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$ are the direction cosines of **a**. The direction cosines of a none zero vector **a** are simply the components of the unit vector $\left(\frac{1}{\|\mathbf{a}\|}\right)\mathbf{a}$.

$$\left(\frac{1}{\|\mathbf{a}\|}\right)\mathbf{a} = \frac{a_1}{\|\mathbf{a}\|}\mathbf{i} + \frac{a_2}{\|\mathbf{a}\|}\mathbf{j} + \frac{a_3}{\|\mathbf{a}\|}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}.$$

Since the magnitude of $\left(\frac{1}{\|\mathbf{a}\|}\right)\mathbf{a}$ is 1, it follows from the last equation that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Example 5 Fine the direction cosines of the vector $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

Solution: Form $\|\mathbf{a}\| = \sqrt{2^2 + 5^2 + 4^2} = \sqrt{45} = 3\sqrt{5}$, we see that the direction cosines are

$$\cos\alpha = \frac{2}{3\sqrt{5}},\ \cos\beta = \frac{5}{3\sqrt{5}},\ \cos\gamma = \frac{4}{3\sqrt{5}}.$$

Observe in Example 5 above that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{4}{45} + \frac{25}{45} + \frac{16}{45} = 1.$$

2.3 Cross (Vector) Product

In this section, we introduce the cross (vector) product of two vectors and its applications. The cross product is the other special product of two vectors, which yields vector unlike that of the dot (scalar) product.

Definition 2.7 The **cross product** $\mathbf{a}\times\mathbf{b}$ of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in $\mathbf{R}^3$ is defined by

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\ .$$

An easy way to remember the last equation is to write it in a determinant form i.e.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Then evaluating it by repeating the firs and second columns and multiplying it as follows

$$\mathbf{a}\times\mathbf{b} = \left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}\right.$$

i.e subtract the sum of the product of the "southwest" diagonals form that of the product of the "southeast" diagonals.

Example 1 Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{v} = \mathbf{j} + 4\mathbf{k}$. Determine the cross product **u**×**v** and **v**×**u**

Solution: From the definition of cross product we have

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 0 & 1 & 4 \end{vmatrix} = [-2(4) - 3(1)]\mathbf{i} + [3(0) - 1(4)]\mathbf{j} + [1(1) - (-2)0] = -11\mathbf{i} - 4\mathbf{j} + \mathbf{k}.$$

and

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 4 \\ 1 & -2 & 3 \end{vmatrix} = [1(3) - (-2)4]\mathbf{i} + [4(1) - (0)3]\mathbf{j} + [0(-2) - (1)1]\mathbf{k} = 11\mathbf{k} + 4\mathbf{j} - \mathbf{k}.$$

Notice that the vector **u**×**v** and **v**×**u** in Example 1 are negatives of each other. This in not a coincidence; in fact it directly follows from the definition of cross product of two vectors as we may see in the theorem below. The proof of the theorem employs properties of determinants which we will discus thoroughly in Chapter 2.

Theorem 2.8 Let $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$ in R^3. Then

a) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

b) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

c) $\mathbf{a}\cdot(\mathbf{a} \times \mathbf{b}) = 0$

d) $\mathbf{b}\cdot(\mathbf{a} \times \mathbf{b}) = 0$

Proof: a) By definition of cross product we have

$$\mathbf{a}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{0}$$

since a determinant with two equal rows is zero (Notice **0** is vector).

b) Further, using properties of determinants, we get

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \approx -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -(\mathbf{b}\times\mathbf{a})$$

Since interchanging rows leads us to the negative of the original determinant.

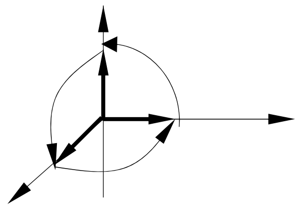
c) Using the definition of dot product, cross product and determinant we have

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = [a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}]\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

It can be proved similarly that $\mathbf{b}\cdot(\mathbf{a} \times \mathbf{b}) = 0$.

Note: From c) and d) of theorem 1.15 we conclude that the vector **a**×**b** is perpendicular to both **a** and **b.**

Example 2 The cross product of any pair of vectors in the **i, j,** and **k** can be obtained by the circular pattern illustrated in Fig. Below that is



i × j = k | **j × i = – k**
j × k = i | **k × j = – i**
k × i = j | **i × k = – j**

Example 2 Let a = i – j + 3k and b = 2i – 3j – k. Find a vector perpendicular to a and b.
Solution: By theorem 1.15 the cross product **a×b** is one such vector thus the vector that is perpendicular to both **a** and **b** is given by

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 3\\ 2 & 3 & -1\end{vmatrix}=(1-9)\mathbf{i}+(6+1)\mathbf{j}+(3+2)\mathbf{k}=-8\mathbf{i}+7\mathbf{j}+5\mathbf{k}.$$

Other properties of the cross product that follow readily from the definition of cross product are

ca × b = c(a × b) = a × (cb) **a × (b + c) = (a × b) + (a × c)**
(a + b) × c = (a × c) + (b × c).

Theorem 2.9 Let **a** and **b** be vectors in $\mathbf{R}^3$. Then $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, where θ (0≤θ≤π) is the angle between **a** and **b**.
Proof: Let **a** = a_1**i** + a_2**j** + a_3**k** and **b** = b_1**i** + b_2**j** + b_3**k** using the definition of norm of a vector, we get

$$\|\mathbf{a}\times\mathbf{b}\|^2=(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2$$

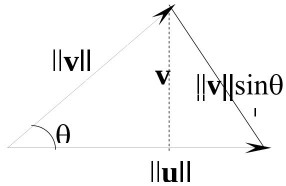
On expanding the squares, this can be rewritten as

$$\begin{aligned}&=(a_1^{\,2}+a_2^{\,2}+a_3^{\,2})(b_1^{\,2}+b_2^{\,2}+b_3^{\,2})-(a_1b_1+a_2b_2+a_3b_3)^2\\ &=\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2\\ &=\|\mathbf{a}\|^2\|\mathbf{b}\|^2-(\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta)^2\\ &=\|\mathbf{a}\|^2\|\mathbf{b}\|^2-\|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2\theta\\ &=\|\mathbf{a}\|^2\|\mathbf{b}\|^2(1-\cos^2\theta)\\ &=\|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta\end{aligned}$$

since sinθ≥0 for (0≤θ≤π), we can take the square root of each side of the equation and obtain

$$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta.$$

Corollary 2.10 Two nonzero vectors **a** and **b** are parallel if and only if **a** × **b** = **0**.
Proof: Left as exercise.
The result of theorem 1.16 leads to the area of a triangle that is defined by two vectors. Consider the triangle whose edges are the vectors **u** and **v**. See the fig below.



Area of triangle = (½) base × height

= (½)||**u**||||**v**||sinθ

= (½) ||**u**×**v**||

Area of a triangle with edges **u** and **v** =(½) ||**u**×**v**||

Example 3 Determine the area of the triangle having vertices A(3,-1,2), B(1,-1,-3), and C(4, -3, 1).

Solution: The points B and C define the following edge vectors, starting from point A.

$$\overrightarrow{AB} = <1,-1,-3> - <3,-1,2> = <-2,0,-5>$$

$$\overrightarrow{AC} = <4,-3,1> - <3,-1,2> = <1,-2,-1>$$

and $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = -10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}.$$

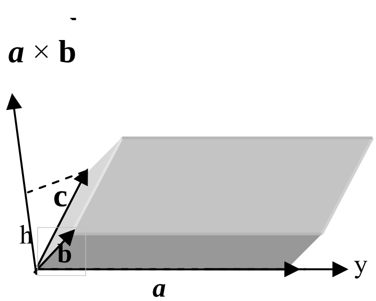
Thus, the area of the triangle = (1/2)|| $\overrightarrow{AB} \times \overrightarrow{AC}$ ||

= (1/2)||-10**i** –7**j** + 4**k**||

$$= \frac{1}{2}\sqrt{10^2 + 7^2 + 4^2} = \frac{1}{2}\sqrt{165}.$$

The other important application of the vector (cross) products is in finding the volume of a parallelepiped.

Consider the parallelepiped whose edges are defined by the vectors **a**, **b**, and **c**. See fig below.



The area of the base is twice of the area of the triangle defined by vectors **a** and **b**. Thus, area of base = ||**a**×**b**||. Further, volume = ||**a**×**b**||×*h*, where *h* is the height. Observe that

$h = \|\mathbf{Pr}_{\mathbf{a}\times\mathbf{b}}{}^{\mathbf{c}}\|$

$$= \left\| \frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{\|\mathbf{a}\times\mathbf{b}\|^2}(\mathbf{a}\times\mathbf{b}) \right\|$$

$$= \frac{|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|}{\|\mathbf{a}\times\mathbf{b}\|}$$

Thus the volume of a parallelepiped with adjacent edges **a**, **b**, and **c** = $|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})|$.

The expression $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ is called the **triple scalar product** of **a**, **b**, and **c**. It can be conveniently written as a determinant. Let

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}, \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}, \quad \mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

Then $$\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Thus

The volume **V** of a parallelogram with edges **a**, **b**, and **c**

$$\mathbf{V}=\text{absolute value of } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4 Find the volume of the parallelepiped having adjacent edges defined by the points A(1,1,3), B(3,7,1), C(-2,3,3), D(1,2,8).

Solution: The points A, B, C, and D define the following three adjacent edge vectors.

$$\overrightarrow{AB} =< 3,7,1 > - < 1,1,3 >=< 2,6,-2 >$$

$$\overrightarrow{AC} =< -2,3,3 > - < 1,1,3 >=< -3,2,0 >$$

$$\overrightarrow{AD} =< 1,2,8 > - < 1,1,3 >=< 0,1,5 >$$

The volume of the parallelepiped is thus

$$= \text{absolute value of } \begin{vmatrix} 2 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 1 & 5 \end{vmatrix}$$

$= \text{absolute value of } (116) = 116.$

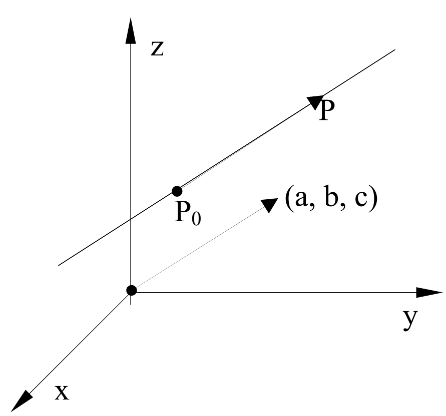
We have also other triple products for instance $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}, (\mathbf{a}\times\mathbf{b})\times\mathbf{c}, \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are the useful ones. The first is called a **triple scalar product** and the last two are called **triple vector products,** since the products are vectors. See Exercise 1 for important relations due to triple products.

2.4 Lines and Planes in R^3

2.4.1 Equations of Lines in Space

Consider a line through the point $P_0(x_0, y_0, z_0)$ in the direction defined by the vector $< a, b, c >$. See the fig below. Let $P(x, y, z)$ be any other point on the line. We get

$$\overrightarrow{P_0P} =< x - x_0, y - y_0, z - z_0 >$$



The vector $\overrightarrow{P_0P}$ and <a, b, c> are parallel. Thus there exists a scalar t such that

$\overrightarrow{P_0P}$ = t <a, b, c> or

$$< x - x_0, y - y_0, z - z_0 >= t < a, b, c > \qquad (1)$$

This is called the **vector equation** of the line. Comparing the components of the vectors on the left and right of this equation gives

$$x - x_0 = ta, \qquad y - y_0 = tb, \qquad z - z_0 = tc$$

Rearranging these equations as follows gives the **parametric equations** of a line in $\mathbf{R}^3$.

$$x = x_0 + ta, \qquad y = y_0 + tb, \qquad z = z_0 + tc \qquad -\infty < t < \infty \qquad (2)$$

in this last equation we get the points on the line as *t* varies.

Example 1 Find a vector equation for the line through (1,2,5) in the direction of <4,3,2>. Give also the parametric equation of the line. Determine any two points on the line.

Solution: Let <a,b,c>=<4, 3, 1> and $(x_0, y_0, z_0) = (1, 2, 5)$, then from equation (1) we can write the vector equation of the line as

$$< x - 1, y - 2, z - 5 >= t < 4,3,1 > .$$

And from equation (2) we give the parametric equation of the line by

$$x = 1 + 4t, \qquad y = 2 + 3t \qquad z = 5 + t \qquad -\infty < t < \infty.$$

To find to points on the line we give *t* two arbitrary values, for instance t =1leads to the point (5,5,6), and t = –1 leads to the point (–3, –1, 4).

Example 2 Find the parametric equation of the line through the points (-1, 2, 6) and (1, 5, 4).

Solution: Let $(x_0, y_0, z_0) = (-1, 2, 6)$. The direction of the line is given by the vector

$$< a, b, c >=< 1,5,4 > - < -1,2,6 >=< 2,3,-2 > .$$

Consequently the parametric equations of the line are given by

$$x = -1 + 2t, \qquad y = 2 + 3t, \qquad z = 6 - 2t \qquad -\infty < t < \infty.$$

Symmetric Equations of a Line: From equation (2) we can clear the parameter t by writing it as

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

provided that the three numbers a, b, and c are nonzero. The resulting equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

are said to be symmetric equations for the line through P_0 and P.

Example 3 find the symmetric equations for the line through (4,10, -6) and (7,9,2).

Solution: First let us find the reference vector as below

$$< a, b, c >=< 4,10,-6 > - < 7,9,2 >=< -3,1,-8 > .$$

Then if we let $(x_0, y_0, z_0) = (7, 9, 2)$ the symmetric equation of the line is given by

$$\frac{x - 7}{-3} = \frac{y - 9}{1} = \frac{z - 2}{-8}.$$

Note: If one of the numbers *a*, *b*, or *c* is zero in (2), we use the remaining two equations to eliminate the parameter t. For example if $a = 0$, $b \neq 0$, $c \neq 0$, then (2) yields the symmetric equations for the line to be

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

2.4.2 Equations of Planes in R3

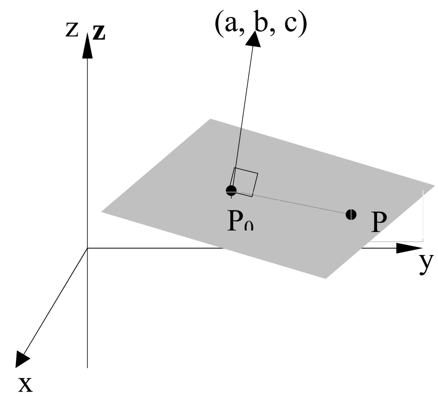
Let $P_0(x_0, y_0, z_0)$ be a point in a plane. Let <a, b, c> be a vector perpendicular to the plane, called a normal to the plane. These two quantities, namely a point in a plane and a normal vector to the plane characterize the plane. There is only one plane through a given point and having a given normal. We will now drive the equation of a plane passing through the point $P_0(x_0, y_0, z_0)$ and having normal <a, b, c>. Let P(x, y, z) be any arbitrary point in the plane. We get

$$\overrightarrow{P_0P} = < x, y, z > - < x_0, y_0, z_0 >$$
$$= < x - x_0, y - y_0, z - z_0 >$$

The vector $\overrightarrow{P_0P}$ lies in the plane. Thus the vector
PP
<a, b, c> and $\overrightarrow{P_0P}$ are orthogonal. Their dot product is zero. This observation leads to a vector equation of the plane

$$< a, b, c > \cdot \overrightarrow{P_0P} = 0.$$

or $< a, b, c > \cdot < x - x_0, y - y_0, z - z_0 > = 0.$



Specifically the last equation yields the point-normal form of the equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (3)$$

and expanding the last equation and putting $d = ax_0 + by_0 + cz_0$ we obtain the **general** form of the equation of the plane

$$ax + by + cy = d \qquad (4)$$

Observe that the components of <a, b, c> appear as coefficients in (3) and (9), ane the coordinates of the points $P_0(x_0, y_0, z_0)$ in the plane appear inside the parenthesis in (3).

Example 1 Find the point-normal and general forms of the equation of the plane passing through the point (1,2,3) and having normal <-1,4,6>.

Solution: Let $(x_0, y_0, z_0) = (1,2,3)$ and $< a, b, c > = < -1,4,6 >$. Then the point normal form equation of the plane is given by

– (x –1) + 4(y –2) + 6(z –3) =0

multiplying and simplifying the last equation we get the general form

- x + 4y + 6z = 25.

Example 2 Determine the equation of the plane through the three points P(2,-1,1), Q(-1,1,3) and R(2,0,-3).

Solution: The vectors $\overrightarrow{PQ}$ *and* $\overrightarrow{PR}$ lie in the plane. Thus $\overrightarrow{PQ} \times \overrightarrow{PR}$ will be normal to the plane. So since

$$\overrightarrow{PQ} =< -1,13 > - < 2,-1,1 >=< -3,2,2 >$$

$$\overrightarrow{PQ} =< 2,0,-3 > - < 2,-1,1 >=< 0,1,-4 >$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = -10\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}$

finally putting $(x_0, y_0, z_0) = (2,-1,1)$ *and* $< a,b,c >=< -10,-12,-3 >$ we give the point normal equation by

$$-10(x-2) - 12(y+1) - 3(z-1) = 0$$

or the general equation by

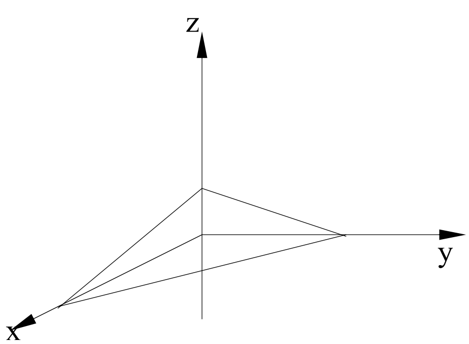
$$-10x - 12y - 3z = -11.$$

Example 3 The normal vector to the plane $3x - 4y + 10z = 8$, can be given by taking the coefficients of x, y, and z and forming a vector i.e 3**i** – 4**j** + 10**k** is the normal vector to our plane.

Example 4 (Graph of a plane)

Graph the equation $2x + 3y + 6z = 12.$

Solution: setting



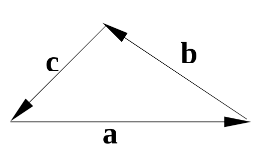
y = 0, z = 0 gives x = 6
x = 0, z = 0 gives y = 4
x = 0, y = 0 gives z = 2

The x, y, and z-intercepts are, 6, 4, and 2 respectively. As shown in the figure to the left. We use the points (6, 0, 0), (0, 6, 0) and (0, 0, 3) to draw the graph of the plane in the first octant.

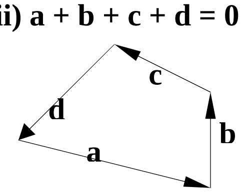
Exercise 1

1. Let $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$ *and* $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$. Graph **a + b** and **a – b**.
2. Use the given figures to prove the given result

   i) a + b + c = 0



   ii) a + b + c + d = 0



3. Sketch position vectors for **a**, **b**, 2**a**, -3**b**, **a** +**b**, and **a-b**.

   i) **a** = <2,3,4> **b** = <1,-2,2>

   ii) **a** = –**i** + 2**j** +3**k** **b** = -2**j** + **k**
4. Determine the scalar *c* so that the vectors $\mathbf{a} = 2\mathbf{i} - c\mathbf{j} + 3\mathbf{k}$ *and* $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ are orthogonal.

5. Verify the vector $\mathbf{c} = \mathbf{b} - \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$ is orthogonal to the vector **a**.
6. Determine a scalar c so that the angle between $\mathbf{a} = \mathbf{i} + c\mathbf{j}$ *and* $\mathbf{b} = \mathbf{i} + \mathbf{j}$ *is* $45°$.
7. Find the angle θ between $\mathbf{a} = 3\mathbf{i} - \mathbf{k}, \mathbf{b} = 2\mathbf{i} + 2\mathbf{k}$.
8. Find the direction cosines of the vector $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
9. Find $\mathbf{v} = < x_1, y_1, 1 >$ that is orthogonal to both $\mathbf{a} = < 3,1,-1 >$ *and* $\mathbf{a} = < -3,2,2 >$.
10. Let $\mathbf{a} = \sqrt{3}\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. $\mathbf{Pr_a}^{\mathbf{b}}$.
11. prove

    $a) |\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$

    $b) \|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$

    $c) \|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$
12. Find the area of the triangle with vertices A(1,2,1), B(-3,4,6), and C(1,8,3).
13. Find the volume and surface area of the parallelepiped having adjacent edges defined by A(1,2,5), B(4,8,1), C(-3,2,3), D(0,3,9).
14. Show that $(\mathbf{a} + \mathbf{b})\times(\mathbf{a} - \mathbf{b})$
15. Let **a**, **b**, and **c** be vectors in $\mathbf{R}^3$. Prove that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.
16. Let **a**, **b**, and **c** be vectors in $\mathbf{R}^3$. Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.
17. Let **a**, **b**, and **c** be vectors in $\mathbf{R}^3$. Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\times\mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0$
18. Find parametric equations and symmetric equations for the line through the points (5, 3, 1) and (2, 1, 1).
19. Fine the equation of the line through the point (1, 2, -4), parallel to the line x = 4 +2t, y = -1 + 3t, z = 2 + t, where -∞<t<∞
20. Find the equation of the line through the point (2,-3,1) in a direction orthogonal to the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z+2}{5}$
21. Show that there are many planes that contain the three points (3, -5, 5), (-1, 1, 3) and (5, -8, 6). Interpret your conclusion geometrically.
22. Find an equation for the line through the point (4, -1, 5), in the direction perpendicular to the line x = 1 – t, y = 3 + 2t, z = 5 – 4t, where –∞ <t< ∞.
23. Show that the line x = 1+t, y = 14 – t, z = 2 – t, where –∞ <t< ∞, lies in the plane 2x –y +3z + 6 = 0.
24. Prove that the line x = 4 +2t, y = 5 + t, z = 7 + 2t, where –∞ <t< ∞, never intersects the plane 3x + 2y – 4z + 7 = 0.
25. Find an equation of the line through the point (5, -1, 2) in a direction perpendicular to the line x = 5 –2t, y = 2 + 3t, z = 2t, where –∞ <t< ∞.
26. Find the line of intersection of the two planes

    x – 4y + 2z + 7 = 0,

    and 3x + 3y –z – 2 = 0.

3 Limit and Continuity

3.1 Definition of Limits

Until know we have been evaluating the limit of a function by using its intuitive definition. That is we have said that limit of $f(x)$ as x approaches to a is L and write

$$\lim_{x \to a} f(x) = L$$

if we can make $f(x)$ close enough to L by choosing x close enough to a but distinct from a. Although this intuitive definition is sufficient for solving limit problems it is not prices enough. In this section we see the formal definition of limit, which we call the $\varepsilon - \delta$ definition of limit.

Definition 3.1 (Formal definition of limit)

The limit of $f(x)$ as x approach a is L, written

$$\lim_{x \to a} f(x) = L$$

if every ε > 0, there exists a δ > 0 such that |f(*x*) - L| < ε whenever 0 < |*x* - *a*| < δ.

In Definition 3.1 above we should not that

I. The absolute value symbol is read as " the distance between" for instance **|*x* - *a*|** is the distance between x and *a*.

II. Notice that **|x - a| >0.** In other words *x* is not equal to *a*.

So with this in mind we can read the definition as:

" The distance between $f(x)$ and L can be made smaller than any positive number ε, whenever the distance between x and *a* is less than some number δ and x does not equal *a*." Fig 3.1 below represents this idea pictorially.

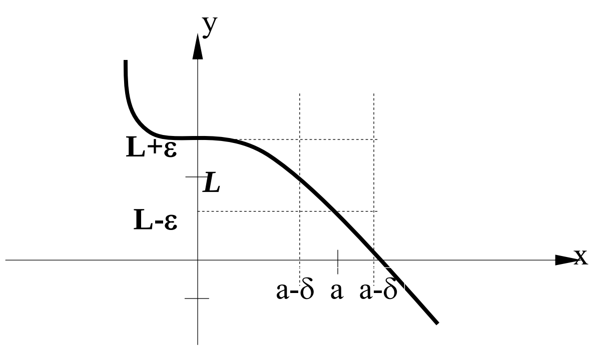


Fig 3.1

If we wish to use a form of Definition 3.1 that does not contain absolute value symbols we can have the following alternative definition of limit.

Definition 3.2 $\lim_{x \to a} f(x) = L$ if and only if for every ε>0, there is a δ>0 such that if x is in the open interval (a – δ, a +δ) and x ≠ a then f(x) is in the open interval (L –ε, L + ε).

Using either of the definitions of limit given above we can prove the following theorem.

Theorem 3.3 If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$ then L=M.

The above theorem tells us that if a limit of a function f(x) at a exists then it must be *unique*.

3.2 Examples on limit

Even if it is very difficult to us the formal definition of limit to handle all limit problems let us see how we can use it for evaluating same important limits that may help us in developing rules by the way of which we can evaluate limits with out using the formal definition.

Example 1 Assume that $\lim_{x\to 2} 5x-7=3$. By using properties of inequalities, determine a δ> 0 such that

$$\text{if } 0<|x-2|<\delta \quad then \quad |(5x-7)-3|<0.01.$$

Solution: By considering $|(5x-7)-3|<0.01$ we can see that

$$\begin{aligned} |(5x-7)-3|<0.01 &\Leftrightarrow |5x-10|<0.01 \\ &\Leftrightarrow 5|x-2|<0.01 \\ &\Leftrightarrow |x-2|<0.002 \end{aligned}$$

so now it is clear that if we choose δ = 0.002 statement holds but to check our result holds we proceed as follows:

$$\begin{aligned} if \quad 0<|x-2|<\delta \quad then \ |x-2|<0.002 \\ \Rightarrow |x-2|<0.01/5 \\ \Rightarrow 5|x-2|<0.01 \\ \Rightarrow |5x-10|<0.01 \\ \Rightarrow |(5x-7)-3|<0.01 \end{aligned}$$

Thus we have shown that the choice of δ = 0.002 satisfies the statement

$$\text{if } 0<|x-2|<\delta \quad then \quad |(5x-7)-3|<0.01.$$

This example is for the specific ε = 0.01. The general case can be seen as follows.

Example 2 Show that

$$\lim_{x\to 2} 5x-7=3$$

Solution:

We need to show that given **ε > 0** then there exists **δ > 0** such that

$$\text{if } 0<|x-2|<\delta \quad then \quad |(5x-7)-3|<\varepsilon$$

To choose an appropriate δ we start with $|(5x-7)-3|<\varepsilon$ then we have

$$\begin{aligned} |5x-10|<\varepsilon \ \Rightarrow 5|x-2|<\varepsilon \\ \Rightarrow |x-2|<\frac{\varepsilon}{5} \end{aligned}$$

Hence, we let

$$\delta=\frac{\varepsilon}{5}$$

this proves that $\lim_{x\to 2} 5x-7=3$.

Example 3 Prove that $\lim_{x\to 5} 7=7$.

Solution:

Begin by letting ε > **0** be given. Find δ > **0** so that if , $0<|x-5|<\delta$ then $|f(x)-7|<\varepsilon$, i.e., $|7-7|<\varepsilon$ i.e., $|0|<\varepsilon$. But this trivial inequality is always true, no matter what value is chosen for δ. For example, $\delta=\frac{1}{2}$ will work. Thus, if $0<|x-5|<\delta$, then it follows that $|f(x)-7|<\varepsilon$. This completes the proof.

A similar proof as example 2 shows us that for any number *a* and k

$$\lim_{x\to a} k = k \qquad (1)$$

Example 4 Prove that $\lim_{x\to a} kx = ka$ for any real number k.

Solution: from (1) it is clear that if c = 0

$$\lim_{x\to a} kx = \lim_{x\to 0} 0 = 0 = 0.a = k.a.$$

If k≠0, letting ε>0 we must find a δ>0 so that

$$0<|x-a|<\delta \Rightarrow |kx-ka|<\varepsilon$$

since

$$|kx-ka|<\varepsilon \Rightarrow |k||x-a|<\varepsilon$$

$$\Rightarrow |x-a|<\frac{\varepsilon}{|k|}$$

choose $\delta=\frac{\varepsilon}{|k|}$.

Example 5 Prove that $\lim_{x\to 1}(x^2+3)=4$.

Solution: Begin by letting $\varepsilon>0$ be given. Find $\delta>0$ (which depends on ε) so that if $0<|x-1|<\delta$, then $|f(x)-4|<\varepsilon$. Begin with $|f(x)-4|<\varepsilon$ and "solve for" |*x*-1| . Then,

$$|f(x)-4|<\varepsilon \text{ iff } |(x^2+3)-4|<\varepsilon$$

$$\text{iff } |x^2-1|<\varepsilon$$

$$\text{iff } |(x-1)(x+1)|<\varepsilon$$

$$\text{iff } |x-1|\,|x+1|<\varepsilon.$$

We will now "replace" the term |*x*+1| with an appropriate constant and keep the term |*x*-1| , since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta\le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x-1| < \delta \leq 1$ implies that -1 < x-1 < 1 and 0 < x < 2 so that 1 < |x+1| < 3 (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.)

$$|x-1|\,|x+1| < |x-1|\,(3) < \epsilon$$

iff $|x-1|\,(3) < \epsilon$

iff $|x-1| < \frac{\epsilon}{3}$.

Now choose $\delta = min\{1, \frac{\epsilon}{3}\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-1| < \delta$, it follows that $|f(x) - 4| < \epsilon$. This completes the proof.

Example 6 Prove that $\lim_{x \to 3} \frac{2}{x+3} = \frac{1}{3}$.

Solution: Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x-3| < \delta$, then $\left|f(x) - \frac{1}{3}\right| < \epsilon$. Begin with $\left|f(x) - \frac{1}{3}\right| < \epsilon$ and "solve for" | x - 3 | . Then,

$\left|f(x) - \frac{1}{3}\right| < \epsilon$ iff $\left|\frac{2}{x+3} - \frac{1}{3}\right| < \epsilon$

iff $\left|\frac{3}{3}\,\frac{2}{x+3} - \frac{1}{3}\,\frac{x+3}{x+3}\right| < \epsilon$

iff $\left|\frac{6-(x+3)}{3(x+3)}\right| < \epsilon$

iff $\frac{|3-x|}{|3|\,|x+3|} < \epsilon$

iff $\frac{|(-1)(x-3)|}{|3|\,|x+3|} < \epsilon$

iff $\dfrac{|-1|\;|x-3|}{|3|\;|x+3|} < \epsilon$

iff $\dfrac{1}{3}\;\dfrac{|x-3|}{|x+3|} < \epsilon$

iff $\dfrac{1}{3}\;|x-3|\;\dfrac{1}{|x+3|} < \epsilon$.

We will now "replace" the term $|x+3|$ with an appropriate constant and keep the term $|x-3|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta \le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x-3| < \delta \le 1$ implies that $-1 < x-3 < 1$ and $2 < x < 4$ so that $5 < |x+3| < 7$ and $\dfrac{1}{7} < \dfrac{1}{|x+3|} < \dfrac{1}{5}$ (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.)

$$\frac{1}{3}\;|x-3|\;\frac{1}{|x+3|} < \frac{1}{3}\;|x-3|\;\frac{1}{5} < \epsilon$$

iff $\dfrac{1}{3}\;|x-3|\;\dfrac{1}{5} < \epsilon$

iff $\dfrac{1}{15}\;|x-3| < \epsilon$

iff $|x-3| < 15\epsilon$.

Now choose $\delta = min\{1, 15\epsilon\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-3| < \delta$, it follows that $\left|f(x) - \dfrac{1}{3}\right| < \epsilon$. This completes the proof.

Example 7 Prove that $\lim\limits_{x \to 9}\,(2+\sqrt{x}) = 5$.

Solution: Begin by letting $\epsilon > 0$ be given. Find $\delta > 0$ (which depends on ϵ) so that if $0 < |x - 9| < \delta$, then $|f(x) - 5| < \epsilon$. Begin with $|f(x) - 5| < \epsilon$ and "solve for" $| x - 9 |$. Then,

$$|f(x) - 5| < \epsilon \text{ iff } |(2 + \sqrt{x}) - 5| < \epsilon$$

$$\text{iff } |\sqrt{x} - 3| < \epsilon$$

(At this point, we need to figure out a way to make $| x-9 |$ "appear" in our computations. Appropriate use of the conjugate will suffice.)

$$\text{iff } \left|(\sqrt{x} - 3)\ \frac{(\sqrt{x} + 3)}{(\sqrt{x} + 3)}\right| < \epsilon$$

(Recall that $(A - B)(A + B) = A^2 - B^2$.)

$$\text{iff } \left|\frac{x - 9}{\sqrt{x} + 3}\right| < \epsilon$$

$$\text{iff } \frac{|x - 9|}{|\sqrt{x} + 3|} < \epsilon.$$

$$\text{iff } |x - 9|\ \frac{1}{|\sqrt{x} + 3|} < \epsilon.$$

We will now "replace" the term $|\sqrt{x} + 3|$ with an appropriate constant and keep the term $|x-9|$, since this is the term we wish to "solve for". To do this, we will arbitrarily assume that $\delta \le 1$ (This is a valid assumption to make since, in general, once we find a δ that works, all smaller values of δ also work.) . Then $|x - 9| < \delta \le 1$ implies that $-1 < x-9 < 1$ and $8 < x < 10$ so that $\sqrt{8} + 3 < |\sqrt{x} + 3| < \sqrt{10} + 3$ and $\frac{1}{\sqrt{10} + 3} < \frac{1}{|\sqrt{x} + 3|} < \frac{1}{\sqrt{8} + 3}$ (Make sure that you understand this step before proceeding.). It follows that (Always make this "replacement" between your last expression on the left and ϵ. This guarantees the logic of the proof.)

$$|x - 9|\ \frac{1}{|\sqrt{x} + 3|} < |x - 9|\ \frac{1}{\sqrt{8} + 3} < \epsilon$$

$$\text{iff } |x-9|\,\frac{1}{\sqrt{8}+3} < \varepsilon$$

$$\text{iff } |x-9| < (\sqrt{8}+3)\varepsilon.$$

Now choose $\delta = min\{1, (\sqrt{8}+3)\varepsilon\}$ (This guarantees that both assumptions made about δ in the course of this proof are taken into account simultaneously.). Thus, if $0 < |x-9| < \delta$, it follows that $|f(x) - 3| < \varepsilon$. This completes the proof.

Example 8:

$$\lim_{x\to 0} 3x\sin\frac{1}{x} = 0$$

Solution:

We need to show that given ε > **0** then there exists δ > **0** such that

$0 < |x - 0| < \delta$ implies $\left|3x\sin\frac{1}{x} - 0\right| < \varepsilon$

Looking for δ:

$$\left|3x\sin\frac{1}{x}\right| < \varepsilon$$

$$3|x|\cdot\left|\sin\frac{1}{x}\right| < \varepsilon$$

$$3|x|\cdot 1 < \varepsilon$$

$$|x| < \frac{\varepsilon}{3}$$

Hence, we let

$$\delta = \frac{\varepsilon}{3}$$

Negation of the Existence of a Limit

Next we present an example of a function that does not have a limit at a certain point. For a function f not to have a limit at a means that for every real number L, the statement "L is the limit of f at a" is false. What does it mean for that statement to be false? By Definition 3.1, "L is the limit of f at a" means that

For every ε>0 there is a number δ> 0 such that if   0<|x – a|<δ, then |f(x) – L|<ε

for this statement to be false, there must be some ε>0 such that for ever δ> 0 it is false that

if   0<|x – a|<δ, then | *f*(x) – L|<ε   (2)

But to say that (2) is false is the same as to say that there must be a number x such that

0<|x – a|<δ and | *f*(x) – L| ≥ ε

Thus to say that the statement $\lim_{x \to a} f(x) = L$ is false is the same as to say that there is some ε>0 such that for ever δ> 0 there is a number x satisfying

0<|x – a|<δ and | *f*(x) – L| ≥ ε.

Example 8 Let *f* be defined by

$$f(x) = \begin{cases} x^2 & for \quad x > 0 \\ -1 & for \quad x \le 0 \end{cases}$$

Solution: Let L be any number. We will prove tht the statement "L is the limit of f at 0" is false by letting ε=1/2 and showing that for any δ>0 there is an x satisfying

0<|x – a|<δ and |f(x) – L| ≥½ = ε

Let δ be any positive number. If L ≤ – ½ , then we let x =δ/2 and note that f(x) = x^2 so that

$$|f(x) - L| = \left| \frac{\delta^2}{4} - L \right| \ge \left| \frac{\delta^2}{4} + \frac{1}{2} \right| > \frac{1}{2} = \varepsilon$$

If L ≥ – ½, then we let x = – δ/2 and note that f(x) = -1, so that

$$|f(x) - L| = |-1 - L| = |-1||1 + L| \ge \left| 1 - \frac{1}{2} \right| = \frac{1}{2} = \varepsilon$$

In either case we have shown that for any δ>0 there is an x satisfying

0<|x – a|<δ and |f(x) – L| ≥½ = ε

There for f has no limit at 0.

Class work

Using the ε-δ definition of limit, prove that

1. $\lim_{x \to 1} 2x - 1 = 1$ 2. $\lim_{x \to 2} \sqrt{x-1} = 1$ 3. $\lim_{x \to 2} x^2 = 4$ 4. $\lim_{x \to 1} 2x - 1 \ne 3$

3.3 One-Sided Limits

The notion of limit discussed in the preceding sections can be extended to one-sided limit as we can see from the definition below.

Definition 3.4 a) A number L is the **right-hand limit of *f* at *a*** denoted by $\lim_{x \to a^+} f(x) = L$ if for every ε > 0 there is a number δ>0 such that

$$if \quad 0 < x - a < \delta, \; then \quad |f(x) - L| < \varepsilon$$

b) A number L is the **left-hand limit of *f* at *a*** denoted by $\lim_{x \to a^-} f(x) = L$ if for every ε > 0 there is a number δ>0 such that

$$if \quad -\delta < x - a < 0, \; then \quad |f(x) - L| < \varepsilon .$$

Example 9 Show that $\lim_{x \to 1^+} \sqrt{x-1} = 0$

Solution: Let ε > 0 be given we need to show there is a δ>0 such that

$$if \quad 0 < x - 1 < \delta, \; then \quad \left| \sqrt{x-1} - 0 \right| < \varepsilon$$

form $\left|\sqrt{x-1}\right| < \varepsilon$ sequaring both sides we get $0 < x-1 < \varepsilon^2$, hence choose $\delta = \varepsilon^2$.
Then

$$if\ \ 0 < x-1 < \delta,\ then\ \left|\sqrt{x-1} - 0\right| = \left|\sqrt{x-1}\right| < \sqrt{\delta} = \varepsilon.$$

Below we give a theorem that relates one sided limit with a general limit the student can see Robert Ellis and Denny Gulick for the proof of the theorem.
Theorem 3.5 $\lim\limits_{x \to a} f(x)$ exists and is equal to L if and only if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist and both are equal to L.
Example 2 Observe that in Example 1 even if the right hand side of *f* at 1 exists since the left side limit of f at 1 does not exist, as the function is not defined for $x < 1$ then $\lim\limits_{x \to 1} \sqrt{x-1}$ does not exist.

3.4 Infinite Limits and Infinite Limits at infinity

According to Definition 3.1 if a function *f* has a limit L at *a* then L is a real number, so if the value of a function *f* becomes larger and larger in absolute value as *x* approach *a* from the right or from the left of *a* then f has no limit *a*. Now we introduce a definition that addresses such a case.
Infinite Limits
Definition 3.6 Let f be defined on some open interval (a, c).

a. If $\forall N, \exists \delta > 0$ such that

$$if\ \ 0 < x - a < \delta\ then\ \ f(x) > N$$

$$then\ \lim_{x \to a^+} f(x) = \infty$$

b. If $\forall N, \exists \delta > 0$ such that

$$if\ \ 0 < x - a < \delta\ then\ \ f(x) < N$$

$$then\ \lim_{x \to a^+} f(x) = -\infty$$

c. In either case (a) or (b) the vertical line $x = a$ is called a **vertical asymptote** of the graph of f, and we say that *f* has an **infinite right-hand limit at *a***.

There are analogous definitions for the limits

$$\lim_{x \to a^-} f(x) = \infty\ \ and\ \ \lim_{x \to a^-} f(x) = -\infty$$

Note if $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = \infty$ then we right simply $\lim\limits_{x \to a} f(x) = \infty$ for the common expression and say that the limit of f(x) as x approaches a is ∞ and that *f* has an infinit limit at a.
Example 10 Show that $\lim\limits_{x \to a} 1/x^2 = \infty$. Show also that the line $x = 0$ is a vertical asymptote of the graph of $1/x$.
Solution: Observe that for any N>0,

$$if\ \ 0 < x < \frac{1}{\sqrt{N}},\ then\ \ \frac{1}{x^2} > N$$

Thus $\lim\limits_{x \to a^+} 1/x^2 = \infty$, and thus the line $x = 0$ is a vertical asymptote of the gragh of $1/x^2$.

Once more for any N>0

$$if \ -\frac{1}{\sqrt{N}} < x < 0, \ then \ \frac{1}{x^2} > N$$

Thus $\lim_{x\to a^-} 1/x^2 = \infty$, again x = 0 is a vertical asymptote of the gragh of $1/x^2$.

Finaly since $\lim_{x\to a^-} 1/x^2 = \infty = \lim_{x\to a^+} 1/x^2$, $\lim_{x\to a} 1/x^2 = \infty$.

Limits at Infinity

Until now the limits we have seen have been limits of a function *f* at a number *a*. Now we consider the limit of f as x becomes larger and larger in absolute value.

Definition 3.7 a) $\lim_{x\to\infty} f(x) = L$ **if for every ε>0 there is a number M such that**

$$if \quad x > M, \ then \ |f(x) - L| < \varepsilon$$

b) $\lim_{x\to-\infty} f(x) = L$ if for every ε>0 there is a number M such that

$$if \quad x < M, \ then \ |f(x) - L| < \varepsilon$$

c) If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then we call the horizontal line y = L a horizontal asymptote of the graph of *f*.

Example 11 Show that $\lim_{x\to\infty} 1/x^2 = 0 \ and \ \lim_{x\to-\infty} 1/x^2 = 0$.

Solution: Let ε>0. To show that $\lim_{x\to\infty} 1/x^2 = 0$ **we must find an M such that**

$$if \quad x > M, \ then \ \left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \varepsilon$$

But then

$$\text{if } x > \frac{1}{\sqrt{\varepsilon}}, \ then \ \frac{1}{x^2} < \varepsilon$$

Therefore we let $M = 1/\sqrt{\varepsilon}$ and conclude that $\lim_{x\to\infty} 1/x^2 = 0$. To show that $\lim_{x\to-\infty} 1/x^2 = 0$

We simply choose $M = -1/\sqrt{\varepsilon}$ Then M<0, and thus

$$if \quad x < M, \ then \ \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \frac{1}{M^2} = \varepsilon$$

this proves that $\lim_{x\to-\infty} 1/x^2 = 0$.

Note here that y = 0 is the horizontal asymptote of the graph of $1/x^2$.

Infinite Limits at infinity

We now see the last possible formal definiton of limit that is not considered yet.

Defintion 3.8 $\lim_{x\to\infty} f(x) = \infty$ if for any real number N there is some number M such that

$$if \ x > M, \ then \ f(x) > N.$$

Note the definition of

$\lim_{x\to\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = \infty$, *and* $\lim_{x\to-\infty} f(x) = -\infty$ are completely analogues.

Example 12 Show that $\lim_{x\to\infty} x^3 = \infty.$

Solution: We use the fact that $x^3 > x$ for $x > 1.$ For any N, choose M so that M>1 and M > N. Then it follows that

$$if\ \ x > M, \ then\ x^3 > x > M > N$$

therefore by Definition 3.8

$$\lim_{x\to\infty} x^3 = \infty.$$

Similarly, we conclude that for any positive integer n,

$$\lim_{x\to\infty} x^n = \infty.$$

3.5 Limit Theorems

Even if we have developed important techniques of solving limit problems by using the formal definition, I hope by now we have realized that it is not that easy to use this definition to solve each and every problem. Nevertheless the student had encountered in his or her earlier studies of calculus rather easy ways of evaluating limits by the help of different rules. Here we state and prove some of them by using Definition 3.1 and use them to evaluate more complex limit cases.

Theorem 3.9 Assume that $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ and c is a constant then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$
4. $if\ \ \lim_{x\to a} g(x) \neq 0 \ and\ \lim_{x\to a}\dfrac{f(x)}{g(x)}$ exists then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$

Proof: Here we proof (1). Statement (2), (3), and (4) are left as exercise.

Let $\lim_{x\to a} f(x) = L \ \ and\ \lim_{x\to a} g(x) = M$ we need to show for every ε>0 there is some δ>0 such that if $0 < |x-a| < \delta,\ then\ |f(x)+g(x)-(L+M)| < \varepsilon.$ Observe that $\lim_{x\to a} f(x) = L$ iff for every ε/2 >0 there is a δ_1>0 such that $if\ \ 0 < |x-a| < \delta_1,\ then\ |f(x)-L| < \varepsilon/2.$

Similarly $\lim_{x\to a} g(x) = M$ iff for every ε/2 >0 there is a δ_2>0 such that

$$if\ \ 0 < |x-a| < \delta_2,\ then\ |g(x)-L| < \varepsilon/2.$$

Let δ = min {δ_1, δ_2} then we can see that

$$if\ \ 0 < |x-a| < \delta,\ then\ |f(x)+g(x)-(L+M)| < |f(x)-L| + |g(x)-M| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M.$

In addition to these rules you have also seen that for instance if *f* is a polynomial or a rational function and *a* is in the domain of *f*, then

$\lim_{x\to a} f(x) = f(a)$ **etc**

Now let as quickly go through some important limit finding techniques that would require a little bit of caution before applying the rules in Theorem 3.9.

Example 13 Find $\lim_{x\to 2}\frac{x^2-4}{x-2}$

Solution: Direct substitution of 2 in $\frac{x^2-4}{x-2}$ implies that we have 0/0 which is indeterminate thus we cannot use Theorem 3.9 (4) but for x≠0 simplification of the rational expression would lead us to

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2$$

thus

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}(x+2)=4.$$

Example 14 Find $\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}$

Solution: Again here we cannot use Theorem 3.9 (4), as we get from direct substitution the indeterminate 0/0. But for x≠0 rationalizing the denominator we have:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}&=\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+1}-1}\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\\ &=\lim_{x\to 0}\frac{x^2\sqrt{x^2+1}+1}{(x^2+1)-1}=\lim_{x\to 0}\frac{x^2\sqrt{x^2+1}+1}{x^2}\\ &=\lim_{x\to 0}\left(\sqrt{x^2+1}+1\right)=2.\end{aligned}$$

Example 15 Find $\lim_{x\to 0}x|x|$

Solution: Observe that

$$x|x|=\begin{cases}x^2 & if\ \ x\geq 0\\ -x^2 & if\ \ x<0\end{cases}$$

Since $x|x|=x^2\ \ for\ \ x>0$**, we have**

$$\lim_{x\to 0^+}x|x|=\lim_{x\to 0^+}x^2=0$$

and also since $x|x|=-x^2\ \ for\ \ x<0,$ we have

$$\lim_{x\to 0^-}x|x|=\lim_{x\to 0^-}-x^2=0$$

therefore we conclude that

$$\lim_{x\to 0}x|x|=0.$$

Example 16 Prove that $\lim_{x\to -1}\frac{x+1}{|x+1|}$ **does not exist**

Solution:

$$\lim_{x\to -1^+}\frac{x+1}{|x+1|}=\lim_{x\to -1^+}\frac{x+1}{x+1}=\lim_{x\to -1^+}1=1\ \ and\ \ \lim_{x\to -1^-}\frac{x+1}{|x+1|}=\lim_{x\to -1^-}\frac{x+1}{-(x+1)}=\lim_{x\to -1^-}-1=-1$$

Consequently

$$\lim_{x\to -1^+}\frac{x+1}{|x+1|}\neq \lim_{x\to -1^-}\frac{x+1}{|x+1|}$$

Thus $\lim\limits_{x\to -1}\frac{x+1}{|x+1|}$ does not exist.

Example 17 Find $\lim\limits_{x\to\infty}\frac{x-2x^2}{x^2-1}$ *and* $\lim\limits_{x\to -\infty}\frac{x-2x^2}{x^2-1}$

Solution: Deveiding the numerator and the denominator of $\frac{x-2x^2}{x^2-1}$ by x^2 in the limit we have

$$\lim_{x\to\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to\infty}\frac{1/x-2}{1-1/x}=-2$$

similarly

$$\lim_{x\to -\infty}\frac{x-2x^2}{x^2-1}=\lim_{x\to -\infty}\frac{1/x-2}{1-1/x}=-2.$$

Observe here that y = -2 is the horizontal asymptote of the graph of $f(x)=\frac{x-2x^2}{x^2-1}$.

Example 18 Let $f(x)=\frac{x-2x^2}{x^2-1}$. Find all vertical asymptotes of the graph of *f*.

Solution: Since f is not defined at x =1 and x = –1 they are the possible vertical asymptotes but to confirm our claim we use limit:
Since

$$\lim_{x\to 1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to 1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=-\infty \quad \textit{and}$$

$$\lim_{x\to -1^+}\frac{x-2x^2}{x^2-1}=\lim_{x\to -1^+}\frac{x}{x+1}\frac{1-2x}{x-1}=\infty$$

it follows that x =1 and x = –1 are the vertical asymptotes of the graph of *f*.

The next theorems give two additional properties of limits. For their proofs the student may refer any major calculus books.

Theorem 3.10 If $f(x)\le g(x)$ for all x in an open interval that contains a (except possibly at *a*) and the limits of *f* and g both exist as x approaches *a*, then

$$\lim_{x\to a} f(x)\le \lim_{x\to a} g(x).$$

Theorem 3.11 (The Squeezing Theorem) If $f(x)\le g(x)\le h(x)$ for all x in an a open interval that contains *a* (except a possibly at *a*) and

$$\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$$

then

$$\lim_{x\to a} g(x)=L.$$

I don't think the student is new for these theorems and for the special limit that is the consequence of especially the Squeezing Theorem. i.e.

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Example 19 Find $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

Solution: Since $-1 \le \sin\frac{1}{x} \le 1$, $\forall x \neq 0$, we have

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2 \quad \forall x \neq 0$$

Moreover $\lim_{x\to 0} -x^2 = 0$ *and* $\lim_{x\to 0} x^2 = 0$, **thus by the squeezing theorem we have**

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Example 20 Find $\lim_{x\to\infty}\frac{x^4-x^2}{x+1}$

Solution: Simplifying $\frac{x^4-x^2}{x+1}$ **we can evaluate the limit as below**

$$\lim_{x\to\infty}\frac{x^4-x^2}{x+1}=\lim_{x\to\infty}\frac{x^2(x^2-1)}{x+1}=\lim_{x\to\infty}\frac{x^2(x-1)(x+1)}{x+1}$$
$$=\lim_{x\to\infty}x^2(x-1)=\infty.$$

Class work

Evaluate each of the following limit as a real number, ∞, – ∞, if it exists.

1. $\lim_{x\to -1}\frac{x^2-x-2}{x+1}$
2. $\lim_{x\to 0}\frac{\sin 2x}{x}$
3. $\lim_{x\to 3^-}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$
4. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$
5. $\lim_{x\to 1} f(x)$ *where* $f(x)=\begin{cases} x^3 & \text{if } x<1 \\ (x-2)^2 & \text{if } x>1 \end{cases}$

3.6 Continuity of a Function and the Intermediate Value Theorem

Definition 3.11 A function *f* is **continuous** at a number *a* in its domain if

$$\lim_{x\to a} f(x) = f(a)$$

f is said to be **discontinuous** at *a* if *f* is not continuous at *a*.

Notice that definition 3.11 implicitly requires three things if *f* is continuous at *a*:

1. *f*(*a*) is defined (that is, *a* is in the domain of *f*)
2. $\lim_{x\to a} f(x)$ exists (so *f* must be defined on an open interval that contains *a*).
3. $\lim_{x\to a} f(x) = f(a)$

Example 21 Let $f(x)=\lim_{x\to -1}\frac{x^2-x-2}{x+1}$. Determine the number at which f is not discontinuous.

Solution: Notice that f is a rational function. Since the denominator of f is 0 for x = -1, f is defined for all x except at –1. Thus f is discontinuous only at x = -1 else where it is continuous in its' domain.

Example 22 If we redefine the function *f* in Example 21 as:

$$f(x)=\begin{cases}\dfrac{x^2-x-2}{x+1} & if\ \ x\neq -1\\ -3 & if\ x=-1\end{cases}$$

then since

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\frac{x^2-x-2}{x+1}=\lim_{x\to -1}\frac{(x-2)(x+1)}{x+1}=-3$$

and hence $\lim_{x\to -1} f(x)=-3=f(-1)$

f is continuous.

Notice that we are able to make *f* in Example 21 to be continuous by redefining it at –1 as in Example 22. Such discontinuity points like –1 in our example are called **removable** discontinuities because we can remove the discontinuity of the function by redefining the function just at the discontinuity point.

Example 23 Let $f(x)=\dfrac{1}{x^2}$ *and* $g(x)=\begin{cases}1 & if \quad x\leq 0\\ 0 & if \quad x>0\end{cases}$ then we can see that, *f* is not defined at 0 and $\lim_{x\to 0} f(x)=\infty,$ g is defined at 0 but $\lim_{x\to 0} g(x)$ does not exist as $\lim_{x\to 0^-} g(x)=1$ and $\lim_{x\to 0^+} g(x)=0.$ Thus both functions are not continuous at 0. We say we have **infinite** discontinuity at 0 in case of *f* while we say we have **jump** discontinuity at 0 in case of *g*.

Clearly combinations of continuous functions follow immediately from the corresponding results for limits.

Theorem 3.12 If f and g are continuous at *a* and c is a constant, than the following functions are also continuous at *a*.

i. *f* + *g* ii. *f* – *g* iii. *cf* iv. *fg* v. *f*/*g* if *g*(a) ≠ 0.

So using theorem 3.12 we can show that every polynomial function is continuous over **R** every rational function is continuous everywhere except at numbers where the denominator is 0.

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f\circ g$. This fact is a consequence of the following theorem.

Theorem 3.13 If *f* is continuous at b and $\lim_{x\to a} g(x)=b$ then

$$\lim_{x\to a} f(g(x))=f(b)=f\left(\lim_{x\to a} g(x)\right)$$

The following theorem tells us that the composition of two continuous functions at a given number is continuous.

Theorem 3.14 If *g* is continuous at a and *f* is continuous at g(*a*), then $f\circ g(x)=f(g(x))$ is continuous at *a*.

Class work

Where are the following functions continuous

a) $f(x)=|x|$ b) $h(x)=\dfrac{1}{\sqrt{x^2+3}-2}$

One-Sided Continuity

Definition 3.15 A function f is continuous from the right at a point *a* in its domain if

$$\lim_{x\to a^+} f(x) = f(a)$$

A function f is continuous from the left at a point *a* in its domain if

$$\lim_{x\to a^-} f(x) = f(a).$$

Example 24 the step function $g(x)=\begin{cases}1 & if \quad x\le 0\\ 0 & if \quad x>0\end{cases}$ is continuous from the left at 0

Since $\lim_{x\to 0^-} g(x)=1=g(0)$ but it is not continuous from the right at 0 as $\lim_{x\to 0^+} g(x)\neq g(0)$ verify.

Continuity on interval

Definition 3.16 a) A function is continuous on (*a*,b), if it is continuous at every point in *(a,b).*

b. A function is continuous on [*a*, b] if it is continuous on (*a*,b) and is also continuous from the right at *a* and continuous from the left at b.

Class Work

Let $f(x)=\sqrt{1-x^2}$ .Show that f is continuous on [-1, 1].

An important property of continuous functions is expressed by the following theorem.

Theorem 3.17 (The Intermediate Value Theorem)

Suppose that f is continuous on the closed interval [a, b] and let N be any number strictly between *f*(*a*) and *f*(*b*). Then there exists a number *c* in (a, b) such that *f*(*c*) = N.

Example 24 Show that there is a root of the equation

$$4x^3-6x^2+3x-2=0$$

between 1 and 2.

Solution: Let $f(x)=4x^3-6x^2+3x-2$. We are looking for a solution of the given equation, that is, a number *c* between 1 and 2 such that *f*(c) = 0. Therefore we take a = 1, b = 2, and N = 0 in Theorem 3.17. We have

$$f(1)=4-6+3-2=-1<0 \quad and \quad f(2)=32-24+6-2=12>0$$

Thus $f(1)<0<f(2)$, that is, N=0 is a number between *f*(1) and *f*(2). Now *f* is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number *c* between 1 and 2 such that $f(c)=0$. In other words, the equation $4x^3-6x^2+3x-2=0$ has a root c in the interval (1, 2).

Class Work

1. Find *A* that makes the function

$$f(x)=\begin{cases}x^2-2 & \text{if } x<1\\ Ax-4 & \text{if } 1\le x\end{cases}$$

continuous at *x*=1.

2. Demonstrate that the equation $\cos x + x = 0$ has at least one solution.

4 Derivatives

4.1 Definition and Properties of Derivative; the Chain Rule

In your previous calculus course you were introduced with the definition of the derivative of a function, properties of derivatives, the chain rule and important application of the derivative. Here our aim is to revise some of these concepts and introduce the derivatives of some more functions.

Definition 4.1 The **derivative** of a function f at a number ***a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad (1)$$

if this limit exists.

If we write x = a + h, then x –a =h and x approaches a iff h approaches to 0. Therefore an equivalent way of stating the definition of the derivative is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \qquad (2)$$

This last definition is more convenient for finding the derivative of a function.

Example 1 Find the derivative of the function $f(x) = x^2 + 3x + 2$ at –1.

Solution: By definition

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}.$$

thus

$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{[(-1+h)^2 + 3(-1+h) + 2] - [(-1)^2 + 3(-1) + 2]}{h} \\ &= \lim_{h \to 0} \frac{1 - 2h + h^2 - 3 + 3h + 2}{h} \\ &= \lim_{h \to 0} \frac{h^2 + h}{h} = \lim_{h \to 0}(h+1) = 1. \end{aligned}$$

I hope the student remembers that the slope of the tangent line to the graph of the function f at a point (a, f(a)) is given by the derivative of f at a i.e $f'(a)$ consequently using the point-slope form of the equation of a line, we have the equation of the tangent line to the curve y = *f*(x) at a point (a, f(a)) is given by $y - f(a) = f'(a)(x-a)$. For instance the equation of the tangent line to the graph of $f(x) = x^2 + 3x + 2$ at (-1, 0) in our Example 1 is given by $y - f(-1) = f'(-1)(x-(-1))$ *or* $y - 0 = 1(x+1)$ or simply y = x+1.

Given a function f, we associate with it a new function f', called the **derivative** of *f*, defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point (x, $f(x)$).

EXAMPLE 2

Find the derivative of $f(x) := x + \sqrt{x+1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The definition of derivative.

$$\lim_{h \to 0} \frac{\left[x + h + \sqrt{(x+h)+1}\right] - \left(x + \sqrt{x+1}\right)}{h}$$

Replace f(x+h) and f(x) by the corresponding expressions.

$$\lim_{h \to 0} \frac{h + \sqrt{(x+h)+1} - \sqrt{x+1}}{h}$$

Simplify.

$$\lim_{h \to 0} \frac{h + \left[\sqrt{(x+h)+1} - \sqrt{x+1}\right] \cdot \dfrac{\left[\sqrt{(x+h)+1} + \sqrt{x+1}\right]}{\left[\sqrt{(x+h)+1} + \sqrt{x+1}\right]}}{h}$$

Rationalize the radicals.

Some steps are omitted — see the **complete solution**

$$\lim_{h \to 0} \left[1 + \frac{1}{\sqrt{(x+h)+1} + \sqrt{x+1}}\right]$$

Divide both the numerator and denominator by h (h is nonzero).

$$1 + \frac{1}{2 \cdot \sqrt{x+1}}$$

Therefore, the derivative of $x + \sqrt{x+1}$ is $1 + \dfrac{1}{2 \cdot \sqrt{x+1}}$

Here, the domain of the function is $x \geq -1$, while the derivative is defined for all values of $x > -1$.

Definition 4.2 A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or (-∞, a) or (-∞, ∞)] if it is differentiable at every number in the interval.

Example 3 Show that $f(x) = |x|$ is not differentiable at 0.

Solution: Observe that

$$|x| = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}$$

then for x >0 using (1) we have

$$\lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

and for x<0

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1.$$

which implies

$$\lim_{x \to 0} \frac{|x| - |0|}{x - 0} \text{ does not exist}$$

thus f is not differentiable at 0.

Theorem 4.3 If f is differentiable at a, then f is continuous at a.

Proof: To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$.

We do this by showing that the difference $f(x) - f(a)$ approaches 0.

For x ≠ a we can divide and multiply by x – a

We did this in order to involve the difference quotient. Thus we can use the Product Law of limits to write

$$\begin{aligned} \lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) \\ &= f'(a) \cdot 0 = 0. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + f(x) - f(a)] \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a). \end{aligned}$$

and so f is continuous at a.

Note: the converse of Theorem 4.3 is false: that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0).$$

But as we have seen in Example 3 that *f* is not differentiable at 0.

Let me remind u some of the differentiation rules that u have developed in your previous calculus course. I advice the student to check on these results using the definition of derivative.

The power rule: If $f(x)=x^n$ for any real number n is given by $f'(x)=nx^{n-1}$.

Derivatives of sine and cosine: $(\sin x)'=\cos x$ *and* $(\cos x)=-\sin x.$

Derivatives of exponential and logarithmic functions: $(e^x)'=e^x$ *and* $(\ln x)=\dfrac{1}{x}.$

etc.

We also need to revise the rules of finding the derivatives of combined functions as in the table below.

Let f and g be differentiable then

1. $(cf)'=cf'$
2. $(f+g)'=f'+g'$
3. $(f-g)=f'-g'$
4. $(fg)'=f'g+fg'$
5. $\left(\dfrac{f}{g}\right)'=\dfrac{f'g+fg'}{g^2}$
6. $(c)'=0$

Class work

Find the derivative of each of the following functions

1. $f(x)=x^{25}+5x^5+25$
2. $f(x)=x^2-\dfrac{1}{x^2}$
3. $f(x)=x^4+\sqrt[4]{x}$
4. $f(x)=x\sqrt{x}+\dfrac{1}{x^2\sqrt{x}}$
5. $f(x)=x\sin x$
6. $f(x)=\sin x\cos x$
7. $f(x)=\tan x$
8. $f(x)=\csc x$
9. $f(x)=\dfrac{\sec x}{1+\tan x}$
10. $f(x)=\dfrac{x^2\tan x}{\sec x}$

The Chain Rule

The rules that we have introduced till now are not enough to find composition of functions thus we need to develop an appropriate to handle these cases. The Chain Rule is such a rule.

Theorem 4.5 If the derivatives $g'(x)$ and $f'(g(x))$ both exist, then $(f\circ g)'(x)=f'(g(x))g'(x)$

Example 4 Find $h'(x)$ *if* $h(x)=\cos 2x$

Solution: Let $f(x)=\cos x$ *and* $g(x)=2x$. Then $h=f\circ g$. Since

$$g'(x)=2 \text{ and } f'(x)=-\sin x$$

we conclude that

$$h'(x)=f'(g(x))g'(x)=(-\sin 2x)(2)=-2\sin 2x.$$

Example 2 Find $h'(x)$ *if* $h(x)=\sqrt{1+x^2}$

Solution: Let $g(x)=1+x^2$ *and* $f(x)=\sqrt{x}$ **consequently** $h=f\circ g$.**Then**

$$g'(x)=2x \text{ and } f'(x)=\frac{1}{2\sqrt{x}} \text{ for } x>0.$$

Therefore

$$h'(x) = f'(g(x))g'(x) = \frac{1}{2\sqrt{1+x^2}}2x = \frac{x}{\sqrt{1+x^2}}.$$

Class Work

Find the derivative of the functions

1. $y = (x^5 + 2x^2 + 3)^{50}$ 2. $y = \dfrac{1}{\sqrt[3]{x^6 + 2x + 1}}$

3. $y = \cos(\sin(\tan x))$ 4. $y = \sqrt{\cos(\sin^2 x)}$

Find the equation of the tangent line to the curve at the given point

5. $(x^3 - x^2 + x - 1)^{10}$, $(1,0)$ 6. $y = \sqrt{x + 1/x}$, $(1, \sqrt{2})$

4.2 Inverse Functions and Their Derivatives

In pre-calculus mathematics courses we defined a function *f* as a relation in which no two elements of the relation have the same first coordinate. Also we have seen that for same of the functions the relation that is found by interchanging the entries of the ordered pairs can be again a function and we called such a function the *inverse* of the original function. In this section we discuss general properties of inverses and their derivatives.

4.2.1 Inverse Functions

In order to define the inverse of a function, it is essential that different numbers in the domain always give different values of *f*. Such functions are called one-to-one functions.

Definition 3.1 A function *f* with domain D and range R is **one-to-one function** if whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

Note from Definition 3.1 we see that every *strictly increasing* function is one-to-one, because if $a < b$, then $f(a) < f(b)$, and if $b < a$, then $f(b) < f(a)$ in short if $a \neq b$, then $f(a) \neq f(b)$. Similarly, every strictly decreasing function is one-to-one. We now give the definition of inverse functions in terms of one-to-one function.

Definition 3.2 Let *f* be a *one-to-one function* with domain D and range R. A function g with domain R and range D is the **inverse function of *f***, provided the following condition is true for every *x* in D and every *y* in R:

$$y = f(x) \text{ if and only if } x = g(y).$$

If a function *f* has an inverse function g, we often denote g by f^{-1}. Of course we must note here that almost always f^{-1} is different from 1/*f*.

If *f* is a one-to-one function with domain D and range R, then for each number y in R, there is exactly one number x in D such that $y = f(x)$. Since x is unique, we may define a function g from R to D by means of the rule $x = g(y)$. g reverses the correspondence given by *f*. We call g the inverse function of *f*. In summary, *a function f has an inverse if and only if it is one-to-one*. This conclusion is especially easy to apply to differentiable functions whose domains are intervals. We know that a function *f* is strictly increasing on I (and hence has an inverse) if $f'(x) > 0$ for all *x* in I or if $f'(x) \geq 0$ for all *x* in I and $f'(x) = 0$ for at most finitely many values of x. Similarly, f is strictly decreasing on I (and hence has an inverse) if $f'(x)<0$ for all x in I or if $f'(x)\le 0$ for all x in I and $f'(x)=0$ for at most finitely many values of x.

Example 1 Let $f(x)=2x^7+3x^5+6x-4$ then since $f'(x)=17x^6+15x^2+6>0$ f is strictly increasing consequently it is invertible.

Properties of Inverses

From Definition 3.2 and the theories we developed above we can drive the following elementary relationships between f and f^{-1}.

I. Domain of f^{-1} = range of f and range of f^{-1} = domain of f.

II. $(f^{-1})^{-1}=f$

III. $f^{-1}(f(x))=x$ for all x in the domain of f.

IV. $f(f^{-1}(y))=y$ for all y in the range of f.

In some cases we can find the inverse of a one-to-one function by solving the equation $y=f(x)$ for x in terms of y, obtaining an equation of the form $x=f^{-1}(y)$. The following guidelines summarize this procedure.

Guidelines for finding f^{-1} is simple cases

1. Verify that f is a one-to-one function (or that f is increasing or is decreasing) throughout its domain.
2. Solve the equation $y=f(x)$ for x in terms of y, obtaining an equation of the form $x=f^{-1}(y)$.

The success of this method depends on the nature of the equation $y=f(x)$, since we must be able to solve for x in terms of y.

Example 2 Let $f(x)=2x+3$. Find the inverse of f.

Solution: Following the guidelines, first since $f'(x)=2>0$, f is increasing for all real number x and thus f^{-1} exists for all real number x.

Now as guideline 2, we consider the equation

$$y=2x+3$$

and solving for x in terms y, we obtain

$$x=\frac{y-3}{2}$$

we now let

$$f^{-1}(y)=\frac{y-3}{2}$$

Since we customarily use *x* as the independent variable, we replace *y* by *x* to obtain

$$f^{-1}(x)=\frac{x-3}{2}.$$

Example 3 Let $f(x)=x^2-3$ for $x\ge 0$. The inverse function of f.

Solution: The domain of f is $[0,\infty)$, and the range is $[-3,\infty)$. Since f is increasing, it is one-to-one and hence has an inverse function f^{-1} that has domain $[-3,\infty)$ and range $[0,\infty)$.

As in guideline 2, we consider the equation

$$y = x^2 - 3$$

and solve for *x*, obtaining

$$x = \pm\sqrt{y+3}.$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ **and let**

$$f^{-1}(y) = \sqrt{y+3}, \text{ or equivalently, } f^{-1}(x) = \sqrt{x+3}.$$

Graphs of Inverse Functions

There is an interesting relationship between the graphs of a functions *f* and f^{-1}. We first not that b = *f*(*a*) is equivalent to *a* = f^{-1}(b). These equations imply that the point (a,b) is on the graph of *f* if and only if the point (b,a) is on the graph of f^{-1}. But (a,b) and (b,a) are symmetric with respect to the line *y* = *x*. Thus the graph of f^{-1} is obtained by simply reflecting the graph of f through the line *y* = *x*.

Example 4 For each function *f*, sketch the graph of *f* and f^{-1} on the same coordinate system.

a) $f(x) = 2x + 3$ c) $f(x) = x^2 - 3$ c) $f(x) = \sin x$

Solution: In each case the graph of f^{-1} is obtained by reflecting the graph of f through the line *y* = *x*. The graphs appear in fig 3.1 below.

Exercise 4.1

I Determine whether the given function has an inverse. If an inverse exists, give the domain and range of the inverse and graph the function and its inverse.

1. $f(x) = 4x + 3$
2. $f(x) = \sqrt{9 - x^2}$, $0 \le x \le 3$
3. $f(x) = x - \sin x$
4. $f(x) = \ln(3 - x)$
5. $f(x) = \dfrac{2x}{x-2}$
6. $f(x) = \sqrt[3]{x} + 1$

II Show *f* has an inverse if

7. $f(x) = \int_0^x \sqrt{1+t^4}\,dt \;\; for\;\; all\;\; x.$
8. $f(x) = \int_0^x \sin^4(t^2)\,dt \;\; for\;\; all\;\; x.$

4.2.2 Continuity and Differentiability of Inverse Functions

If f is continuous, then the graph of f has no breaks or holes, and hence the same is true for the (reflected) graph of f^{-1}. Thus we see intuitively that if f is continuous on [*a*,b], then f^{-1}

continuous on $[f(a), f(b)]$. We can also show that if f is increasing, then so is f^{-1}. These facts are stated in the next theorem that is given with out a proof.

Theorem 3.3 If f is continuous and increasing on [a,b], then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

We can also prove the analogous result obtained by replacing the word increasing in Theorem 3.3 by decreasing.

The next theorem provides us a method of finding of the derivative of an inverse function.

Theorem 3.4 Suppose that f has an inverse and is continuous on an open interval I containing a. Assume also that $f'(a)$ exists, $f'(a) \neq 0,$ and $f(a) = c.$ Then $(f^{-1})'(c)$ exists, and

$$(f^{-1})'(c) = \frac{1}{f'(a)} \tag{1}$$

Proof Using the fact that $f^{-1}(c) = a$ and definition of the derivative, we find that

$$(f^{-1})'(c) = \lim_{y \to c} \frac{f^{-1}(y) - f^{-1}(c)}{y - c} = \lim_{y \to c} \frac{f^{-1}(y) - a}{f(f^{-1}(y)) - f(a)} \tag{2}$$

provided that the latter limit exists. We will simultaneously show that it does exist and find its value. First notice that f^{-1} is continuous at c by theorem 3.3. Therefore

$$\lim_{y \to c} f^{-1}(y) = f^{-1}(c) = a$$

so that if $x = f^{-1}(y),$ then x approaches a as y approaches c. Moreover, the fact that f has an inverse and $f^{-1}(c) = a$ implies that $f^{-1}(y) \neq a$ for $y \neq c.$ Consequently (2) and the Substitution Theorem for Limits (with x substituting for $f^{-1}(y)$)imply that

$$(f^{-1})'(c) = \lim_{y \to c} \frac{f^{-1}(y) - a}{f(f^{-1}(y)) - f(a)} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)}$$

$$= \frac{1}{\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}} = \frac{1}{f'(a)}.$$

It is convenient to restate Theorem 3.4 as follows.

Corollary 3.5 If f^{-1} is the inverse function of a differentiable function f and if $f'(f^{-1}(x)) \neq 0,$ then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3}$$

Example 1 Let $f(x) = x^7 + 8x^3 + 4x - 2.$ Find $(f^{-1})'(-2).$

Solution: In order to use (1), we must first find the value of a for which f(a) = –2. But f(0) = –2, so a = 0. Since $f'(x) = 7x^6 + 24x^2 + 4,$ it follows that $f'(0) = 4.$ Thus we conclude from (1) that

$$(f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{4}$$

Example 2 If $f(x) = x^3 + 2x - 1,$ prove that *f* has an inverse function f^{-1}, and find the slope of the tangent line to the graph of f^{-1} at the point P(2,1).

Solution: Since $f'(x) = 3x^2 + 2 > 0$ for every x, f is increasing and hence is one-to-one. Thus, f has an inverse function f^{-1}. Since *f*(1) = 2, it follows that $f^{-1}(2) = 1,$ and consequently the point P(2,1) is on the graph of f^{-1}. It would be difficult to find f^{-1} using Guidelines, because we would have to solve the equation $y = x^3 + 2x - 1,$ for x in terms of y. However, even if we cannot find f^{-1} explicitly, we can find the slope $f^{-1}(2)$ of the tangent line to the graph of g at P(2,1). Thus, by Theorem 3.4

$$f^{-1}(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{5}.$$

An easy way to remember Corollary 3.5 is to let $y = f(x).$ If f^{-1} is the inverse function of f, then $f^{-1}(y) = f^{-1}(f(x)) = x.$ Then

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}$$

or, in differential notation,

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

Example 3 Let *f* be the function in example 2 then let $y = x^3 + 2x - 1 \ and \ x = f^{-1}(y)$.

Then $$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3x^2 + 2};$$

That is $$(f^{-1})'(y) = \frac{1}{3x^2 + 2} = \frac{1}{3(f^{-1}(y))^2 + 2}.$$

Or using x

$$(f^{-1})'(x) = \frac{1}{3(f^{-1}(x))^2 + 2}.$$

Consequently, to find $(f^{-1})'(x)$ it is necessary to know $f^{-1}(x),$ just as in corollary 3.5.

Exercise 4.2

I Find $(f^{-1})'(c)$.

1. $f(x) = x^3 + 7;\ c = 6$
2. $f(x) = x + \sin x;\ c = 0$
3. $f(x) = x + \sqrt{x};\ c = 2$
4. $f(x) = x \ln x;\ c = 2e^2$

II a) Use f' to prove that *f* has an inverse function. b) Find the slope of the tangent line at the point P on the graph of f^{-1}.

5. $f(x) = x^5 + 3x^2 + 2x - 1;\quad P(5,1)$
6. $f(x) = 4x^5 - (1/x^3);\ x > 0;\quad P(3,1)$

III Find dx/dy

7. $f(x) = 4 - x^2,\quad x \geq 0$
8. $f(x) = \ln(x^3 + 1)$

4.2.3 Inverse Trigonometric Functions

Since the trigonometric functions are not one-to-one, they do not have inverse functions. By restricting their domains, however, we may obtain one-to-one functions that have the same values as the trigonometric functions and that do have inverse over these restricted domains.

The Arcsine Function

If we restrict the domain of the sine function to $[-\pi/2, \pi/2]$, then the resulting function is strictly increasing (because its derivative is positive except $-\pi/2$ *and* $\pi/2$.) Hence the restricted function which is called **arcsine function** has domain [–1,1], and range $[-\pi/2, \pi/2]$. Its value at x is usually written $\arcsin x$ or $\sin^{-1}x$. As a consequence,

$$\arcsin x = y \ \ if \ \ and \ \ only \ \ if \ \ \sin y = x$$

$$for \ -1 \le x \le 1 \ \ and \ -\pi/2 \le y \le \pi/2$$

We also see from the property of inverse functions that

$i.$ $\arcsin(\sin x) = x$ *for* $-\pi/2 \le x \le \pi/2$ $\quad$ $ii.$ $\sin(\arcsin x) = x$ *for* $-1 \le x \le 1.$

Example 1 Evaluate

a) $\sin\left(\arcsin\frac{1}{2}\right)$ $\quad$ b) $\arcsin\left(\sin\frac{\pi}{4}\right)$ $\quad$ c) $\arcsin\left(\sin\frac{5\pi}{6}\right)$

Solution:

a) $\sin\left(\arcsin\frac{1}{2}\right) = \frac{1}{2}$ since $-1 < \frac{1}{2} < 1$

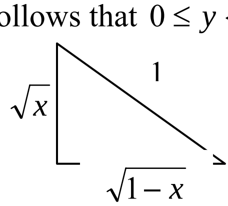
b) $\arcsin\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$ since $-\frac{\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}$

c) $\arcsin\left(\sin\frac{5\pi}{6}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.$

In Example 1c) $5\pi/6$ is not between $-\pi/2$ *and* $\pi/2$, and hence we cannot use ii. Instead we use properties of special angles to first evaluate $\sin(5\pi/6)$ and then find $\arcsin(1/2)$.

Example 2 Simplify the expression $\sec(\arcsin\sqrt{x})$

Solution: We will evaluate $\sec(\arcsin\sqrt{x})$ by evaluating $\sec y$ for the value of y in $(-\pi/2, \pi/2)$ such that $\arcsin\sqrt{x} = y$, that is, $\sin y = \sqrt{x}$. Since $\sin y = \sqrt{x} \ge 0$, it follows that $0 \le y < \pi/2$. Applying the Pythagorean Theorem to the triangle in Fig (3.2)



We find $\sec y = \frac{1}{\sqrt{1-x}}$. Therefore

$$\sec(\arcsin\sqrt{x}) = \sec y = \frac{1}{\sqrt{1-x}}.$$

The Arccosine Function

If the domain of the cosine

one continuous decreasing function that has a continuous decreasing inverse function. We call the inverse function of cosine **arccosine function**. The domain of the $\arccos ine$ is [–

1,1], and its range is $[0,\pi]$. Its value at x is usually written $\arccos x$ *or* $\cos^{-1} x$. As a consequence,

$$\arccos x = y \text{ if and only if } \cos y = x$$
$$\text{for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi$$

Since cos and arccos are inverse functions of each other, we obtain the following properties.

$$i.\ \arccos(\cos x) = x \text{ for } 0 \le x \le \pi \qquad ii.\ \cos(\arccos x) = x \text{ for } -1 \le x \le 1.$$

Example 2 Evaluate

a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$ b) $\arccos\left(\cos\frac{2\pi}{3}\right)$ c) $\arccos\left[\cos\left(-\frac{1}{2}\right)\right]$

Solution:

a) $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$ since $-1 < -\frac{1}{2} < 1$

b) $\arccos\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ since $0 < \frac{2\pi}{3} < \pi$

c) $\arccos\left[\cos\left(-\frac{\pi}{4}\right)\right] = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Note that in the c) part of the preceding Example 2, $-\pi/4$ is not between 0 and π, and hence we cannot use property ii. above. Instead, we first evaluate $\cos(-\pi/4)$ and then find $\cos^{-1}(\sqrt{2}/2)$.

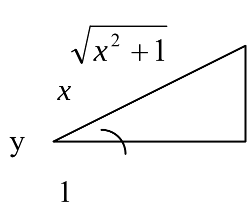
Example 3 Simplify the expression $\cos(\arctan x)$.

Solution: Let $y = \arctan x$. Then $\tan y = x$ and $-\pi/2 < y < \pi/2$. We want to find $\cos y$ but, since $\tan y$ is known, it is easier to find $\sec y$ first:

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$
$$\sec y = \sqrt{1+x^2} \qquad (as \ \ \sec y > 0 \ \ for \ -\pi/2 < y < \pi/2)$$

Thus $$\cos(\arctan x) = \cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1+x^2}}.$$

Note instead of using trigonometric identities as in the solution above, it is easy to use a triangular diagram. If we let $y = \arctan x$ then $\tan y = x$, and using the right triangle below we can read from the fig that



$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1+x^2}}.$$

The Arctangent Function

To find an inverse for the tangent function, we restrict the tangent function to $(-\pi/2, \pi/2)$. The resulting inverse function is called the **arctangent function**. Its domain is $(-\infty,\infty)$, and its range is $(-\pi/2, \pi/2)$. We usually write its value at x as $\arctan x$ or $\tan^{-1} x$. As a consequence,

$$\arctan x = y \textit{ if and only if } \tan y = x$$
$$\textit{for any } x \textit{ and for } -\pi/2 < y < \pi/2$$

Thus for any x, $\arctan x$ is the number y between $-\pi/2$ *and* $\pi/2$ whose tangent is x.

As with arcsin and arccox, we have the following properties of arctan

$i.\ \arctan(\tan x) = x \textit{ for } -\pi/2 \le x \le \pi/2 \qquad ii.\ \tan(\arctan x) = x \textit{ for every } x.$

Example 4

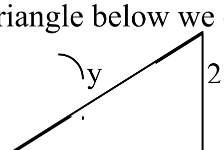
a) $\tan(\arctan 99) = 99$

b) $\arctan(\tan\frac{\pi}{4}) = \frac{\pi}{4}$

c) $\arctan(\tan\pi) = \arctan 0 = 0$

Example 5 Evaluate $\sec(\arctan\frac{2}{3})$

Solution: If we let $y = \arctan\frac{2}{3}$, then $\tan y = \frac{2}{3}$. We wish to find $\sec y$. Since $-\pi/2 < \arctan x < \pi/2$ for every x and $\tan y > 0$, it follows that $0 < y < \pi/2$ and from the triangle below we obtain that



$$\sec\left(\arctan\frac{2}{3}\right) = \sec y = \frac{\sqrt{13}}{3}.$$

The remaining trigonometric funct tly and are summarized here as below:

$$y = \csc^{-1} x(|x| \ge 1) \iff \csc y = x \quad and \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \sec^{-1} x(|x| \ge 1) \iff \sec y = x \quad and \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2]$$

$$y = \cot^{-1} x(x \in \Re) \iff \cot y = x \quad and \quad y \in (0, \pi)$$

Of these functions only the arcsecant function appears with any frequency in the sequel.

Exercise 4.3

I Find the exact value of the expression, whenever it is defined.

1. $\arcsin(-\sqrt{2}/2)$ 2. $\arccos(-1/2)$
3. $\arctan(-\sqrt{3})$ 4. $\sin(\arcsin 2/3)$
5. $\arcsin(\sin 5\pi/4)$ 6. $\arccos(\cos 5\pi/4)$
7. $\cos[\arctan(-3/4) - \arcsin(4/5)]$ 8. $\tan[\arctan(3/4) + \arccos(8/17)]$

II Rewrite as an algebraic expression in x for $x > 0$.

9. $\sec(\arcsin(x/3)$ 10. $\tan(arc\csc(x/2))$
11. $\cos(2\arcsin x)$ 12. $\sin(2\arcsin x)$

Derivatives and Integrals

We know see the derivatives and integrals of the inverse trigonometric functions in the following two theorems.

Theorem 3.1

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Proof

To proof $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, put $y = \sin^{-1} x$ so that $\sin y = x$ whenever $-1 < x < 1$ and $-\pi/2 < y < \pi/2$. Then differentiating $\sin y = x$ implicitly, we have

$$\cos y \frac{dy}{dx} = 1$$

and hence $$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos y}$$

Since $-\pi/2 < y < \pi/2$, $\cos y$ is positive and, therefore,

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}.$$

Thus, $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

For $|x| < 1$. Observe that the inverse sine function is not differentiable at ± 1.

Since the inverse tangent function is differentiable at every real number, let us consider the equivalent equation

$$y = \arctan x \quad and \quad \tan y = x$$

for $-\pi/2 \le y \le \pi/2$. Differentiating $\tan y$ and trigonometric identities we have

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\frac{d \tan y}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

In other words, $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

The rest of the formulas can be obtained in similar fashion.

Example 1 Find

a) $\frac{d}{dx}\arcsin 3x^2$ b) $\frac{d}{dx}\arccos(\ln x)$ c) $\frac{d}{dx}\arctan e^{2x}$ d) $\frac{d}{dx} arc\sec 3x^2$

Solution: Using Theorem 3.1 along the Chain Rule, we have

a) $$\frac{d}{dx}(\arcsin 3x^2) = \frac{1}{\sqrt{1-(3x^2)^2}}\frac{d}{dx}(3x^2) = \frac{6x}{\sqrt{1-9x^4}}$$

b) $$\frac{d}{dx}\arccos(\ln x) = -\frac{1}{\sqrt{1-(\ln x)^2}}\frac{d}{dx}(\ln x) = -\frac{1}{x\sqrt{1-(\ln x)^2}}$$

c) $$\frac{d}{dx}\arctan e^{2x} = \frac{1}{1+(e^{2x})^2}\frac{d}{dx}(e^{2x}) = \frac{2e^{2x}}{1+(e^{2x})^2}$$

d) $\frac{d}{dx} arc\sec 3x^2 = \frac{1}{3x^2\sqrt{(3x^2)^2-1}}\frac{d}{dx}(3x^2) = \frac{2}{x\sqrt{9x^4-1}}$

Each of the formulas in Theorem 3.1 gives rise to an integration formula. The three most useful of these are given in the following theorem.

Theorem 3.2

i) $\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\frac{u}{a} + C$

ii) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{u}{a} + C$

iii) $\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a} arc\sec\frac{u}{a} + C$

The proof of the above example is left as exercise.

Example 2 Evaluate $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

Solution: If we let $u = e^{2x}$ so that $du = 2e^{2x}dx$, the integral may be written as in Theorem (i) as below.

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx = \int \frac{1}{\sqrt{1-u^2}}\frac{1}{2}du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin e^{2x} + C.$$

Example 3 Evaluate $\int \frac{x^2}{4+x^6}dx.$

Solution: The integral may be written as in the second formula of Theorem (3.2) by letting $a^2 = 4$ and using the substitution

$$u = x^3, \qquad du = 3x^2dx$$

and proceed as follows:

$$\int \frac{x^2}{4+x^6}dx = \int \frac{1}{4+u^2}\left(\frac{du}{3}\right) = \frac{1}{3}\int \frac{1}{2^2+u^2}du$$

$$= \frac{1}{3}\cdot\frac{1}{2}\arctan\frac{u}{2} + C$$

$$= \frac{1}{6}\arctan\frac{u}{2} + C.$$

Example 4 Evaluate $\int \frac{1}{x\sqrt{x^4-9}}dx$

Solution: The integral may be written as in Theorem 3.2(iii) by letting $a^2 = 9$ and using the substitution

$$u = x^2, \qquad du = 2xdx,$$

we introduce $2x$ is the integrand by multiplying numerator and denominator by $2x$ and then proceed as follows:

$$\int \frac{1}{x\sqrt{x^4-9}}dx = \int \frac{1}{2x \cdot x\sqrt{(x^2)^2-3^2}} 2xdx$$
$$= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-3^2}}du$$
$$= \frac{1}{2}\cdot\frac{1}{3} arc\sec\frac{u}{3}+C$$
$$= \frac{1}{6} arc\sec\frac{x^2}{3}+C.$$

Exercise 3.3

I Find the derivative of the function. Simplify where possible.

1. $f(x)=\sin^{-1}(2x-1)$
2. $f(x)=(1+x^2)\arctan x$
3. $f(x)=\tan^{-1}(x-\sqrt{1+x^2})$
4. $f(x)=\cos(x^{-1})+(\cos x)^{-1}+\cos^{-1} x$
5. $f(x)=(\tan x)^{\arctan x}$
6. $f(x)=(\tan^{-1} 4x)e^{\arctan 4x}$

II Evaluate the integral

7. $\int_0^4 \frac{1}{x^2+16}$
8. $\int \frac{\cos x}{\sqrt{9-\sin^2 x}}dx$
9. $\int \frac{1}{\sqrt{e^{2x}-25}}dx$
10. $\int \frac{1}{\sqrt{x}(1+x)}dx$

4.2.4 Hyperbolic Functions

The exponential expressions

$$\frac{e^x-e^{-x}}{2} \text{ and } \frac{e^x+e^{-x}}{2}$$

occur in advanced applications of calculus. Their properties are similar in many ways to those of $\sin x$ and $\cos x$, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine** and **hyperbolic cosine**. We also define the rest of the hyperbolic functions in terms of these functions.

Definition 3.3

$$\sinh x=\frac{e^x-e^{-x}}{2} \qquad \csc hx=\frac{1}{\sinh x}$$
$$\cosh x=\frac{e^x+e^{-x}}{2} \qquad \sec hx=\frac{1}{\cosh x}$$
$$\tanh x=\frac{\sinh x}{\cosh x} \qquad \coth x=\frac{1}{\tanh x}$$

The hyperbolic functions satisfy a number of identities that are analogues of well-known trigonometric identities. We least some of the as below

Hyperbolic Identities

$\sinh(-x) = -\sinh x$ $\qquad$ $\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$ $\qquad$ $\cosh^2 x + \sinh^2 x = \cosh 2x$

$\sinh 2x = 2\sinh x \cosh x$ $\qquad$ $1 - \tanh^2 x = \sec h^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

The proof the above identities are left as exercise.

The derivatives of the hyperbolic functions are easily computed. For example,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

We list the differentiation formulas for the hyperbolic functions as below. The remaining proofs are left as exercises. Note the analogy with the differentiation formulas for trigonometric

Theorem 3.4

$$\frac{d}{dx}\sinh x = \cosh x \qquad \frac{d}{dx}\csc hx = -\csc hx \coth x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sec hx = -\sec hx \tanh x$$

$$\frac{d}{dx}\tanh x = \sec h^2 x \qquad \frac{d}{dx}\coth x = -\csc h^2 x$$

Example 1 If $f(x) = \cosh(e^{2x} + x)$, find $f'(x)$.

Solution: Applying Theorem 3.4, with the chain rule, we obtain

$$f'(x) = [\sinh(e^{2x} + x)] \cdot [(2e^{2x} + 1)] = (2e^{2x} + 1)\sinh(e^{2x} + x)$$

The integration formulas that correspond to the derivative formulas in theorem 3.4 are as follows.

Theorem 3.5

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \sec h^2 x dx = \tanh x + C \qquad \int \csc h^2 x dx = -\coth x + C$$

$$\int \sec hx \tanh x = -\sec h + C \qquad \int \csc hx \coth x = -\csc h + C$$

Example 2 Evaluate $\int x^2 \sinh x^3 dx$.

Solution: If we let $u = x^3$, then $du = 3x^2 dx$ and

$$\begin{aligned} \int x^2 \sinh x^3 dx &= \int \sinh u(\tfrac{1}{3}du) \\ &= \frac{1}{3}\cosh u + C \\ &= \frac{1}{3}\cosh x^3 + C. \end{aligned}$$

Exercise 3.4

I Find $f'(x)$ if $f(x)$ it the given expression.

1. $e^x \sinh x$
2. $\cosh(x^4)$
3. $\cos(\sinh x)$
4. $e^{\tanh x} \cosh(\cosh x)$

II Evaluate the integral

5. $\int \tanh 3x \sec h 3x dx$
6. $\int \sinh x \sec h^2 dx$
7. $\int \sec hx dx$
8. $\int \tanh x dx$

III. Verify the identity.

9. $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
10. $\sinh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{2}$

4.2.5 Inverse Hyperbolic Functions

The hyperbolic sine function is continuous and increasing for every x and hence, has a continuous, increasing inverse, function, denoted by $\sinh^{-1}$. Since $\sinh x$ is defined in terms of e^x, we might expect that $\sinh^{-1}$ can be expressed in terms of the inverse, ln, of the natural exponential function. The first formula of the next theorem shows that this is the case.

Theorem 3.6

1. $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$
2. $\cosh^{-1} x = \ln(x + \sqrt{x^2+1}),\ \ x \geq 1$
3. $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x},\ |x| < 1$
4. $\sec h^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x},\ 0 < x \leq 1$

Proof: To prove (1), let $y = \sinh^{-1} x.$ **Then**

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

then $$e^y - 2x - e^{-y} = 0.$$

Multiplying by e^y, we have

$$e^{2y} - 2xe^y - 1 = 0$$

which is a quadratic equation in e^y:

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $x - \sqrt{x^2+1} < 0$ **and** $e^y > 0$**, we must have**

$$e^y = x + \sqrt{x^2+1}.$$

The equivalent logarithmic form is

$$y = \ln(x + \sqrt{x^2+1})$$

that is, $$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

The proofs of the formulas 2-4 in theorem 3.6 are left as exercise.

The inverse hyperbolic functions are all differentiable because the hyperbolic functions are differentiable. The formulas in theorem 3.7 below can be proved by the method for inverse functions or by differentiating the formulas in theorem 3.6.

Theorem 3.7

1. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ 2. $\frac{d}{dx}(\operatorname{csc}h^{-1}x) = \frac{1}{|x|\sqrt{x^2+1}}$

3. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$ 4. $\frac{d}{dx}(\operatorname{sec}h^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$

5. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$ 6. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Proof: To proof (1) let $y = \sinh^{-1} x.$ Then $\sinh y = x$ and $\frac{dx}{dy} = \cosh y$. Since $\cosh y \geq 0$ and $\cosh^2 y - \sinh^2 y = 1,$ we have $\cosh y = \sqrt{1+\sinh^2 y}$. Then applying the method for inverse functions, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

Observe that we could have done the proof (1) by using formula (1) of Theorem 3.6 as below.

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{d}{dx}\ln(x+\sqrt{x^2+1})$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

Example 1 Find $\frac{d}{dx}\sinh^{-1}(\tan x).$

Solution: Using Theorem 3.7 and the Chain rule, we have

$$\frac{d}{dx}\sinh^{-1}(\tan x) = \frac{1}{\sqrt{\tan^2 x+1}}\frac{d}{dx}\tan x = \frac{1}{\sqrt{\sec^2 x}}\sec^2 x$$

$$= \frac{1}{|\sec^2 x|}\sec^2 x = |\sec x|.$$

Example 2 Evaluate $\int_0^{1/2} \frac{1}{1-x^2}dx$.

Solution: Referring to Theorem 3.7 we can see that the antiderivative of $1/1-x^2$ is $\tanh^{-1} x.$ Therefore

$$\int_0^{1/2} \frac{1}{1-x^2}dx = \tanh^{-1} x\Big|_0^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\Big|_0^{1/2}$$

$$= \frac{1}{2}\ln 3$$

Exercise 3.5

I Find $f'(x)$ *if* $f(x)$ is the given expression.

1. $\sinh^{-1} 5x$
2. $\sqrt{\cosh^{-1} x}$
3. $x\tanh^{-1} x + \ln\sqrt{1-x^2}$
4. $\operatorname{sec}h^{-1}\sqrt{1-x^2}, x > 0$

II Evaluate the integral

5. $\int \sinh 2x dx$
6. $\int \frac{\sinh x}{1+\cosh x}dx$
7. $\int \frac{e^x}{\sqrt{e^x - 16}}$
8. $\int \frac{\sin x}{\sqrt{1+\cos^2 x}}dx$

III Prove that the formulas 2,3, and 4 in Theorem 3.6.

4.2.6 L'Hôpital's Rule

While we study limits in the previous course of calculus we considered limits of quotients such as

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \qquad \text{and} \qquad \lim_{x\to 0}\frac{\sin x}{x}$$

and calculated the limits by using algebraic, geometric, and trigonometric methods even if the limits have the undefined form 0/0. In this section we develop another technique that employs the derivatives of the numerator and denominator of the quotient. This new technique is called L'Hôpital's rule. For the proof of this rule we need the following generalization of the Mean Value Theorem.

Theorem 4.1 (Cauchy's formula)

If *f* and *g* are continuous on [a,b] and differentiable on (a,b) and if $g'(x) \neq 0$ for every *x* in (a,b), then there is a number c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{f'(c)}$$

Proof: We first note that $g(b)-g(a) \neq 0,$ because otherwise g(a)=g(b) and, by Rolle's Theorem, there is a number c is (a,b) such that $g'(c) = 0,$ contrary to our assumption about g'.

Let us introduce a new function h as follows:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

for ever x in [a,b]. It follows that h is continuous on [a,b] and differentiable on (a,b) and that h(a) = h(b). By Rolle's Theorem, there is a number c in (a,b) such that $h'(c) = 0$; that is,

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0.$$

This is equivalent to Cauchy's formula.

The Indeterminate Form 0/0

If $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$, then we say that $\lim_{x\to a^+} f(x)/g(x)$ has the **indeterminate form 0/0**. The same notion applies if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}, \lim_{x\to c}, \lim_{x\to\infty}, or \lim_{x\to-\infty}$. The limits

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \qquad and \qquad \lim_{x\to 0}\frac{\sin x}{x}$$

therefore have the indeterminate form 0/0. Our first version of L'Hôpital's rule is concerning the indeterminate form 0/0.

Theorem 4.2 (L'Hôpital's rule)

Let L be a real number or ∞ or $-\infty$.

a. Suppose *f* and g are differentiable on (a,b) and $g'(x) \neq 0$ for $a < x < b$. If

$$\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \qquad and \qquad \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c}$, where c is any number in (a,b). In the letter case *f* and g need not be differentiable at c.

b. Suppose f and g are differentiable on (a,∞) and $g'(x) \neq 0$ for $x > a$. If

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \qquad and \qquad \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to\infty}$ *is replaced by* $\lim_{x\to-\infty}$.

Proof: We establish the formula involving the right-hand limits in (a). Define F and G on [a,b) by

$$F(x) = \begin{cases} f(x) & for \quad a < x < b \\ 0 & for \quad x = b \end{cases}$$

$$G(x) = \begin{cases} g(x) & for \quad a < x < b \\ 0 & for \quad x = a \end{cases}$$

Then

$$\lim_{x\to a^+} F(x) = \lim_{x\to a^+} f(x) = 0 = F(a)$$

so that F is continuous [a,b). The same is true of G. Moreover, F and G are differentiable on (a,b), since they agree with f and g, respectively, on (a,b). Consequently if x is any number in (a,b), the F and G are continuous on [a,x] and differentiable on (a,x). By the Generalized Mean Value Theorem, this means that there is a number c(x) in (a,x) such that

$$\frac{F(x)}{G(x)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F'(c(x))}{G'(c(x))}.$$

Because F = f and G =g on (a,b), this means that

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))}.$$

Since $a < c(x) < x,$ we know that

$$\lim_{x\to a^+} c(x) = a$$

so we can use the Substitution Theorem with y = c(x) to conclude that

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(c(x))}{g'(c(x))} = \lim_{y\to a^+}\frac{f'(y)}{g'(y)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L.$$

This proves the equation involving right-hand limit in (a). The results involving left-hand and two-sided limits are proved analogously. Part (b) is more difficult to prove, and we omit its proof.

Example 1 Evaluate $\lim\limits_{x\to 0}\dfrac{1-3^x}{x}$.

Solution: Both the numerator and the denominator have the limit 0 as $x \to 0$. Hence the quotient has the indeterminate form 0/0 at x = 0. By L'Hôpital's rule

$$\lim_{x\to 0}\frac{1-3^x}{x} = \lim_{x\to 0}\frac{-3^x \ln 3}{1} = -\ln 3$$

Example 2 Evaluate $\lim\limits_{x\to 0}\dfrac{\ln(1-x^2)}{\ln\cos 2x}$

Solution: Observe that

$$\lim_{x\to 0}\ln(1-x^2) = 0 = \lim_{x\to 0}\ln\cos 2x$$

thus by applying L'Hôpital's rule we get

$$\lim_{x\to 0}\frac{\ln(1-x^2)}{\ln\cos 2x} = \frac{\dfrac{-2x}{1-x^2}}{-2\tan 2x} = \lim_{x\to 0}\left(\frac{1}{1-x^2}\cdot\frac{-2x}{-2\tan 2x}\right)$$

$$= \lim_{x\to 0}\frac{x}{\tan 2x}, \quad \sin ce \lim_{x\to 0}\frac{1}{1-x^2} = 1$$

$$= \lim_{x\to 0}\left[\frac{x}{\sin 2x}\cdot(\cos 2x)\right]$$

$$= \lim_{x\to 0}\frac{x}{\sin 2x}\cdot\lim_{x\to 0}(\cos x) = \frac{1}{2}$$

Example 3 Evaluate $\lim\limits_{x\to\infty}\dfrac{(\pi/2)-\arctan x}{1/x}$.

Solution: Since $\lim_{x\to\infty} \arctan x = \pi/2,$ we have

$$\lim_{x\to\infty}\left(\frac{\pi}{2} - \arctan x\right) = 0 = \lim_{x\to\infty}\frac{1}{x}$$

Hence by L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{(\pi/2) - \arctan x}{1/x} = \lim_{x\to\infty}\frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{x^2}{1+x^2} = 1.$$

In same limits we need to apply L'Hôpital's rule several times in succession. The next example is one.

Example 4 Evaluate $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{2x - \sin 2x}$

Solution: The given quotient has the indeterminate form 0/0. By L'Hôpital's rule we have

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{2x - \sin 2x} = \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{2 - 2\cos 2x}$$

provided the second limit exists. Because the last quotient has the indeterminate form 0/0, we apply L'Hôpital's rule again, to obtain

$$\lim_{x\to 0}\frac{e^x + e^{-x} - 2}{2 - 2\cos 2x} = \lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin 2x}$$

still the last quotient has the indeterminate form 0/0, hence applying the L'Hôpital's rule for third time we get

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{4\sin 2x} = \lim_{x\to 0}\frac{e^x + e^{-x}}{8\cos 2x} = \frac{2}{8} = \frac{1}{4}.$$

The Indeterminate Form ∞/∞

Our second version of L'Hôpital's rule involves limits with indeterminate form ∞/∞. We give it now, with out proof.

Theorem 4.3 (L'Hôpital's rule)

Let L be a real number or ∞ *or* $-\infty$.

a. Suppose *f* and g are differentiable on (a,b) and $g'(x) \neq 0$ *for* $a < x < b$. If

$$\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty,\ \lim_{x\to a^+} g(x) = \infty \text{ or } -\infty,\ \text{and } \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim_{x\to a^+}$ is replaced by $\lim_{x\to b^-}$ or by $\lim_{x\to c},$ where c is any number in (a,b). In the letter case *f* and g need not be differentiable at c.

b. Suppose f and g are differentiable on (a,∞) and $g'(x) \neq 0$ for $x > a$. If

$$\lim_{x\to\infty} f(x) = \infty \text{ or } -\infty,\ \lim_{x\to\infty} g(x) = \infty \text{ or } -\infty,\ \text{and } \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $\lim\limits_{x\to\infty}$ *is replaced by* $\lim\limits_{x\to-\infty}$.

Evaluate 5 Evaluate $\lim\limits_{x\to(\pi/2)^-}\dfrac{4\tan x}{1+\sec x}$.

Solution: Observe that the limit has the indeterminate form ∞/∞. Then by L'Hôpital's rule we have

$$\lim_{x\to(\pi/2)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-}\frac{4\sec^2 x}{\sec x\tan x} = \lim_{x\to(\pi/2)^-}\frac{4\sec x}{\tan x}.$$

The last quotient again has the indeterminate form ∞/∞ at $x=\pi/2$; however, additional applications of L'Hôpital's rule always produce the form ∞/∞. In this case the limit may be found by using trigonometric identities to change the quotient as follows:

$$\frac{4\sec x}{\tan x} = \frac{4/\cos x}{\sin x/\cos x} = \frac{4}{\sin x}$$

Consequently

$$\lim_{x\to(\pi/2)^-}\frac{4\tan x}{1+\sec x} = \lim_{x\to(\pi/2)^-}\frac{4}{\sin x} = \frac{4}{1} = 4.$$

Example 6 Evaluate $\lim\limits_{x\to\infty}\dfrac{e^{2x}}{x^2}$.

Solution: Since the limit has the indeterminate form ∞/∞ by applying L'Hôpital's rule we have

$$\lim_{x\to\infty}\frac{e^{2x}}{x^2} = \lim_{x\to\infty}\frac{2e^{2x}}{2x} = \lim_{x\to\infty}\frac{e^{2x}}{x}.$$

The last quotient has the indeterminate form ∞/∞, so we apply L'Hôpital's rule for a second time, to obtain

$$\lim_{x\to\infty}\frac{e^{2x}}{x} = \lim_{x\to\infty}\frac{2e^{2x}}{1} = \infty.$$

Particularly in a similar fashion we can show that

$$\lim_{x\to\infty}\frac{e^{x}}{x^n} = \infty \quad \textit{for every real number } n.$$

Other Indeterminate Forms

Various indeterminate forms, such as $0\cdot\infty, 0^0, 1^\infty, \infty^0$, *and* $\infty-\infty$, can usually be converted into the indeterminate form 0/0 or ∞/∞ and then evaluated by one of the versions of L'Hôpital's rule given in Theorem 4.2 and 4.3.

Example 7 Find $\lim\limits_{x\to0^+} x^2\ln x$

Solution: Since $\lim\limits_{x\to0^+} x^2 = 0$ *and* $\lim\limits_{x\to0^+}\ln x = -\infty$ the given limit is of the form $0\cdot\infty$ (more precisely, $0\cdot(-\infty)$). However, we can transform it into the indeterminate form ∞/∞ by writing it as

$$\lim_{x\to0^+}\frac{\ln x}{1/x^2}$$

and apply the L'Hôpital's rule we get

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3}.$$

The last quotient has the indeterminate form ∞/∞; however, further application of L'Hôpital's rule would again lead to ∞/∞. In this case we simplify the quotient algebraically and find the limit as follows:

$$\lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{x^3}{-2x} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Example 8 Find $\lim_{x\to 0^+} x^x$.

Solution: The limit evidently has the indeterminate form 0^0. But then since $x^x = e^{x\ln x}$ and consequently

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}.$$

Since the exponential function is continuous, it follows that

$$\lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+}(x\ln x)}.$$

it the limit on the right side exists. But since

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+}(-x) = 0$$

by L'Hôpital's rule,

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1.$$

Example 9 Show that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Solution: Observe that the limit has the indeterminate form 1^∞. As in example 8, since $\left(1+\frac{1}{x}\right)^x = e^{\ln\left(1+\frac{1}{x}\right)^x}$ first let us evaluate $\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x$. But in doing so we have

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x}.$$

This expression is now prepared for L'Hôpital's rule as the limit has 0/0 form. As a result

$$\lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x} = \lim_{x\to\infty}\frac{\frac{1}{(1+1/x)}\left(-\frac{1}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{1}{1+1/x} = 1.$$

Thus

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^0 = 1.$$

Example 10 Find $\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$.

Solution: The limit has the indeterminate form $\infty-\infty$; however; if the difference is written as a single fraction, then

$$\lim_{x\to 0^+}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}.$$

This gives us the indeterminate form 0/0. It is necessary to apply L'Hôpital's rule twice, since the first application leads to the indeterminate form 0/0. Thus,

$$\lim_{x\to 0^+}\frac{x-e^x+1}{xe^x-x}=\lim_{x\to 0^+}\frac{1-e^x}{xe^x+e^x-1}$$

$$=\lim_{x\to 0^+}\frac{-e^x}{xe^x+2e^x}=-\frac{1}{2}.$$

Exercise

I Find the limit

1. $\lim\limits_{x\to -1}\dfrac{x^6-1}{x^4-1}$ 2. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$ 3. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}$

4. $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$ 5. $\lim\limits_{x\to\infty}\dfrac{\tan 1/x}{1/x}$ 6. $\lim\limits_{x\to 0}\dfrac{\tan^{-1}(2x)}{3x}$

7. $\lim\limits_{x\to 0}\dfrac{e^x-1-x-(x^2/2)}{x^3}$ 8. $\lim\limits_{x\to -\infty}xe^x$ 9. $\lim\limits_{x\to\infty}(x-\sqrt{x^2-1})$

10. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)$ 11. $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x$ 12. $\lim\limits_{x\to\infty}x^2\left(1-x\sin\dfrac{1}{3}\right)$

13. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$ 14. $\lim\limits_{x\to\pi/4}(\tan x)^{\tan 2x}$ 15. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right)$

16. Why is the following "application" of L'Hôpital's rule invalid?

$$\frac{1}{\pi/2}=\lim_{x\to\pi/2}\frac{\sin x}{x}=\lim_{x\to\pi/2}\frac{\cos x}{1}=0$$

17. Evaluate $\lim\limits_{x\to 0}\dfrac{1}{x^3}\displaystyle\int_0^x \sin(t^2)\,dt.$

4.3 Implicit Differentiation Problems

The following problems require the use of implicit differentiation. Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives. The majority of differentiation problems in first-year calculus involve functions y written EXPLICITLY as functions of x . For example, if

$$y=3x^2-\sin(7x+5)$$

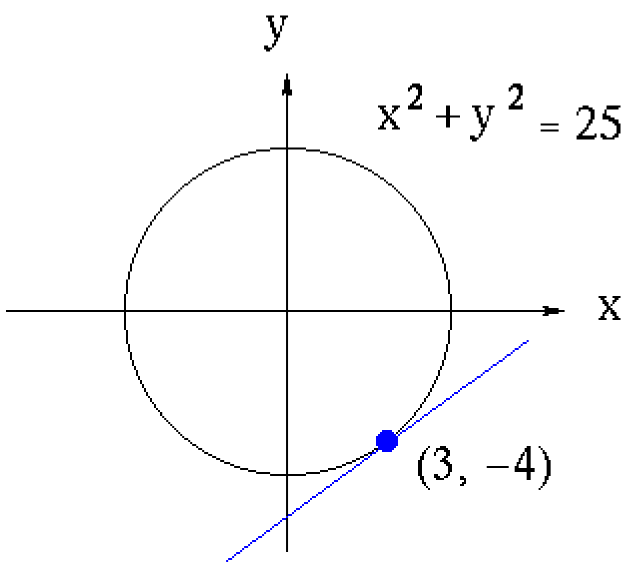
then the derivative of y is

$$y'=6x-7\cos(7x+5)$$

However, some functions y are written IMPLICITLY as functions of x . A familiar example of this is the equation

$$x^2+y^2=25\ ,$$

which represents a circle of radius five centered at the origin. Suppose that we wish to find the slope of the line tangent to the graph of this equation at the point (3, -4) .



How could we find the derivative of y in this instance ? One way is to first write y explicitly as a function of x . Thus,

$$x^2 + y^2 = 25 ,$$
$$y^2 = 25 - x^2 ,$$

and

$$y = \pm\sqrt{25 - x^2}$$

where the positive square root represents the top semi-circle and the negative square root represents the bottom semi-circle. Since the point (3, -4) lies on the bottom semi-circle given by

$$y = -\sqrt{25 - x^2} ,$$

the derivative of y is

$$y' = -(1/2)(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}} ,$$

i.e.,

$$y' = \frac{x}{\sqrt{25 - x^2}} .$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{(3)}{\sqrt{25 - (3)^2}} = \frac{3}{4} .$$

Unfortunately, not every equation involving x and y can be solved explicitly for y . For the sake of illustration we will find the derivative of y WITHOUT writing y explicitly as a function of x . Recall that the derivative (D) of a function of x squared, $(f(x))^2$, can be found using the chain rule :

$$D\{(f(x))^2\} = 2f(x)\ D\{f(x)\} = 2f(x)f'(x) .$$

Since y symbolically represents a function of x, the derivative of y^2 can be found in the same fashion :

$$D\{y^2\} = 2y\, D\{y\} = 2yy'$$

.

Now begin with $x^2 + y^2 = 25$. Differentiate both sides of the equation, getting

$$D(x^2 + y^2) = D(25),$$
$$D(x^2) + D(y^2) = D(25),$$

and

$$2x + 2yy' = 0,$$

so that

$$2yy' = -2x,$$

and

$$y' = \frac{-2x}{2y} = \frac{-x}{y},$$

i.e.,

$$y' = \frac{-x}{y}.$$

Thus, the slope of the line tangent to the graph at the point (3, -4) is

$$m = y' = \frac{-(3)}{(-4)} = \frac{3}{4}.$$

This second method illustrates the process of implicit differentiation. It is important to note that the derivative expression for explicit differentiation involves *x* only, while the derivative expression for implicit differentiation may involve BOTH *x* AND *y* .

The following problems range in difficulty from average to challenging.

Example 1 Assume that *y* is a function of *x* . Find $y' = dy/dx$ for $x^3 + y^3 = 4$.

SOLUTION: Begin with $x^3 + y^3 = 4$. Differentiate both sides of the equation, getting

$$D(x^3 + y^3) = D(4),$$
$$D(x^3) + D(y^3) = D(4),$$

(Remember to use the chain rule on $D(y^3)$.)

$$3x^2 + 3y^2 y' = 0,$$

so that (Now solve for y' .)

$$3y^2 y' = -3x^2,$$

and

$$y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$$

Exercise 2 Assume that *y* is a function of *x* . Find $y' = dy/dx$ for $(x-y)^2 = x + y - 1$.

SOLUTION: Begin with $(x-y)^2 = x + y - 1$. Differentiate both sides of the equation, getting

$$D(x-y)^2 = D(x + y - 1),$$
$$D(x-y)^2 = D(x) + D(y) - D(1),$$

(Remember to use the chain rule on $D(x-y)^2$.)

$$2(x-y)\, D(x-y) = 1 + y' - 0,$$

$$2(x-y)(1-y') = 1 + y',$$

so that (Now solve for y' .)

$$2(x-y) - 2(x-y)y' = 1 + y',$$
$$-2(x-y)y' - y' = 1 - 2(x-y),$$

(Factor out y' .)

$$y' [- 2 (x-y) - 1] = 1 - 2 (x-y) ,$$

and

$$y' = \frac{1-2(x-y)}{-2(x-y)-1} = \frac{2y-2x+1}{2y-2x-1} .$$

Example 3 Assume that y is a function of x . Find $y' = dy/dx$ for $y = \sin(3x+4y)$.

SOLUTION: Begin with $y = \sin(3x+4y)$. Differentiate both sides of the equation, getting

$$D(y) = D(\sin(3x+4y)) ,$$

(Remember to use the chain rule on $D(\sin(3x+4y))$.)

$$y' = \cos(3x+4y)\ D(3x+4y) ,$$

$$y' = \cos(3x+4y)(3+4y') ,$$

so that (Now solve for y' .)

$$y' = 3\cos(3x+4y) + 4y' \cos(3x+4y) ,$$

$$y' - 4y' \cos(3x+4y) = 3\cos(3x+4y) ,$$

(Factor out y' .)

$$y'[1 - 4\cos(3x+4y)] = 3\cos(3x+4y) ,$$

and

$$y' = \frac{3\cos(3x+4y)}{1-4\cos(3x+4y)}$$

Example 4 Assume that y is a function of x . Find $y' = dy/dx$ for $y = x^2 y^3 + x^3 y^2$.
SOLUTION: Begin with $y = x^2 y^3 + x^3 y^2$. Differentiate both sides of the equation, getting

$$D(y) = D (x^2 y^3 + x^3 y^2) ,$$

$$D(y) = D (x^2 y^3) + D (x^3 y^2) ,$$

(Use the product rule twice.)

$$y' = \{x^2 D(y^3) + D(x^2) y^3\} + \{x^3 D(y^2) + D(x^3) y^2\} ,$$

(Remember to use the chain rule on $D (y^3)$ and $D (y^2)$.)

$$y' = \{x^2 (3y^2 y') + (2x) y^3\} + \{x^3 (2yy') + (3x^2) y^2\} ,$$

$$y' = 3x^2 y^2 y' + 2x y^3 + 2x^3 y y' + 3x^2 y^2 ,$$

so that (Now solve for y' .)

$$y' - 3x^2 y^2 y' - 2x^3 y y' = 2x y^3 + 3x^2 y^2 ,$$

(Factor out y' .)

$$y' [1 - 3x^2 y^2 - 2x^3 y] = 2x y^3 + 3x^2 y^2 ,$$

and

$$y' = \frac{2xy^3 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$$
.

PROBLEM 5 Assume that y is a function of x . Find $y' = dy/dx$ for $e^{xy} = e^{4x} + e^{5y}$.
SOLUTION: Begin with $e^{xy} = e^{4x} - e^{5y}$. Differentiate both sides of the equation, getting

$$D(e^{xy}) = D(e^{4x} + e^{5y}),$$
$$D(e^{xy}) = D(e^{4x}) + D(e^{5y}),$$
$$e^{xy} D(xy) = e^{4x} D(4x) + e^{5y} D(5y),$$
$$e^{xy}(xy' + (1)y) = e^{4x}(4) + e^{5y}(5y'),$$

so that (Now solve for y' .)

$$xe^{xy}y' + y e^{xy} = 4e^{4x} + 5e^{5y}y',$$
$$xe^{xy}y' - 5e^{5y}y' = 4e^{4x} - y e^{xy},$$

(Factor out y' .)

$$y'[xe^{xy} - 5e^{5y}] = 4e^{4x} - y e^{xy},$$

and

$$y' = \frac{4e^{4x} - ye^{xy}}{xe^{xy} - 5e^{5y}}$$

Example 6 Assume that y is a function of x . Find $y' = dy/dx$ for $\cos^2 x + \cos^2 y = \cos(2x + 2y)$
.

SOLUTION: Begin with $\cos^2 x + \cos^2 y = \cos(2x + 2y)$. Differentiate both sides of the equation, getting

$$D(\cos^2 x + \cos^2 y) = D(\cos(2x + 2y))$$
,
$$D(\cos^2 x) + D(\cos^2 y) = D(\cos(2x + 2y))$$
,
$$(2\cos x)D(\cos x) + (2\cos y)D(\cos y) = -\sin(2x + 2y)D(2x + 2y)$$
,
$$2\cos x(-\sin x) + 2\cos y(-\sin y)(y') = -\sin(2x + 2y)(2 + 2y')$$
,

so that (Now solve for y' .)

$$-2\cos x\sin x - 2y'\cos y\sin y = -2\sin(2x + 2y) - 2y'\sin(2x + 2y)$$
,
$$2y'\sin(2x + 2y) - 2y'\cos y\sin y = -2\sin(2x + 2y) + 2\cos x\sin x$$
,

(Factor out y' .)

$$y'[2\sin(2x + 2y) - 2\cos y\sin y] = 2\cos x\sin x - 2\sin(2x + 2y)$$
,
$$y' = \frac{2\cos x\sin x - 2\sin(2x + 2y)}{2\sin(2x + 2y) - 2\cos y\sin y}$$
,
$$y' = \frac{2[\cos x\sin x - \sin(2x + 2y)]}{2[\sin(2x + 2y) - \cos y\sin y]}$$
,

and

$$y' = \frac{\cos x \sin x - \sin(2x + 2y)}{\sin(2x + 2y) - \cos y \sin y}$$

Example 7 Assume that y is a function of x . Find $y' = dy/dx$ for $x = \sqrt{x^2 + y^2}$.

SOLUTION: Begin with $x = \sqrt{x^2 + y^2}$. Differentiate both sides of the equation, getting

$$D(x) = D(\sqrt{x^2 + y^2})$$
,

$$1 = (1/2)(x^2 + y^2)^{-1/2} D(x^2 + y^2),$$
$$1 = (1/2)(x^2 + y^2)^{-1/2} (2x + 2y y'),$$

so that (Now solve for y' .)

$$1 = \frac{(1/2)(2)(x + yy')}{\sqrt{x^2 + y^2}},$$

$$1 = \frac{x + yy'}{\sqrt{x^2 + y^2}},$$

$$\sqrt{x^2 + y^2} = x + yy',$$

$$\sqrt{x^2 + y^2} - x = yy',$$

and

$$y' = \frac{\sqrt{x^2 + y^2} - x}{y}.$$

Exercise 8: Assume that y is a function of x . Find $y' = dy/dx$ for $\frac{x - y^3}{y + x^2} = x + 2$.

SOLUTION: Begin with $\frac{x - y^3}{y + x^2} = x + 2$. Clear the fraction by multiplying both sides of the equation by $y + x^2$, getting

$$\frac{x - y^3}{y + x^2}(y + x^2) = x + 2(y + x^2)$$

or $$x - y^3 = xy + 2y + x^3 + 2x^2 .$$

Now differentiate both sides of the equation, getting

$$D(x - y^3) = D(xy + 2y + x^3 + 2x^2),$$
$$D(x) - D(y^3) = D(xy) + D(2y) + D(x^3) + D(2x^2),$$

(Remember to use the chain rule on $D(y^3)$.)

$$1 - 3y^2 y' = (xy' + (1)y) + 2y' + 3x^2 + 4x,$$

so that (Now solve for y' .)

$$1 - y - 3x^2 - 4x = 3y^2 y' + xy' + 2y',$$

(Factor out y' .)

$$1 - y - 3x^2 - 4x = (3y^2 + x + 2) y',$$

and

$$y' = \frac{1 - y - 3x^2 - 4x}{3y^2 + x + 2}$$

.

Class Work

PROBLEM 9 : Assume that y is a function of x . Find $y' = dy/dx$ for $\frac{y}{x^3} + \frac{x}{y^3} = x^2 y^4$.

Example 10 Find an equation of the line tangent to the graph of $(x^2+y^2)^3 = 8x^2y^2$ at the point (-1, 1) .

SOLUTION Begin with $(x^2+y^2)^3 = 8x^2y^2$. Now differentiate both sides of the equation, getting

$$D\,(x^2+y^2)^3 = D\,(\,8x^2y^2\,)\,,$$
$$3\,(x^2+y^2)^2\,D\,(x^2+y^2) = 8x^2\,D\,(y^2\,) + D\,(\,8x^2\,)\,y^2\,,$$

(Remember to use the chain rule on $D\,(y^2\,)$.)

$$3\,(x^2+y^2)^2\,(\,2x + 2\,y\,y'\,) = 8x^2\,(2\,y\,y'\,) + (\,16\,x\,)\,y^2\,,$$

so that (Now solve for y' .)

$$6x\,(x^2+y^2)^2 + 6\,y\,(x^2+y^2)^2\,y' = 16\,x^2\,y\,y' + 16\,x\,y^2\,,$$
$$6\,y\,(x^2+y^2)^2\,y' - 16\,x^2\,y\,y' = 16\,x\,y^2 - 6x\,(x^2+y^2)^2\,,$$

(Factor out y' .)

$$y'\,[\,6\,y\,(x^2+y^2)^2 - 16\,x^2\,y\,] = 16\,x\,y^2 - 6x\,(x^2+y^2)^2\,,$$

and

$$y' = \frac{16xy^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 16x^2y}$$

.

Thus, the slope of the line tangent to the graph at the point (-1, 1) is

$$m = y' = \frac{16(-1)(1)^2 - 6(-1)((-1)^2 + (1)^2)^2}{6(1)((-1)^2 + (1)^2)^2 - 16(-1)^2(1)} = \frac{8}{8} = 1$$

,

and the equation of the tangent line is

$$y - (\,1\,) = (1)\,(\,x - (\,-1\,)\,)$$

or

$$y = x + 2$$

Example 11 Find an equation of the line tangent to the graph of $x^2 + (y-x)^3 = 9$ at x=1.

SOLUTION: Begin with $x^2 + (y-x)^3 = 9$. If x=1 , then

$$(1)^2 + (\,y-1\,)^3 = 9$$

so that

$$(\,y-1\,)^3 = 8\,,$$
$$y-1 = 2\,,$$
$$y = 3\,,$$

and the tangent line passes through the point (1, 3) . Now differentiate both sides of the original equation, getting

$$D\,(\,x^2 + (y-x)^3\,) = D\,(\,9\,)\,,$$
$$D\,(\,x^2\,) + D\,(y-x)^3 = D\,(\,9\,)\,,$$
$$2x + 3\,(y-x)^2\,D\,(y-x) = 0\,,$$
$$2x + 3\,(y-x)^2\,(y'-1) = 0\,,$$

so that (Now solve for y' .)

$$2x + 3\,(y-x)^2\,y' - 3\,(y-x)^2 = 0\,,$$
$$3\,(y-x)^2\,y' = 3\,(y-x)^2 - 2x\,,$$

and

$$y' = \frac{3(y-x)^2 - 2x}{3(y-x)^2}$$

.

Thus, the slope of the line tangent to the graph at (1, 3) is

$$m = y' = \frac{3(3-1)^2 - 2(1)}{3(3-1)^2} = \frac{10}{12} = \frac{5}{6}$$

and the equation of the tangent line is

$$y - (3) = (5/6)(x - (1)),$$

or

$$y = (7/6)x + (13/6).$$

Finally let us see how to find the second derivative of a function that is defined implicitly.

Example 12 Find y'' *if* $x^4 + y^4 = 25$

Solution: Differentiating the equation implicitly with respect to x, we get

$$4x^3 + 4y^3 y' = 0$$

solving for y'gives

$$y' = -\frac{x^3}{y^3} \qquad (1)$$

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x:

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3 D(x^3) - x^3 D(y^3)}{(y^3)^2}$$

$$= -\frac{y^3 .3x^2 - x^3(3y^2 y')}{y^6}$$

If we now substitute Equation 1 into this expression we get

$$y'' = \frac{3x^2 y^3 - 3x^3 y^2\left(\frac{-x^3}{y^3}\right)}{y^6}$$

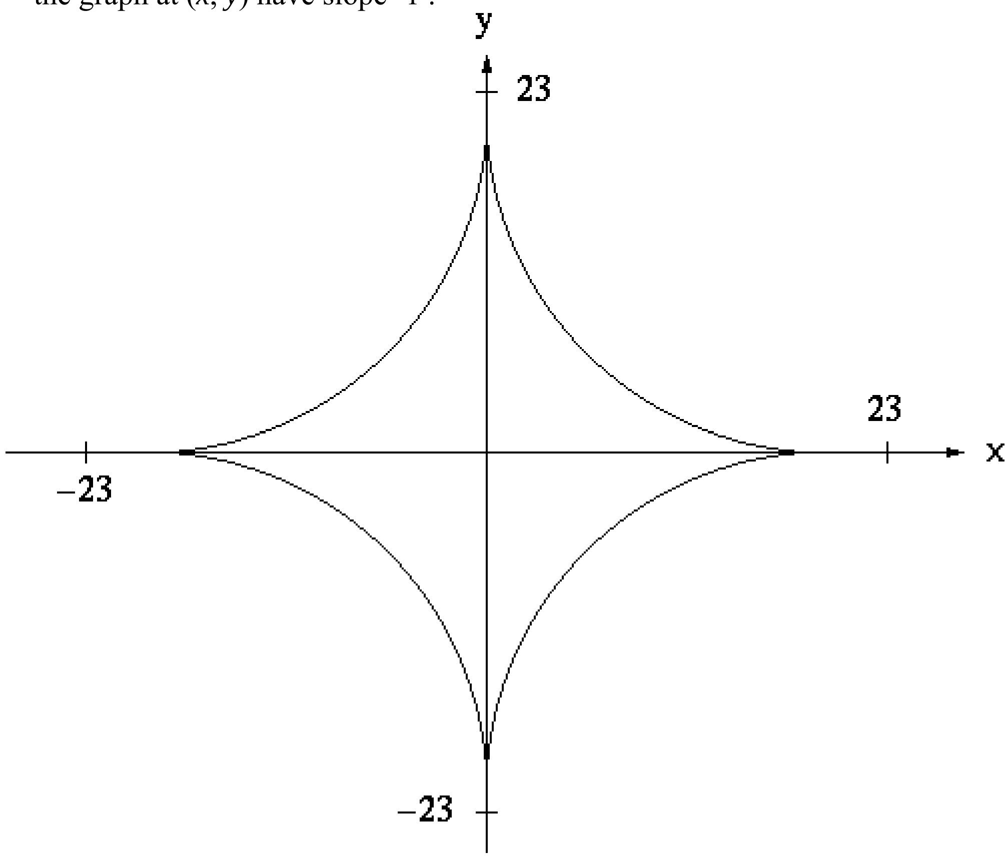
$$= -\frac{3(x^2 y + x^6)}{y^7} = -\frac{3x(y^4 + x^4)}{y^7}$$

But the values of x and y must satisfy the original equation $x^4 + y^4 = 25$. So that answer simplifies to $y'' = -\frac{3x^2(25)}{y^7} = -75\frac{x^2}{y^7}$

Class Work

1. Find the slope and concavity of the graph of $x^2y + y^4 = 4 + 2x$ at the point (-1, 1) .
2. Consider the equation $x^2 + xy + y^2 = 1$. Find equations for y' and y" in terms of x and y
3. Find all points (x, y) on the graph of $x^{2/3} + y^{2/3} = 8$ (See diagram.) where lines tangent to

the graph at (x, y) have slope -1 .



4.Find y'' by implicit differentiation.

a. $x^3 + y^3 = 1$

b. $x^2 + 6xy + y^2 = 8$

c. $\sqrt{x} + \sqrt{y} = 1.$

4.4 Application of the derivative

4.4.1 Extrema of a function

Definition 1 A function f has an **absolute maximum** at ***c*** if f©≥ f(x) for all x in D, where D is the domain of f. The number f© is called the **maximum value of f** on D. Similarly, f has an **absolute minimum** at c if f© ≤ f(x) for all x in D and the number f© is called the **minimum value of f** on D. The maximum and minimum values of are called the **extreme** values of f.

Definition 2 A function f has a **local maximum** (or relative maximum) at c if there is an open interval I containing c such that f©≥ f(x) for all x in I. Similarly, f has a **local minimum** at c if there is an open interval I containing c such that f© ≤ f(x) for all x in I.

Example 1 If $f(x) = x^2$, then $f(x) \geq f(0)$ because $x^2 \geq 0$ for all x. therefore f(0) = 0 is the absolute (and local) minimum value of f. this corresponds to the fact that the origin is that lowest point on the parabola $y = x^2$. However, there is no highest point on the parabola and so this function has no maximum value.

Example 2 From the graph of the function f(x) = x^3 we see that this function has neither an absolute maximum value nor an absolute minimum value. In fact, it has no local extreme values either.

Theorem 3 If f has a relative (local) extremum (that is, maximum or minimum) at c, and that $f'(c)$ exists, then $f'(c) = 0$.

Definition 4 A number c in the domain of a function f is a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 3 Find the critical numbers of f $f(x) = 4x^{3/5} - x^{8/5}$.

Solution The derivative of f is given by

$$f'(x) = \frac{12}{5}x^{-2/5} - \frac{8}{5}x^{3/5} = \frac{12-8x}{5x^{2/5}}$$

Therefore $f'(c) = 0$ if 12-8x = 0, that is, x = 3/2 and $f'(x)$ does not exist when x =0. Thus the critical numbers are 3/2 and 0.//

To find the absolute extreme value of a function on a closed interval a similar theorem to Theorem 3 is given bellow.

Maximum-Minimum Theorem

Theorem 5 Let f be continuous on a closed interval [a, b]. Then f has a maximum and a minimum value on [a, b].

Note that according to Maximum-Minimum Theorem an extreme value can be taken on more than once.

The following Theorem will simplify our effort of searching for an extreme value on a closed interval.

Theorem 6 Let f be defined on [a, b]. If an *absolute* extreme value of f on [a, b] occurs at a number c in (a, b) at which f has a derivative, then $f'(c) = 0$.

In using theorem 5 to find the extreme value we followthe three-step procedure bellow.

1. Find the values of f at the critical numbers of f in (a,b)
2. Find the values of f(a) and f(b).
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 2 \qquad \tfrac{1}{2} \le x \le 3$$

Solution: Since f is continuous on [-½,3], we can use the procedure outlined above:

Since

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, x = 0 or x = 2. Notice that each of these critical numbers lies in the interval [-½,3]. The values of f at these critical numbers are

f(0) = 2 f(2) = -3

The values of f at the endpoints of the interval are

$$f(-\tfrac{1}{2}) = \tfrac{1}{8} \qquad f(3) = -2$$

Comparing these four numbers, we see that the absolute maximum value is f(0)=f(3)=2 and the absolute minimum value is f(2) = –2.

Class Work

1. Find the critical numbers of each function a) $f(x) = x^3 - 6x + 1$ b) $f(x)=|x|$ c) $\cos\sqrt{x}$ d) $f(x) = \dfrac{1}{\sqrt{x^2+1}}$
2. Find all extreme values (if any) of the given function on the given interval. Determine at which numbers in the interval these values occur. a) $f(x)=x^2 - 2x + 2$, [0,3]   b) $f(x)=x^2 + 2/x$, [1/2, 2] c) $f(x)=x^{2/3}$, [-8,8].
3. Show that 0 is a critical number of the function $f(x) = x^5$ but f does not have a local extremum at 0.
4. Prove that the function $f(x) = x^{51} + x^{21}+x+1$ has neither a local maximum nor a local minimum.

4.4.2 The Mean Value Theorem

Theorem 6 Let f be continuous on [a, b] and differentiable on (a, b). Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b-a).$$

Example 5 Let $f(x) = x^3 - 8x + 5$. Find a number c in (0,3) that satisfies the Mean Value Theorem.

Solution Since f is continuous on [0,3] and $f'(c)$ should satisfy the condition

$$f'(c) = \frac{f(3) - f(0)}{3-0} = \frac{8-5}{3} = 1$$

we seek a number c in (0,3) such that $f'(c) = 1$. But

$$f'(x) = 3x^2 - 8$$

so that c must satisfy

$$3c^2 - 8 = 1$$

$$c = \pm\sqrt{3}$$

Since $-\sqrt{3} \notin (0,3)$, the value of c that satisfies the mean value theorem in the interval (0,3) is $\sqrt{3}$.

Class Work

1. Verify that the function bellow satisfies the hypothesis of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   a) $f(x)=1 - x^2$, [0, 3]   b) $f(x) = 3\left(x + \dfrac{1}{x}\right), [\tfrac{1}{3}, 3]$ c) $f(x) = \sqrt{x}, [1,4]$
2. Let f(x)=|x-1|. Show that there is no value of c such that f(3)-f(0)=f'©(3-0). Why does this not contradict the Mean Value Theorem?
3. Show that the equation $x^5+10x+3=0$ has exactly one real root.
4. Show that the equation $x^4+4x+c=0$ has at most two real roots.

4.4.3 First and Second Derivative Tests; Curve sketching

I hope you remember that a function that is increasing or decreasing on an interval I is called **monotonic** on I and we used the test stated in the theorem bellow to identify whether a function is monotonic or not on a given interval.

Theorem 7 Suppose f is continuous on [a,b] and differentiable on (a,b).

a) If $f'(x) > 0$ for all x in (a,b), then f is increasing on [a,b].
b) If $f'(x) < 0$ for all x in (a,b), then f is decreasing on [a,b].

Theorem 7 lays the bases for the proof of the first derivative test stated as follows.

Theorem 8 (The first derivative test)

Suppose that c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.
b) If f' changes from negative to positive at c, then f has a local minimum at c.
c) If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 6 Find the local extrema of $x^{1/3}(8-x)$ and sketch its graph.

Solution By the product rule we have

$$f'(x) = \tfrac{1}{3}x^{-2/3}(8-x) - x^{1/3}$$

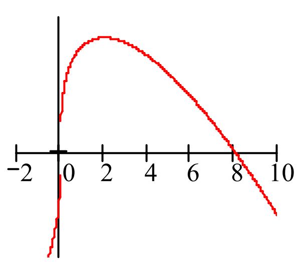
$$= \frac{8-x-3x}{3x^{2/3}} = \frac{4(2-x)}{3x^{2/3}}$$

The derivative $f'(x) = 0$ when x =2 more over f'(x) does not exist when x = 0. So the critical numbers are 0 and 2.

Bellow we give the sign chart for f'(x).

	0	2	
4(2-x)	+	+	–
$3x^{2/3}$	+	+	+
f'(x)	+	+	–
f	*f* is increasing	*f* is increasing	*f* is decreasing

Then the function does not have an extreme value at 0. Since f' does not change sign at 0. But f has a local maximum at 2 since f' changes sign from positive to negative and the local maximum value is given by f(2)=$2^{1/3}$(8 –2)=$6\sqrt[3]{2}$. Then using the sign chart and the extreme value we sketch the graph as bellow.



Class Work

If $f(x) = x^{2/3}(x^2 - 8)$, find the local extrema, and sketch the graph of *f*.

As the first derivative is useful to sketch the graph of a function the second derivative gives also additional information that enables us to sketch a better picture of the graph. The tests that we give below involve second derivative the student can consult advanced books for there proofs.

Theorem 9 (The Test For Concavity) Suppose f is twice differentiable on an interval I.

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 10 A point (a,b) on a curve is called a **point of inflection** if the curve changes from concave upward to concave downward or from concave downward to concave upward at (a,b).

Example 7 Determine where the curve $y = x^3 - 3x + 1$ is concave upward and where it is concave downward. Find the inflection points and sketch the curve.

Solution If $f(x) = x^3 - 3x + 1$, then

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Since $f'(x) = 0$ when $x^2 = 1$, the critical numbers are ±1. Also

$$f'(x) < 0 \quad \Leftrightarrow \quad x^2 - 1 < 0 \quad \Leftrightarrow x^2 < 1 \quad \Leftrightarrow \quad |x| < 1$$

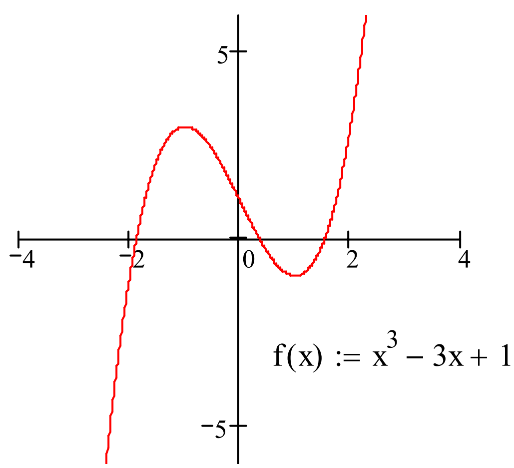
$$f'(x) < 0 \quad \Leftrightarrow \quad x^2 < 1 \quad \Leftrightarrow \quad x > 1 \quad or \quad x < -1$$

Therefore f is increasing on the interval (-∞, -1] and [1,∞) and is decreasing on [-1,1]. By the first derivative test, f(-1) = 3 is local maximum value and f(1) = -1 is a local minimum value.

To determine the concavity we compute the second derivative:

$$f''(x) = 6x$$

Thus $f''(x) > 0$ when x>0 and $f''(x) < 0$ when x<0. The Test for concavity then tells us that the curve is concave downward on (-∞, 0) and concave upward on (0, ∞). Since the curve changes from concave downward to concave upward when x = 0, the point (0, 1) is a point of inflection. We use this information to sketch the curve in Fig below.



Another Application of the second derivative is in finding maximum and minimum values of a function.

Theorem 11 Suppose f'' is continuous on an open interval of a function.

a) If $f'(x) = 0$ *and* $f''(c) > 0$, then f has a local minimum at c.

b) If $f'(x) = 0$ *and* $f''(c) < 0$, then f has a local maximum at c.

EXAMPLE

Use the Second Derivative Test to find relative extrema of

$f(x) = 3\cdot x^4 + 8\cdot x^3 + 4$.

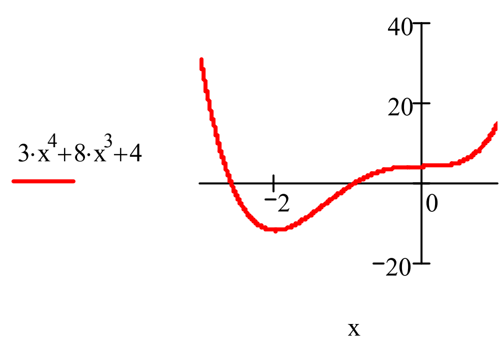
Solution

$f'(x) = 12\cdot x^3 + 24\cdot x^2$	**Find critical numbers of f.**
$= 12\cdot x^2\cdot(x+2)$	**(Note that the Second Derivative Test can only be applied at critical numbers where f '=0.)**
Critical numbers: x = 0, x = -2	
$f''(x) = 36\cdot x^2 + 48\cdot x$	**Find f ". Evaluate f " at the critical numbers where f ' = 0.**
$f''(-2) = 48 > 0$	**Relative minimum**
$f''(0) = 0$	**The Second Derivative Test fails in this case.**

In the last case, x = 0 could still be a relative maximum, relative minimur or neither; but the Second Derivative Test fails to produce any useful information.

If you used the First Derivative Test, you would find out that x = 0 is not relative extremum (there is an inflection point there instead).

The graph on the right illustrates these findings.



Class Work

Find (a) the intervals of increasing or decreasing, b) the local maximum and minimum values of the points of inflection. Then use this information to sketch the graph.

a) $f(x)=x^3 - x$ b) $f(x) = x\sqrt{x+1}$ c) $f(x) = x^{1/3}(x+3)^{2/3}$

4.4.3 Curve Sketching

Now we apply the knowledge that we have developed in this chapter for sketching the graphs of different functions. The table below lists the items that are most important in graphing a function f.

Property	Test
f has y intercept c	*f*(0)=c
f has x intercept c	*f*©=c
symmetric with respect to the $\begin{cases} y\ axis \\ origin \end{cases}$	$f(-x) = f(x)$ $f(-x) = -f(x)$
f has a relative maximum value at c	$\begin{cases} f'(c) = 0\ and\ \ f'changes\ from\ v+ \ to\, v- \\ f'(c) = 0\ and\ f''(c) < 0 \end{cases}$
f has a relative minimum value at c	$\begin{cases} f'(c) = 0\ and\ \ f'changes\ from\ v- \ to\, v+ \\ f'(c) = 0\ and\ f''(c) > 0 \end{cases}$
f is strictly increasing on an open interval I	f'>0 for all except finitely many x in I
f is strictly decreasing on an open interval I	f'<0 for all except finitely many x in I
Graph of f is concave upward on I	$f''(x) < 0 \quad \forall x \in I$
Graph of f is concave downward on I	$f''(x) > 0 \quad \forall x \in I$
(c, f©) is an inflection point	f'' changes sign at c and usually $f''(c) = 0$
f has a vertical asymptote $x = c$	$\lim_{x\to c^+} f(x) = \pm\infty\ \ or\ \ \lim_{x\to c^-} f(x) = \pm\infty$
f has a horizontal asymptote $x = d$	$\lim_{x\to\infty} f(x) = d\ \ or\ \ \lim_{x\to-\infty} f(x) = d$

Example If $g(x) = \dfrac{x^2}{1-x^2}$, discuss and sketch the graph of *g*.

Solution:

1. **Analyze the first derivative.**

$$g'(x) := \frac{2\cdot x}{\left(1-x^2\right)^2}$$

This has a root at x = 0. Possible local maximum or minimum here.

Notice that neither g(x) nor its derivative are defined at x = 1 and x = –1. The derivative is negative for x < 0, except at x = –1, where it is not defined. It is positive for x > 0, except at x = 1, where it is not defined.

2. **Analyze the second derivative.**

$$g''(x) := \frac{2+6\cdot x^2}{\left(1-x^2\right)^3}$$

There are no values of x where the second derivative equals zero, so the graph of g has no inflection points.

$g''(0) = 2$

At x = 0, a critical number, the second derivative is positive, so the graph is concave up at this point, and has a local minimum.

3. Find horizontal asymptotes.

$$\lim_{x \to \infty} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad -1$$

$$\lim_{x \to -\infty} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad -1$$

$h(x) := -1$ is a horizontal asymptote.

4. Find vertical asymptotes.

Since g is undefined at 1 and –1, examine the limits of g as x approaches these values.

$$\lim_{x \to 1^+} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad -\infty$$

$$\lim_{x \to 1^-} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad \infty$$

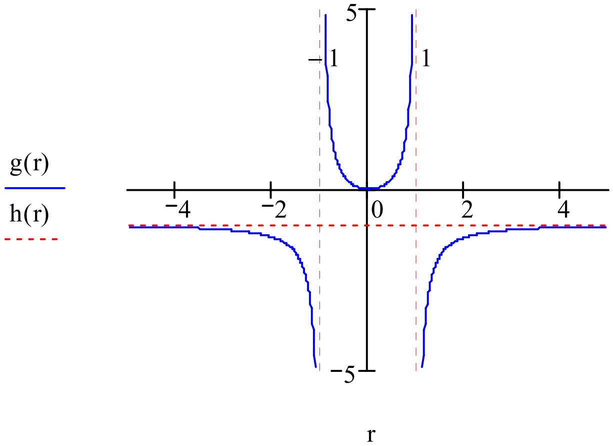
$$\lim_{x \to -1^+} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad \infty$$

$$\lim_{x \to -1^-} \frac{x^2}{1-x^2} \quad \text{simplifies to} \quad -\infty$$

g has vertical asymptotes at x = 1 and x = –1.

5. Put it all together.

$r := -5, -4.99 .. 5$ **Range for graphing**



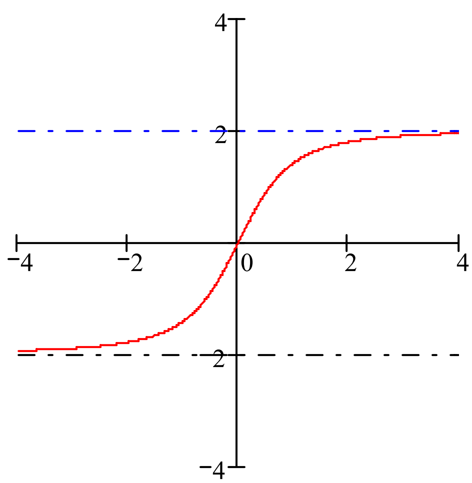
Notice that all the aspects of the graph you found in your analysis are present: a local minimum at x = 0, vertical asymptotes at x = 1 and x = –1, a horizontal asymptote at y = –1, downward sloping when x < 0, upward sloping when x > 0.

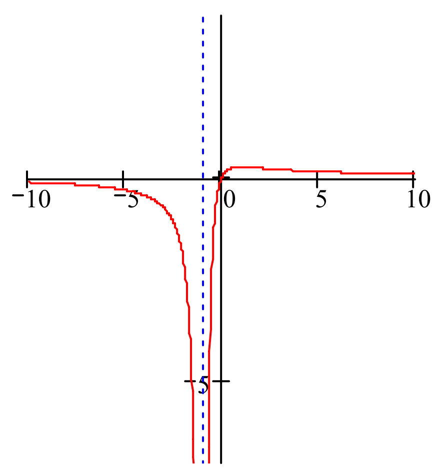
Class Work

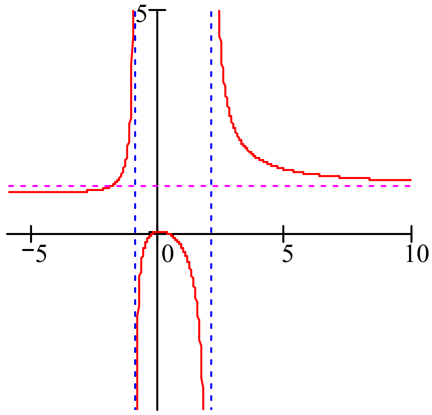
Discuss and sketch the graph of *f* if

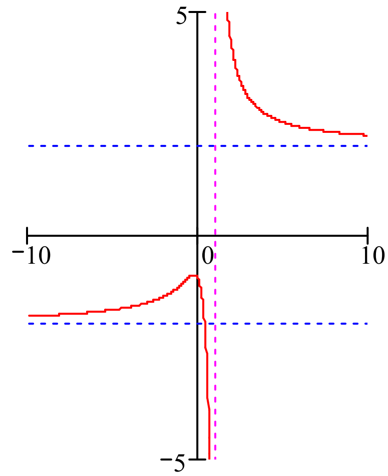
a) $f(x) = \dfrac{x^2}{x^2 - x - 2}$ b) $f(x) = \dfrac{x}{(x+1)^2}$ c) $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$ d) $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$

Note your sketch should look like one of the graphs bellow.









5 Review of Techniques of Integration

5.0 Introduction

Before we see techniques of integration let us revise the integrals of important functions in the following table.

Derivative	Indefinite integral
$D_x(x) = 1$	1. $\int 1dx = \int dx = x + c$
$D_x\left(\dfrac{x^{r+1}}{r+1}\right) = x^r \ (r \neq -1)$	2. $\int x^r dx = \dfrac{x^{r+1}}{r+1} + c \ \ (r \neq -1)$
$D_x(\sin x) = \cos x$	3. $\int \cos xdx = \sin x + c$
$D_x(-\cos x) = \sin x$	4. $\int \sin xdx = -\cos x + c$
$D_x(\tan x) = \sec^2 x$	5. $\int \sec^2 xdx = \tan x + c$
$D_x(-\cot x) = \csc^2 x$	6. $\int \csc^2 xdx = -\cot x + c$
$D_x(\sec x) = \sec x \tan x$	7. $\int \sec x \tan xdx = \sec x + c$
$D_x(-\csc x) = \csc x \cot x$	8. $\int \csc x \cot xdx = -\csc x + c$
$D_x(e^x) = e^x$	9. $\int e^x dx = e^x + c$
$D_x\left(\dfrac{a^x}{\ln a}\right) = a^x$	10. $\int a^x dx = \dfrac{a^x}{\ln a} + c$
$D_x(\ln\lvert x\rvert) = \dfrac{1}{x}$	11. $\int \dfrac{1}{x} dx = \ln\lvert x\rvert + c$

$D_x(\sin^{-1}\frac{x}{a}) = \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} D_x(\frac{x}{a}) = \frac{1}{\sqrt{a^2-x^2}}$	12. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$
$D_x(\cos^{-1}\frac{x}{a}) = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} D_x(\frac{x}{a}) = -\frac{1}{\sqrt{a^2-x^2}}$	13. $\int \frac{1}{\sqrt{a^2-x^2}}dx = -\cos^{-1}\frac{x}{a} + c$
$D_x(\frac{1}{a}\tan^{-1}\frac{x}{a}) = (\frac{1}{a})\frac{1}{1+(\frac{x}{a})^2} D_x(\frac{x}{a}) = \frac{1}{a^2+x^2}$	14. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
$D_x\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2-a^2}}$	15. $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

Table 1.0

I hope the student does not forget how to evaluate the definite integral by using the following fundamental theorem of calculus:

Theorem 1.0 (Fundamental theorem of calculus)
Suppose f is continuous on a closed interval [a,b].
Part I If the function G is defined by

$$G(x) = \int_a^x f(t)dt$$

for every x in [a,b], then G is an antiderivative of f on [a,b].
Part II If F is any antiderivative of f on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 1.0 Evaluate $\int_{-2}^{3}(6x^2-5)dx.$

Solution: An antiderivative of $6x^2 - 5$ is $F(x) = 2x^3 - 5x.$ Then

$$\int_{-2}^{3}(6x^2-5)dx = 2x^3 - 5x\Big|_{-2}^{3}$$

$$= [2(3)^3 - 5(3)] - [2(-2)^3 - 5(-2)] = 45.$$

5.1 Integration by Substitution

The formulas for indefinite integrals in Table (1.0) are limited in scope, because we cannot use them directly to evaluate such as

$$\int \sqrt{2x-5}dx \;\; or \;\; \int \sin 3x dx$$

In this section we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (1.0).

Method of Substitution

If the integral to be evaluated is of the form

$$\int f(g(x))g'(x)dx$$

we substitute $u = g(x)$ and $du = g'(x)dx,$ then the integral becomes $\int f(u)du.$

Example 1 Evaluate $\int \sqrt{2x-5}dx.$

Solution: We let u = 2x – 5 and calculate du:

u = 2x – 5 , du = 2dx

Since du contains the factor 2, the integral is not in the proper form $\int f(u)du$ required in the method of substitution given above. However, we can introduce the factor 2 into the integrand, provided we also multiply by $\frac{1}{2}$. Doing this and property of integral we have

$$\int \sqrt{2x-5}dx = \int \sqrt{2x-5}\,\tfrac{1}{2}2dx$$
$$= \tfrac{1}{2}\int \sqrt{2x-5}\,2dx$$

We now substitute and use the power rule for integration:

$$\int \sqrt{2x-5}dx = \tfrac{1}{2}\int \sqrt{u}du$$
$$= \tfrac{1}{2}\int u^{\frac{1}{2}}du$$
$$= \tfrac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \tfrac{1}{3}u^{\frac{3}{2}} + c$$
$$= \tfrac{1}{3}(2x-5)^{3/2} + c$$

Example 2 Evaluate $\int \sin 2x dx.$

Solution: We make the substitution

$$u = 2x,\ du = 2dx.$$

Since du contains the factor 2, we adjust the integrand by multiplying by 2 and compensate by multiplying the integral by $\frac{1}{2}$ before substituting:

$$\int \sin 2x dx = \tfrac{1}{2}\int (\sin 2x)2dx$$
$$= \tfrac{1}{2}\int \sin u\, du$$
$$= -\tfrac{1}{2}\cos u + c$$
$$= -\tfrac{1}{2}\cos 2x + c$$

It is not always easy to decide what substitution u = g(x) is needed to transform an indefinite integral into a form that can be readily evaluated. It may be necessary to try several different possibilities before finding a suitable substitution. In most cases no substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for changing variables in indefinite integrals

1. Decide on a reasonable substitution $u = g(x)$.
2. Calculate $du = g'(x)dx.$
3. Using 1 and 2, try to transform the integral into a form that involves only the variable u. If necessary, introduce a constant factor k into the integrand and compensate by 1/k. If any part of the resulting integrand contains the variable x, use a different substitution in 1.
4. Evaluate the integral obtained in 3, obtaining an antiderivative involving u.

5. Replace u in the antiderivative obtained in guideline 4 by $g(x)$. The final result should contain only the variable x.

The following examples illustrate the use of the guidelines.

Example 3 Evaluate $\int x^2(3x^3+2)^{10}\,dx$.

Solution: If an integrand involves an expression raised to a power, such as $(3x^3+2)^{10}$, we often substitute u for the expression. Thus, we let

$$u = 3x^3+2, \quad du = 9x^2dx \Leftrightarrow \tfrac{1}{9}du = x^2dx.$$

Comparing $du = 9xdx$ with x^2dx in the integral suggests that we introduce the factor 9 into the integrand. Doing this and compensating by multiplying the integral by 1/9, we obtain the following:

$$\begin{aligned}\int x^2(3x^3+2)^{10}dx &= \int u^{10}\tfrac{1}{9}du\\ &= \tfrac{1}{9}\int u^{10}du\\ &= \frac{1}{9}\left(\frac{u^{11}}{11}\right)+c\\ &= \tfrac{1}{99}(3x^2+2)^{11}+c.\end{aligned}$$

Example 4 Evaluate $\int x\sqrt{3x-1}dx.$

Solution: To simplify the expression $\sqrt{3x-1}$, we let

$$u = 3x-1, \ so\ that \ \ du = 3dx.$$

Then

$$\int x\sqrt{3x-1}dx = \int x\overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du}$$

Thus we still need to find x in terms of u. From the equation u=3x –1 we deduce that

$$x = \frac{1}{3}(u+1).$$

Therefore

$$\begin{aligned}\int x\sqrt{3x-1}dx &= \int \overbrace{x}^{\frac{1}{3}(u+1)}\overbrace{\sqrt{3x-1}}^{\sqrt{u}}\overbrace{dx}^{\frac{1}{3}du} = \int \tfrac{1}{3}(u+1)\sqrt{u}\,\tfrac{1}{3}du\\ &= \tfrac{1}{9}\int\left(u^{3/2}+u^{1/2}\right)du\\ &= \frac{1}{9}\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+c\\ &= \frac{2}{45}(3x-1)^{5/2}+\frac{2}{27}(3x-1)^{3/2}+c.\end{aligned}$$

Sometimes there is more than one substitution that will work. For instance, in Example 4 we could have let $u=\sqrt{3x-1}$, then $u^2 = 3x-1$ *or* $x = \tfrac{1}{3}(u^2+1)$ and

$$2udu = 3dx \ so \ \tfrac{2}{3}udu = dx,$$

As a result,

$$\int x\sqrt{3x-1}dx = \int \overbrace{x}^{\frac{1}{3}(u^2+1)} \overbrace{\sqrt{3x-1}}^{u} \tfrac{2}{3}udu = \int \tfrac{1}{3}(u^2+1)u\tfrac{2}{3}udu$$
$$= \tfrac{2}{9}\int (u^2+1)u^2du$$
$$= \tfrac{2}{9}\int (u^4+u^2)du = \tfrac{2}{9}(\tfrac{1}{5}u^5+\tfrac{1}{3}u^3)+c$$
$$= \frac{2}{45}(3x-1)^{5/2} + \frac{2}{27}(3x-1)^{3/2}+c.$$

Even though we used a different substitution, the final answer remains the same. **Example 5** Evaluate $\int xe^{x^2}dx$

Solution: We let

$$u = x^2, \text{ so that } du = 2xdx.$$

Then

$$\int xe^{x^2}dx = \int \overbrace{e^{x^2}}^{e^u} \overbrace{xdx}^{\frac{1}{2}du} = \int e^u \tfrac{1}{2}du$$
$$= \frac{1}{2}e^u + c$$
$$= \frac{1}{2}e^{x^2} + c.$$

Example 6 Evaluate $\int \frac{1}{x}(1+\ln x)^4 dx.$

Solution: We let

$$u = 1+\ln x, \text{ so than } du = \frac{1}{x}dx.$$

Then

$$\int \frac{1}{x}(1+\ln x)^4 dx = \int u^4 du = \frac{1}{5}(1+\ln x)^5 + c.$$

Example 7 Evaluate $\int \tan xdx.$

Solution: First write the integral in the following form

$$\int \tan xdx = \int \frac{\sin x}{\cos x}dx.$$

Now let $u = \cos x, \text{ so that } du = -\sin xdx.$

Then $\int \tan xdx = \int \frac{\sin x}{\cos x}dx = \int \frac{1}{u}(-du) = -\ln|u|+c = -\ln|\cos x|+c.$

Example 8 Evaluate $\int \sec xdx.$

Solution: We first put the integral in the form

$$\int \sec xdx = \int \sec x\frac{\sec x+\tan x}{\sec x+\tan x}dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx$$

If we now let $u = \sec x+\tan x, \text{ so that } du = (\sec x\tan x+\sec^2 x)dx,$ then

$$\int \sec x dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
$$= \int \frac{1}{u} du = \ln|u| + c$$
$$= \ln|\sec x + \tan x| + c.$$

Example 9 Evaluate $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx.$

Solution: The integral may be written as in the first formula 12 of table 1.0 by letting a=1 and using the substitution

$$u = e^{2x}, \qquad du = 2e^{2x} dx.$$

Then

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{2}\sin^{-1} u + c = \frac{1}{2}\sin^{-1} e^{2x} + c.$$

Example 10 Evaluate $\int_0^{\pi/2} \sin x \cos^4 x dx$

Solution: Let $u = \cos x$, *then* $du = -\sin x dx$ hence

$$\int_0^{\pi/2} \sin x \cos^4 x dx = \int u^4 (-du) = -\frac{1}{5}u^5 = -\frac{1}{5}\cos^5 x \Big|_0^{\frac{\pi}{2}}$$
$$= -\frac{1}{5}\cos^5 \frac{\pi}{2} + \frac{1}{5}\cos^5 0 = \frac{1}{5}.$$

Exercise 1.2 Evaluate the following integrals.

1. $\int \sin^2 x dx$
2. $\int \csc x dx$
3. $\int \frac{x}{(x^2+5)^3} dx$
4. $\int \frac{x}{\sqrt[3]{1-2x^2}} dx$
5. $\int x^5 \sqrt{x^2-1} dx$
6. $\int_{-1}^{2} \frac{t^2}{\sqrt{t+2}} dt$
7. $\int_1^2 \frac{e^{3/x}}{x^2} dx$
8. $\int \frac{1}{x(\ln x)^2} dx$
9. $\int \frac{3\sin x}{1+2\cos x} dx$
10. $\int_0^{\sqrt{2}/2} \frac{x}{\sqrt{1-x^4}} dx$

5.2 Integration by parts

If we try to evaluate integrals of the type

$$\int xe^x dx, \text{ and } \int \ln x dx$$

by using the method of substitution we obviously fail. But don’t worry the next formula will enable us to evaluate not only these, but also many other types of integrals.

Integration by parts formula

If $u = f(x)$ *and* $v = g(x)$ and if f' *and* g' are continuous, then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \text{ or using } u \text{ and } v$$

$$\int udv = uv - \int vdu.$$

Proof By the product rule,

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

or equivalently, $f(x)g'(x) = [f(x)g(x)]' - g(x)f'(x).$

Integrating both sides of the last equation gives us

$$\int f(x)g'(x)dx = \int [f(x)g(x)]'dx - \int g(x)f'(x)dx.$$

The first integral on the right side equals f(x)g(x)+c. Since another constant of integration is obtained from the second integral, we may omit c in the formula; that is

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \qquad (1)$$

Since $dv = g'(x)dx$ and $du = f'(x)dx$, we may write the preceding formula as

$$\int udv = uv - \int vdu.$$

Since applying (1) involves splitting the integrand into two parts, the use of (1) is referred to as **integrating by parts**. A proper choice for *dv* is crucial. We usually let *dv* equal the most complicated part of the integrand that can be readily integrated. The following examples illustrate this method of integration.

Example 1 Evaluate $\int xe^x dx.$

Solution: The integrand xe^x can be split into two parts x and e^x. We let

$$u = x \quad and \quad dv = e^x dx$$

Then $$du = x \quad and \quad v = \int e^x dx = e^x$$

Consequently integration by parts yields

$$\int \overbrace{x}^{u}\overbrace{e^x dx}^{dv} = \overbrace{x}^{u}\overbrace{e^x}^{v} - \int \overbrace{e^x}^{v}\overbrace{dx}^{du} = xe^x - e^x + c.$$

Example 2 Evaluate

a) $\int x\sec^2 xdx$ b) $\int_0^{\pi/3} x\sec^2 xdx$

Solution: a) We let here

$$u = x \quad and \quad dv = \sec^2 xdx$$

then $$du = dx \quad and \quad v = \tan x.$$

Hence integration by parts yields

$$\int x\sec^2 xdx = x\tan x - \int \tan xdx = x\tan x - (-\ln|\cos x|) + c$$

$$= x\tan x + \ln|\cos x| + c.$$

b) The indefinite integral obtained in part (a) is an antiderivative of $x\sec^2 x$. Using the fundamental theorem of calculus (and dropping the constant of integration c), we obtain

$$\int_0^{\pi/3} x\sec^2 x dx = \left[x\tan x + \ln|\cos x|\right]_0^{\pi/3}$$

$$= \left(\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\cos\frac{\pi}{3}\right|\right) - (0 + \ln 1)$$

$$= \left(\frac{\pi}{3}\sqrt{3} + \ln\frac{1}{2}\right) - (0 + 0)$$

$$= \frac{\pi}{3}\sqrt{3} - \ln 2.$$

Example 3 Evaluate $\int \ln x dx.$

Solution: Let $u = \ln x$ *and* $dv = dx$

Then $du = \frac{1}{x}dx$ *and* $v = x$

and integrating by parts yields:

$$\int \ln x dx = x\ln x - \int x(\frac{1}{x}dx) = x\ln x - \int dx = x\ln x - x + c.$$

Sometimes it is necessary to usse integration by parts more than once in the same problem. This is illustrated in the next example.

Example 4 Evaluate $\int_0^{\pi/2} x^2 \sin 2x dx.$

Solution: Let

$u = x^2$ *and* $dv = \sin 2x dx$

Then $du = 2x dx$ *and* $v = -\frac{1}{2}\cos 2x$.

Thus using integration by parts we have;

$$\int_0^{\pi/2} x^2 \sin 2x dx = \left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} - \int_0^{\pi/2} 2x\left(-\frac{1}{2}\cos 2x\right)dx$$

$$= \left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} + \int_0^{\pi/2} x\cos 2x dx$$

but then since

$$\left[-\frac{1}{2}x^2\cos 2x\right]_0^{\pi/2} = -\frac{1}{2}\left(\frac{\pi}{2}\right)^2 \cos 2(\tfrac{\pi}{2}) - 0 = \frac{\pi^2}{8}$$

and

$$\int_0^{\pi/2} x\cos 2x dx = \left[x\frac{\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin 2x}{2}dx$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\sin 2(\frac{\pi}{2}) - 0\right] - \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\cos 2(\frac{\pi}{2}) - \cos 0\right] = \frac{1}{4}[-1-1] = -\frac{1}{2}$$

Hence,

$$\int_0^{\pi/2} x^2 \sin 2x dx = \frac{\pi^2}{8} - \frac{1}{2}.$$

The following example illustrates another device for evaluating an integral by means of two applications of the integration by parts formula.

Example 5 Evaluate $\int e^x \cos x dx.$

Solution: We could either let $dv = \cos x dx$ or let $dv = e^x dx,$ since each of these expression is readily integrable. Let us choose

$$u = e^x \quad and \quad dv = \cos x dx$$

so that $\quad du = e^x dx \quad and \quad v = \sin x$

Then by integrating by parts we have;

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \qquad (1)$$

We next apply integration by parts to the integral of the right side of equation (1). Since we chose a trigonometric form for *dv* in the first integration by parts, we shall also choose a trigonometric form for the second. Letting

$$u = e^x \quad and \quad dv = \sin x dx \quad so\ that$$

$$du = e^x dx \quad and \quad v = -\cos x$$

integrating by parts, we have

$$\int e^x \sin x dx = e^x(-\cos x) - \int (-\cos x)e^x dx$$

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx. \qquad (2)$$

If we now use equation (2) to substitute on the right side of equation (1), we obtain

$$\int e^x \cos x dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x dx\right]$$

or $\quad \int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx\,.$

Adding $\int e^x \cos x dx$ to both sides of the last equation gives us

$$2\int e^x \cos x dx = e^x(\sin x + \cos x).$$

Finally, dividing both sides by 2 and adding the constant of integration yiels

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + c.$$

We could have evaluated the given integral by using $dv = e^x dx$ for both the first and second applications of the integration by parts formula.

In conclusion we remark that integration by parts is effective with integrals involving a polynomial and either an exponential, a logarithmic, or a trigonometric function. More specifically, integration by parts is especially well adapted to integrals of the form

$$\int (polynomial)\sin ax dx, \qquad \int (polynomial)\cos ax dx,$$

$$\int (polynomial)e^{ax} dx, \qquad \int (polynomial)\ln x dx.$$

In all except $\int (polynomial)\ln x dx$, the most effective choice of u is the polynomial, since the derivatives of a polynomial are simpler than the polynomial itself, while the choice $u = \ln x$ is effective for $\int (polynomial)\ln x dx$.

Example 6 Evaluate $\int \sin^{-1} x dx.$

Solution: Let

$u = \sin^{-1} x$ *and* $dv = dx$ so that $du = \dfrac{1}{\sqrt{1-x^2}}dx$ *and* $v = x$.

Then

$$\int \sin^{-1} x dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

Now we use substitution to solve the integral to the right. That is let

$$w = \sqrt{1-x^2} \text{ or } w^2 = 1-x^2 \text{ so that } 2wdw = -2xdx$$

we then have

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{wdw}{w} = -\int dw = -w + c = -\sqrt{1-x^2} + c$$

Consequently

$$\int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

Integration by parts may sometimes be employed to obtain **reduction formulas** for integrals. We now find reduction formulas of $\int \sin^n x dx$ *and* $\int \cos^n x dx$ with the help of integration by parts.

Example 7 Find a reduction formula for $\int \sin^n x dx.$

Solution: First write $\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$ and let

$u = \sin^{n-1} x$ *and* $dv = \sin x dx$ so that

$$du = (n-1)\sin^{n-2} x\cos x dx \text{ and } v = -\cos x$$

then using integration by parts we have:

$$\int \sin^n x dx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x dx$$

since $\cos^2 x = 1-\sin^2 x,$ we may write

$$\int \sin^n x dx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx.$$

Consequently,

$$\int \sin^n x dx + (n-1)\int \sin^n x dx. = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx.$$

The left side of the last equation reduces to $n\int \sin^n x dx$. Dividing both sides by n, we obtain

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

In a similar fashion we can show the reduction formula for $\int \cos^n x dx$ is given by:

$$\int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx.$$

Example 8 Evaluate $\int \sin^5 x dx.$

Solution: Using the reduction formula for sine with n = 5 gives us

$$\int \sin^5 x dx = -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x dx$$

A second application of the reduction formula, to $\int \sin^5 x dx$, yields

$$\int \sin^3 x dx = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x dx$$
$$= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C_1$$

Consequently

$$\int \sin^5 x dx = -\frac{1}{5}\cos x \sin^4 x + \frac{4}{5}\left(-\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C_1\right)$$
$$= -\frac{1}{5}\cos x \sin^4 x - \frac{4}{15}\cos x \sin^2 x - \frac{8}{15}\cos x + C.$$

Exercise 1.2

Evaluate the integral

1. $\int xe^{-x} dx$
2. $\int x \ln x dx$
3. $\int \sec^3 x dx$
4. $\int x2^x dx$
5. $\int x \tan x \sec x dx$
6. $\int_0^{\pi/2} 2t \sin 2t dt$
7. $\int (x+1)^{10}(x+2) dx$
8. $\int \sin(\ln x) dx$ (Hint: Let $u = \sin(\ln x)$)
9. $\int \tan^{-1} x dx$
10. $\int \cos\sqrt{x} dx$

Evaluate the integral with the help of the reduction formulas

11. $\int_0^{\pi/2} \cos^3 \frac{x}{2} dx$
12. $\int \cos^5 x dx$

5.3 Integration by Partial Fractions

An expression for rational function is called a ***proper fraction*** if the degree of the numerator is strictly less than the degree of the denominator; otherwise it is called an ***improper fraction***. In case of improper fraction we actually divide the numerator by the denominator and the improper fraction is expressed in terms of a polynomial and a proper fraction. For example,

$$\frac{2x+1}{x-3} = 2 + \frac{7}{x-3} \quad \text{and} \quad \frac{4x^3 - 3x^2 + 2x - 1}{x^2 + 9} = 4x - 3 - \frac{34x - 26}{x^2 + 9}$$

Let us consider a proper fraction $\dfrac{P(x)}{Q(x)}$ where P and Q are polynomials in *x*, then it can be proved that

$$\frac{P(x)}{Q(x)} = F_1 + F_2 + \cdots + F_r$$

Such that each term F_k of the sum has one of the forms

$$\frac{A}{(ax+b)^n} \quad or \quad \frac{Ax+B}{(ax^2+bx+c)^n}$$

for real numbers A and B and a nonnegative integer n, where $ax^2 + bx + c$ is **irreducible** in the sense that this quadratic polynomial has no real zeros (that is, $b^2 - 4ac < 0$). In this case, $ax^2 + bx + c$ cannot be expressed as a product of two first-degree polynomials with real coefficients.

The sum $F_1 + F_2 + \cdots + F_r$ is the partial **fraction decomposition** of P(*x*)/Q(*x*), and each F_k is a **partial fraction**. We state guidelines for obtaining this decomposition.

Guidelines for partial fraction decompositions of P(*x*)/Q(*x*)

1. If the degree of *P*(*x*) is not lower than the degree of Q(*x*), use long division to obtain the proper form.
2. Express Q(x) as a product of linear factors a*x* + b or irreducible quadratic factors $ax^2 + bx + c$, and collect repeated factors so that Q(*x*) is a product of different factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ for a nonnegative integer n.
3. Apply the following rules.

   Rule a For each factor $(ax+b)^n$ with $n \geq 1$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

where each numerator A_k is a real number.

Rule b For each factor $(ax^2+bx+c)^n$ with $n \geq 1$, and with ax^2+bx+c irreducible, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2+bx+c)^n},$$

where each A_k and B_k is a real number.

Example 1 Evaluate $\displaystyle\int \frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} dx.$

Solution: We may factor the denominator of the integrand as follows:

$$x^3 + 2x^2 - 3x = x(x^2 + 2x + 3) = x(x+3)(x-1)$$

Each factor has the form stated in Rule (a) of the guideline, with n = 1. Therefore the partial fraction decomposition has the form

$$\frac{4x^2 + 13x - 9}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}.$$

Multiplying by the LCM of the denominators gives us

$$4x^2 + 13x - 9 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3). \qquad (*)$$

In a case such as this, in which the factors are linear and nonrepeated, the values of A, B and C can be found by substituting values for *x* that make the various factors zero. If we let x = 0 in (*), then

$$-9 = -3A, \quad or \quad A = 3.$$

Letting x = 1 in (*) gives us

$$8 = 4C, \quad or \quad C = 2.$$

Finally, if x = -3 in (*), we have

$$-12 = 12B, \quad or \quad B = -1.$$

The partial fraction decomposition is, therefore,

$$\frac{4x^2 + 13x - 9}{x^3 + 2x^2 - 3x} = \frac{3}{x} + \frac{-1}{x+3} + \frac{2}{x-1}.$$

Integrating and letting C denote the sum of the constants of integration we have

$$\begin{aligned}\int \frac{4x^2 + 13x - 9}{x(x+3)(x-1)}dx &= \int \frac{3}{x}dx + \int \frac{-1}{x+3}dx + \int \frac{2}{x-1}dx.\\ &= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C\\ &= \ln|x^3| - \ln|x+3| + \ln|x-1|^2 + C\\ &= \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + C.\end{aligned}$$

Another technique for finding A, B, and C is to expand the right-hand side of (*) and collect like powers of x as follows:

$$4x^2 + 13x - 9 = (A + B + C)x^2 + (2A - B + 3C)x - 3A$$

We now use the fact that if two polynomials are equal, then coefficients of like powers of *x* are the same. It is convenient to arrange our work in the following way, which we call **comparing coefficients of *x***.

Coefficients of x^2:	$A + B + C = 4$
Coefficients of x:	$2A - B + 3C = 13$
Constant terms:	$-3A = -9$

We may show the solution of this system of equations is A = 3, B = -1, and C = 2.

Example 2 Evaluate $\int \frac{13 - 7x}{(x+2)(x-1)^3}dx$

Solution: By Rule (a) of the Guidelines the partial fraction of the integrand has the form

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+2}$$

Multiplying both sides by $(x+2)(x-1)^3$ gives us

$$13-7x = A(x-1)^2(x+2)+B(x-1)(x+2)+C(x+2)+D(x-1)^3 \qquad (*)$$

Two of the unknown constants may be determined easily as follows.
Let x = 1 in ($*$) the we obtain $13-7=C(1+2)$ or C=2.

Similarly, letting x = -2 in ($*$) yields $13+14=D(-2-1)^3$ *or* $D=-1$.

The remaining constants may be found by comparing coefficients. So comparing the coefficients of x^3 on both sides of ($*$), gives

0=A+D or A = - D = 1.

And comparing the constant terms on both sides of ($*$), gives

$$13 = 2A-2B+2C-D \quad or \quad B=\frac{1}{2}(2+4+1-13)=-3.$$

Therefore

$$\frac{13-7x}{(x+2)(x-1)^3}=\frac{1}{x-1}+\frac{-3}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{-1}{x+2}.$$

Thus

$$\int\frac{13-7x}{(x+2)(x-1)^3}dx=\int\frac{1}{x-1}dx-\int\frac{3}{(x-1)^2}dx+\int\frac{2}{(x-1)^3}dx-\int\frac{1}{x+2}dx$$

$$=\ln|x-1|+\frac{3}{x-1}-\frac{1}{(x-1)^2}-\ln|x+2|+C$$

$$=\ln\left|\frac{x-1}{x+2}\right|+\frac{3}{x-1}-\frac{1}{(x-1)^2}+C.$$

Example 3 Evaluate $\int\frac{x^2+2x+7}{x^3+x^2-2}dx.$

Solution: The denominator of the integrand may be factored as follows:

$$x^3+x^2-2=(x-1)(x^2+2x+2)$$

Applying Rule (b) of the Guidelines to the irreducible quadratic factor x^2+2x+2 we have

$$\frac{x^2+2x+7}{x^3+x^2-2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+2}$$

This leads to

$$x^2+2x+7=A(x^2+2x+2)+(Bx+C)(x-1) \qquad (*)$$

As in previous examples, substituting $x=1$ in ($*$) gives us

$10=A(5)$ or $A=2$

The remaining constants may be found by combining like powers of x:

$$x^2+2x+7=(2+B)x^2+(4+C-B)x+(4-C) \qquad (**)$$

and comparing coefficients in ($**$).

Coefficients of x^2: $1=2+B$ or B = -1

Constant terms: 7=4-C or C = -3

Thus the partial fraction decomposition of the integrand is

$$\frac{x^2+2x+7}{x^3+x^2-2}=\frac{2}{x-1}+\frac{-x-3}{x^2+2x+2}.$$

Consequently

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx-\int\frac{x+3}{x^2+2x+2}dx$$

To evaluate the right-hand integral, we first complete the square in the denominator to obtain $x^2+2x+2=(x+1)^2+1$

and substitute $u=x+1$, so that du = dx and x +3 = u + 2

Therefore

$$\int\frac{x+3}{x^2+2x+2}dx=\int\frac{x+3}{(x+1)^2+1}dx=\int\frac{u+2}{u^2+1}du$$

$$=\int\frac{u}{u^2+1}du+2\int\frac{1}{u^2+1}du$$

$$=\frac{1}{2}\int\frac{2u}{u^2+1}du+2\int\frac{1}{u^2+1}du$$

$$=\frac{1}{2}\ln(u^2+1)+2\arctan u+C$$

$$=\frac{1}{2}\ln((x+1)^2+1)+2\arctan(x+1)+C.$$

Hence

$$\int\frac{x^2+2x+7}{x^3+x^2-2}dx=\int\frac{2}{x-1}dx-\int\frac{x+3}{x^2+2x+2}dx$$

$$=2\ln|x-1|-\tfrac{1}{2}\ln((x+1)^2+1)-2\arctan(x+1)+C$$

Example 4 Evaluate $\int\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx.$

Solution: Applying Rule b) of the Guidelines, with n = 2, yields

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}.$$

Multiplying by both sides of the equation by $(x^2+1)^2$ gives

$$5x^3-3x^2+7x-3=(Ax+B)(x^2+1)+Cx+D$$

$$5x^3-3x^2+7x-3=Ax^3+Bx^2+(A+C)x+(B+D)$$

We next compare coefficients as follows:

coefficients of x^3: 5 = A

coefficients of x^2: -3 = B

coefficients of x : 7 = A + C or C = 2

constant terms : -3 = B + D or D = 0

Therefore

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}=\frac{5x-3}{x^2+1}+\frac{2x}{(x^2+1)^2}$$

so that

$$\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx = \int \frac{5x}{x^2+1}dx - \int \frac{3}{x^2+1}dx + \int \frac{2x}{(x^2+1)^2}dx$$

$$= \frac{5}{2}\ln(x^2+1) - 3\arctan(x^2+1) - \frac{1}{x^2+1} + C.$$

Example 5 Evaluate $\int \frac{dx}{\sin x(2+\cos^2 x)}$.

Solution: Since

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin x^2(2+\cos^2 x)}$$

substituting

$u = \cos x$ *and* $du = -\sin x dx,$ we get

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \int \frac{\sin x dx}{\sin x^2(2+\cos^2 x)} = \int \frac{-du}{(1-u^2)(2+u^2)} = \int \frac{du}{(u^2-1)(u^2+2)}$$

But then the partial fraction representation for the integrand of the last integral has the form

$$\frac{1}{(u^2-1)(u^2+2)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+2}$$

Then by similar procedure as the above examples we have

$$1 = A(u+1)(u^2+2) + B(u-1)(u^2+2) + (Cu+D)(u-1)(u+1) \qquad (*)$$

Then putting u=1 gives us 1=6A or A=1/6

Putting u=-1 gives us 1=-6B or B=-1/6

We now compare coefficients to find the remaining two constants

Coefficients of x^3: 0=A+B+C or C=0

Constant terms: 1=2A- 2B-D or D=-1/3 Therefore

$$\frac{1}{(u^2-1)(u^2+2)} = \frac{1/6}{u-1} + \frac{-1/6}{u+1} + \frac{-1/3}{u^2+2}$$

so that

$$\int \frac{du}{(u^2-1)(u^2+2)} = \frac{1}{6}\int \frac{1}{u-1}du - \frac{1}{6}\int \frac{1}{u+1}du + \frac{1}{3}\int \frac{1}{u^2+2}du$$

$$= \frac{1}{6}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C$$

Consequently resubstituting cos*x* for *u* we have

$$\int \frac{dx}{\sin x(2+\cos^2 x)} = \frac{1}{6}\ln\left|\frac{\cos x-1}{\cos x+1}\right| - \frac{1}{3\sqrt{2}}\arctan\frac{\cos x}{\sqrt{2}} + C.$$

Exercise 1.3

1. $\int \frac{x^2}{x^2-1}dx$
2. $\int \frac{2x^2-12x+4}{x^3-4x^2}dx$
3. $\int_{-1}^{0} \frac{x^2+x+1}{x^2+1}dx$
4. $\int \frac{-x^3+x^2+x+3}{(x+1)(x^2+1)^2}dx$

5. $\int \frac{x^2-1}{x^3+3x+4}dx$

6. $\int \frac{\sqrt{x}+1}{x+1}dx$; ($H$ int : *Substitute* $u=\sqrt{x}$)

7. $\int \frac{\sin^2 x\cos x}{\sin^2 x+1}dx$

8. $\int_0^{\pi/4} \tan^3 xdx$; ($H$ int : *Substitute* $u=\tan x$)

9. $\int \frac{e^x}{1-e^{3x}}dx$

10. $\int \frac{dx}{1+3e^x+2e^{2x}}$

5.4 Trigonometric Integrals

Integrals such as

$$\int \sin^5 x\cos^3 xdx, \qquad \int \tan^2 x\sec^3 xdx, \qquad and \quad \int \sin 3x\cos 4xdx$$

are called **trigonometric integrals** because their integrands are combinations of trigonometric functions. This section is devoted to trigonometric integrals especially those in which the integrands are composed of the basic trigonometric functions.

Guidelines for evaluating integrals of the form $\int \sin^m x\cos^n xdx$

1. **If m is an odd integer**: Write the integrals as $\int \sin^m x\cos^n xdx = \int \sin^{m-1} x\cos^n x\sin xdx$ and express $\sin^{m-1} x$ in terms of $\cos x$ by using the trigonometric identity $\sin^2 x = 1-\cos^2 x$. Make the substitution

$$u=\cos x, \quad du=-\sin xdx$$

and evaluate the resulting integral.

2. **If n is an odd integer**: write the integral as

$$\int \sin^m x\cos^n xdx = \int \sin^m x\cos^{n-1} x\cos xdx$$

and express $\cos^{n-1} x$ in terms of $\sin x$ by using the trigonometric identity $\cos^2 x = 1-\sin^2 x$. Make the substitution

$$u=\sin x, \qquad du=\cos xdx$$

and evaluate the resulting integral.

3. **If m and n are even**: Use half-angle formulas for $\sin^2 x = \frac{1-\cos 2x}{2}$ *and* $\cos^2 x = \frac{1+\cos 2x}{2}$ and the identity

$$\sin x\cos x = \frac{1}{2}\sin 2x$$

to reduce the exponents by one-half.

Example 1 Evaluate $\int \sin^3 x\cos^2 xdx.$

Solution: By guideline 1

$$\int \sin^3 x\cos^2 xdx = \int \sin^2 x\cos^2 x\sin xdx$$
$$= \int (1-\cos^2 x)\cos^2 x\sin xdx.$$

If we let $u=\cos x$, then $du=-\sin xdx$, and the integral may be written

$$\int \sin^3 x \cos^2 x dx = \int (1-u^2)u^2(-du) = \int (u^4 - u^2)du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C$$
$$= \tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C.$$

Example 2 Evaluate $\int \sin^2 x \cos^4 x dx.$

Solution: By guideline 3 we have

$$\int \sin^2 x \cos^4 x dx = \int (\sin^2 x \cos^2 x)\cos^2 x dx$$
$$= \int (\sin x \cos x)^2 \cos^2 x dx$$
$$= \int \left(\frac{1}{2}\sin 2x\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$
$$= \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cos 2x dx$$

Putting $\sin^2 2x = \dfrac{1-\cos 4x}{2}$ and $u = \sin 2x$ *so that* $du = 2\cos 2x dx$ in the first and second integrals of the right of the last equation we get:

$$= \frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{8}\int \frac{1}{2}u^2 du$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}u^3 + C$$
$$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

An alternative way to evaluate $\int \sin^m x \cos^n x dx$ when m and n are even is to use the identity $\sin^2 x + \cos^2 x = 1,$ but this time we transform the integral into integrals of the form $\int \sin^k x dx$ or of the form $\int \cos^k x dx$, which can be evaluated by the reduction formulas.

Guidelines for evaluating integrals of the form $\int \tan^m x \sec^n x dx$

1. **If m is an odd integer**: Write the integrals as $\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x dx$ and express $\tan^{m-1} x$ in terms of $\sec x$ by using the trigonometric identity $\tan^2 x = \sec^2 x - 1.$ Make the substitution

$$u = \sec x, \quad du = \sec x \tan x dx$$

and evaluate the resulting integral.

2. **If n is an even integer**: write the integral as

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x \sec^2 x dx$$

and express $\sec^{n-2} x$ in terms of $\tan x$ by using the trigonometric identity $\sec^2 x = 1 + \tan^2 x.$ Make the substitution

$$u = \tan x, \qquad du = \sec^2 x dx$$

and evaluate the resulting integral.

3. **If m is even and n is odd**: Reduce to powers of $\sec x$ alone by using the identity $\tan^2 x = \sec^2 x - 1$.

Example 3 Evaluate $\int \tan^3 x \sec^5 x dx.$

Solution: By guideline 1 above

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x(\sec x \tan x)dx$$
$$= \int (\sec^2 x - 1)\sec^4 x(\sec x \tan x)dx.$$

Substituting $u = \sec x$ **and** $du = \sec x \tan x dx,$ **we obtain**

$$\int \tan^3 x \sec^5 x dx = \int (u^2 - 1)u^4 du$$
$$= \int (u^6 - u^4)du.$$
$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C$$
$$= \tfrac{1}{7}\sec^7 x - \tfrac{1}{5}\sec^5 x + C$$

Example 4 Evaluate $\int \tan^3 x \sec^4 x dx.$

Solution: By guideline 2 above

$$\int \tan^3 x \sec^4 x dx = \int \tan^3 x \sec^2 x \sec^2 x dx$$
$$= \int \tan^3 x(1 + \tan^2 x)\sec^2 x dx$$

If we let $u = \tan x,$ *then* $du = \sec^2 x dx$, and

$$\int \tan^3 x \sec^4 x dx = \int u^3(1 + u^2)du$$
$$= \int (u^5 + u^3)du$$
$$= \tfrac{1}{6}u^6 + \tfrac{1}{4}u^4 + C$$
$$= \tfrac{1}{6}\tan^6 x + \tfrac{1}{4}\tan^4 x + C.$$

Integrals of the form $\int \cot^m x \csc^n x dx$ may be evaluated in similar fashion.

Finally, the evaluation of integrals of the form $\int \sin ax \cos bx dx$ depends on the trigonometric identity

$$\sin x \cos y = \frac{1}{2}\sin(x - y) + \frac{1}{2}\sin(x + y)$$

With the appropriate replacements, this identity becomes

$$\sin ax \cos bx = \frac{1}{2}\sin(a - b)x + \frac{1}{2}\sin(a + b)x \qquad (*)$$

Notice that $\tfrac{1}{2}\sin(a - b)x$ *and* $\tfrac{1}{2}\sin(a + b)x$ are easy to integrate by substitution.

Example 5 Evaluate $\int \sin 4x \cos 2x dx.$

Solution: Using (*) with $a = 4$ and b = 2, we find that

$$\int \sin 4x \cos 2x dx = \int \left(\frac{1}{2} \sin 2x + \frac{1}{2} \sin 6x \right) dx$$
$$= -\frac{1}{4} \cos 2x - \frac{1}{12} \sin 6x + C.$$

Note that integrals of the form

$$\int \sin ax \sin bx dx \qquad and \qquad \int \cos ax \cos bx dx$$

can be found by similar techniques.

Exercise 1.4 Evaluate the following integrals.

1. $\int \sin^3 x \cos^4 x dx$
2. $\int_0^{\pi/2} \sin^2 x \cos^5 x dx$
3. $\int \sqrt{\sin x} \cos^3 x dx$
4. $\int (\tan x + \cot x)^2 dx$
5. $\int \tan^3 x \csc^4 x dx$
6. $\int \cot^3 x \csc^3 x dx$
7. $\int \sin 5x \sin 3x dx$
8. $\int_0^{\pi/4} \cos x \cos 5x dx$
9. $\int_0^{\pi/3} \tan x \sec^{3/2} x dx$
10. $\int \tan^6 x dx$

5.5 Trigonometric Substitutions

Observe that the trigonometric substitution $x = a \sin \theta$ simplifies the expression $\sqrt{a^2 - x^2}$, with a>0, into a trigonometric expression without radical i.e

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a\sqrt{1 - \sin^2 \theta} = a \cos \theta .$$

We can use a similar procedure for $\sqrt{a^2 + x^2}$,and $\sqrt{x^2 - a^2}$. This technique is useful for eliminating radicals from these types of integrands. The substitutions are listed in the table 1.1.

When making a trigonometric substitution we shall assume that θ is in the range of the corresponding inverse trigonometric function. Thus, for the substitution $x = a \sin \theta$, we have $-\pi/2 \le \theta \le \pi/2$, In this case, $\cos \theta \ge 0$.

Trigonometric Substitutions

Expressions in integrand	Trigonometric substitution	Interval(s)
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$-\pi/2 \le \theta \le \pi/2,$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$-\pi/2 < \theta < \pi/2,$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$0 \le \theta < \pi/2, or\ \pi \le \theta < \frac{3\pi}{2}$

Table 1.1

Example 1 Evaluate $\int \frac{1}{x^2 \sqrt{16 - x^2}} dx.$

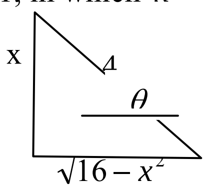
Solution: Since $\sqrt{16 - x^2} = \sqrt{4^2 - x^2}$, we substitute

$x = 4\sin\theta$, so that $dx = 4\cos\theta\, d\theta$ for $-\pi/2 < \theta < \pi/2$.

Then

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \int \frac{1}{16\sin^2\theta 4\sqrt{1-\sin^2\theta}}(4\cos\theta)d\theta$$
$$= \frac{1}{16}\int \frac{1}{\sin^2\theta}d\theta = \frac{1}{16}\int \csc^2\theta d\theta$$
$$= -\frac{1}{16}\cot\theta + C.$$

In order to write the answer in terms of the original variable *x*, we draw the triangle as fig1.1, in which $x = 4\sin\theta$.



$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Fig 1.1

Thus

$$\int \frac{1}{x^2\sqrt{16-x^2}}dx = -\frac{1}{16}\cot\theta + C = -\frac{\sqrt{16-x^2}}{16x} + C.$$

Example 2 Evaluate $\int_{-5/2}^{5/2} \sqrt{25-4x^2}\,dx$

Solution: Because $\sqrt{25-4x^2} = \sqrt{5^2-(2x)^2}$, we are led to substitute

$$2x = 5\sin\theta, \text{ so that } x = \frac{5}{2}\sin\theta, \text{ and thus } dx = \frac{5}{2}\cos\theta d\theta$$

For the limits of integration we notice that

if $x = -\frac{5}{2}$ *then* $\theta = -\frac{\pi}{2},$ *and if* $x = \frac{5}{2}$ *then* $\theta = \frac{\pi}{2}$.

Therefore

$$\int_{-5/2}^{5/2} \sqrt{25-4x^2}\,dx = \int_{-5/2}^{5/2} \sqrt{5^2-(2x)^2}\,dx$$
$$= \int_{-\pi/2}^{\pi/2} \sqrt{5^2-5^2\sin^2\theta}\left(\frac{5}{2}\cos\theta\right)d\theta$$
$$= \frac{25}{2}\int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\theta}(\cos\theta)d\theta$$
$$= \frac{25}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta$$
$$= \frac{25}{2}\left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Bigg|_{-\pi/2}^{\pi/2}$$
$$= \frac{25}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right) = \frac{25}{4}\pi$$

Example 3 Evaluate $\int \frac{1}{x^2\sqrt{16+x^2}}dx.$

Solution: The denominator of the integrand has an expression of the form $\sqrt{a^2+x^2}$ with $a=4$. Hence, using table 1.1, we make the substitution

$$x = 4\tan\theta,\ dx = 4\sec^2\theta d\theta.$$

Consequently

$$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = 4\sqrt{1+\tan^2\theta} = 4\sqrt{\sec^2\theta} = 4\sec\theta$$

and
$$\int \frac{1}{x^2\sqrt{16+x^2}}dx = \int \frac{1}{16\tan^2\theta(4\sec\theta)}4\sec^2\theta d\theta$$
$$= \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$
$$= -\frac{1}{16\sin\theta}$$

To give the answer in terms of *x*, we use the triangle in Fig 1.2, with $x = 4\tan\theta$. We then find that

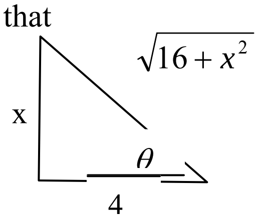


Fig 1.1

$$\sin\theta = \frac{x}{\sqrt{16+x^2}} \quad and$$
$$\int \frac{1}{x^2\sqrt{16+x^2}}dx = -\frac{1}{16\sin\theta} = \frac{\sqrt{16+x^2}}{16x}+C.$$

Example 4 Evaluate $\int_{-6}^{-3} \frac{\sqrt{x^2-9}}{x}dx.$

Solution: The domain of the integrand consists of $(-\infty,-3]$ and $[3,\infty)$, but since the interval over which we must integrate is [-6,-3], we seek an antiderivative whose domain is contained in $(-\infty,-3]$. Since $\sqrt{x^2-9}=\sqrt{x^2-3^2}$, we let

$$x = 3\sec\theta,\ \textit{so that}\ dx = 3\sec\theta\tan\theta d\theta$$

and notice that $\sqrt{x^2-9}=\sqrt{9\sec^2\theta-9}=3\tan\theta.$ For the limits of integration we observe that

$$\textit{if}\ \ x=-6\ \textit{then}\ \theta=\sec^{-1}(-2)=\frac{4\pi}{3},\ \textit{and}\ \ \textit{if}\ \ x=-3\ \ \textit{then}\ \ \theta=\pi.$$

Therefore

$$\int_{-6}^{-3}\frac{\sqrt{x^2-9}}{x}dx=\int_{4\pi/3}^{\pi}\frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}(3\sec\theta\tan\theta)d\theta$$

$$=\int_{4\pi/3}^{\pi}\frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)d\theta=3\int_{4\pi/3}^{\pi}\tan^2\theta d\theta$$

$$=3\int_{4\pi/3}^{\pi}(\sec^2\theta-1)d\theta=3(\tan\theta-\theta)\Big|_{4\pi/3}^{\pi}$$

$$=\pi-3\sqrt{3}.$$

Integrals containing $\sqrt{bx^2+cx+d}$

By completing the square in bx^2+cx+d we can express $\sqrt{bx^2+cx+d}$ in terms of $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, *or* $\sqrt{x^2-a^2}$ for suitable $a>0$. Then a trigonometric substitution eliminates the square root as before.

Example 5 Evaluate $\displaystyle\int\frac{1}{\sqrt{x^2+8x+25}}dx.$

Solution: We complete the square for the quadratic expression as follows:

$$x^2+8x+25=(x^2+8x\qquad)+25$$

$$=(x^2+8x+16)+25-16$$

$$=(x+4)^2+9$$

Thus,

$$\int\frac{1}{\sqrt{x^2+8x+25}}dx=\int\frac{1}{\sqrt{(x+4)^2+9}}dx.$$

If we make the trigonometric substitution

$$x+4=3\tan\theta,\quad dx=3\sec^2\theta\,d\theta$$

then

$$\sqrt{(x+4)^2+9}=\sqrt{9\tan^2\theta+9}=3\sqrt{\tan^2\theta+1}=3\sec\theta$$

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \int \frac{1}{3\sec\theta} 3\sec^2\theta\, d\theta$$

and $$= \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C.$$

Using our formulas for $\tan\theta$ *and* $\sec\theta$, we conclude that

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C.$$

Exercise 1.4 In Exercises1-10 evaluate the integral.

1. $\int \frac{1}{x^2\sqrt{9-x^2}}dx$
2. $\int \frac{(1-x^2)^{3/2}}{x^6}dx.$
3. $\int_0^1 \frac{1}{(3x^2+2)^{5/2}}dx$
4. $\int_1^{\sqrt{2}} \frac{1}{\sqrt{2x^2-1}}dx$
5. $\int_{3\sqrt{2}}^{6} \frac{1}{x^4\sqrt{x^2-9}}dx$
6. $\int_{\sqrt{2}}^{2} arc\sec dx$
7. $\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}dx$
8. $\int \frac{1}{\sqrt{4x-x^2}}dx$
9. $\int \frac{1}{x^2-2x+2}dx$
10. $\int \frac{x+5}{9x^2+6x+17}dx$

5.6 Improper integrals

The definite integral $\int_a^b f(x)dx$ has meaning only when *f* is continuous on [a,b] consequently bounded on [a,b]. We say *f* is bounded on an interval **I** if there is a constant M such that $\left|f(x)\right| \le M$ for all x in **I**. In this section, we shall extend the definition of the definite integral when either the integrand or the interval of integration is unbounded. Such integrals are called **improper integrals**.

1.6.1 Integrals Over Unbounded Intervals

If *f* is continuous on $[a,\infty)$, then the improper integral $\int_a^\infty f(x)dx$ **converges** if

$\lim_{t\to\infty}\int_a^t f(x)dx$ exists. In that case

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \qquad (1)$$

If the limit does not exist, the improper integral **diverges**.
Again if *f* is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x)dx = \lim_{t\to-\infty}\int_{t}^{a} f(x)dx \qquad (2)$$

provided the limit exists.

Example 1 Determine whether the integral converges or diverges, and if it converges, find its value.

(a) $\int_0^\infty \frac{1}{(x+1)^2}dx$ (b) $\int_0^\infty \frac{1}{x+1}dx$

Solution: (a) Following the discussion above and equation (1) we have

$$\int_0^\infty \frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+1)^2}dx = \lim_{t\to\infty}\left[\frac{-1}{x+1}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{t+1}+\frac{1}{0+1}\right] = 0+1 = 1$$

Thus, the improper integral converges and has the value 1.

(b) Using equation (2)

$$\int_0^\infty \frac{1}{x+1}dx = \lim_{t\to\infty}\int_0^t \frac{1}{x+1}dx$$

$$= \lim_{t\to\infty}\left[\ln(x+1)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\ln(t+1)-\ln(0+1)\right]$$

$$= \lim_{t\to\infty}\left[\ln(t-1)\right] = \infty.$$

Since the limit does not exist, the improper integral diverges.

Example 2 Determine whether the integral $\int_{-\infty}^{1} e^x dx$ converges or diverges, and if it converges, find its value.

Solution: As in Example 1;

$$\int_{-\infty}^{1} e^x dx = \lim_{t\to-\infty}\int_{t}^{1} e^x dx = \lim_{t\to-\infty}\left[e^x\right]_t^1$$

$$= \lim_{t\to-\infty}\left[e^1 - e^t\right] = e.$$

Thus, the integral converges and has the value e.

Finally, for integrals over the range $(-\infty,\infty)$, we write

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx \qquad (3)$$

provided both of the improper integrals on the right converge.

If either of the integrals on the right in (3) diverges, then $\int_{-\infty}^{\infty} f(x)dx$ is said to **diverge**. It can be shown that (3) does not depend on the choice of the real number a.

Example 3 Determine whether $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ diverges.

Solution: Using (3), with $a=0$, we have

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx.$$

Next, applying (2)

$$\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \lim_{t\to-\infty}\int_{t}^{0}\frac{1}{1+x^2}dx = \lim_{t\to-\infty}\left[\arctan x\right]_t^0$$

$$= \lim_{t\to-\infty}\left[\arctan 0 - \arctan t\right] = 0-(-\frac{\pi}{2}) = \frac{\pi}{2}.$$

Similarly, we may show, by using (1) that

$$\int_{0}^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{2}.$$

Consequently the given improper integral converges and has the value $\frac{\pi}{2}+\frac{\pi}{2}=\pi.$

1.6.2 Integrals with Unbounded Integrands

We now consider a function *f* that is continuous at every point in $(a,b]$ and unbounded near a. By assumption *f* is continuous on the interval $[t,b]$ for any *t* in (a,b), so that $\int_t^b f(x)dx$ is defined for such *t*. If the one-sided limit

$$\lim_{t\to a^+}\int_t^b f(x)dx$$

exists, then we define $\int_a^b f(x)dx$ to be the limit. This idea leads us to the following definitions:

(i) If *f* is continuous on $[a,b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx, \qquad (4)$$ provided the limit exists.

(ii) If *f* is continuous on $(a,b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx, \qquad (5)$$

provided the limit exists.

As in the preceding section, the integrals defined in (4) and (5) are referred to as improper integrals and they converge if the limits exist. The limits are called the values of the improper integrals. If the limits do not exist, the improper integrals diverge.

Example 4 Evaluate $\int_1^2 \frac{1}{\sqrt{2-x}}dx.$

Solution: Since the integrand has an infinite discontinuity at $x = 2$, we apply (4) and have

$$\int_1^2 \frac{1}{\sqrt{2-x}}dx = \lim_{t\to 2^-}\int_1^t \frac{1}{\sqrt{2-x}}dx$$
$$= \lim_{t\to 2^-}[-2\sqrt{2-x}]_1^t$$
$$= \lim_{t\to 2^-}[-2\sqrt{2-t} - (-2\sqrt{2-1}] = 2.$$

Example 5 Determine whether the improper integral $\int_0^1 \frac{1}{x}dx$ converges or diverges.

Solution: The integrand is unbounded near 0. Applying (5) gives us

$$\int_0^1 \frac{1}{x}dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx = \lim_{t\to 0^+}[\ln x]_t^1 = \lim_{t\to 0^+}[\ln 1 - \ln t] = \infty.$$

Consequently the improper integral diverges, since the limit does not exist.

We give the definition of another improper integral as follows.

If f has a discontinuity at a number c in the open interval (a,b) but continuous elsewhere on $[a,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \qquad (6)$$

provided both of the improper integrals on the right converge. If both converge, then the value of the improper integral $\int_a^b f(x)dx$ is the sum of the two values.

Example 6 Determine whether the improper integral $\int_0^4 \frac{1}{(x-3)^2}dx$ converges or diverges.

Solution: The integrand is undefined at $x = 3$. Since this number is in the interval (0,4), we use (6), with c = 3:

$$\int_0^4 \frac{1}{(x-3)^2}dx = \int_0^3 \frac{1}{(x-3)^2}dx + \int_3^4 \frac{1}{(x-3)^2}dx$$

For the integral on the left to converge, both integrals on the right must converge. However, since

$$\int_0^3 \frac{1}{(x-3)^2}dx = \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}dx$$
$$= \lim_{t\to 3^-}\left[\frac{-1}{x-3}\right]_0^t$$
$$= \lim_{t\to 3^-}\left[\frac{-1}{t-3} - \frac{1}{3}\right] = \infty$$

the given improper integral diverges.

The other kind of improper integral is found if f is continuous in $(a,\text{b})$and is unbounded near both a and b. We say that $\int_a^b f(x)dx$ **converges** if for some point c in (a,b) both the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. Otherwise we say that the integral is divergent.

Example 7 Determine whether $\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}dx$ diverges.

Solution: The integrand is unbounded near both the endpoints 0 and 1 and is continuous on (0,1). Consequently the integral is of the type under consideration. If we let $c=\frac{3}{4}$,then we need to analyze the convergence of

$$\int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx \quad \text{and} \quad \int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx$$

For $0<t<\frac{3}{4}$ we have

$$\int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx = \lim_{t\to 0^+}\int_t^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx = \lim_{t\to 0^+}[2\sqrt{x-x^2}\,]\Big|_t^{3/4}$$

$$= \lim_{t\to 0^+}[2(\sqrt{\tfrac{3}{16}}-\sqrt{t-t^2}\,)] = \tfrac{\sqrt{3}}{2}$$

A similar computation shows that the second improper integral also converges and that

$$\int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx = -\frac{\sqrt{3}}{2}.$$

Therefore the original integral converges, and

$$\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}dx = \int_0^{3/4} \frac{1-2x}{\sqrt{x-x^2}}dx + \int_{3/4}^1 \frac{1-2x}{\sqrt{x-x^2}}dx = \frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=0.$$

Exercise 1.6

Determine whether the integral converges or diverges, and if it converges, find its value.

1. $\int_0^{\infty} \frac{x}{1+x^2}dx$
2. $\int_{-\infty}^0 \frac{1}{(x+3)^2}dx$
3. $\int_1^{\infty} \frac{1}{\sqrt{x^2-1}}dx$
4. $\int_{-\infty}^{\infty} xe^{-x^2}dx$
5. $\int_0^9 \frac{1}{\sqrt{x}}dx$
6. $\int_0^{\pi/2} \sec^2 x dx$
7. $\int_{-2}^0 \frac{1}{\sqrt{4-x^2}}dx$
8. $\int_0^{\pi} \sec x dx$
9. $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}dx$
10. $\int_0^1 \frac{3x^2-1}{x^3-x}dx$

5.7 Application of the Integral

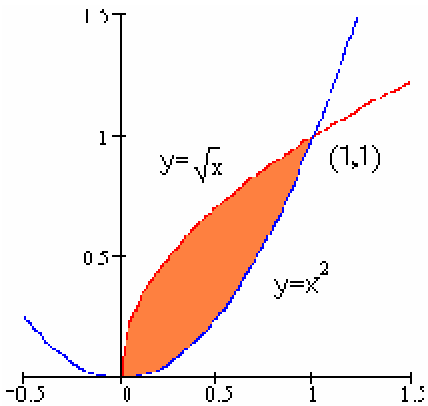
Area (Review)

Definition: Let f and g be continuous on [a,b], with $f(x) \ge g(x)$ *for* $a \le x \le b$. The area **A** of the region between the graphs of f and g on [a,b] is given by

$$\mathbf{A} = \int_a^b [f(x) - g(x)]dx$$

Example 1 Find the area of the region bounded by the graphs of the equations $y = x^2$ *and* $y = \sqrt{x}$.

Solution: First sketch the graphs on the same plane. And find the intesection of the two graphs by putting $x^2 = \sqrt{x}$. Observe that



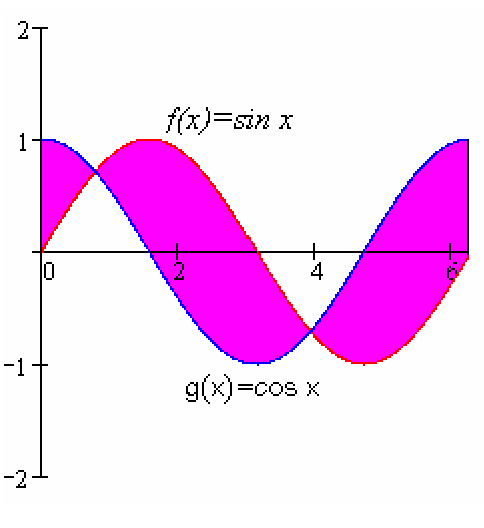
$$x^2 = \sqrt{x} \Rightarrow x^4 = x \Leftrightarrow x^4 - x = 0$$

Hence x=0 or $\Leftrightarrow x(x-1)(x^2+x+1)$ x=1 since $x^2 + x + 1 > 0$ for every real x the two graphs intersect at (0,0) and (1,1). Moreover $x^2 \le \sqrt{x}$ on [0,1]. Thus the area A of the region bounded by the graphs is given by

$$\mathbf{A} = \int_0^1 (\sqrt{x} - x^2) = \frac{2}{3}x^{3/2} - \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

Example 2 Let $f(x) = \sin x$ *and* $g(x) = \cos x$. Find the area A of the region between the graphs of *f* and *g* on [0,2π].

Solution: $\sin x = \cos x$, on [0,2π] implies that $\tan x = 1$ on [0,2π]. And $\tan x = 1$ on [0,2π] for $x = \dfrac{\pi}{4}$ *and* $x \dfrac{5\pi}{4}$. Thus the two graphs intersect at $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ *and* $\left(\dfrac{5\pi}{4}, -\dfrac{\sqrt{2}}{2}\right)$ and the region bounded by the two graphs on [0,2π] is as below.



Observe that $\sin x \ge \cos x$ *on* $[0, \pi/4]$,
$\sin x \ge \cos x$ *on* $[\pi/4, 5\pi/4]$ *and* $\sin x \ge \cos x$ [5π/4,2π] and it follows that

$$\mathbf{A} = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)dx$$

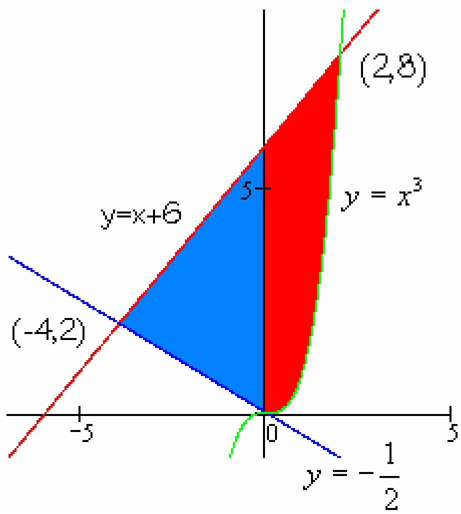
$$= (\sin x + \cos x)_0^{\pi/4} + (-\cos x - \sin x)_{\pi/4}^{5\pi/4}$$

$$(\sin x + \cos x)_{5\pi/4}^{2\pi}$$

$$= (\sqrt{2} - 1) + 2\sqrt{2} + (1 + \sqrt{2}) = 4\sqrt{2}$$

Example 3 Find the area of the region bounded by the graph of y-x = 6, $y-x^3=0$ and 2y+x=0.

Solution: First we graph the region as follows. We divid the region in two two regions R_1 and R_2 as in the plote to the right



$$\mathbf{A}_1 = \int_{-4}^{0}[x+6+\tfrac{1}{2}x]dx = \tfrac{3}{4}x^2+6x\Big|_{-4}^{0} = 12$$

and

$$\mathbf{A}_2 = \int_{0}^{2}(x+6-x^3)dx = 10$$

Thus the area **A** of the entire region **R** is

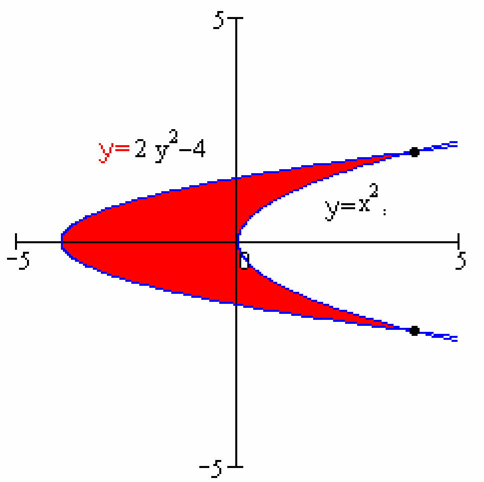
A=A$_1$+A$_2$=22.

Reversing the roles of x and y

Instead of considering a region **R** that is bounded between the graphs of two functions of x, it is sometimes convenient to consider R as the region between the graphs of two functions of y. Then the area is computed by integrating along the y-axis, instead of along the x-axis.

Example 4 Find the area of the region bounded by the graphs of the equations $2y^2=x+4$ and $y^2=x$.

Solution: First we sketch the region as below



'e can see that

points (4,-2)

y^2=x+4 lays

ence the area

the two graphs

$$\mathbf{A} = \int_{-2}^{2}[y^2-(2y^2-4)]dy = 4y-\frac{1}{3}y^3\Big|_{-2}^{2}$$

$$= \frac{32}{3}.$$

Class Work

Find the area A of the regions bounded between the graphs of the equations bellow.

a) $y=x^2+1$ and y=2x+9 b) $x=y^2-y$ and $x=y-y^2$

Volume

The cross –section method

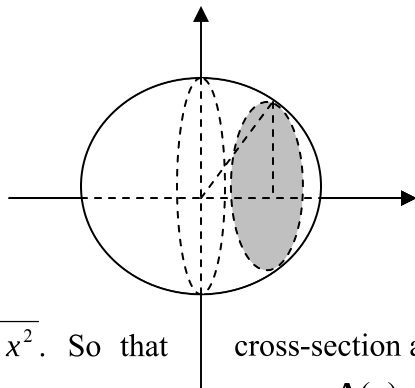
If a solid region D has cross-sectional area **A**(x) for $a \le x \le b$, and if **A** is continuous on [a,b] , then we define the volume V of D by the fromula

$$V = \int_a^b \mathbf{A}(x)dx.$$

Example 1 Show that the volume of a sphere of radius r is

$$V = \tfrac{4}{3}\pi r^3$$

Solution: If we place the sphere so that its center is at the origin then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean theorem) is



$y = \sqrt{r^2 - x^2}$. So that cross-section area is

$$\mathbf{A}(x) = \pi y^2 = \pi(r^2 - x^2)$$

Using the formula with $a = -r$ *and* $b = r$, we have

$$V = \int_{-r}^{r} A(x)dx = \int_{-r}^{r} \pi(r^2 - x^2)dx$$

$$= \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

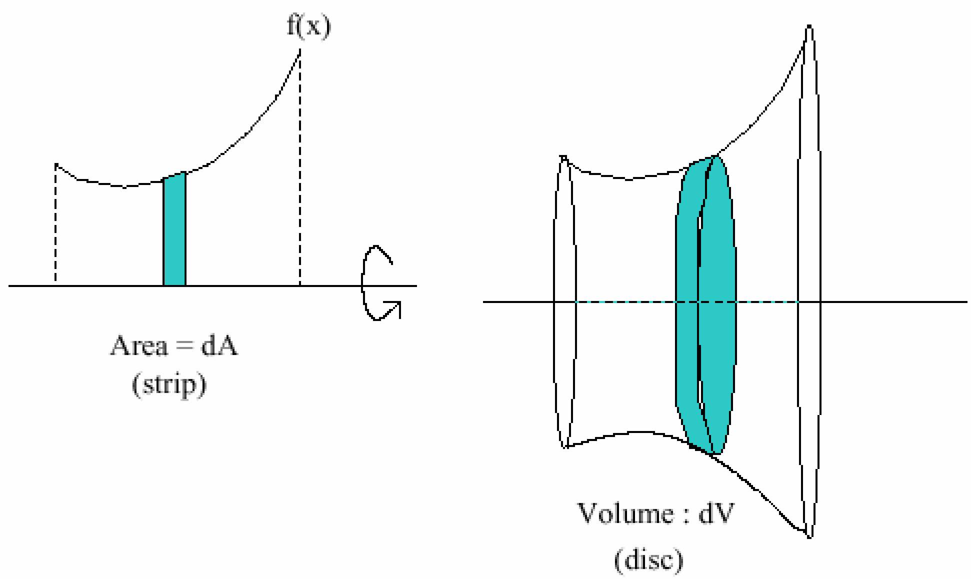
$$= \tfrac{4}{3}\pi r^3$$

Class work

Suppose a pyramid is 4 units tall and has a squere base 3 units on a side. Find the volume V of the pyramid.

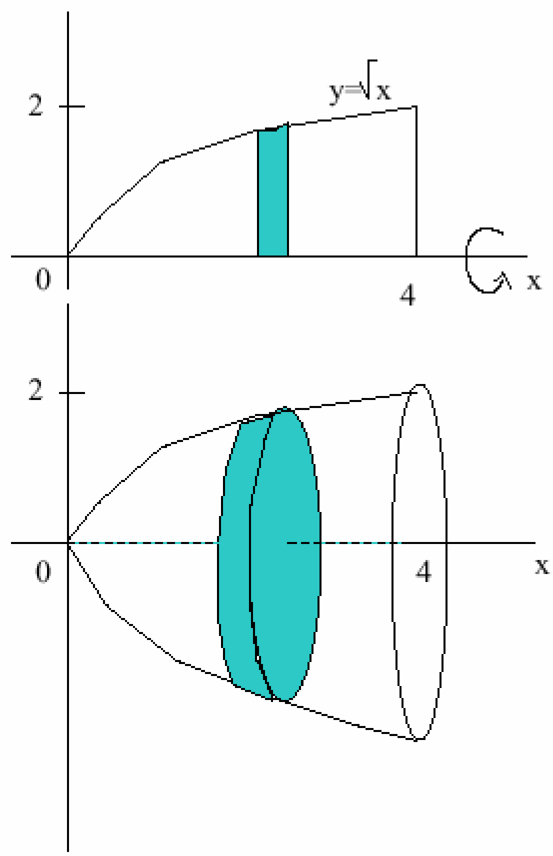
The Disc Method

We now move on to yet another application of definite integrals: **volumes of revolution**. Volumes of revolution are solids whose shapes can be generated by revolving some curve(s) about some axis in three-space. If we can set things up so that a solid of revolution is generated by revolving the region between the graph of a continuous function $f(x), a \leq x \leq b$ and the x axis, and the axis of rotation is the x axis (see diagram below), we can then calculate the volume in the following way:



The steps to follow are very familliar: (1) sketch the region to be revolved
(2) Draw a small strip perpendicular to the axis of revolution ,then revolve it about the axis of rotation and calculate the volume that it generates, say dV (see Fig 6)
(3) Integrate dV to find the entire volume

Example 2 Find the volume generated by revolving the region bounded by $y = \sqrt{x}, y = 0$, and $x = 4$ about the x-axis.
Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):



Rotating the strip about the x-axis we see that we get something of the form:

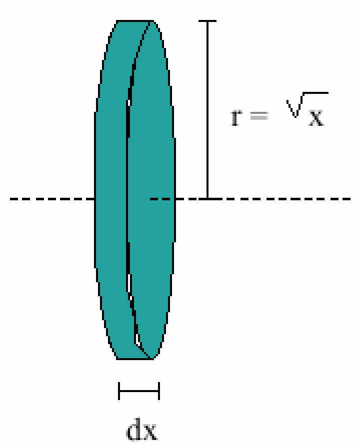


Figure 8: $volume = \pi r^2 h$

This is clearly a cylindrical shape and so has volume given by the classical formula: $v = \pi r^2 h$, where r is the radius of the cylinder, and h is the height. Looking at the specific solid generated by the strip here, we see that $h = dx$ and r=the length of the strip =the y-value of the curve =$\sqrt{x}$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi r^2 h \\ &= \pi(\sqrt{x})^2 dx \\ &= \pi x dx \end{aligned}$$

We also see from the sketch that x varies from 0 to 4 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^4 \pi x dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 \\ &= \pi(8-0) \\ &= 8\pi \end{aligned}$$

The Washer Method

The next examples illustrate the above process which is sometimes called the **method of washers**, for a soon obvious reason (the strip generates a solid resembling a washer).

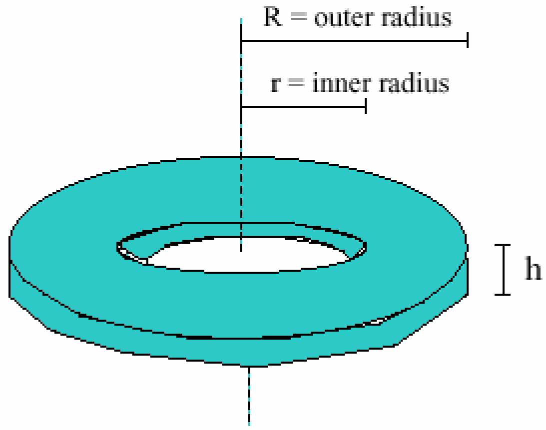


Figure 9: washer volume $= \pi(R^2 - r^2)h$

To find the volume dV, of such an animal, we simply find the volume of the large disc as if it were solid $(\pi R^2 h)$and then subtract the volume of the hole $(\pi r^2 h)$. This gives us the formula:

$$\boxed{dV = \pi(R^2 - r^2)h}$$

The use of the above formula is better illustrated through some examples:

Example 7. Find the volume generated by revolving the region bounded by $y = x^2 + 2$, $y = 1$, $x = 0$ and $x = 2$ about the x-axis.

Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):

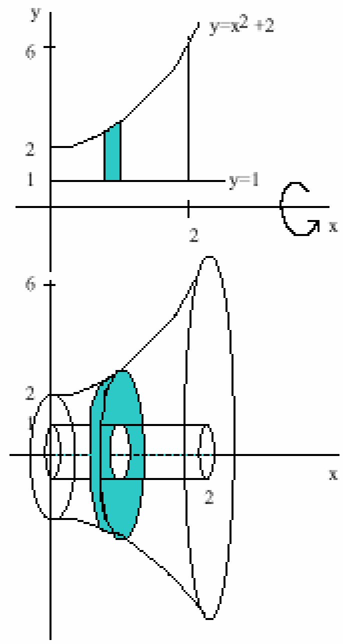
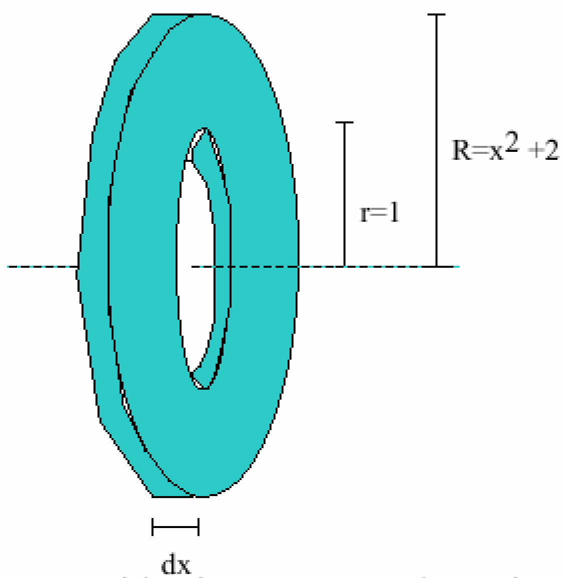


Figure 10:

Rotating the strip about the x-axis we see that we get something resembling figure 11.



The volume generated by the strip is one of a washer with $R =$ (the distance from the x-axis to the outer edge of the strip) $= x^2 + 2$, $r =$ (the distance from the x-axis to the inner edge of the strip) $= 1$, and $h = dx$. So the volume generated by the strip is given by:

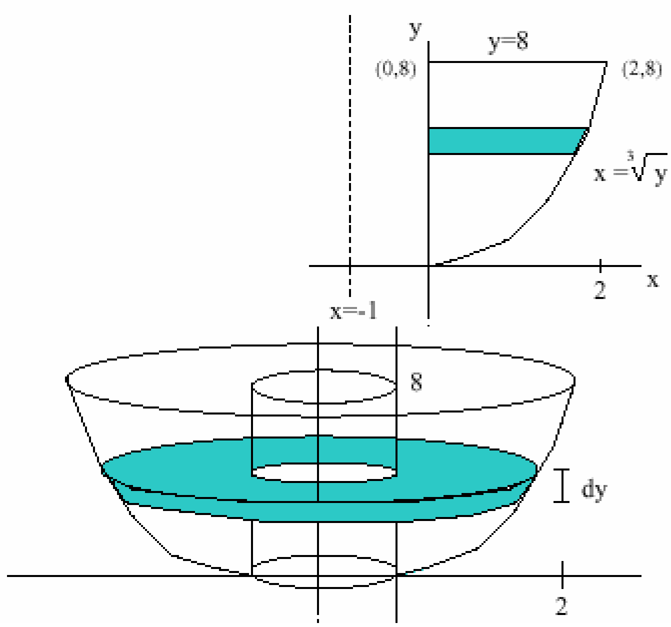
$$
\begin{aligned}
dV &= \pi(R^2 - r^2)h \\
&= \pi[(x^2+2)^2 - (1^2)]dx \\
&= \pi(x^4 + 4x^2 + 3)dx
\end{aligned}
$$

We also see from the sketch that x varies from 0 to 2 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$
\begin{aligned}
V &= \int_0^2 \pi[(x^2+2)^2 - (1^2)]dx \\
&= \pi \int_0^2 (x^4 + 4x^2 + 3)dx \\
&= \pi \left[\tfrac{1}{5}x^5 + \tfrac{4}{3}x^3 + 3x\right]_0^2 \\
&= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6) - 0\right] \\
&= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6)\right] \approx 72.5
\end{aligned}
$$

Just as with areas, we sometimes use horizontal strips for finding volumes. This comes about since the method we learned above requires the strips to be perpendicular to the axis of rotation, so if we revolve a region about, say, the y-axis then our strips must be horizontal. All other mechanics of such a problem are business as usual as we shall see:

Example 8. Find the volume generated by revolving the region bounded by $y = x^3$, $y = 8$,and $x = 0$ about the line $x = -1$.
Solution: We first sketch the region in question, and draw our small strip (with width dy) perpendicular to the axis of rotation :



Rotating the strip about the axis of rotation we see that we get something resembling figure 13.

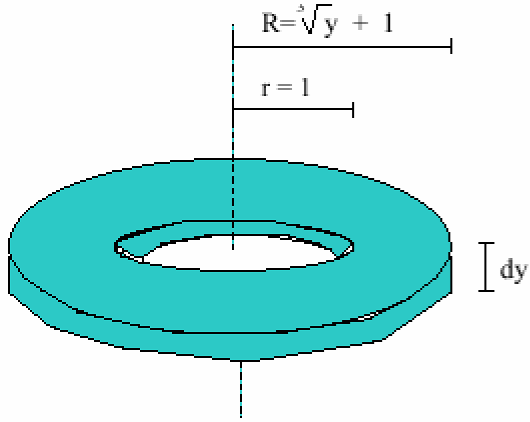


Figure 13:

The volume generated by the strip is one of a washer with $R =$ (the distance from the axis of rotation to the outer edge of the strip)=(1+the x value of the outer curve) $= 1+y^{\frac{1}{3}}$, $r =$ (the distance from the axis of rotation to the inner edge of the strip) $= 1$, and $h = dy$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi(R^2 - r^2)h \\ &= \pi[(1 + y^{\frac{1}{3}})^2 - (1^2)]dy \\ &= \pi(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \end{aligned}$$

We also see from the sketch that y varies from 0 to 8 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^8 \pi[(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi \int_0^8 (y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi \left[\tfrac{3}{5}y^{\frac{5}{3}} + \tfrac{3}{2}y^{\frac{4}{3}}\right]_0^8 \\ &= \pi \left[\tfrac{3}{5}(8)^{\frac{5}{3}} + \tfrac{3}{2}(8)^{\frac{4}{3}}\right] \\ &= \pi \left[(\tfrac{96}{5} + 24\right] \approx 135.7 \end{aligned}$$

Class Work

Let f(x)=5x and g(x)=x^2 and let R be the region between the graphs of f and g on [0,3]. Find the volume of the solid obtained by revolving R about the x-axis.

6 Sequences and Series

In this chapter we first study sequences, which by definition are functions since they are helpful in the study of series. Series can be used to represent many of the differentiable functions such as polynomial, exponential, logarithmic etc. functions. A major advantage of the series representation of functions is that it allows us to evaluate integrals of the form say $\int \sin\sqrt{x}\,dx$ *and* $\int e^{-x^2}dx$ and also approximate numbers such as e, π, and $\sqrt{2}$.

6.1 Definition and Notions of Sequence

An ordered set of numbers such as $a_1, a_2, a_3, ..., a_n, ...$ is called a *sequence* and usually designated briefly by $\{a_n\}$. Each number a_k is a **term** of the sequence. In particular the **nth term** of a sequence is denoted by a_n. We may also define a sequence as a function.

Definition 6.1 A sequence is a function whose domain is the collection of all integers greater than by or equal to a given integer m(usually 0 or 1).

Observe if we define a function by

$$f(n) = a_n \quad for\ n \geq 1 \tag{1}$$

then the ordered set of numbers $a_1, a_2, a_3, ..., a_n, ...$ determines a sequence. As a result we normally suppress the symbol *f* and just write $\{a_n\}_{n=1}^{\infty}$ for the sequence defined in (1). Similarly if

$$f(n) = a_n \quad for\ n \geq m$$

then we would write $\{a_n\}_{n=m}^{\infty}$ for the sequence.

Example 1 List the first four terms and the tenth term of each sequence:

(a) $\left\{(-1)^{n-1}\dfrac{2n}{n+1}\right\}_{n=1}^{\infty}$ (b) $\{2+(0.1)^n\}_{n=0}^{\infty}$ (c) $\left\{\left(\dfrac{1}{2}\right)^n\right\}_{n=0}^{\infty}$ (d) $\{2\}_{n=1}^{\infty}$

Solution: To find the first four terms, we substitute, successively, n = 1, 2, 3, and 4 in the formula for a_n. The tenth term is found by substituting 10 for n. Doing this and simplifying gives us the following:

Sequence	**nth term a_n**	The first four terms	Tenth term
(a) $\left\{(-1)^{n-1}\dfrac{2n}{n+1}\right\}_{n=1}^{\infty}$	$(-1)^{n-1}\dfrac{2n}{n+1}$	$1, -\dfrac{4}{3}, \dfrac{3}{2}, -\dfrac{8}{5}$	$-\dfrac{20}{11}$
(b) $\{2+(0.1)^n\}_{n=0}^{\infty}$	$2+(0.1)^{n-1}$	2,2.1,2.01,2.001	2.000000001
(c) $\left\{\left(\dfrac{1}{2}\right)^n\right\}_{n=0}^{\infty}$	$\left(\dfrac{1}{2}\right)^{n-1}$	$1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}$	$\dfrac{1}{2^9}$
(d) $\{2\}_{n=1}^{\infty}$	2	2,2,2,2	2

When the first few terms of a sequence are given, the general term is obtained by inspection.

Example 2 Obtain the nth term for each of the sequences:

a) 1, 4, 9, 16, 25, . . . b) 3, 7, 11, 19, 23, . . .

Solution: (a) The terms of the sequence are the squares of the positive integers; the nth term is n^2.

(b) This is an arithmetic progression having first term $A_1 = 3$ and common difference d=4. The nth term is $A_1 + (n - 1)d = 4n - 1$. Note, however, that the nth term can be obtained about as easily by inspection.

Exercise 2.1

1. Write the first five terms and the tenth term of the sequence whose nth term is:

   a) $4n - 1$ b) $\dfrac{(-1)^{n-1}}{n+1}$ c) $\dfrac{n+1}{n!}$

2. Write the nth term for each of the following sequences:

   a) 2,4,6,8,10,12,… b) 2,-5,8,-11,14,… c) 3,4,5/2,1,7/24,…

6.2 Convergence of Sequences

A sequence $\{a_n\}$ may have the property that as n increase, a_n gets very close to some real number L. For instance in the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=0}^{\infty}$ the nth term $\left(\frac{1}{2}\right)^n$ can be made arbitrary close to 0 by choosing n sufficiently large. This concept leads us to the following definition of convergence of a sequence.

Definition 6.2 A sequence $\{a_n\}_{n=1}^{\infty}$ **has the limit L**, or **converges to L**, denoted by either

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \ as\ n \to \infty,$$

if for every $\varepsilon > 0$ there exists a positive number N such that

$$|a_n - L| < \varepsilon \ \text{ whenever } \ n > N.$$

If such a number L does not exist, the sequence **has no limit**, or diverges.

If we can make a_n as large as desired by choosing n sufficiently large, then the sequence $\{a_n\}_{n=1}^{\infty}$ diverges, but we still use the limit notation and write $\lim_{n\to\infty} a_n = \infty$. A more precise way of specifying this follows.

Definition 6.3 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If for every number M there is an integer N such that

$$\text{If } n \geq N,\ then\ \ a_n > M$$

we say that $\{a_n\}_{n=1}^{\infty}$ **diverges to** ∞, and we write

$$\lim_{n\to\infty} a_n = \infty.$$

Similarly, if for every number M there is an integer N such that

$$\text{If } n \geq N,\ then\ \ a_n < M$$

we say that $\{a_n\}_{n=1}^{\infty}$ **diverges to** $-\infty$, and we write

$$\lim_{n\to\infty} a_n = -\infty.$$

But referring to Definition 6.3 to show that a sequence diverges, or referring to Definition 6.2 to show that a sequence converges or diverges is tedious. One way to avoid constantly using Definition 6.2 and Definition 6.3 arises from the fact that the definition of $\lim_{n\to\infty} a_n = L$ is analogues with the definition of $\lim_{n\to\infty} f(x) = L$ and similarly, that the definition of $\lim_{n\to\infty} a_n = \infty\ (or -\infty)$ is analogues with the definition of $\lim_{n\to\infty} f(x) = \infty(or - \infty)$. These observations lead us to the following theorem.

Theorem 6.4 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, L a number, and f a function defined on [m,∝) such that $f(n) = a_n \ \ for \ n \geq m.$ If $\lim\limits_{x\to\infty} f(x) = L,$ then $\{a_n\}_{n=1}^{\infty}$ converges and $\lim\limits_{n\to\infty} a_n = L$. If $\lim\limits_{n\to\infty} f(x) = \infty(or \ \lim\limits_{n\to\infty} f(x) = -\infty),$ then $\{a_n\}_{n=1}^{\infty}$ diverges, and $\lim\limits_{n\to\infty} a_n = \infty \ \ (\lim\limits_{n\to\infty} a_n = -\infty).$

Thus $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(x).$$

The following example illustrates the use of Theorem 2.4.

Example 1 Determine whether the sequence $\left\{2 + \dfrac{1}{n^2}\right\}_{n=1}^{\infty}$ converges or diverges.

Solution: We let

$$f(x) = 2 + \frac{1}{x^2} \ \ for \ \ x \geq 1$$

Then $f(n) = 2 + \dfrac{1}{n^2} \ \ for \ \ n \geq 1$. Since $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(2 + \dfrac{1}{x^2}\right) = \lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty}\dfrac{1}{x^2} = 2$. We then conclude from theorem 2.4 that

$$\lim_{n\to\infty}\left(2 + \frac{1}{n^2}\right) = 2.$$

Thus the sequence converges to 2.

Example 2 Determine whether the sequence converges or diverges:

(a) $\{n^3 + 2\}_{n=1}^{\infty}$ (b) $\{(-1)^n\}_{n=1}^{\infty}$

Solution: If we let

$$f(x) = x^3 + 2 \ \text{ for every } \ x \geq 1$$

then $f(n) = n^3 + 2$ for every $n \geq 1$. Since $\lim\limits_{x\to\infty}(x^3 + 2) = \infty$, by Theorem 6.4

$$\lim_{n\to\infty}(n^3 + 2) = \infty.$$

Hence the sequence diverges.

(b) Letting n=1,2,3, . . . , we see that the terms of $(-1)^n$ oscillate between 1 and –1 as follows:

$$-1 \ , 1, -1, 1, -1, \ldots$$

Thus, $\lim\limits_{n\to\infty}(-1)^n = \infty$ does not exist, so the sequence diverges.

Example 3 Let r be any number. Show that the sequence $\{r^n\}_{n=1}^{\infty}$ diverges for $|r| > 1$ and $r = -1$. show that for all other values of r the sequence converges, with

$$\lim_{n\to\infty} r^n = \begin{cases} 1 & for \ r = 1 \\ 0 & for \ |r| < 1 \end{cases}$$

Solution: First we consider nonnegative values of r. Let

$$f(x) = r^n \ \ \ for \ \ \ x \geq 1$$

so

that $f(n) = r^n \ \ \ for \ \ n \geq 1.$ It follows from our analysis of exponential functions that

$$\lim_{x\to\infty} r^x = \begin{cases} 0 & for\ 0 \le r < 1 \\ 1 & for\ r = 1 \\ \infty & for\ r > 1 \end{cases}$$

By theorem 6.4 this means that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & for\ 0 \le r < 1 \\ 1 & for\ r = 1 \\ \infty & for\ r > 1 \end{cases} \qquad (*)$$

Thus $\{r^n\}_{n=1}^{\infty}$ diverges for $r > 1$ and converges for $0 \le r \le 1$. Next we consider negative values of r. If r = –1, then $\{r^n\}_{n=1}^{\infty}$ becomes $\{(-1)^n\}_{n=1}^{\infty}$, which diverges by Example 2 (b). If $r \ne -1$, the since $|r^n| = |r|^n$, we know from $(*)$ that

$$\lim_{n\to\infty}|r^n| = \lim_{n\to\infty}|r|^n = \begin{cases} 0 & for -1 < r < 0 \\ \infty & for\ r < -1 \end{cases}$$

It follows that $\lim_{n\to\infty} r^n = 0\ \ when\ -1 < r < 0$ and that $\lim_{n\to\infty} r^n$ does not exist when $r < -1$.

Example 4 Show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Solution: Notice

$$\sqrt[n]{n} = n^{1/n} = e^{(1/n)\ln n}$$

Thus we let

$$f(x) = e^{(1/x)\ln x} \text{ for } x \ge 1$$

so that f is continuous and $f(n) = e^{(1/n)\ln n}$ for $x \ge 1$. Since

$$\lim_{n\to\infty} x = \infty = \lim_{n\to\infty} \ln$$

by l'Hôpital's Rule implies that

$$\lim_{n\to\infty}\frac{\ln x}{x} = \lim_{n\to\infty}\frac{1/x}{1} = \lim_{n\to\infty}\frac{1}{x} = 0$$

and thus $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{(1/x)\ln x} = e^0 = 1$

6.2.1 Convergence Properties of Sequences

Limit theorems that are analogous to those stated for real valued functions can be established for sequences. That is if $\{a_n\}_{n=m}^{\infty}$ and $\{b_n\}_{n=m}^{\infty}$ are convergent sequences, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad (provided\ \lim_{n\to\infty} b_n \ne 0)$$

Example 5 Find $\lim_{n\to\infty} \dfrac{n^3}{2n^3+4n}$.

Solution: We divide both the numerator and denominator by n^3 and apply the above limit theorems as follows:

$$\lim_{n\to\infty} \frac{n^3}{2n^3+4n} = \lim_{n\to\infty} \frac{1}{2+(4/n^2)} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}[2+(4/n^2)]}$$

$$= \frac{1}{\lim_{n\to\infty} 2 + \lim_{n\to\infty} 4/n^2} = \frac{1}{2+0} = \frac{1}{2}.$$

The version of the squeezing Theorem for sequences is as follows:

Theorem 6.5 If $\{a_n\}_{n=m}^{\infty}$, $\{b_n\}_{n=m}^{\infty}$, and $\{c_n\}_{n=m}^{\infty}$ are sequences and $a_n \le b_n \le c_n$ for every n and if

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then
$$\lim_{n\to\infty} b_n = L.$$

Example 6 Find the limit of the sequence $\left\{\dfrac{\sin^2 n}{2^n}\right\}$.

Solution: Since $0 \le \sin^2 n \le 1$ for every positive integer n,

$$0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}.$$

Applying Example 3 with $r = \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$$

Moreover, $\lim_{n\to\infty} 0 = 0.$ Hence it follows from the squeezing theorem that

$$\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0.$$

Hence the limit of the sequence is 0.

6.2.2 Bounded Monotone Sequences

In analogy with boundedness for a function we say that a sequence $\{a_n\}_{n=m}^{\infty}$ is **bounded** if there is a number M such that $|a_n| \le M$ for every $n \ge m$. Otherwise we say that the sequence is unbounded. For instance the sequences $\{1/n\}_{n=1}^{\infty}$ *and* $\{(-1)^n\}_{n=1}^{\infty}$ are bounded, whereas the sequence $\{n^2\}_{n=1}^{\infty}$ is unbounded.

The following theorem gives as important criteria boundedness and divergence of sequences

Theorem 2.6 a. If $\{a_n\}_{n=m}^{\infty}$ converges, then $\{a_n\}_{n=m}^{\infty}$ is bounded

b. If $\{a_n\}_{n=m}^{\infty}$ is unbounded, then $\{a_n\}_{n=m}^{\infty}$ is divergent.

Example 7 Since $\{1/n\}_{n=1}^{\infty}$ is a convergent sequence the by Theorem6. 6 $\{1/n\}_{n=1}^{\infty}$ is bounced. And the unboundedness of $\{n^2\}_{n=1}^{\infty}$ implies by again Theorem 6.6 that $\{n^2\}_{n=1}^{\infty}$ is divergent.

Note: The above theorem does not imply that all bounded sequences converge, and indeed that is not the case. For example the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but it diverges.

Definition 6.7 A sequence $\{a_n\}_{n=m}^{\infty}$ is said to be an **increasing** sequence if $a_n \leq a_{n+1}$ for each $n \geq m.$ Similarly, $\{a_n\}_{n=m}^{\infty}$ is said to be a decreasing sequence if $a_n \geq a_{n+1}$ for each $n \geq m.$

Example 8 The sequences $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ and $\{1/n\}_{n=1}^{\infty}$ are increasing and decreasing sequences respectively by definition 2.7.

The other way of showing whether a sequence $\{a_n\}_{n=m}^{\infty}$ is first to find a continuous real valued function f, if possible, such that $f(n) = a_n$ and show that whether f is increasing or decreasing by using the first derivative test, and consequently decide that the sequence is increasing or decreasing. In Example 8 above if we let $f(x) = \frac{x}{x+1}$ *then* $f(n) = \frac{n}{n+1}$ and $f'(x) = \frac{1}{(x+1)^2} > 0$ for every x thus f is an increasing function for every x consequently the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is decreasing. Similarly we can show that $\{1/n\}_{n=1}^{\infty}$ is decreasing.

Definition 6.8 A sequence which is either increasing or decreasing is said to be **monotonic** sequence

Theorem 6.9 A bounded monotonic sequence $\{a_n\}_{n=m}^{\infty}$ converges. If the sequence is increasing, then the limit is the smallest number L such that $a_n \leq L$ *for* $n \geq m.$ If the sequence is decreasing, then the limit is the largest number L such that $a_n \geq L$ *for* $n \geq m.$

Example 8 Let

$$a_n = \frac{1}{n+2} \quad for \;\; n \geq 1.$$

Show that $\{a_n\}_{n=1}^{\infty}$ is convergent.

Solution: Since

$$a_{n+1} = \frac{1}{(n+1)+2} = \frac{1}{n+3} < \frac{1}{n+2} = a_n$$

The sequence is decreasing. Moreover

$$0 \leq \frac{1}{n+2} \leq \frac{1}{3} \quad for \;\; n \geq 1$$

so $\{a_n\}_{n=1}^{\infty}$ is bounded. Hence $\{a_n\}_{n=1}^{\infty}$ is a bounded monotonic sequence consequently it is convergent by Theorem 6.9.

Exercise 6.2 In problems 1-4 evaluate the limit a number ∞ *or* $-\infty$.

1. $\lim\limits_{n\to\infty}(3-2n)$
2. $\lim\limits_{n\to\infty} e^{-1/n}$
3. $\lim\limits_{n\to\infty}\left(1+\dfrac{0.05}{n}\right)^n$
4. $\lim\limits_{n\to\infty}\ln\dfrac{1}{n}$

II Determine whether the sequence converges of diverges, and if it converges, find the limit.

5. $\left\{2(-\tfrac{4}{5})^n\right\}_{n=1}^{\infty}$
6. $\left\{(-1)^n n^3 2^{-n}\right\}_{n=1}^{\infty}$
7. $\left\{2^{-n}\cos n\right\}_{n=1}^{\infty}$
8. $\left\{\sqrt{n+1}-\sqrt{n}\right\}_{n=1}^{\infty}$
9. $\left\{e^{-n}\ln n\right\}_{n=1}^{\infty}$
10. $\left\{\cos \pi n\right\}_{n=1}^{\infty}$

6.3 Subsequence and Limit Points

Definition 6.10 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers such that $n_k < n_{k+1}$ for each k, that is, $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. Then the sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$.

Example 1 Consider the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$. If we let $n_k = 2k$ for each positive integer k, the corresponding subsequence of $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ is $\left\{\dfrac{1}{2k}\right\}_{k=1}^{\infty} = \left\{\dfrac{1}{2n}\right\}_{n=1}^{\infty}$. Furthermore if we let $\{n_k\}_{k=1}^{\infty}$ be any strictly increasing sequence of positive integers, then the sequence $\left\{\dfrac{1}{n_k}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

Note: If $\{a_n\}_{n=1}^{\infty}$ is a sequence, then $\{a_n\}_{n=1}^{\infty}$ is a trivial subsequence of itself.

Theorem 6.11 If the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, then every subsequence of the sequence $\{a_n\}_{n=1}^{\infty}$ also converges to L.

Example 2 Observe that the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ in Example 1 above, converges to 0, consequently by Theorem 2.11 the subsequences $\left\{\dfrac{1}{2n-1}\right\}_{n=1}^{\infty}$ and $\left\{\dfrac{1}{2n}\right\}_{n=1}^{\infty}$ converge to 0.

Definition 6.12 If the sequence $\{a_n\}_{n=1}^{\infty}$ diverges but does not diverge to positive infinity or to minus infinity, then the sequence $\{a_n\}_{n=1}^{\infty}$ is said to oscillate to be an oscillating sequence.

Example 3 The sequence $\left\{(-1)^n\right\}_{n=1}^{\infty}$ is an oscillating sequence on the other hand even if the terms of the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ go up and down, it does not oscillate as it is a convergent sequence.

Definition 6.13 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, the number L is called a limit point of $\{a_n\}_{n=1}^{\infty}$ if and only if a subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example 4 Since the sequences $\left\{(-1)^{2n}\right\}_{n=1}^{\infty}$ *and* $\left\{(-1)^{2n-1}\right\}_{n=1}^{\infty}$ are subsequences of $\left\{(-1)^n\right\}_{n=1}^{\infty}$ which converge to 1 and –1, 1 and –1 are the limit points of $\left\{(-1)^n\right\}_{n=1}^{\infty}$.

Exercise 6.3 Give the subsequence(s) and limit point(s) of the following sequences.

1. $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

2. $\{\cos n\pi\}_{n=1}^{\infty}$

2. $\left\{\sin\left(\frac{n\pi}{4}\right)\right\}_{n=1}^{\infty}$

4. $\left\{\frac{n^2+1}{n}\right\}_{n=1}^{\infty}$

Worksheet

1. Write the first four terms of the sequence and determine whether it is convergent or divergent. If the sequence converges, find its limit.

   a. {sinh n}

   b. $\left\{\sqrt{n^2+n}-\sqrt{n^2}\right\}$

   c. $\left\{\frac{n}{c^n}\right\}$ $c > 1$

2. Find $\lim\limits_{n\to\infty} a_n$ if it exists where

   a. $a_n = \left(1+\frac{3}{n}\right)^n$

   b. $a_n = \sqrt[3]{n+2}-\sqrt[3]{n+1}$

   c. $a_n = \left(\frac{n^2-1}{n}\right)^n$

   d. $a_n = \frac{n}{n+1}\sin\frac{n\pi}{2}$

   e. $a_n = \tanh n$

   f. $a_n = \frac{n^2(n!)}{(n+2)!}$

3. Determine whether the given sequence is monotonic or not, and convergent or not.

   a. $\left\{\frac{3^n}{2+3^n}\right\}$

   b. $\left\{\frac{n!}{n^n}\right\}$

   c. $\{\cos n\pi\}$

   d. $\left\{\ln\left(\frac{3n}{n+1}\right)\right\}$

   e. $\left\{\sqrt[n]{3^n+4^n}\right\}$

   f. $\left\{\left(1+\frac{1}{n}\right)^2\right\}$

g. $\left\{\frac{\ln(n+2)}{n+2}\right\}$ h. $\left\{\frac{3^n}{(n+1)^2}\right\}$

4. Determine the convergence or divergence of the sequence $\{a_n\}$ if

a. $a_n = \frac{\ln n}{n+1}$ b. $\left(1+\frac{1}{n^2}\right)^{3n}$ c. $\frac{1+3+5+\ldots+(2n-1)}{n^2}$

d. $a_n = \sum_{k=0}^{n} 2^{-k}$ e. $a_n = \frac{1-\left(1-\frac{1}{n}\right)^a}{1-\left(1-\frac{1}{n}\right)^b}$, where a and b are constants and $b \neq 0$.

5. i. Using $\lim_{n\to\infty} r^n = \begin{cases} 0 & if \quad |r|<1 \\ \infty & if \quad |r|>1 \end{cases}$

a. $\lim_{n\to\infty} \frac{1+\left(\frac{1}{3}\right)^n}{2}$ b. $\lim_{n\to\infty} \frac{1+(-1)^n}{e^{2n}}$ c. $\lim_{n\to\infty} \sqrt[n]{4^n+5^n}$

6. Let $\{a_n\}$ be sequence with $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$, $n \geq 1$.

a. Show that $\{a_n\}$ is convergent
b. Find $\lim_{n\to\infty} a_n$

7. Let i. $a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$. Show that $\lim_{n\to\infty} a_n = 1$.

ii. $b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$. Show that $\lim_{n\to\infty} b_n = 1$.

8. Using the limit theorem sequence it can be show that if $a_n \to 0$ and $\{b_n\}$ is bounded then $\lim_{n\to\infty} a_n b_n = 0$. Use this result to show the following sequences converge to 0.

i. $\left\{\frac{\cos n}{n}\right\}$ ii. $\left\{\frac{1}{n^2}\ln\left[1+\frac{1+(-1)^{n+1}}{n!}\right]\right\}$ iii. $\left\{\frac{2^n+(-1)^n}{e^{2n}}\right\}$

10. Find two limit points and subsequences, which converge to each of these points for

a. $\left\{(-1)^n\left(2-\frac{1}{n}\right)\right\}$ b. $\left\{\sin\frac{n\pi}{4}\right\}$

6. 4 Real Series

6.4.1 Definition and Notations of Infinite Series

We may use sequences to define expressions of the form

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots.$$

We call such an expression an infinite series. Since only finite sums may be added algebraically, we must define what is meant by this "infinite sum." As we shall see, the key to the definition of infinite series is to consider the sequence of partial sums $\{S_n\}$, where S_k is the sum of the first k number of the infinite series. For the infinite sum above the partial sums are given by

$$S_1=\frac{1}{2}$$

$$S_2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$

and so on. Thus, the sequence of partial sums $\{S_n\}$ may be written as

$$\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\ldots.$$

It follows that

$$S_n\to 1 \quad as \quad n\to\infty.$$

From intuitive point of view the more numbers of the infinite series that we add, the closer the sum gets to 1.Thus we write

$$1=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$$

and call 1 the sum of the infinite series.

Definition 6.14 An **infinite series** (or simply a **series**) is an expression of the form

$$a_1+a_2+a_3+\cdots+a_n+\ldots$$

or in summation notation,

$$\sum_{n=1}^{\infty}a_n \text{ or } \sum a_n$$

Each number a_k is a term of the series, and a_n is the **nth term**.

Definition 6.15 i) The **kth partial** sum S_k of the series $\sum a_n$ is

$$S_k=a_1+a_2+a_3+\ldots+a_k.$$

ii) The **sequence of partial sums** of the series $\sum a_n$ is

$$S_1, S_2, S_3,\ldots, S_n, \ldots$$

Definition 6.16 A series $\sum a_n$ is **convergent** (or **converges**) if its sequence of partial sums $\{S_n\}$ converges ,that is ,if

$$\lim_{n\to\infty}S_n=L \text{ for some real number L.}$$

Otherwise we say that the series $\sum a_n$ is **divergent** (or **diverges)**.

Note: Almost all series we will consider is of the form $\sum_{n=1}^{\infty} a_n$ or $\sum_{n=0}^{\infty} a_n$ .Thus for $\sum_{n=1}^{\infty} a_n$ the jth partial sum is $S_j = a_1 + a_2 + a_3 + \ldots + a_j$ and for $\sum_{n=0}^{\infty} a_n$ the jth partial sum is $S_j = a_0 + a_1 + a_3 + \ldots a_{j-1}$.

Example 1: Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots$$

converges and find its sum.

Solution: Using the partial sum representation of a_n we have

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{1+n} \quad for \quad n \geq 1$$

consequently the jth partial sum $S_j = \sum_{n=1}^{j} S_n$ of the series is given by

$$S_j = (1 - \frac{1}{2}) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{j-1} - \frac{1}{j}\right) + \left(\frac{1}{j} - \frac{1}{j+1}\right).$$

Since adjacent pairs of numbers cancel each other we have

$S_j = 1 - \frac{1}{j+1}$ and thus $\lim_{n\to\infty} S_j = \lim_{j\to\infty}\left(1 - \frac{1}{j+1}\right) = 1$.

Thus the sequence converges and the sum of the series is 1. ◊

The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is called **telescopic series**.

Example 2: The series $\sum (-1)^{n-1}$ diverges.

Solution: Since we can write S_n as

$$S_n = \begin{cases} 1 \;\; if \;\; n \; is \; odd \\ 0 \;\; if \;\; n \;\; is \;\; even \end{cases}$$

the sequence of partial sums $\{S_n\}$oscillates between 1 and 0, it follows that $\lim_{n\to\infty} S_n$ does not exist. Hence the series diverges.

Example 4: Show that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Solution: Grouping the terms of the series as

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)+\cdots$$

we can see that

$$S_2=1+\frac{1}{2}$$

$$S_{2^2}=S_4=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1+2\left(\frac{1}{2}\right)$$

$$S_{2^3}=S_8=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\geq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=1+3\left(\frac{1}{2}\right)$$

In general we arrange the making up $S_2^{\,j}$ into several groups and then substituting smaller values for the terms so that each group has sum $\frac{1}{2}$. Consequently we get

$$S_{2^j}\geq 1+j\left(\frac{1}{2}\right) \text{ and } \lim_{j\to\infty} S_j \geq \lim_{j\to\infty}\left[1+j\left(\frac{1}{2}\right)\right]=\infty .$$

Hence the sequence $\{S_j\}_{j=1}^{\infty}$ of partial sums is unbounded, as we wished to prove, and thus $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. ◊

Definition 6.17 The divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$ is called **harmonic series**.

6.4.2 Divergence Test and Properties of Convergent Series

Our next theorem which some times is called a **divergence test** (or **nth term test**) will tell us immediately that certain series diverge.

Theorem 6.18 a. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n$ =0.

b. If $\lim_{n\to\infty} a_n$ is not zero (or does nor exist),then $\sum_{n=1}^{\infty} a_n$ diverges.

Question: Is the converse of the statement in theorem 2.5a. above always true? if not give example.

The next illustration shows how to apply the nth-term test to a series.

SERIES	nTH-TERM TEST	CONCLUSION
$\sum_{n=1}^{\infty} 1+\frac{1}{n}$	$\lim_{n\to\infty} 1+\frac{1}{n} = 1 \neq 0$	Diverges by thm.2.5
$\sum_{n=1}^{\infty} \frac{1}{n^2}$	$\lim_{n\to\infty} \frac{1}{n^2} = 0$	Further investigation is needed.
$\sum_{n=1}^{\infty} \frac{1}{n}$	$\lim_{n\to\infty} \frac{1}{n} = 0$	Further investigation is needed.
$\sum_{n=1}^{\infty} \frac{2^n}{n}$	$\lim_{n\to\infty} \frac{2^n}{n} = \infty$	Diverges.

We have seen that the third series is the harmonic divergent series and we shall see in the next section that the second series is converges.

One of the very important series in solutions of applied problems is the **geometric series** which is of the form $\sum_{n=m}^{\infty} cr^n$, where r and c are constants and $c \neq 0$ .The convergence of geometric series $\sum_{n=m}^{\infty} cr^n$ depends entirely on the choice of r, as we see in the following theorem.

Theorem 6.19 Let r be any number, and let $c \neq 0$ and $m \geq 0$. Then the geometric series $\sum_{n=m}^{\infty} cr^n$ converges if and only if $|r| < 1$ and

$$\sum_{n=m}^{\infty} cr^n = \frac{cr^m}{1-r}.$$

Proof: We consider the cases $|r| \geq 1$ *and* $|r| < 1$ separately .If $|r| \geq 1, |cr^n| \geq |c|$ for all $n \geq m$ thus $\lim_{n\to\infty} cr^n \neq 0$ consequently by theorem 2.18(b) the series diverges. If $|r| < 1$, then we use the identity

$$(1-r)(1+r+r^2+\cdots+r^{j-1})=1-r^j$$

which implies that

$$S_j = cr^m + cr^{m+1} + \cdots + cr^{m+j-1} = cr^m(1+r+r^2+\cdots+r^{j-1})$$

$$=cr^m\left(\frac{1-r^j}{1-r}\right).$$

Since $\lim_{j\to\infty} r^j = 0$ as $|r| < 1$, it follows that

$$\lim_{j\to\infty} S_j = \frac{cr^m}{1-r}\lim_{j\to\infty}(1-r^j) = \frac{cr^m}{1-r}. \quad \Diamond$$

The number **r** is called the **common ratio** of the geometric series. By the Geometric Series theorem the sum of s convergent geometric series is equal to the first term (cr^m) divided by 1-r.

Example 5 Show that $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$

Solution: The common ratio of the series is $r = \frac{1}{2}$; hence the series is convergent by theorem 6.19. Since the first term is $(\frac{1}{2})^0 = 1$ we have

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2. \qquad \Diamond$$

Example 6 Determine whether or not the series $\sum_{n=0}^{\infty}(-1)^n \frac{3^{n+3}}{5^{n-1}}$ converges, and if so, find its sum.

Solution: Since $\sum_{n=0}^{\infty}(-1)^n \frac{3^{n+3}}{5^{n-1}} = \sum_{n=0}^{\infty}135\left(-\frac{3}{5}\right)^n$ and so the series is a geometric series with common ration $r = -\frac{3}{5}$, it is convergent. Since the first term is $135\left(\frac{3}{5}\right)^0 = 135$, the sum is given by

$$\sum_{n=0}^{\infty}(-1)^n \frac{3^{n+3}}{5^{n-1}} = \frac{135}{1-(-\frac{3}{5})} = \frac{675}{8}. \qquad \Diamond$$

Combination of Series

The proof of the next theorem follows directly from Definition (6.3)

Theorem 6.20 If $\sum a_n$ *and* $\sum b_n$ are convergent series, then

i. $\sum(a_n + b_n)$ converges and $\sum(a_n + b_n) = \sum a_n + \sum b_n$
ii. $\sum ca_n$ converges and $\sum ca_n = c\sum a_n$
iii. $\sum(a_n - b_n)$ converges and $\sum(a_n - b_n) = \sum a_n - \sum b_n$

Example 7 Show that the series $\sum_{n=1}^{\infty}\frac{8}{3^n} + \frac{6}{n(n+1)}$ converge, and find its sum.

Solution: The series $\sum_{n=1}^{\infty}\frac{8}{3^n}$ is a convergent geometric series of common ratio $r = \frac{1}{3}$, first term $\frac{8}{3}$ and sum $\sum_{n=1}^{\infty}\frac{8}{3^n} = \frac{\frac{8}{3}}{1-\frac{1}{3}} = 2$ .Moreover the series $\sum_{n=0}^{\infty}\frac{6}{n(n+1)}$ is the convergent telescopic series with sum $\sum_{n=0}^{\infty}\frac{6}{n(n+1)}$=6.

Consequently from theorem (2.20) we have

$$\sum_{n=1}^{\infty}\frac{8}{3^n}+\frac{6}{n(n+1)} = \sum_{n=1}^{\infty}\frac{8}{3^n} + \sum_{n=0}^{\infty}\frac{6}{n(n+1)} = 4+6=10. \quad \Diamond$$

Exercise 6.4

I. Compute the third, fourth and nth partial sums and find the sum of the series, if it converges.

1. $\sum_{n=1}^{\infty} 1$
2. $\sum_{n=1}^{\infty} (-1)^n$
3. $\sum_{n=1}^{\infty} \frac{-2}{(2n+5)(2n+3)}$
4. $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

II. Use the nth term test to determine whether the series diverges or needs further investigation.

1. $\sum_{n=1}^{\infty} \frac{2n}{4n-1}$
2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)$
3. $\sum_{n=1}^{\infty} \tan\left(\frac{\pi}{2}+\frac{1}{n}\right)$
4. $\sum \frac{1}{\sqrt[n]{e}}$

III. Express the repeating decimals as a fraction.

1. $0.\overline{2}$
2. $2.\overline{32}$

IV. Determine whether the following series converge and if so find its sum.

1. $\sum_{n=1}^{\infty} \frac{3}{(n+3)(n+4)}$
2. $\sum_{n=1}^{\infty} \frac{3^{n+3}}{5^{n-1}}$
3. $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}-\frac{1}{n}\right)$
4. $\sum_{n=0}^{\infty} \frac{3^n+4^n}{3^n 4^n}$

6.5 Convergence Tests for Nonnegative Terms Series

In this section we will develop tests for convergence or divergence of a series $\sum a_n$ that employ the nth term. These tastes will not give us the sum S of the series, but instead will tell us only whether the sum exists. For the present we will restrict our attention to **nonnegative series**, that is, to series whose terms are nonnegative. For simplicity we assume that the initial index is 1. The sequence of partial sums $\{S_j\}_{j=1}^{\infty}$ of the nonnegative series $\sum a_n$ form an increasing sequence:

$$S_j = a_1 + a_2 + \ldots + a_j \le a_1 + a_2 + \ldots + a_j + a_{j+1} = S_{j+1} \quad \textit{for} \quad j \ge 1$$

Consequently if $\{S_j\}_{j=1}^{\infty}$ is bounded, then $\lim_{j\to\infty} S_j$ exists, so $\sum_{n=1}^{\infty} a_n$ converges. By contrast, if $\{S_j\}_{j=1}^{\infty}$ is unbounded, then $\lim_{j\to\infty} S_j$ cannot exist, so $\sum_{n=1}^{\infty} a_n$ diverges.

We now discuss four important convergence tests in the following theorems.

6.5.1 The Integral Test

Theorem 6.21 Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative sequence, and let f be a continuous, decreasing function defined on $[1,\infty)$ such that

$$f(n) = a_n \quad \text{for } n \ge 1$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges.

Let us see how we can use theorem (6.21) in solving problems.

Example 1 Use the integral test to prove that the harmonic series

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges.

Solution: Let $f(x)=\frac{1}{x}$ for $x\geq 1$, then $f(n)=\frac{1}{n}$. Since f is nonnegative valued function and decreasing we can apply the integral test: So

$$\int_1^{\infty}\frac{1}{x}dx=\lim_{t\to\infty}\int_1^t\frac{1}{x}dx=\lim_{t\to\infty}[\ln x]_1^t=\lim_{t\to\infty}[\ln t-\ln 1]=\infty$$

thus the series diverges.

Example 2 Determine whether the infinite series $\sum_{n=1}^{\infty}ne^{-n^2}$ converges.

Solution: Let $f(x)=xe^{-x^2}$ for $x\geq 1$ then $f(n)=ne^{-n^2}$. f is nonnegative-valued and since $f'(x)=e^{-x^2}-2x^2e^{-x^2}=e^{-x^2}(1-2x^2)<0$, f is decreasing on $[1,\infty)$. We may therefore apply the integral test as follows:

$$\int_1^{\infty}xe^{-x^2}dx=\lim_{t\to\infty}\int_1^t xe^{-x^2}dx=\lim_{t\to\infty}\left[(-\tfrac{1}{2})e^{-x^2}\right]_1^t=-\frac{1}{2}\lim_{t\to\infty}\left[\frac{1}{e^{t^2}}-\frac{1}{e}\right]=\frac{1}{2e}$$

Hence the series converges.

Definition 6.22 A **p-series**, or a **hyperharmonic series**, is a series of the form

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots,$$

where p is a positive real number.

Example 3 Show that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if p>1.

Solution: If $p\leq 0$, then $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)\neq 0$ and, by theorem (6.18), the series diverges. If $p=1$, we have the divergent harmonic series. Hence from here on in the proof, we assume that $p>0$ *and* $p\neq 1$. We shall employ the integral test, defining the ideal function f by

$$f(x)=\frac{1}{x^p}\quad for\ x\geq 1.$$

Since f is continuous, $f(n)=\frac{1}{n^p}$, $f'(x)=-px^{-p-1}<0$, and hence f is decreasing f satisfies the conditions stated in the integral test. Thus we have

$$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{t\to\infty}\int_1^t x^{-p}dx=\lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t=\frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1).$$

If p>1, then p-1>0 and the last expression may be written

$\frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1)=\frac{1}{1-p}(0-1)=\frac{1}{1-p}$, hence the improper integral converges consequently the p-series converges by theorem (6.8) if p>1.

If 0<p<1, then 1-p>0 and

$$\frac{1}{1-p}\lim_{t\to\infty}(t^{1-p}-1)=\infty.$$

Hence, by theorem (6.8), the p-series diverges. ◊

ILLUSTRATION

	p-SERIES	VALUE OF p	CONCLUSION
I	$\sum_{n=1}^{\infty}\frac{1}{n^3}$	p=3	Converges
II	$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$	p=1/3	Diverges

Class Work

Use the integral test to determine whether the series converges or diverges for the series:

a. $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ b. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ c. $\sum_{n=1}^{\infty}n^2e^{-n^3}$

d. $\sum_{n=1}^{\infty}\frac{\arctan n}{1+n^2}$

6.5.2 Basic Comparison Tests

The next theorem allows us to use known convergent (divergent) series to establish the convergence (divergence) of other series.

Theorem 6.23(Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be nonnegative-term series.

i. If $\sum b_n$ converges and $a_n \le b_n$ for every positive integer n, then $\sum a_n$ converges.

ii. If $\sum b_n$ diverges and $a_n \ge b_n$ for every positive integer n, then $\sum a_n$ diverges.

Example 4 Show that $\sum_{n=1}^{\infty}\frac{1}{3+4^n}$ converges.

Solution: Since $\frac{1}{3+4^n}\le\frac{1}{4^n}$ *for* $n\ge 1$, and the series $\sum_{n=1}^{\infty}\frac{1}{4^n}$ is a convergent Geometric series it follows from theorem 6.23(a) that $\sum_{n=1}^{\infty}\frac{1}{3+4^n}$ converges.

Example 5 Determine whether the series below converges or diverges.

a. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$ b. $\sum_{n=0}^{\infty}\frac{3\sin^2 n}{n!}$ c. $\sum_{n=1}^{\infty}\frac{1}{2^n-1}$

Solution: a) Since $\frac{1}{\sqrt{n}-1}\ge\frac{1}{\sqrt{n}}$ *for* $n\ge 2$ and the series $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$ is the divergent p-series with p=1/2, it follows from theorem 6.23(b) that the series $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$ diverges.

b) Since $\frac{3\sin^2 n}{n!} \le \frac{3}{n!}$ $\quad for \;\; n \ge 0$ and $\sum_{n=0}^{\infty}\frac{3}{n!} = 3\sum_{n=0}^{\infty}\frac{1}{n!} = 3e$,it follows from theorem 6.23 b) that the series $\sum_{n=0}^{\infty}\frac{3\sin^2 n}{n!}$ converges.

c) It might be tempting to compare the give series with the convergent series $\sum_{n=1}^{\infty}\frac{1}{2^n}$.

However,

$$\frac{1}{2^n - 1} \ge \frac{1}{2^n} \qquad for \;\; n \ge 1$$

and thus it is impossible to determine the convergence or divergence of the given series by comparing it with the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$. But we can see that

$$\frac{1}{2^n - 1} \le \frac{1}{2^n - 2^{n-1}} = \frac{1}{2^{n-1}(2-1)} = \frac{1}{2^{n-1}} \quad for \;\; n \ge 1$$

and since $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ converges, the comparison test implies that the given series converges.

Class work: Determine whether the following series converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{1}{n^4 + n^2 + 1}$ $\qquad$ b. $\sum_{n=1}^{\infty}\frac{\arctan n}{n}$ $\qquad$ c. $\sum_{n=1}^{\infty}\frac{n^2}{n^3 + 1}$

Theorem 6.24(Limit Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be nonnegative series.

a. If $\lim_{n\to\infty}\frac{a_n}{b_n}$=L>0, then either both series converge or both diverge.

b. If $\lim_{n\to\infty}\frac{a_n}{b_n}$=0 and $\sum b_n$ converges, then $\sum a_n$ also converges.

c. If $\lim_{n\to\infty}\frac{a_n}{b_n} = \infty$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

To find a suitable series $\sum b_n$ to use in the limit comparison test when a_n is a quotient, a good procedure is to delete all terms in the numerator and denominator of a_n except those that have the greatest effect on the magnitude. We may also replace any constant factor c by 1.

Example 6 Determine whether the series converges or diverges:

a. $\sum_{n=1}^{\infty}\frac{2n^2 + 1}{n^5 + 7n^3 - 2}$ $\qquad$ b. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2 + 1}}$

Solution: a. Let $a_n = \frac{2n^2 + 1}{n^5 + 7n^3 - 2}$ then deleting terms of least magnitude both from the numerator and denominator we get $\frac{2n^2}{n^5} = \frac{2}{n^3}$. If we choose $b_n = \frac{1}{n^3}$, the series

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ will be a convergent p-series (with p=3). Since

$$\lim_{n\to\infty} \frac{\frac{2n^2+1}{n^5+7n^3-1}}{\frac{1}{n^3}} = 2 > 0,$$

it follows from limit comparison test (a) that the series $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^5+7n^3-2}$ converges.

b. Let $a_n = \frac{1}{\sqrt[3]{n^2+1}}$ *we then may choose* $b_n = \frac{1}{n^{\frac{2}{3}}}$. Since the series $\sum \frac{1}{\sqrt[3]{n^2}}$ is a divergent p-series (with p=2/3) and

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n^2+1}}}{\frac{1}{\sqrt[3]{n^2}}} = 1 > 0$$

it follows from limit comparison test (a) that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ diverges.

Example 7 Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence.

Solution: Let $a_n = \ln/n$ *and* $b_n = 1/n$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln/n}{1/n} = \lim_{n\to\infty} \ln n = \infty$$

We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent so, by part (c) of the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent.

Class Work: Determine whether the series converges or diverges:

a. $\sum_{n=1}^{\infty} \frac{n+\ln n}{n^3+2n+3}$ b. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ c. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

6.5.3 The Ratio Test and the Root Test

The ratio test and the root test are tests that involve only the terms of the series being tested; it is not necessary to manufacture another series, an improper integral, or anything else against which to compare the given series.

Theorem 6.25 (Ratio Test)

Let $\sum a_n$ be a nonnegative series. Assume that $a_n \neq 0$ for all n and that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r \quad (possibly\ \infty)$$

a. If $0 \le r < 1$ then $\sum a_n$ converges.

b. If r>1, then $\sum a_n$ diverges.

If r=1, then from this test alone we cannot draw any conclusion about the convergence of $\sum a_n$.

Example 1 Determine whether the series converges or diverges.

a. $\sum_{n=0}^{\infty}\frac{n!}{2^n}$ b. $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ c. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: a. Applying the ratio test we have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!}=\lim_{n\to\infty}\frac{n+1}{2}=\infty .$$

Since $r\notin[0,1)$, the series diverges by theorem (2.25).

b. Since $r=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^2/2^{n+1}}{n^2/2^n}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^2=\frac{1}{2}$ and r<1 by the ration test the series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ converges.

c. Since $r=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}=\frac{1}{e}<1$

consequently from theorem 6.25 that the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges.

Theorem 6.26 (Root Test)

Let $\sum a_n$ be a nonnegative series. Assume that

$$\lim_{n\to\infty}\sqrt[n]{a_n}=r \quad (possibly\ \infty)$$

a. If $0\le r<1$ then $\sum a_n$ converges.

b. If r>1, then $\sum a_n$ diverges.

If r=1, then from this test alone we cannot draw any conclusion about the convergence of $\sum a_n$.

Example 2 Determine the convergence or divergence of

a. $\sum_{n=1}^{\infty}\left(\frac{n}{\ln n}\right)^n$ b. $\sum_{n=1}^{\infty}\frac{n}{3^n}$ c. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ $d.\sum n\left(\frac{\pi}{4}\right)^n$

Solution: a. Applying the root test we have

$$r=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{\ln n}\right)^n}=\lim_{n\to\infty}\left(\frac{n}{\ln n}\right)\{\tfrac{\infty}{\infty}\}=\lim_{n\to\infty}\frac{1}{1/n}=\infty>1.$$

Since r>1, the series diverges.

b. Applying the root test and the fact that $\lim_{n\to\infty}\sqrt[n]{n}=1$ we have

$$r=\lim_{n\to\infty}\left(\frac{n}{3^n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n}}{3}=\frac{1}{3}<1.$$

Hence the series converges. ◊

The c) and d) are left as exercise.

6.6 Alternating Series Test

If the terms in a series are alternately positive and negative, we call the series an **alternating series**. For instance, the series

$$\sum_{n=1}^{\infty}(-1)^n=-1+1-1+1-\cdots$$

and
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}=1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\cdots$$

are alternating series.

Theorem 6.27 (Alternating Series Test)

Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty}a_n=0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converge. Furthermore, for either series the sum S and the sequence of partial sums $\{S_j\}_{j=1}^{\infty}$ satisfy the inequality

$$\left|S-S_j\right|\le a_{j+1}$$

Example 1 Show that the **alternating harmonic** series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

converges.

Solution: Let $a_n=1/n$, since $a_n=1/n > 1/n+1 = a_{n+1}$ the sequence $\{a_n\}$ is a decreasing, nonnegative sequence such that $\lim_{n\to\infty}a_n=0$. Therefore the alternating harmonic series satisfies the conditions of the Alternating Series Test and consequently must converge.

Example 2 Determine the convergence or divergence of the alternating series:

a. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n^2-3}$ b. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{4n-3}$

Solution: a. Let $a_n=\frac{2n}{4n^2-3}$.

In applying the alternating series test, we must show that

i. $\{a_n\}$ is decreasing

ii. $\lim_{n\to\infty}a_n=0$

There are several ways to prove (i). One method is to show that the ideal function to a_n, $f(x)=\frac{2x}{4x^2-3}$ is decreasing for $x\ge1$. So finding the derivative of f we have

$$f'(x) = \frac{(4x^2 - 3)2 - 2x(8x)}{(4x^2 - 3)^2}$$
$$= \frac{-8x^2 - 6}{(4x^2 - 3)^2} < 0.$$

Hence f is decreasing and, therefore, f(n)≥f(n+1); that is, $a_n \geq a_{n+1}$ for every positive integer k.

To prove (ii), we see that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n^2 - 3} = 0$$

Thus, by the alternating series test the series is convergent.

b. we may show that $\left\{\frac{2n}{4n-3}\right\}$ is decreasing; however,

$$\lim_{n\to\infty} \frac{2n}{4n-3} = \frac{1}{2} \neq 0,$$

and hence the series diverges.

Example 3 Prove that the series

$$1 - \frac{1}{3!} + \frac{1}{5!} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)!} + \ldots$$

is convergent, and approximate its sum S to five decimal places.

Solution: The *n*th term $a_n = 1/(2n-1)!$ has the limit 0 as $n \to \infty$, and $a_k > a_{k+1}$ for every positive integer *k*. Hence the series converges, by the alternating series test. If we use S_n to approximate *S*, then, by Theorem (6.27), the error involved is less than or equal to $a_{n+1} = 1/(2n+1)!$. Calculating several values of a_{n+1}, we find that for n = 4,

$$a_5 = \frac{1}{9!} \approx 0.0000028 < 0.000005.$$

Hence the partial sum S_4 approximates *S* to vive decimal places. Since

$$S_4 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}$$
$$= 1 - \frac{1}{60} + \frac{1}{120} - \frac{1}{5040} \approx 0.841468,$$

we have $S \approx 84147$.

It follows from the next section that the sum of the given series is $\sin 1$, and hence $\sin 1 \approx 84147$.

Class Work

Determine whether the series converges or diverges.

a. $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ b. $\sum_{n=1}^{\infty} (-1)^n \coth n$. c. $\sum_{n=3}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$

6.7 Absolute and Conditional Convergence

The following theorem is very useful in investigating the convergence of a series that is neither nonnegative nor alternating. It allows us to use tests that are applicable for nonnegative term series to establish convergence for other types of series.

Theorem 6.28 If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Proof If we let $b_n = a_n + |a_n|$ and use the property $-|a_n| \le a_n \le |a_n|$ we have

$$0 \le a_n + |a_n| \le 2|a_n|, \text{ or } 0 \le b_n \le 2|a_n|.$$

If $\sum_{n=1}^{\infty}|a_n|$ is convergent then from the convergent properties of series we know that $\sum_{n=1}^{\infty}2|a_n|$ is convergent. If we apply the basic comparison test, it follows that $\sum_{n=1}^{\infty}b_n$ is convergent. And again by the convergent properties of series $\sum_{n=1}^{\infty}(b_n - |a_n|)$ is convergent. Since $b_n - |a_n| = a_n$, $\sum_{n=1}^{\infty}a_n$ is convergent.

Example 1 Prove that the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)^3 = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$$

converges.

Solution: Since $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\left(\frac{1}{n}\right)^3\right| = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^3$ is a convergent p-series with p=3, from Theorem (6.28) the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)^3$ converges. ◊

Example 2 Show that the series $\sum_{n=1}^{\infty}\frac{\cos n\pi/4}{n^2}$ converges.

Solution: Calculating a few values of $\cos n\pi/4$, we can see that the series is neither nonnegative nor alternating. Thus none of the earlier tests applies directly to it. However, since

$$\left|\frac{\cos n\pi/4}{n^2}\right| \le \frac{1}{n^2} \qquad \text{for } n \ge 1$$

and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because it is a p-series with p=2, we know by the comparison test that the series $\sum_{n=1}^{\infty}\left|\frac{\cos n\pi/4}{n^2}\right|$ converges. Consequently Theorem (6.28) tells us that the given series is convergent. ◊

Oral Question: Is the converse of theorem (6.28) always true?

The following definition gives us two special names of series.

Definition 6.29 Let $\sum_{n=1}^{\infty} a_n$ be convergent series. If $\sum_{n=1}^{\infty} |a_n|$ converges, we say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely**. If $\sum_{n=1}^{\infty} |a_n|$ diverges, we say that the series $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 3 The alternate harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n}\right)$ is conditionally convergent while the series $\sum_{n=1}^{\infty} \left|\frac{\cos n\pi/4}{n^2}\right|$ is absolutely convergent. Note also that all convergent nonnegative series converge absolutely.

Class Work 2. Determine the following series converges conditionally or absolutely:

a. $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{3n+4}$ b. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)}$ c. $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

6.8 Generalized Convergence Tests

By combining Theorem (2.15) with our tests for nonnegative series, we obtain convergence tests that apply to any series, nonnegative or not.

Theorem 6.30 (GENERALIZED CONVERGENCE TESTS)

Let $\sum_{n=1}^{\infty} a_n$ be a series.

a. *Generalized Comparison Test.* If $|a_n| \le |b_n|$ for $n \ge 1$, and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

b. *Generalized limit Comparison Test.* If $\lim_{n\to\infty} |a_n / b_n| = L$, where L is a positive number, and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges(absolutely).

c. *Generalized Ratio Test*. Suppose that $a_n \neq 0$ *for* $n \ge 1$ and that

$$\lim_{n\to\infty} \left|\frac{a_n + 1}{a_n}\right| = r \quad (\textit{possibly } \infty)$$

If r<1, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely). If r>1, then $\sum_{n=1}^{\infty} a_n$ diverges.

If r=1, then from this test alone we cannot draw any conclusion about the convergence of the series.

d. *Generalized Root Test*. Suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = r \quad (possibly\ \infty)$$

If r<1, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely). If r>1, then $\sum_{n=1}^{\infty} a_n$ diverges.
If r=1, then from this test alone we cannot draw any conclusion about the convergence of the series.

Example 3 Show that the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges absolutely for $|x|<1$, converges conditionally for x=-1 and diverges for x=1 and for $|x|>1$.

Solution: If x=0, the series converges. If $x \neq 0$, then

$$\lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)}{x^n/n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}x\right| = |x|.$$

Therefore the Generalized Ratio Test implies that the given series converges for $|x|<1$ and diverges for $|x|>1$. For x=1 the series becomes the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges. For x=-1 the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

This is the negative of the alternating harmonic series and consequently converges. Since

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges, we conclude that $\sum_{n=1}^{\infty}(-1)^n/n$ converges conditionally. ◊

Example 4 Show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

converges absolutely for $|x|<1$, converges conditionally for $|x|=1$, and diverges for $|x|>1$.

Solution: If x=0, the series converges. If $x \neq 0$, then we have

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n}{2(n+1)+1}x^{2(n+1)+1}}{\frac{(-1)^n}{2n+1}x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}x^2\right| = \left|x^2\right|$$

Consequently the Generalized Ratio Test implies that the series converges absolutely for $|x|<1$ and diverges for $|x|>1$. It remains to consider the cases in which $|x|=1$. For $x=-1$ The series becomes

$$\sum_{n=0}^{\infty}\frac{-(-1)^n}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$$

and this converges by the Alternating Series Test. For x=1 the series reduces to

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

which also converges by the Alternating Series Test. It is easy to show by using the Integral Test or the Limit Comparison Test that

$$\sum_{n=0}^{\infty}\frac{1}{2n+1}$$

diverges. Hence the given series converges conditionally for $|x| = 1$. ◊

Corollary 6.18 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < 1 \quad or \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = r < 1$$

then

$$\lim_{n\to\infty} a_n = 0$$

Class Work: **1.** Show that $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ for all x.

2. Show that the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$ converges for $|x| < e$.

Worksheet VII

1. Give the fourth, fifth, and nth partial sums of

   a. $\sum_{n=0}^{\infty}(-1)^n$ b. $\sum_{n=1}^{\infty}\frac{5}{(5n+2)(5n+7)}$ c. $\sum_{n=1}^{\infty}\sin\left(\frac{\pi}{2}-\frac{1}{n}\right)$ $d.\sum_{n=1}^{\infty}(-1)^{n-1}4^{-n}$

2. Use the nth term test to determine whether the series diverges or needs further investigation.

   a. $\sum_{n=1}^{\infty}\sin n\pi$ $b.$ $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)$ $c.$ $\sum_{n=1}^{\infty}n\sin\frac{1}{n}$

   d. $\sum_{n=1}^{\infty}\frac{n}{\ln(n+1)}$ e. $\sum_{n=1}^{\infty}\ln\left(\frac{2n}{7n-5}\right)$

3. Determine whether the following series converge and if so find its sum.

   a. $\sum_{n=1}^{\infty}\frac{-1}{9n^2+3n-2}$ b. $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$ c. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$

   d. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ e. $\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{2^{n+2}}$ f. $\sum_{n=1}^{\infty}\left[\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right]$

   g. $\sum_{n=1}^{\infty}\frac{3^n+2^n}{6^n}$ h. $\sum_{n=2}^{\infty}\ln\left(\frac{n^2-1}{n^2}\right)$ i. $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$

4. If the nth partial sum of a series $\sum_{n=1}^{\infty}a_n$ is

   $$s_n=\frac{n-1}{n+1} \qquad \text{find } a_n \text{ and } \sum_{n=1}^{\infty}a_n .$$

5. Find the value of x for which the series converges, and find the sum of the series for

   a. $1-x+x^2-x^3+\cdots+(-1)^{n-1}x^{n-1}+\cdots$

   b. $\frac{1}{2}+\frac{(x-3)}{4}+\frac{(x-3)^2}{8}+\cdots+\frac{(x-3)^n}{2^{n+1}}+\cdots$

6. Use CT, LCT, IT, RaT, or Root Test to determine whether the series below converges or diverges:

   a. $\sum_{n=1}^{\infty}\frac{1}{e^{n^2}}$ b. $\sum_{n=1}^{\infty}\frac{n}{5^n}$ c. $\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{3/2}}$ d. $\sum_{n=1}^{\infty}\sin\frac{1}{n^2}$

   e. $\sum_{n=1}^{\infty}\frac{n^2+2^n}{n+3^n}$ f. $\sum_{n=1}^{\infty}\ln(1+\frac{1}{2^n})$ g. $\sum_{n=1}^{\infty}\frac{\sin n+2^n}{n+5^n}$ h. $\sum_{n=1}^{\infty}\frac{2n!}{(n!)^2}$

   i. $\sum_{n=1}^{\infty}\left(\frac{n!}{n^n}\right)^n$ j. $\sum_{n=1}^{\infty}\frac{1\cdot 4\cdot 7\cdots(3n-2)}{2\cdot 4\cdot 6\cdots(2n)}$ k. $\sum_{n=1}^{\infty}\frac{\sin 1/n!}{\cos 1/n!}$ l. $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^n$

7. Find every real number k for which the series below converge.

a. $\sum_{n=1}^{\infty} \frac{1}{n^k \ln n}$ b. $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^k}$

8. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

a. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{2n+1}}$ b. $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan n}{n^2}$ c. $\sum_{n=1}^{\infty} \frac{\cos \frac{1}{6}\pi n}{n^2}$

d. $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$ e. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$ d. $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+1} - n\right)$

9. Discuss the convergence of the following series

a. $\frac{1 \cdot 2}{3 \cdot 4 \cdot 5} + \frac{2 \cdot 3}{4 \cdot 5 \cdot 6} + \frac{3 \cdot 4}{5 \cdot 6 \cdot 7} + \cdots$

b. $\frac{1}{2} + \frac{3!}{2 \cdot 4} + \frac{5!}{2 \cdot 4 \cdot 6} + \cdots$

c. $\frac{1}{1+x^2} + \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} + \cdots$ for real of x.

d. $\frac{1}{1+x} + \frac{1}{1+2x^2} + \frac{1}{1+3x^3} + \cdots$ for positive values of x.

e. $\sum_{n=1}^{\infty} c_n,\ where\ c_n = \begin{cases} -\frac{1}{n} & if\ n\ is\ a\ perfect\ square \\ \frac{1}{n^2} & if\ n\ is\ not\ a\ perfect\ squar \end{cases}$

e. $\sum_{n=1}^{\infty} c_n$, where $c_n = \begin{cases} -\frac{1}{n} & if\ \frac{1}{4}n\ is\ an\ integer. \\ \frac{1}{n^2} & if\ \frac{1}{4}n\ is\ not\ an\ integer. \end{cases}$

10. A series $\sum a_n$ is defined recursively by the equations

$$a_1 = 1,\ a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$$

Determine whether $\sum a_n$ converges or diverges.

11. If $\sum a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is divergent, show that the series $\sum (a_n + b_n)$ is divergent.

12. Consider $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$

a) By using the pattern of the partial sums the first four partial sums guess a formula for the nth partial sum.
b) Use mathematical induction to prove your guess. Show that the given infinite sum is convergent and find its sum.