

# A Proof of a Harmonic Number Identity via Analytic Continuation of Finite Differences

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## Abstract

We prove a formula for the  $N$ -th harmonic number  $H_N = \sum_{n=1}^N \frac{1}{n}$  and generalize this approach so it can be applied to other sequences.

## 1 Introduction

### 1.1 Notations

- (i)  $\mathbb{N}$  is the set of the natural numbers  $\{1, 2, \dots\}$
- (ii)  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{N}_- := \{-n : n \in \mathbb{N}\}$
- (iii)  $\Delta = \Delta_n$  is the forward difference operator, i.e.  $\Delta a_n := a_{n+1} - a_n$ .

### 1.2 Overview of the Result

We prove this small neat formula for the  $N$ -th harmonic number  $H_N = \sum_{n=1}^N \frac{1}{n}$ . For that sake, we define the auxiliary function

$$f : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \rightarrow \mathbb{R}, f(n, z) := \cos(\pi z) \frac{\Gamma(z+1)\Gamma(n)}{\Gamma(n+z+1)}. \quad (1)$$

This function analytically continues the forward difference operator applied to the sequence  $\frac{1}{n}$  in the difference order  $k$ , because, for  $k \in \mathbb{N}_0$ , we have  $\Delta^k\left(\frac{1}{n}\right) = f(n, k)$ .

**Theorem 1.2.1.**  $H_N = \sum_{n=1}^N \frac{1}{n} = \lim_{z \rightarrow -1} [f(N+1, z) - f(1, z)]$ .

The formula will be proven using the following result:

**Lemma 1.2.2.** Assume  $g : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \rightarrow \mathbb{R}$  is a function s.t.

- (i)  $z \mapsto g(n, z)$  is continuous on  $(-1, \infty) \cup U$ , where  $U$  is a deleted neighborhood of  $-1$ , for all  $n \in \mathbb{N}$ ,
- (ii)  $g(n+1, z) - g(n, z) = g(n, z+1)$  for all  $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}_-)$ ,
- (iii)  $\lim_{z \rightarrow -1} [g(n, z) - g(m, z)]$  exists for all  $n, m \in \mathbb{N}$ .

Then  $\sum_{n=1}^N g(n, 0) = \lim_{z \rightarrow -1} [g(N+1, z) - g(1, z)]$

*Proof.* Take a  $z \in U$ . Then

$$\sum_{n=1}^N g(n, z) = \sum_{n=1}^N [g(n+1, z-1) - g(n, z-1)] = g(N+1, z-1) - g(1, z-1). \quad (2)$$

Using continuity of  $z \mapsto g(n, z)$  and taking the limit as  $z \rightarrow 0$  yields the claim.  $\square$

## 2 Proof of Theorem 1.2.1

$$\begin{aligned}
\text{Consider } \Delta^0\left(\frac{1}{n}\right) &\stackrel{\text{def}}{=} \frac{1}{n} \\
\Delta^1\left(\frac{1}{n}\right) &= \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} \\
\Delta^2\left(\frac{1}{n}\right) &= -\frac{1}{(n+1)(n+2)} - \left(-\frac{1}{n(n+1)}\right) = \frac{2}{n(n+1)(n+2)} \\
\Delta^3\left(\frac{1}{n}\right) &= -\frac{6}{n(n+1)(n+2)(n+3)} \\
&\dots \\
\Rightarrow \Delta^k\left(\frac{1}{n}\right) &= (-1)^k k! \frac{(n-1)!}{(n+k)!} \text{ follows inductively.}
\end{aligned} \tag{3}$$

Immediately, it is neither possible to set  $k = -1$  to get a primitive sequence nor to take the limit  $\lim_{k \rightarrow -1} \Delta^k\left(\frac{1}{n}\right)$ . The first step is to obtain a continuous version of the identity above using

$$\Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N} \text{ and } (-1)^k = \cos(\pi k), k \in \mathbb{Z}, \tag{4}$$

which motivates the definition of  $f$  in (1). Using (3) and (4), we get

$$\Delta^k\left(\frac{1}{n}\right) = f(n, k) \text{ for } n \in \mathbb{N}, k = 0, 1, 2, \dots \tag{5}$$

Thus it remains to show that  $f$  satisfies (ii) and (iii) to apply Lemma 1.2.2.

**Lemma 2.1.**  $f(n+1, z) - f(n, z) = f(n, z+1)$  for all  $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}_-)$ .

*Proof.* Using the functional equation (4) twice

$$\Gamma(z+2) = (z+1)\Gamma(z+1), \quad \Gamma(n+z+2) = (n+z+1)\Gamma(n+z+1),$$

we obtain

$$\begin{aligned}
f(n+1, z) - f(n, z) &= \cos(\pi z)\Gamma(z+1) \left( \frac{\Gamma(n+1)}{\Gamma(n+z+2)} - \frac{\Gamma(n)}{\Gamma(n+z+1)} \right) \\
&= \cos(\pi z) \frac{\Gamma(z+2)}{z+1} \left( \frac{\Gamma(n)n - \Gamma(n)(n+z+1)}{\Gamma(n+(z+1)+1)} \right) \\
&= \underbrace{-\cos(\pi z)\Gamma(z+2)}_{\cos(\pi(z+1))} \frac{\Gamma(n)}{\Gamma(n+(z+1)+1)} = f(n, z+1).
\end{aligned}$$

□

**Lemma 2.2.** The limit in Theorem 1.2.1 exists for all  $N \in \mathbb{N}$ .

*Proof.* Take any  $n \in \mathbb{N}$ . Then, using Lemma 2.1

$$\begin{aligned}
f(n+1, z) - f(n, z) &= f(n, z+1) = \cos(\pi(z+1)) \frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+2)} \\
&= \cos(\pi(z+1)) \frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+1)(n+z+1)} \\
&\rightarrow \frac{\Gamma(1)\Gamma(n)}{\Gamma(n)} \frac{1}{n} = \frac{1}{n} \text{ for } z \rightarrow -1.
\end{aligned} \tag{6}$$

The rest follows inductively with the telescope sum argument.

□