A Proof of a Harmonic Number Identity via Analytic Continuation of Finite Differences

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Abstract

We prove a formula for the N-th harmonic number $H_N = \sum_{n=1}^{N} \frac{1}{n}$ and generalize this approach so it can be applied to other sequences.

1 Introduction

1.1 Notations

(i) $\mathbb{N}$ is the set of the natural numbers $\{1, 2, \ldots\}$

(ii) $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

(iii) $\mathbb{N}_- := \{-n : n \in \mathbb{N}\}$

(iv) $\Delta = \Delta_n$ is the forward difference operator, i.e. $\Delta a_n := a_{n+1} - a_n$.

1.2 Overview of the result

We prove this small neat formula for the N-th harmonic number $H_N = \sum_{n=1}^{N} \frac{1}{n}$. For that sake, we define the auxiliary function

$$f : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \to \mathbb{R}, f(n, z) := \cos(\pi z) \frac{\Gamma(z+1)\Gamma(n)}{\Gamma(n+z+1)}. \tag{1}$$

This function analytically continues the forward difference operator applied to the sequence $\frac{1}{n}$ in the difference order k, because, for $k \in \mathbb{N}_0$, we have $\Delta^k\left(\frac{1}{n}\right) = f(n, k)$.

Theorem 1.2.1: $H_N = \sum_{n=1}^{N} \frac{1}{n} = \lim_{z \to -1} [f(N+1, z) - f(1, z)]$.

The formula will be proven using the following result:

Lemma 1.2.2: Assume $g : \mathbb{N} \times \mathbb{R} \setminus (\mathbb{N}_-) \to \mathbb{R}$ is a function s.t.

(i) $z \mapsto g(n, z)$ is continuous on $(-1, \infty) \cup U$, where U is a deleted neighborhood of -1, for all $n \in \mathbb{N}$,

(ii) $g(n+1, z) - g(n, z) = g(n, z+1)$ for all $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}_-)$,

(iii) $\lim_{z \to -1} [g(n, z) - g(m, z)]$ exists for all $n, m \in \mathbb{N}$.

Then $\sum_{n=1}^{N} g(n, 0) = \lim_{z \to -1} [g(n+1, z) - g(1, z)]$

Proof: Take a $z \in U$. Then

$$\sum_{n=1}^{N} g(n, z) = \sum_{n=1}^{N} [g(n+1, z-1) - g(n, z-1)] = g(N+1, z-1) - g(1, z-1). \tag{2}$$

Using continuity of $z \mapsto g(n, z)$ and taking the limit on both sides as $z \to 0$, we obtain the desired equality. □

2 Proof of Theorem 1.2.1

Let's look at the forward differences $\Delta^k \frac{1}{n}$.

$$
\begin{aligned}
\Delta^0\left(\frac{1}{n}\right) &\stackrel{\text{def}}{=} \frac{1}{n} \\
\Delta^1\left(\frac{1}{n}\right) &= \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} \\
\Delta^2\left(\frac{1}{n}\right) &= -\frac{1}{(n+1)(n+2)} - \left(-\frac{1}{n(n+1)}\right) = \frac{2}{n(n+1)(n+2)} \\
\Delta^3\left(\frac{1}{n}\right) &= -\frac{6}{n(n+1)(n+2)(n+3)} \\
&\dots \\
\implies \Delta^k\left(\frac{1}{n}\right) &= (-1)^k k! \frac{(n-1)!}{(n+k)!} \text{ follows inductively.}
\end{aligned}
\tag{3}
$$

Immediately, it is neither possible to set $k = -1$ to get a primitive sequence nor to take the limit $\lim_{k\to-1} \Delta^k\left(\frac{1}{n}\right)$. The first step is to obtain a continuous version of the identity above using

$$
\Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N} \text{ and } (-1)^k = \cos(\pi k), k \in \mathbb{Z}, \tag{4}
$$

and this is where the function f defined in (1) comes into play. Using (3) and (4), we obtain

$$
\Delta^k\left(\frac{1}{n}\right) = f(n,k) \text{ for } n \in \mathbb{N}, k = 0, 1, 2, \dots \tag{5}
$$

Now, we only need to deal with the limit defined in Theorem 1.2.1. If the following lemmas hold, then Theorem 1.2.1 is proven.

Lemma 2.1: $f(n+1,z) - f(n,z) = f(n,z+1)$ for all $n \in \mathbb{N}, z \in \mathbb{R} \setminus (\mathbb{N}_-)$.

Proof: Using the functional equation (4) twice

$$
\begin{aligned}
\Gamma(z+2) &= (z+1)\Gamma(z+1) \\
\Gamma(n+z+2) &= (n+z+1)\Gamma(n+z+1)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
f(n+1,z) - f(n,z) &= \cos(\pi z)\Gamma(z+1)\left(\frac{\Gamma(n+1)}{\Gamma(n+z+2)} - \frac{\Gamma(n)}{\Gamma(n+z+1)}\right) \\
&= \cos(\pi z)\frac{\Gamma(z+2)}{z+1}\left(\frac{\Gamma(n)n - \Gamma(n)(n+z+1)}{\Gamma(n+(z+1)+1)}\right) \\
&= \underbrace{-\cos(\pi z)}_{\cos(\pi(z+1))}\Gamma(z+2)\frac{\Gamma(n)}{\Gamma(n+(z+1)+1)} = f(n,z+1).
\end{aligned}
$$

□

Lemma 2.2: The limit in Theorem 1.2.1 exists for all $N \in \mathbb{N}$.

Proof: Take any $n \in \mathbb{N}$. Then, using Lemma 2.1

$$
\begin{aligned}
f(n+1,z) - f(n,z) = f(n,z+1) &= \cos(\pi(z+1))\frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+2)} \\
&= \cos(\pi(z+1))\frac{\Gamma(z+2)\Gamma(n)}{\Gamma(n+z+1)(n+z+1)} \\
&\to \frac{\Gamma(1)\Gamma(n)}{\Gamma(n)}\frac{1}{n} = \frac{1}{n} \text{ for } z \to -1.
\end{aligned}
\tag{6}
$$

The rest follows inductively with the telescope sum argument. □