A Proof of a Harmonic Number Identity via Analytic Continuation of Finite Differences

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Abstract

We prove a well-known formula for the N-th harmonic number $H_N = \sum_{n=1}^{N} \frac{1}{n} = \frac{\Gamma'(N+1)}{\Gamma(N+1)} - \frac{\Gamma'(1)}{\Gamma(1)}$ using a new approach and generalize it so it can be applied to other sequences.

1 Introduction

1.1 Notations

(i) $\mathbb{N}$ is the set of the natural numbers $\{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N}_- := \{-n : n \in \mathbb{N}\}$.

(ii) $\Delta = \Delta_n$ is the forward difference operator, i.e. $\Delta a_n := a_{n+1} - a_n$.

Also recall that $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ with $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

1.2 Overview of the Result

To prove this formula for the N-th harmonic number $H_N = \sum_{n=1}^{N} \frac{1}{n}$, we will need the following auxiliary function

$$f : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \to \mathbb{R}, f(n, z) := \cos(\pi z) \frac{\Gamma(z+1)\Gamma(n)}{\Gamma(n+z+1)}. \tag{1}$$

We show that this function analytically continues the forward difference operator applied to the sequence $\frac{1}{n}$ in the difference order k, because, for $k \in \mathbb{N}_0$, we have $\Delta^k\left(\frac{1}{n}\right) = f(n, k)$.

Theorem 1.2.1. $H_N = \sum_{n=1}^{N} \frac{1}{n} = \lim_{z \to -1}[f(N+1, z) - f(1, z)] = \frac{\Gamma'(N+1)}{\Gamma(N+1)} - \frac{\Gamma'(1)}{\Gamma(1)}$.

The formula will be proven using the following general result:

Lemma 1.2.2. *Assume $g : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \to \mathbb{R}$ is a function s.t.*

(i) $z \mapsto g(n, z)$ is continuous on $(-1, \infty) \cup U$, where U is a deleted neighborhood of -1, for all $n \in \mathbb{N}$,

(ii) $g(n+1, z) - g(n, z) = g(n, z+1)$ for all $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}_-)$,

(iii) $\lim_{z \to -1}[g(n, z) - g(m, z)]$ exists for all $n, m \in \mathbb{N}$.

Then $\sum_{n=1}^{N} g(n, 0) = \lim_{z \to -1}[g(n+1, z) - g(1, z)]$

Proof. Take a $z \in U$. Then

$$\sum_{n=1}^{N} g(n, z) = \sum_{n=1}^{N} [g(n+1, z-1) - g(n, z-1)] = g(N+1, z-1) - g(1, z-1). \tag{2}$$

Using continuity of $z \mapsto g(n, z)$ and taking the limit as $z \to 0$ yields the claim. $\square$

2 Proof of Theorem 1.2.1

$$
\begin{aligned}
\text{Consider } \Delta^0\left(\frac{1}{n}\right) &\stackrel{\text{def}}{=} \frac{1}{n} \\
\Delta^1\left(\frac{1}{n}\right) &= \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} \\
\Delta^2\left(\frac{1}{n}\right) &= -\frac{1}{(n+1)(n+2)} - \left(-\frac{1}{n(n+1)}\right) = \frac{2}{n(n+1)(n+2)} \\
\Delta^3\left(\frac{1}{n}\right) &= -\frac{6}{n(n+1)(n+2)(n+3)} \\
&\dots \\
\implies \Delta^k\left(\frac{1}{n}\right) &= (-1)^k k! \frac{(n-1)!}{(n+k)!} \text{ follows inductively.}
\end{aligned}
\tag{3}
$$

Immediately, it is neither possible to set $k = -1$ to get a primitive sequence nor to take the limit $\lim_{k \to -1} \Delta^k\left(\frac{1}{n}\right)$. The first step is to obtain a continuous version of the identity above using

$$
\Gamma(n) = (n-1)! \ (n \in \mathbb{N}) \qquad (-1)^k = \cos(\pi k) \ (k \in \mathbb{Z}),
\tag{4}
$$

which motivates the definition of f in (1). Using (3) and (4), we get

$$
\Delta^k\left(\frac{1}{n}\right) = f(n,k) \text{ for } n \in \mathbb{N}, k = 0, 1, 2, \dots
\tag{5}
$$

Thus it remains to show that f satisfies (ii) and (iii) to apply Lemma 1.2.2.

Lemma 2.1. *$f(n+1,z) - f(n,z) = f(n,z+1)$ for all $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}\_)$.*

Proof. Using the functional equation (4) twice

$$
\Gamma(z+2) = (z+1)\Gamma(z+1), \ \ \Gamma(n+z+2) = (n+z+1)\Gamma(n+z+1),
$$

we obtain

$$
\begin{aligned}
f(n+1,z) - f(n,z) &= \cos(\pi z)\Gamma(z+1)\left(\frac{\Gamma(n+1)}{\Gamma(n+z+2)} - \frac{\Gamma(n)}{\Gamma(n+z+1)}\right) \\
&= \cos(\pi z)\frac{\Gamma(z+2)}{z+1}\left(\frac{\Gamma(n)n - \Gamma(n)(n+z+1)}{\Gamma(n+(z+1)+1)}\right) \\
&= \underbrace{-\cos(\pi z)}_{\cos(\pi(z+1))}\Gamma(z+2)\frac{\Gamma(n)}{\Gamma(n+(z+1)+1)} = f(n,z+1).
\end{aligned}
$$

□

Lemma 2.2. *The limit in Theorem 1.2.1 exists for all $N \in \mathbb{N}$.*

Proof. Take any $n \in \mathbb{N}$. Then, using Lemma 2.1

$$
\begin{aligned}
f(n+1,z) - f(n,z) = f(n,z+1) &= \cos(\pi(z+1))\frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+2)} \\
&= \cos(\pi(z+1))\frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+1)(n+z+1)} \\
&\to \frac{\Gamma(1)\Gamma(n)}{\Gamma(n)}\frac{1}{n} = \frac{1}{n} \text{ for } z \to -1.
\end{aligned}
\tag{6}
$$

The rest follows inductively with the telescope sum argument. □

Now, we have proven the first part of the identity

$$H_N = \sum_{n=1}^{N} \frac{1}{n} = \lim_{z \to -1} [f(N+1, z) - f(1, z)] = \frac{\Gamma'(N+1)}{\Gamma(N+1)} - \frac{\Gamma'(1)}{\Gamma(1)}. \tag{7}$$

The rest is technical. Observe that

$$\begin{gathered} \frac{\Gamma(n+z+1) - \Gamma(n)}{\Gamma(n+z+1)(z+1)} \to \frac{\Gamma'(n)}{\Gamma(n)}, z \to 0 \\ \text{and} \\ \frac{\Gamma(z+2) - 1}{z+1} \to \Gamma'(1) = \frac{\Gamma'(1)}{\Gamma(1)}, z \to 0 \end{gathered} \tag{8}$$

Thus,

$$\begin{aligned} & f(N, z) - f(1, z) - \left(\frac{\Gamma(N+z+1) - \Gamma(N)}{\Gamma(N+z+1)(z+1)} - \frac{\Gamma(z+2) - 1}{z+1} \right) = \\ & = -\frac{\Gamma(z+1)\Gamma(N)}{\Gamma(N+z+1)} + \underbrace{\frac{\Gamma(z+1)}{\Gamma(z+2)}}_{(z+1)^{-1}} - \frac{\Gamma(N+z+1) - \Gamma(N)}{\Gamma(N+z+1)(z+1)} + \frac{\Gamma(z+2) - 1}{z+1} \\ & = \frac{\Gamma(N+z+1) - \Gamma(z+2)\Gamma(N) + [\Gamma(z+2) - 1]\Gamma(N+z+1) + \Gamma(N)}{\Gamma(N+z+1)(z+1)} \\ & = \underbrace{\frac{\Gamma(z+2) - 1}{z+1}}_{\to \Gamma'(1)} \cdot \underbrace{\frac{\Gamma(n+z+1) - \Gamma(n)}{\Gamma(n+z+1)}}_{\to 0} \to 0, z \to -1. \end{aligned} \tag{9}$$

Thus, we finally arrive at

$$H_N = \sum_{n=1}^{N} \frac{1}{n} = \frac{\Gamma'(N+1)}{\Gamma(N+1)} - \frac{\Gamma'(1)}{\Gamma(1)}. \tag{10}$$

Remark 2.3. $\frac{\Gamma'(1)}{\Gamma(1)} = \gamma$ is known to be the famous Euler-Mascheroni constant.

3 Generalizations

The approach in Lemma 1.2.2 can be used in other cases. As a simple example, consider the sequence $a_n = b^n$, $b \in \mathbb{C} \setminus \{1\}$.

$$\Delta a_n = b^{n+1} - b^n = (b-1)a_n. \tag{11}$$

In this case, $b - 1$ is an eigenvalue of Δ. Thus,

$$\Delta^k a_n = (b-1)^k a_n, \tag{12}$$

and we obtain the sum of the geometric progression by setting $k = -1$ (with no need for limit since everything is continuous and well-defined):

$$\sum_{n=0}^{N} a_n = \Delta^{-1} a_{N+1} - \Delta^{-1} a_1 = \frac{a^{N+1} - a_0}{b-1} = \frac{b^{N+1} - 1}{b-1}. \tag{13}$$