Assignment 2: CS 754

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[In this answer, all norms are ℓ_2 norms unless stated otherwise]
For integer $k = 1, 2, \ldots, n$, the restricted isometry constant(RIC) δ_k of a matrix $\boldsymbol{A}$ of size $m \times n$ is the smallest number such that for any k-sparse vector $\boldsymbol{\theta}$, we have:

$$(1-\delta_k)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_k)||\boldsymbol{\theta}||^2$$

Let $s < t$.
Let Q_s be the set of all s-sparse vectors, and Q_t be the set of all t-sparse vectors. As $s < t$, any s-sparse vector is also a t-sparse vector. Hence, we have $Q_s \subset Q_t$.
δ_s is the smallest number such that for all $\boldsymbol{\theta} \in Q_s$,

$$(1-\delta_s)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_s)||\boldsymbol{\theta}||^2$$

And δ_t is the smallest number such that for all $\boldsymbol{\theta} \in Q_t$,

$$(1-\delta_t)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_t)||\boldsymbol{\theta}||^2$$

We have to prove that $\delta_s \leq \delta_t$.
On the contrary, let's assume $\delta_s > \delta_t$. As $Q_s \subset Q_t$, for all $\boldsymbol{\theta} \in Q_s \subset Q_t$, we have

$$(1-\delta_t)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_t)||\boldsymbol{\theta}||^2$$

As δ_s was defined to be the smallest number satisfying above, it cannot be the case that $\delta_s > \delta_t$.
Hence, it must be the case that $\delta_s \leq \delta_t$.

Question 5

Question 6

Fix a $\lambda > 0$ and consider the LASSO Problem:
$J(\boldsymbol{x}) = ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1$

Suppose $\boldsymbol{r}$ minimizes $J(\cdot)$.

We claim that, if $\epsilon = ||\boldsymbol{y} - \boldsymbol{\Phi r}||_2$ then, $\boldsymbol{r}$ is also a solution to:
P1: $\min_{\boldsymbol{x}} ||\boldsymbol{x}||_1$ s. t. $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$

To prove this, we do as follows:
Firstly, for all $\boldsymbol{x} \neq \boldsymbol{r}$, we will have: $J(\boldsymbol{x}) \geq J(\boldsymbol{r})$.

Consider all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$.
Due to both sides being positive, we also have- $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 \leq \epsilon^2$

which we can write as: $0 \leq \epsilon^2 - ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2$

We do know that following holds true for all $\boldsymbol{x}$, so it does for our constrain too:

$||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1 \geq ||\boldsymbol{y} - \boldsymbol{\Phi r}||_2^2 + \lambda||\boldsymbol{r}||_1$

$||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1 \geq \epsilon^2 + \lambda||\boldsymbol{r}||_1$

Now using our constrain, we can write,

$\lambda(||\boldsymbol{x}||_1 - ||\boldsymbol{r}||_1) \geq \epsilon^2 - ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 \geq 0$

As $\lambda > 0$ by definition, we thus have $||\boldsymbol{x}||_1 - ||\boldsymbol{r}||_1 > 0$ for all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$.

That is, for all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$, $||\boldsymbol{r}||_1 < ||\boldsymbol{x}||_1$

Thus, $\boldsymbol{r}$ is also a solution to P1 for the value of ϵ stated above. $\square$