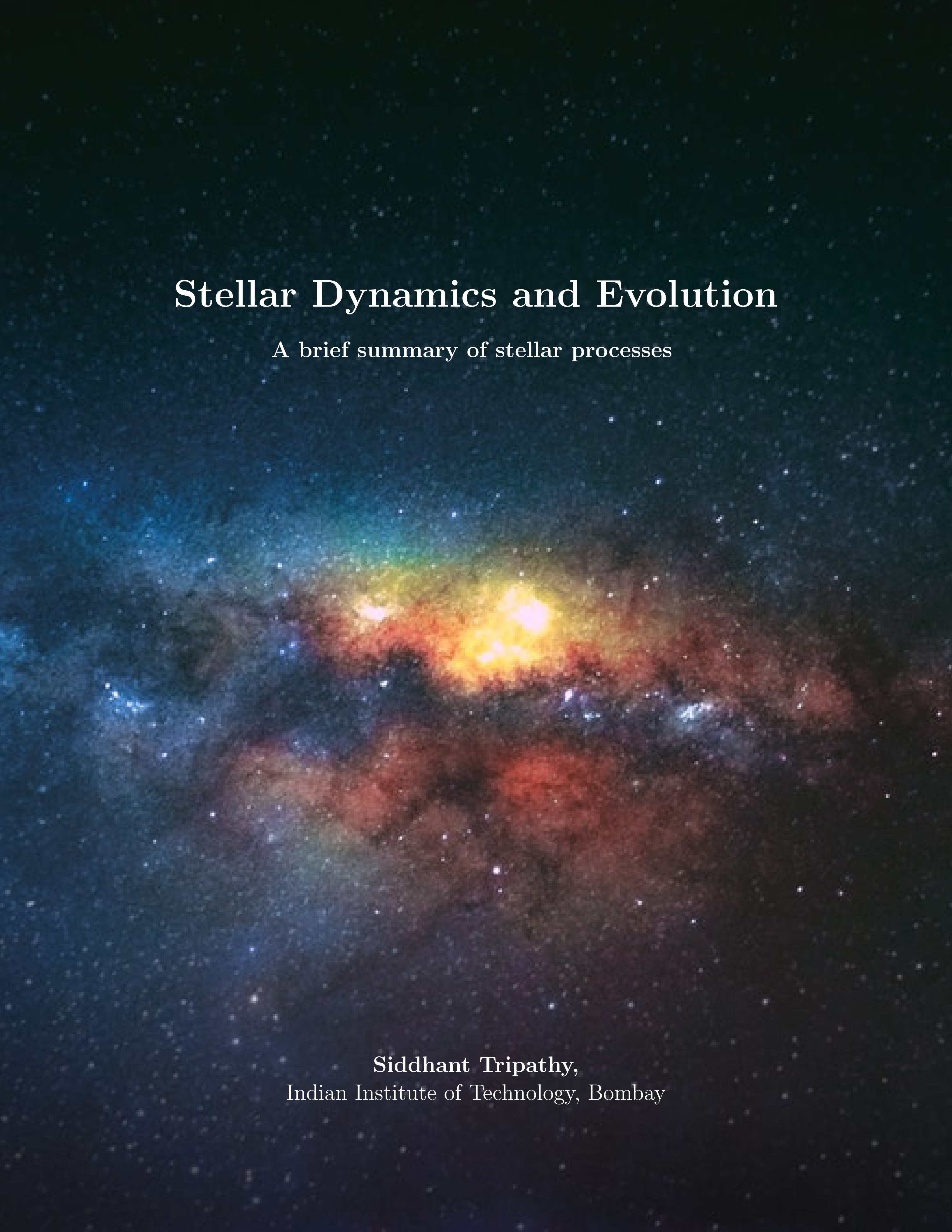

Stellar Dynamics and Evolution

A brief summary of stellar processes

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Contents

1 Introduction

"Stars are hot balls of burning gases", is the one universal truth about stars each and every student is taught since school. Of course, this statement is absolutely correct, but understanding the mechanism that stars employ to "burn" these gases, the reasons why these specific mechanisms are employed, and how they are sustained, is a whole different story altogether.

If you want important insight into how stars work, just go out and look at them for a few nights.. You fill find that they appear to do nothing other than shine steadily. This is certainly true from a historic perspective, taking the Sun as an example,from fossil evidence, we can extend this period of inactivity to nearly five billion years!

The reason for this relative tranquility is that stars are in general very stable objects in that the self-gravitational forces are delicately balanced by steep internal pressure gradients. The latter require high temperatures. In the deep interior of stars, these temperatures are measured to be of the order of millions of Kelvins, and in most cases are sufficiently high to initiate the thermonuclear fusion of light nuclei. The power so produced then laboriously works its way out of the remaining bulk of the star and finally gives rise to the radiation we see streaming off the surface. The vast majority of stars spend much of their active lives in such an equilibrium state, gradually converting Hydrogen into Helium, and it is only this gradual transmutation of elements by the fusion process that eventually causes the structure to change in some marked way.

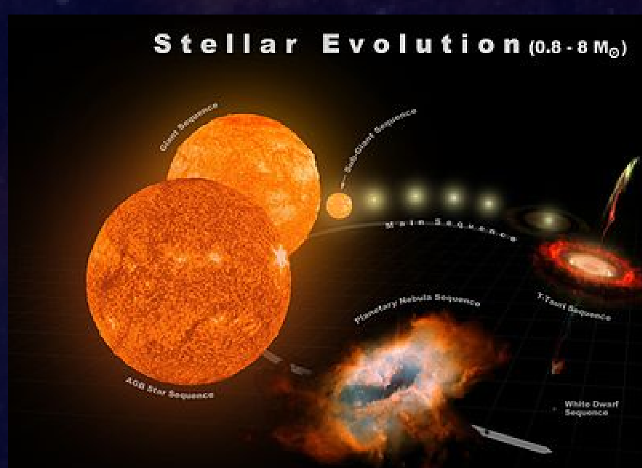

Figure 1: An artist's impression of stellar evolution. Of course, all this shall be elaborated upon later in the article

In this article, I am going to focus on the mechanism responsible for this equilibrium state, as well as those responsible for any changes in it. I am then going to show how these apply specifically to the Sun, White Dwarfs and certain variable stars too. This article requires the user to have a basic knowledge of thermodynamics, including the theory of ideal gases, as well as Newton's laws of Gravitation. I shall not spend much time on the basics, but delve directly into the mathematics that govern stellar dynamics and evolution.

2 The Virial Theorem

The virial theorem is an accepted theorem to explain the delicate mechanism of force balance within a star. Derived only from classical mechanics, it more or less explains the equilibrium of stars as we see them. Without more qualitative description of the theorem, I shall let the mathematics do the talking.

2.1 Setting up conventions

As in any classical theory of physics, the virial theorem too makes incredulous assumptions, but also like many classical theories, more or less agrees with observations. So let us consider the theoretician's dream: a a spherically symmetric, non rotating star on which there are no net forces acting and hence no net acceleration. There may be internal unbalanced forces of course, such as convection, but we assume them to average out to nil on the whole. Further we assume that the stellar material is so constituted that the internal pressures are uniform.

I shall now define some standard variables that I shall be using throughout the article in equations and their derivations:

$$\begin{aligned}
r &: \text{Radial distance from stellar centre (cm)} \\
R &: \text{Total stellar radius measured from the centre (cm)} \\
\rho(r) &: \text{Mass density at a distance } r \text{ (g cm}^{-3}\text{)} \\
T(r) &: \text{Temperature as a function of } r \text{ (K)} \\
P(r) &: \text{Pressure as a function of } r \text{ (dyne cm}^{-2}\text{)} \\
M_r &: \text{Mass contained within a radius } r \text{ (g)} \\
M_R &: \text{Toral stellar mass (g)} \\
L_r &: \text{Luminosity, i.e. energy flow rate at } r \text{ (erg s}^{-1}\text{)} \\
L_R &: \text{Total stellar luminosity (erg s}^{-1}\text{)} \\
g(r) &: \text{Local acceleration due to gravity at } r \text{ (cm s}^{-2}\text{)} \\
G &: \text{The universal gravitational constant} = 6.6726 \times 10^{-8} \text{ g}^{-1} \text{ cm}^{3} \text{ s}^{-2}
\end{aligned}$$

With all this set up, let us finally begin the physics of stellar equilibrium.

2.2 The Hydrostatic Equilibrium

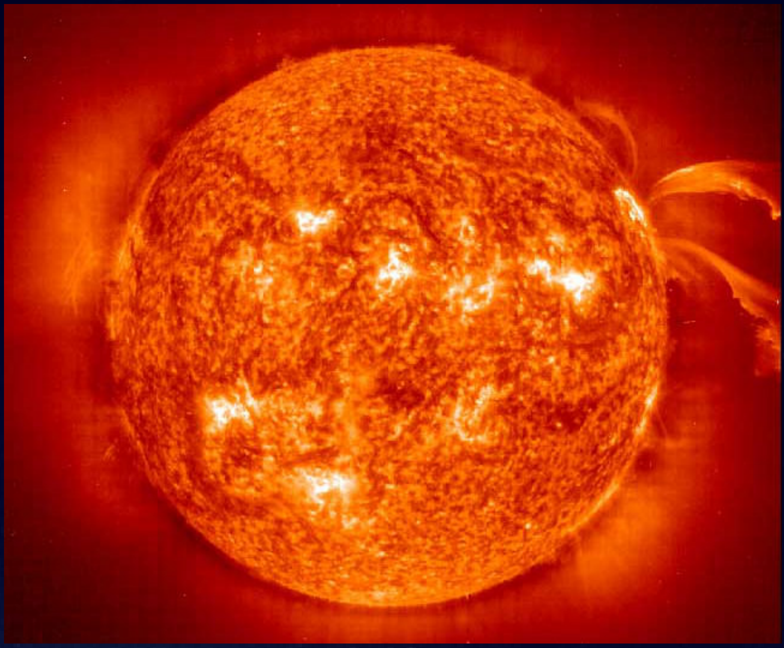

Figure 2: Our sun has been in perfect equilibrium for the past five billion years and is expected to be in this state for yet another five billion years

Let us consider a thin shell at a distance r from the stellar centre, having density $\rho(r)$. Since the volume element for a shell of thickness dr is $4\pi r^2 dr$, we have:

$$dM_r = 4\pi r^2 \rho(r) dr$$
$$M_r = \int_0^r 4\pi r^2 \rho(r) dr \tag{1}$$

This equation is known as the Equation of mass conservation, and is the first fundamental equation of Stellar structure.

Let us now consider a 1 cm^2 area at a radius r. Ler us consider the volume element with this area, and thickness dr. Now, the gravitational force acting inwards is given by:

$$\rho(r)g(r)dr = \rho(r)\frac{GM_r}{r^2}dr$$

In order to prevent collapse of the star, this gravitational force must be balanced by some outwards force. This force is provided by the pressure imbalance in the stellar material which acts radially outwards. Pressure disbalance

$$= P(r) - P(r + dr)$$
$$= -\frac{dP(r)}{dr}dr$$

Now, by Newton's laws, the acceleration of this small volume element is given by:

$$\rho(r)(Area)dr\ddot{r} = -\frac{dP(r)}{dr}dr(Area) - \rho(r)\frac{GM_r}{r_2}(Area)dr$$

Taking Area as 1 cm^2 and cancelling dr on both sides, we get:

$$\rho(r)\ddot{r} = -\frac{dP(r)}{dr} - \rho(r)\frac{GM_r}{r^2}$$

But, in equilibrium, the net acceleration of this small volume element is 0. Hence:

$$\frac{dP(r)}{dr} = -\rho(r)\frac{GM_r}{r^2} \tag{2}$$

This is the second important equation of Stellar structure, known as the Equation of Hydrostatic Equilibrium.

2.3 Using The Energy Principle

We know that equilibrium implies that the net acceleration of any volume element must be 0. However, there is another important result. It states that the energy must also not be changing with time. This essentially means that out star exists in a state which is the stationary point in the star's energy function. For this, we first need to find the energy function.

The first source of energy that comes to mind is the gravitational self energy. This is the negative of the total energy required to assemble the star using material from the universe. This can be calculated by doing the reverse process, mathematically, of the taking about the star, shell by shell. Thus the work on a single shell in moving it from to r to ∞ is:

$$\begin{aligned} d\Omega &= -\int_r^\infty \frac{GM_r}{r^2}dM_r dr \\ &= -\frac{GM_r dM_r}{r} \end{aligned}$$

For the entire star:

$$\Omega = -G\int_0^{M_R} \frac{M_r dM_r}{r} \tag{3}$$

To calculate this, we need M_r or alternatively, ρ (r) as a function of r. However we know that it will be proportional to $\frac{M_R^2}{R}$. Hence:

$$\Omega = q\frac{GM_R^2}{R}$$

where q is some rational number.

Figure 3: Famous Cat's Eye Nebula. Nebulae are the birthplace of stars, where they acquire material and grow

Now, let E be the local specific internal energy per unit mass of the material. This E arises due to the internal forces such as nuclear forces. The total energy W is:

$$W = \int_M E dM_r + \Omega \tag{4}$$

We define total internal energy U as $\int_M E dM_r$.

The equilibrium point of the star is now the stationary point of W, implying that W must be at an extremum for hydrostatic equilibrium. In order to find this point, we perform some perturbations from the original state in an adiabatic fashion. This is because the perturbations must be rapid enough to avoid heat transfer between neighbouring portions of the stellar material and the same time sufficiently slow so as the be able to neglect kinetic energy of the mass elements.

Let δ represent the perturbation operator. At hydrostatic equilibrium,

$$(\delta W)_{ad} = 0$$

Now,

$$(\delta W)_{ad} = (\delta U)_{ad} + (\delta \Omega)_{ad} \tag{5}$$

The resulting U is:

$$\begin{aligned} U + \delta U &= U + \delta \int_M E dM_r \\ &= U + \int_M \delta E dM_r \end{aligned} \tag{6}$$

We need to study δE in terms of ρ, T, etc. For reversible processes, we know from the first and second laws of thermodynamics that:

$$\begin{aligned} dQ &= TdS \\ dQ &= dE + PdV_P \end{aligned}$$

where variables hold their usual meanings from thermodynamics. V_P is the specific volume, i.e. $V_P = \frac{1}{\rho}$. Q, E, and S are also specific in nature. In adiabatic processes, $\delta S = 0$. Hence:

$$\begin{aligned} dE &= -PdV_P \\ (\delta U)_{ad} &= -\int_M P\delta V_P dM_r \end{aligned} \tag{7}$$

Also:

$$\begin{aligned} \frac{dM}{dV} &= \rho \\ \implies \rho &= \frac{dM_r}{4\pi r^2 dr} \\ \implies V_P &= \frac{4\pi r^2 dr}{dM_r} \end{aligned}$$

We assume all perturbations to be only radial:

$$\begin{aligned} \delta V_P &= \frac{d\big(4\pi(r+\delta r)^3/3\big)}{dM_r} - \frac{d(4\pi r^2 \delta r)}{dM_r} \\ &= \frac{d(4\pi r^2 \delta r)}{dM_r} \end{aligned}$$

Using this and equation (7):

$$(\delta U)_{ad} = -\int_M P \frac{d(4\pi r^2 \delta r)}{dM_r} dM_r \tag{8}$$

Let us follow the same treatment for Ω to find $(\delta U)_{ad}$:

$$\begin{aligned} \Omega(r+\delta r) - \Omega(r) &= \int_M \frac{GM_r}{r} dM_r - \int_M \frac{GM_r}{r+\delta r} dM_r \\ &= \int_M \frac{GM_r}{r}\big(1 - (1 - \frac{\delta r}{r})\big) dM_r \\ &= \int_M \frac{GM_r}{r^2} \delta r dM_r \end{aligned} \tag{9}$$

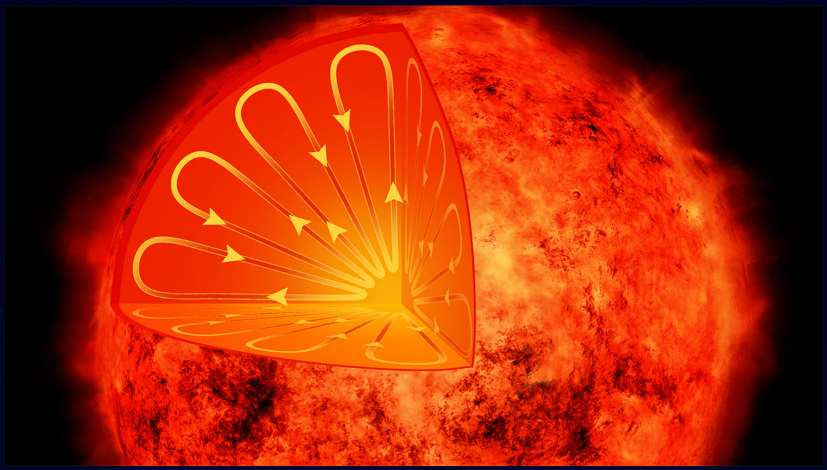

Figure 4: An artist's impression of convection currents in stellar material, an important source of internal disturbance of equilibrium. However they cancel out on average and hnece we ignore them in our calculations.

From equations (5), (8), and (9), we get:

$$(\delta W)_{ad} = \int_M \Big[\frac{dP(r)}{dM_r} 4\pi r^2 + \frac{GM_r}{r^2}\Big] \delta r dM_r \tag{10}$$

Here, as δr is arbitrary, and $(\delta W)_{ad} = 0$ in hydrostatic equilibrium, the only way to achive out desired conditon is the make the integral identically 0:

$$\begin{aligned} \frac{dP(r)}{dM_r} 4\pi r^2 + \frac{GM_r}{r^2} &= 0 \\ \implies \frac{dP(r)}{r} &= -\frac{GM_r}{4\pi r^4} \end{aligned} \tag{11}$$

This is basically the Equation of Hydrostatic Equilibrium again, this time in terms of mass. Take $dM_r = 4\pi r^2 \rho(r) dr$ and we get back equation (2).

2.4 Arriving at the Virial Theorem

Now that we have satisfactorily derived the Equation of Hydrostatic equilibrium by two methods, let us work towards reaching the final Virial theorem. This will require some knowledge of vectors, but it is the final result that is of importance.

Let m_i be a free particle at $\mathbf{r_1}$ with momentum $\mathbf{P_i}$:

$$\begin{aligned} \frac{d}{dt}\sum \mathbf{p_i} \cdot \mathbf{r_i} &= \frac{d}{dt}\sum m_i \dot{\mathbf{r_i}} \cdot \mathbf{r_i} \\ &= \frac{1}{2}\frac{d}{dt}\sum \frac{d}{dt}(m_i \mathbf{r_i}^2) \\ &= \frac{1}{2}\frac{d^2 I}{dt^2} \end{aligned} \tag{12}$$

where U is the moment of Inertia. Alternatively:

$$\begin{aligned}\frac{d}{dt}\sum \mathbf{p_i}\cdot\mathbf{r_i} &= \sum \frac{d\mathbf{p_i}}{dt}\mathbf{r_i} + \sum \mathbf{p_i}\frac{d\mathbf{r_i}}{dt} \\ &= \sum \mathbf{F_i}\cdot\mathbf{r_i} + \sum m_i\dot{\mathbf{r_i}}^2 \\ &= 2K + \sum \mathbf{F_i}\cdot\mathbf{r_i}\end{aligned}$$

Where $K = \frac{1}{2}\sum m_i\dot{\mathbf{r_i}}^2$ is the Kinetic energy. From equation (12):

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \sum \mathbf{F_i}\cdot\mathbf{r_i} \tag{13}$$

$\sum \mathbf{F_i}\cdot\mathbf{r_i}$ is the mutual interaction between all particles. Let $\mathbf{F_{ij}}$ be the gravitational force on particle i due to particle j. $\mathbf{F_{ij}} = -\mathbf{F_{ji}}$

$$\begin{aligned}\sum \mathbf{F_i}\cdot\mathbf{r_i} &= \sum_{i<j}(\mathbf{F_{ij}}\cdot\mathbf{r_i} + \mathbf{F_{ji}}\cdot\mathbf{r_j}) \\ &= \sum \mathbf{F_{ij}}\cdot(\mathbf{r_i} - \mathbf{r_j})\end{aligned}$$

But $\mathbf{F_{ij}} = -\frac{Gm_im_j}{r_{ij}^3}(\mathbf{r_i} - \mathbf{r_j})$, where $r_{ij} = |\mathbf{r_i} - \mathbf{r_j}|$. Therefore:

$$\sum \mathbf{F_{ij}}\cdot(\mathbf{r_i} - \mathbf{r_j}) = -\sum \frac{Gm_im_j}{r_{ij}} = Virial$$

The middle number here is nothing but the Gravitational self energy in discreet form. This means:

$$Virial = \Omega \tag{14}$$

And:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \Omega \tag{15}$$

At last we have arrived at the all famous Virial Theorem! At any general $r < R$,

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \Omega - 3P_sV_s$$

Where P_s and V_s are the pressure and volume at the surface at r.

Figure 5: Algol, an important variable star. Different types of stars can be understood from the equation of state, depending on the type of equilibrium

Let us now pbserve K. $2K = \sum m_i \mathbf{v_i}^2 = \sum \mathbf{p_i} \cdot \mathbf{v_i}$. Assuming an ideal gas configuration and a continuous mass distribution,

$$P = \frac{1}{3} \int_{\mathbf{p}} n(\mathbf{p}) \mathbf{p} \cdot \mathbf{v} d^3 \mathbf{p}$$

Here, $n(\mathbf{p})$ is the number density of particles with momentum $\mathbf{p}$, and the integral is over all momenta. It can be seen that:

$$\begin{aligned} 2K &= 3 \int_V P dV_p \\ &= 3 \int_V \frac{P}{\rho} dM_r \end{aligned}$$

The Virial theorem now becomes,

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 3 \int_V \frac{P}{\rho} dM_r + \Omega \tag{16}$$

This equation now enables us to solve for different stars, given the choice for equilibrium of state.

3 Variable Stars and their Types

Let us take a break from some intensive mathematics and look at a special category of stars. This sections is purely descriptive, and we shall deal with the state dynamics of these stars later in the article.

3.1 Causes of Variation

The light output of virtually every star, including our Sun, varies over time, form a period of minutes to several billion years. Both causes and manifestations are many and varied; a recent count identified more than 70 classes of variable stars, most of the names after the first star (called prototype) in which it was first observed. Let us take a look at some of these types and their probable causes.

3.1.1 Pulsational Variables

These are the most useful variable stars because the length of time in which they brighten and fade again, i.e. their periods, are frequently correlated with their absolute brightness, so that they can be used to measure distances to star clusters anywhere in the Milky Way and to nearby galaxies.

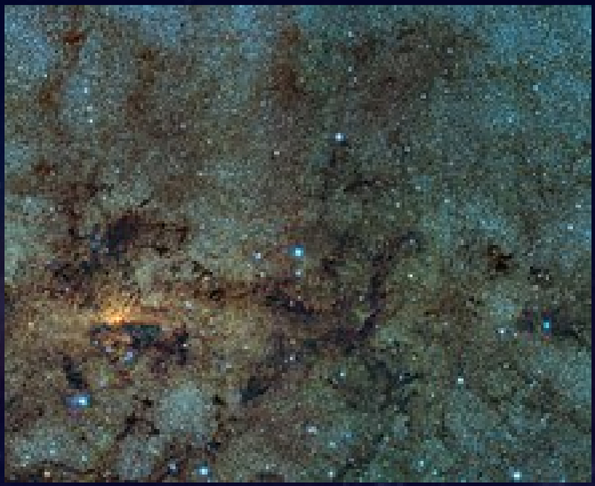

Figure 6: The galactic centre view showing multiple RR Lyrae variables, that help us determine the distance to it

Their pulsation might be driven by some instability, and the restoring force that brings the gas back where it started from can be gravity or pressure or magnetic fields. In this sense, the instability is intrinsic to the star and not due to external influences; hence, such stars are often referred to as intrinsic variables. The type of instability and the restoring force employed by the star can be used to differentiate between different types of variables, some of the more common ones being:

- *Classical Cepheids*, usually just called Cepheids are young, metal rich stars with periods of days to months. They appear to be purely radial pulsators.
- *RR Lyrae* variables were formerly known as cluster type variables or cluster Cepheids because they are common in globular clusters. They have periods of a day down to a couple of hours and are useful in determine distances to globular clusters in out galaxy in nearby galaxies.

- *δ Scruti* variables have periods ranging from about 30 minutes to 8 hours and pulsate in radial and non-radial pressure nodes although gravity may be present. Amplitudes tend to be low.
- *Mira* variables, are luminours red supergiants belonging in the class of *Long Period Variables* with periods ranging from roughly 100 to 700 days. Radial nodes seem to be the norm. *RV Tauri* stars are an extreme version with low mass and large luminosity, so that a second pulse starts before the atmosphere has had time to fall down from the previous one.
- *The Rapidly Oscillating Ap* stars are characterized by low amplitude, short period photometric variations, typically 10 minutes, strong magnetic fields, and enhanced surface abundances of exotic elements such as strontium and europium. The observed light variations are modulated in amplitude by the rotation of the star and it is thought that the pulsations are carried around by an off-axis magnetic field as the star rotates.

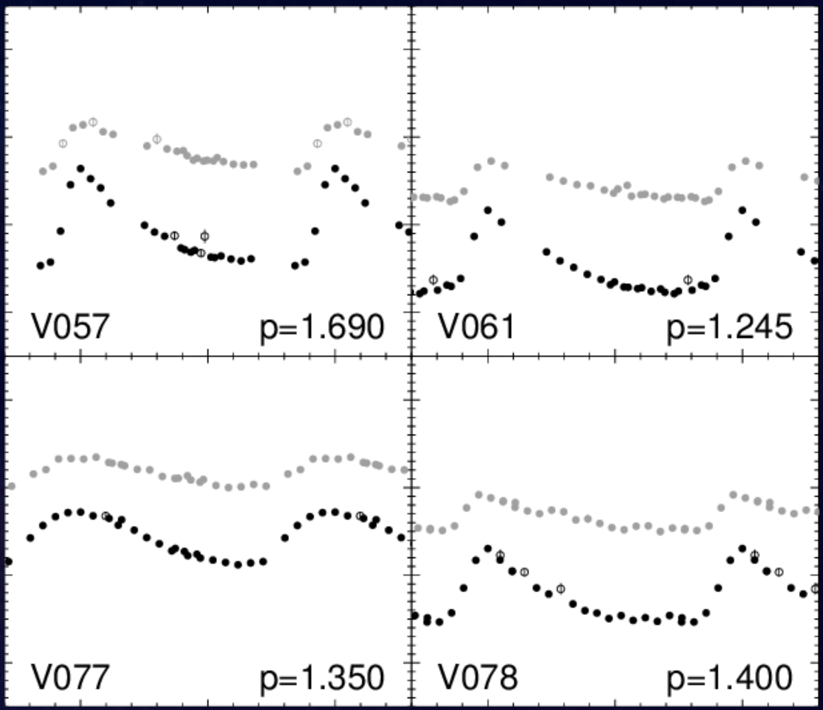

Figure 7: Light curves of four different cepheids showing how the light output varies with time

3.1.2 Explosive variables

These are the stars that release a great deal of nuclear or gravitational energy in a hurry. The cataclysmic variables are close star pairs with a white dwarf in orbit with a main sequence or red giant companion. The white dwarf accepts material from its companion, One sort of variability arises when the rate of acceptance or accretion and therefore the rate of release of gravitational potential energy changes. When enough hydrogen-rich material has accumulated on the surface of the white dwarf it fuses explosively.

Novae

Nova is the name given to these outbursts fueled by degenerate hydrogen ignition.

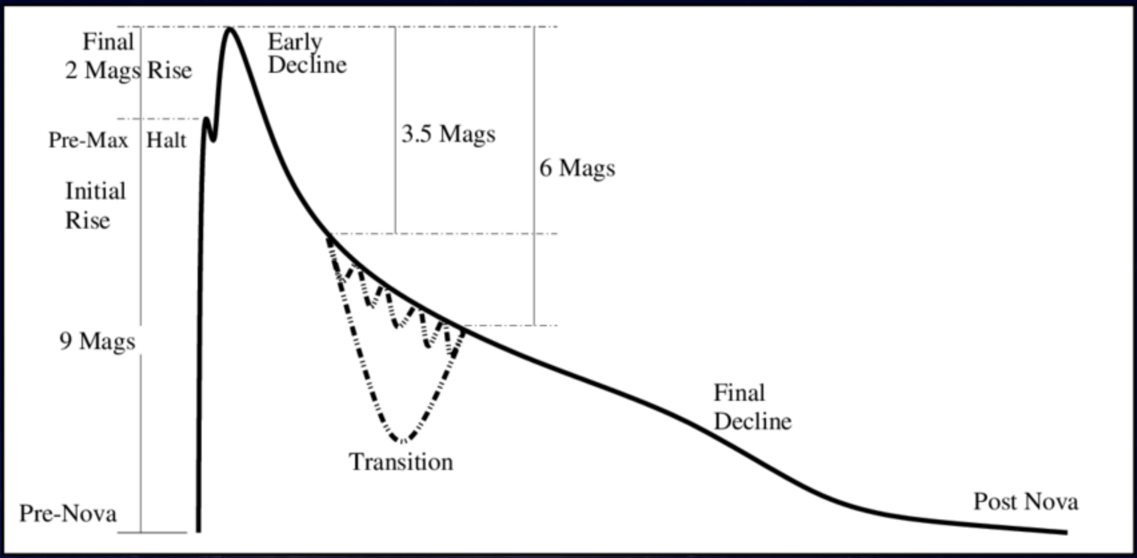

Figure 8: Light output curve for a typical nova as is explained below

As seen above, there is a fast rise lasting perhaps a day, followed by a decline in brightness that may be quite variable from nova to nova. In the "fast" novae the decline may take a couple of weeks to reduce the visual brightness by two magnitudes. "Slow" novae may take a few months to accomplish the same thing. Between the initial rise and eventual decline, there may be a plateau near the peak lasting an hour or so in fast novae but extending over days in slow ones. As indicated in the figure, the decline phase may be interrupted by oscillations or a pronounced trough.

Supernovae

Supernovae are the most spectacular variables of all. At maximum light, they are as bright as a whole, smallish galaxy, and recognizing them for what they are was part of the total process between 1900-1925 CE that sorted out the approximate size of the Milky Way and demonstrated the existence of other galaxies. There is a sort of family resemblance among all supernovae - they get really bright in a matter of days and fade in months to years. Their spectra display very broad features, indicating velocities of thousands of km s^{-1}. And they blow out a solar mass or more of material at these large velocities that can then be seen as supernova remnant for thousands of years thereafter. We shall not get into much more detail because the dynamics of supernovae is beyond the scope of this article.

Figure 9: Crab nebula: a supernova remnant in Taurus

3.2 Other Broad Categories of Variables

As mentioned in the beginning of this section, every star shows some kind of variability upon observation, which may be intrinsic or may be due to purely observational factors. These may not be as "interesting" as pulsational or explosive variables, but they are significant and worth mentioning nonetheless.

3.2.1 Eclipsing and Ellipsoidal Variables

If a pair of stars orbit each other, you, the observer, may be located close enough to the orbital plane to have one pass in front of the other and block its light for a portion of each orbital period. Because the eclipse tells us that the system is nearly edge on, eclipsing binaries are among the sorts particularly useful in measuring stellar masses. Even if there is no eclipse, the gravitational field of one star may distort the shape of its companion into an ellipsoid, so that you see a larger star area when the stars are side-on to you than when they are end on. This will also result in periodic variability, though of a lesser useful sort.

3.2.2 Spotted, Rotating Stars

The sun is an example of this class. Its brightness varies both at its rotation period and through the 11 year susnpot cycle. The variation is, however, only about 0.1%. Curiously though, the sun is brighter when it has more spots because the extra brightness of the bright vein.0like facular regions of the photosphere more than makes up for the darker spots.

3.2.3 T Tauri Stars, FU Orionis Stars and Luminous Blue Variables

These are stars that are very young or very massive and bright or both. They are probably both accreting material from a disk and blowing off material at their poles, and may be heavily spotted as well. The result is non-periodic flaring and variability. Surrounding gas and dust frequently show up in images and spectra of these stars, and very occasionally it is possible to tell which bits are flowing in and which are being ejected, sometimes in jets. Rapid rotation and magnetic fields are also part of the picture.

4 Equations of State

It is now time to go back to the mathematics of stellar dynamics, to derive the equations of state and their applications. This section will assume a background knowledge of statistical mechanics. Some results shall be quoted without proof which will eventually be used to derive equations of state for stellar material consisting of gases, including photons in thermodynamic equilibrium.

4.1 Preliminary results

Following are the two most important results we will need to use further along to describe stellar interiors.

4.1.1 Distribution functions

The "Distribution function" for a species of particle measures the number density of that species in the combined six-dimensional space of coordinates plus momenta. If that function is known for a particular gas composed of a combination of species, then all other thermodynamic variables may be derived given the temperature, density and composition. But before writing down the distribution function for any gas we first introduce what may be an unfamiliar thermodynamic quantity.

Let N_i be the number density of an ith species in the units of number per gram of material with $N_i = n_i/\rho_i$. Ir is the Lagrangian version of n_i and it will prove useful because it remains constant even if volume changes. Another very useful quantity is the *chemical potential*, μ_i, defined by:

$$\mu_i = \left(\frac{\partial E}{\partial N_i}\right)_{S,V}$$

as associated with an ith species in the material.

Having defined μ, let us use a result from statistical mechanics. The relation between the number density of some species of elementary nature in the coordinate-momentum space and its chemical potential in thermodynamics is given by:

$$n(p) = \frac{1}{h^3} \sum_i \frac{g_i}{exp[-\mu + \mathcal{E}_j + \mathcal{E}(p)]/kT \pm 1} \tag{17}$$

We call $n(p)$ the distribution function of the species. The "+" is for Fermi-Dirac particles while the "-" is for Bose-Einstein particles. The various quantities used above are:

$$\begin{aligned}
\mu &: \text{chemical potential of the species} \\
j &: \text{possible energy states of the species} \\
\mathcal{E}_j &: \text{energy of state } j \text{ referenced to some energy level} \\
g_i &: \text{degeneracy of state } j \\
\mathcal{E}(p) &: \text{kinetic energy as a function of momentum } p \\
h &: \text{Planck's constant}
\end{aligned}$$

Because we shall eventually want to consider relativistic particles, the correct form of the kinetic energy, E, for a particle of rest mass m is given by:

$$\mathcal{E}(p) = \left(p^2c^2 + m^2c^4\right)^{1/2} - mc^2 \tag{18}$$

which reduces to $\mathcal{E}(p) = p^2/2m$ for $pc << mc^2$ in the non-relativistic limit, and $\mathcal{E}(p) = pc$ for extremely relativistic particles or those with zero rest mass.

We shall also need an expression for velocity which, from Hamilton's equation is:

$$v = \frac{\partial \mathcal{E}}{\partial p} \tag{19}$$

This is the velocity to be used in the following kinetic theory expression for isotropic pressure:

$$P = \frac{1}{3}\int_p n(p) p v 4\pi p^2 dp \tag{20}$$

Finally, the internal energy is simply:

$$E = \int_p n(p)\mathcal{E}(p) 4\pi p^2 dp \tag{21}$$

4.1.2 Blackbody radiation

Photons are massless bosons of unit spin. Since they travel at c, they only have two states for a given energy and thus the degeneracy factor is $g = 2.\mu$ is 0 for photons and $\mathcal{E} = pc$. Because there is only one energy level, $\mathcal{E}_j$ may be taken as zero. Putting this together, we find that the photon number density is given by:

$$n_\gamma = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{exp(pc/kT) - 1}\ \text{cm}^{-3} \tag{22}$$

Let $x = pc/kT$ and use the integral:

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2\zeta(3)$$

where $\zeta(3)$ is the Riemann Zeta function, to find:

$$n_\gamma = 2\pi\zeta(3)\left(\frac{2kT}{ch}\right)^3 \tag{23}$$

Find, in similar fashion, that the radiation pressure is given by:

$$\begin{aligned} P_{rad} &= \left(\frac{k^4}{c^3h^3}\frac{8\pi^5}{15}\right)\frac{T^4}{3} \\ &= \frac{aT^4}{3} \end{aligned} \tag{24}$$

and that the energy density is:

$$E_{rad} = aT^4 = 3P_{rad} \tag{25}$$

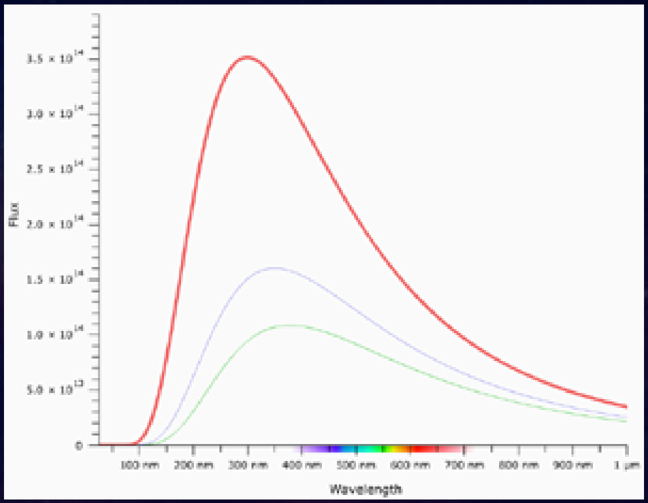

Figure 10: The famous blackbody curve we all know

Let us also define the energy density per unit frequency (ν) or wavelength ()λ in the radiation field. These energy densities are usually designated by μ. Frequency is given by $\nu = \mathcal{E}/h = pc/hh$ and wavelength by $\lambda = c/\nu$. If μ_p is the energy density per unit momentum and μ_ν and μ_λ are the corresponding densities per unit frequency and wavelength, then we can easily see that:

$$\mu_\nu d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu \tag{26}$$

and

$$\mu_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda \tag{27}$$

Associated quantities are the frequency dependent Planck function

$$B_\nu(T) = \frac{c}{4\pi}\mu_\nu \tag{28}$$

and the integrated Planck function

$$B(T) = \int_0^\infty B_\nu(T)d\nu = \frac{ca}{4\pi}T^4 = \frac{\sigma}{\pi}T^4 \tag{29}$$

4.2 Fermi-Dirac Equation of State

The most commonly encountered Fermi-Dirac elementary particles of stellar astrophysics are electrons, protons and neutrons, all have spin one-half. The prime motivation for this section is that the equation of state in the inner regions of many highly evolved stars, including white dwarfs, is dominated by degenerate electrons and, to a great extent, this determines the structure of such stars.

The number density of Fermi-dirac particles is given by equation (17) with the choice of $+1$ and an energy reference level of $\mathcal{E}_0 = mc^2$, where m is the mass of the fermion. For these spin 1/2 particles, the statistical weight is $g = 2$. Transcribing these statements then means that the number density is:

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2dp}{exp[-\mu + mc^2 + \mathcal{E}(p)]/kT + 1} \tag{30}$$

where, in general, from equations (18) and (19),

$$\mathcal{E}(p) = mc^2\left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1\right] \tag{31}$$

and

$$v(p) = \frac{\partial\mathcal{E}}{\partial p} = \frac{p}{m}\left[1 + \left(\frac{p}{mc}\right)^2\right]^{-1/2} \tag{32}$$

We now explore some consequences of the above.

4.2.1 Completely Degenerate Gas

The "completely degenerate" state refers to the unrealistic assumption that the gas is at a temperature of absolute zero, and under certain circumstances, the gas effectively behaves as if it were at zero temperature. Consider a part of equation (17):

$$F(\mathcal{E}) = \frac{1}{exp[\mathcal{E} - (\mu - mc^2)]/kT - 1} \tag{33}$$

where, as $T \to 0$, $F(\mathcal{E})$approaches either zero or unity depending on whether $\mathcal{E}$ is greater or less than $\mu - mc^2$.

The critical kinetic energy at which $F(\mathcal{E})$ is discontinuous is called the "Fermi energy" and we denote it by $\mathcal{E}_F$; that is, where $\mathcal{E}_F = \mu - mc^2$. The situation is depicted in the figure below where, in the unit square corresponding to particle energies $0 \leq \mathcal{E} \leq \mathcal{E}_F$, $F(\mathcal{E})$ is unity. Fermions are contained only in that energy range and not at energies greater than $\mathcal{E}_F$ where the distribution function is zero. In this situation we refer to a "filled Fermi Sea" of fermions because all the fermions present are swimming in that sea and nowhere else.

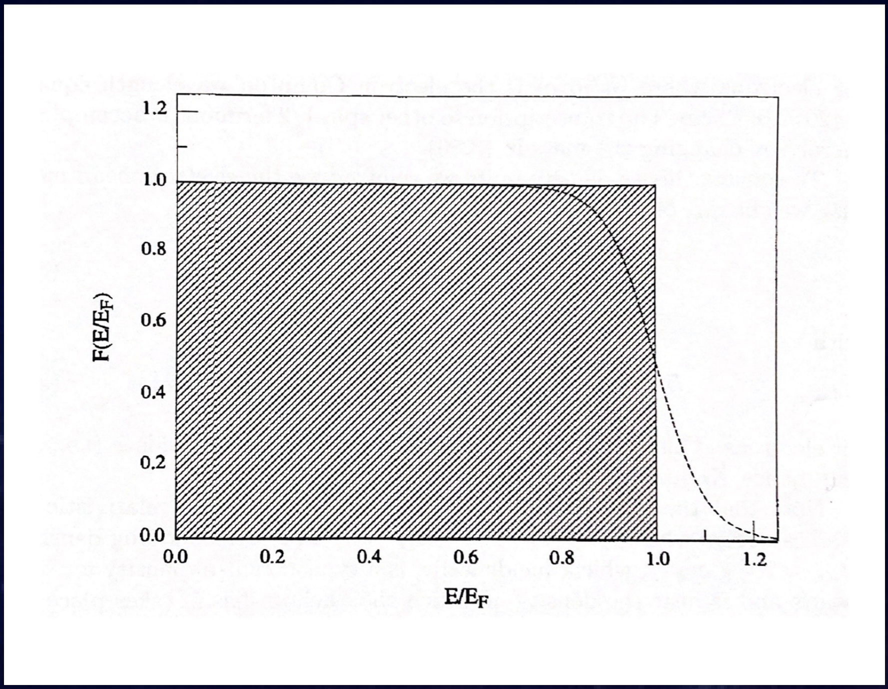

Figure 11: Fermi sea arising from the stepped energy function

The momentum corresponding to the Fermi energy is the Fermi momentum p_F. It is usually reduced to the dimensionless form by setting $x = p/mc$ and defining $x_F = p_F/mc$. Then, from equation (31), we have:

$$\mathcal{E}_F = mc^2\left[(1+x_F^2)^{1/2} - 1\right] \tag{34}$$

Now, the pressure of a completely degenerate electron gas is treated in the same way as that for the number density. It is the integral in equation (20) truncated at the Fermi momentum with $F(\mathcal{E})$ of equation (33) set to unity. A little work on equation (30) now yields:

$$P_{\mathrm{e}} = \frac{8\pi}{3}\frac{m_{\mathrm{e}}^4 c^5}{h^3}\int_0^{x_F} \frac{x^4 dx}{(1+x^2)^{1/2}} = Af(x) \tag{35}$$

where

$$A = \frac{\pi}{3}\left(\frac{h}{m_{\mathrm{e}}c}\right)^{-3} m_{\mathrm{e}}c^2 \tag{36}$$

for electrons and

$$f(x) = x\left(2x^2 - 3\right)\left(1 + x^2\right)^{1/2} + 3\sinh^{-1} x \tag{37}$$

Similarly, the internal energy from equation (31) is given by the integral

$$E_{\mathrm{e}} = 8\pi\left(\frac{h}{m_{\mathrm{e}}c}\right)^{-3} m_{\mathrm{e}}c^2 \int_0^{x_F} x^2\left[\left(1 + x^2\right)^{1/2} - 1\right] dx = Ag(x) \tag{38}$$

with

$$g(x) = 8x^3\left[\left(1 + x^2\right)^{1/2} - 1\right] - f(x) \tag{39}$$

Note that $x << 1$implies non-relativistic particles, and $x >> 1$ is the extreme relativistic limit. Also observe that

$$P_{\mathrm{e}} \propto E_{\mathrm{e}} \propto \begin{cases} (\rho/\mu_{\mathrm{e}})^{5/3}, & x \ll 1 \\ (\rho/\mu_{\mathrm{e}})^{4/3}, & x \gg 1 \end{cases} \tag{40}$$

and the limiting rations of P_{e} and E_{e} are

$$\frac{E_{\mathrm{e}}}{P_{\mathrm{e}}} = \frac{g(x)}{f(x)} = \begin{cases} 3/2 \quad (\gamma = 5/3), & x \ll 1 \\ \;\;3 \quad\; (\gamma = 4/3), & x \gg 1 \end{cases} \tag{41}$$

The values of γ indicate that the completely degenrate non-relativistic electron gas acts like a monoatomic ideal gas whereas, in the extreme relativistic limit, it behaves like a photon gas.

4.2.2 Application to White Dwarfs

As a simple, but important, application of completely degenerate fermion statistics, consider zero temperature stars in hydrostatic equilibrium whose internal pressures are due solely to electron degenerate material and whose densities and composition are constant throughout.

Using the Virial theorem and a little algebra, we get the non-relativistic mass-radius equation:

$$\mathcal{M} = \frac{1}{4}\left(\frac{3}{4\pi}\right)^4\left(\frac{h^2 N_{\mathrm{A}}}{m_{\mathrm{e}}G}\right)^3 \frac{N_{\mathrm{A}}^2}{\mu_{\mathrm{e}}^5}\frac{1}{\mathcal{R}^3} \text{ for constant density.} \tag{42}$$

This relation has the remarkable property that as mass increases, radius decreases, which is a major flaw in this mass-radius relation. The correct way to construct equilibrium degenerate models is to use the general expression for the pressure given by equation (35) along with the relation between ρ/μ_{e} and dimensionless Fermi momentum. This yields a pressure-density relation, which is then put into the equation of hydrostatic equilibrium. The resulting equation is then combined with the equation of mass conservation yielding a second-order differential equation that must be integrated numerically.

In the limit of extreme relativistic degeneracy, you may convince yourself by dimensional analysis that the total stellar mass depends only on μ_e and not on radius. To be precise

$$\frac{\mathcal{M}}{\mathcal{M}_\odot} = \frac{\mathcal{M}_\infty}{\mathcal{M}_\odot} = 1.456\left(\frac{2}{\mu_e}\right)^2 \tag{43}$$

where M_∞ is the Chandrasekhar limiting mass.

The regime intermediate between non-relativistic and full relativistic degeneracy is intractable using simple means. The following is a useful and quite accurate mass-radius relation bridging the two regimes.

$$\frac{\mathcal{R}}{\mathcal{R}_1} = 2.02\left[1-\left(\frac{\mathcal{M}}{\mathcal{M}_\infty}\right)^{4/3}\right]^{1/2}\left(\frac{\mathcal{M}}{\mathcal{M}_\infty}\right)^{-1/3} \tag{44}$$

Here, M/M_∞ is given by equation (43), and R_1 is defined by

$$\frac{\mathcal{R}_1}{\mathcal{R}_\odot} = 5.585\times 10^{-3}\left(\frac{2}{\mu_e}\right) \tag{45}$$

This radius is a typical scale length for electron degenerate objects.

Figure 12: White dwarfs are extremely compact stars balanced by electron degeneracy pressure as we have seen

4.2.3 Effects of Temperature

The crucial step in deriving some of the thermodynamics of the completely degenerate zero temperature fermion gas was the realization that the distribution function becomes a unit step function at a kinetic energy equal to $\mu - mc^2$. If the zero temperature condition is relaxed, the distribution function follows suit. Suppose the temperature is low, but nor zero.

Fermions deep in the Fermi sea, at at energies much less than $\mathcal{E}_F$, need roughly an additional $\mathcal{E}_F$ energy units to move around in energy. That is, if the energy input to the system, as measured by kT, is much smaller than $\mathcal{E}_F$, then low-energy particles are excluded from promotion to already occupied upper energy levels by the Pauli exclusion principle.

Fermions near the top of the Fermi sea don't have that difficulty and they may find themselves elevated into states with energies greater than $\mathcal{E}_F$. Thus as temperature is raised from zero, the stepped end of the distribution function smooths out to higher energies. If temperatures rise high enough, we expect the effects of Fermi-Dirac statistics to be washed out completely and the gas should merge into a Mexwell-Boltzman distribution.

We shall not get into the mathematical details for the effect of temperature as we will not be using in ahead to describe any phenomena.

5 Structure and Evolution of the Sun

Perhaps the first astronomical object we became aware of as children is our sun. Indeed, the sun is the prototype star, and before we can claim to understand the stars, we must claim some mastery of current ideas about own sun's origin, its internal structure, and how it has evolved to this state.

5.1 The Solar Interior

Depending on the type of heat transfer, which in turns depends on the nuclear and thermodynamic processes, the solar interior has several broad layers, the boundaries between them not pronounced but significant over a long range of distance.

5.1.1 The Core

The Sun's core is the central region where nuclear reactions consume hydrogen to form helium. These reactions release the energy that ultimately leaves the surface as visible light. These reactions are highly sensitive to temperature and density. The individual hydrogen nuclei must collide with enough energy to give a reasonable probability of overcoming the repulsive electrical force between these two positively charged particles. The temperature at the very center of the Sun is about 15,000,000° . Both the temperature and the density decrease as one moves outward from the center of the Sun.

In stars like the Sun the nuclear burning takes place through a three step process called the proton-proton or pp chain. In the first step two protons collide to produce deuterium, a positron, and a neutrino. In the second step a proton collides with the deuterium to produce a helium-3 nucleus and a gamma ray. In the third step two helium-3s collide to produce a normal helium-4 nucleus with the release of two protons. In this process of fusing hydrogen to form helium, the nuclear reactions produce elementary particles called neutrinos.

5.1.2 The Radiative Zone

The radiative zone extends outward from the outer edge of the core to the interface layer or tachocline at the base of the convection zone. The radiative zone is characterized by the method of energy transport - radiation. The energy is carried by photons which eventually reach the surface and we see them as solar radiation. However, it is indeed interesting to note, that the opacity of the solar interior is so high, that the mean free path is extremely low and collision frequency is extremely high. As a result, despite travelling at the speed of light, these photons take millions of years to reach the surface.

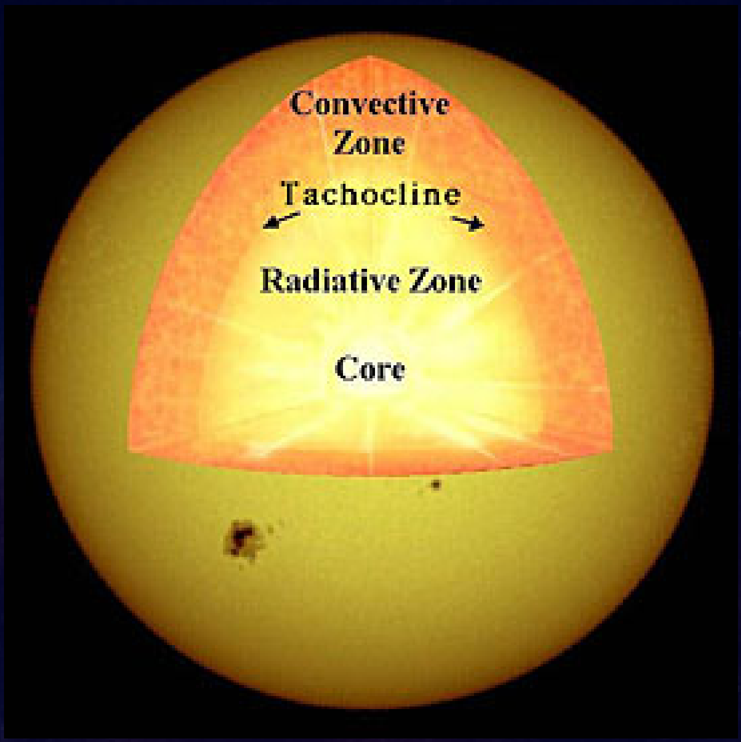

Figure 13: The layered solar interior

5.1.3 The Interface Layer (Tachocline)

The interface layer lies between the radiative zone and the convective zone. The fluid motions found in the convection zone slowly disappear from the top of this layer to its bottom where the conditions match those of the calm radiative zone.

It is now believed that the Sun's magnetic field is generated by a magnetic dynamo in this layer. The changes in fluid flow velocities across the layer (shear flows) can stretch magnetic field lines of force and make them stronger. This change in flow velocity gives this layer its alternative name - the tachocline. There also appears to be sudden changes in chemical composition across this layer.

5.1.4 The Convection Zone

The convection zone is the outer-most layer of the solar interior. At the base of the convection zone the temperature is about 2,000,000° C. This is "cool" enough for the heavier ions (such as carbon, nitrogen, oxygen, calcium, and iron) to hold onto some of their electrons. This makes the material more opaque so that it is harder for radiation to get through. This traps heat that ultimately makes the fluid unstable and it starts to "boil" or convect.

Convection occurs when the temperature gradient gets larger than the adiabatic gradient. Where this occurs a volume of material moved upward will be warmer than its surroundings and will continue to rise further. These convective motions carry heat quite rapidly to the surface. The fluid expands and cools as it rises. At the visible surface the temperature has dropped to 5,700 K and the density is only 0.0000002 gm/cm^3. The convective motions themselves are visible at the surface as granules and supergranules.

5.2 The Solar Neutrino Problem

Solar neutrinos arise from the nuclear reactions that power the Sun. The principal reactions are thought to be those of the proton–proton chain, listed in the table below. According to the standard solar model, these reactions account for 98% of the solar luminosity. There are three branches, each giving rise to an electron neutrino. The dominant branch is the first, which occurs 91% of the time; branches II and III occur approximately 9% and 0.1% of the time, respectively.

Branch I
$E_\nu^{\max} = 0.420$ MeV
$p + p \to d + e^+ + \nu$
$p + d \to {}^3\mathrm{He} + \gamma$
${}^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + p + p$

Branch II	Branch III
$E_\nu = 0.86$ MeV (90%), 0.38 MeV (10%)	$E_\nu^{\max} = 14.06$ MeV
${}^3\mathrm{He} + {}^4\mathrm{He} \to {}^7\mathrm{Be} + \gamma$	${}^7\mathrm{Be} + p \to {}^8\mathrm{B} + \gamma$
${}^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu$	${}^8\mathrm{B} \to {}^8\mathrm{B}^* + e^+ + \nu$
${}^7\mathrm{Li} + p \to {}^4\mathrm{He} + {}^4\mathrm{He}$	${}^8\mathrm{B}^* \to {}^4\mathrm{He} + {}^4\mathrm{He}$

Figure 14: The reactions that result in the production of these neutrinos, as predicted by the Standard Solar Model

These neutrinos travel from the Sun's core to Earth without any appreciable absorption by the Sun's outer layers. In the late 1960s, Ray Davis and John N. Bahcall's Homestake Experiment was the first to measure the flux of neutrinos from the Sun. The expected number of solar neutrinos was computed using the standard solar model, which Bahcall had helped establish. However this experimented detected a distinct deficit in the number of neutrinos reaching the Earth. The experiment used a chlorine-based detector. Many subsequent radiochemical and water Cherenkov detectors confirmed the deficit, including the Kamioka Observatory and Sudbury Neutrino Observatory.

Early attempts to explain the discrepancy proposed that the models of the Sun were wrong, i.e. the temperature and pressure in the interior of the Sun were substantially different from what was believed. For example, since neutrinos measure the amount of current nuclear fusion, it was suggested that the nuclear processes in the core of the Sun might have temporarily shut down. Since it takes thousands of years for heat energy to move from the core to the surface of the Sun, this would not immediately be apparent.

Advances in helioseismology observations made it possible to infer the interior temperatures of the Sun; these results agreed with the well established standard solar model. Detailed observations of the neutrino spectrum from more advanced neutrino observatories produced results which no adjustment of the solar model could accommodate: while the overall lower neutrino flux (which the Homestake experiment results found) required a reduction in the solar core temperature, details in the energy spectrum of the neutrinos required a higher core temperature.

Strong evidence for neutrino oscillation came in 1998 from the Super-Kamiokande collaboration in Japan. It produced observations consistent with muon neutrinos (produced in the upper atmosphere by cosmic rays) changing into tau neutrinos within the Earth: Fewer atmospheric neutrinos were detected coming through the Earth than coming directly from above the detector. These observations only concerned muon neutrinos. No tau neutrinos were observed at Super-Kamiokande. The result made it, however, more plausible that the deficit in the electron-flavor neutrinos observed in the (relatively low-energy) Homestake experiment has also to do with neutrino mass.

The problem is however not completely resolved as any solution involves the assumption that electron neutrinos get lost en-route their journey; to Earth.

6 Summary

- In the first section we learnt about the Virial theorem, and equation that sums up the delicate balance that holds a star together. We learnt about the forces that threaten to collapse a star or blow it apart, and the mathematics behind how these two balance each other out.
- Here we learnt about some unique type of staars that vary in light output, and their classificaatiojn. An important take away however was the fact that each and every star is indeed variable in light output, it is the period that makes some more "interesting' than others.
- In the third section we learnt about the actual interior of a star, the degeneracy pres-

sure, and the state of the gas that makes up the bulk of the material. We also learnt the causes for this state, and its various applications, all of course, in the ideal zero temperature regime.

- Finally we came crawling back to our good old sun, and tried to get a little background in a very qualitative manner, of the interior of the sun, and about the long standing solar neutrino problem.