Assignment 2: CS 754

Krushnakant Bhattad	Devansh Jain
190100036	190100044

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Contents

Question 1

1. δ_{2s} is the smallest number for which -

$$(1-\delta_{2s})\|x\|_{l_2}^2 \leq \|\Phi x\|_{l_2}^2 \leq (1+\delta_{2s})\|x\|_{l_2}^2$$

holds for all 2S-sparse vectors.

If $\delta_{2s}=1$, then there maybe a ~~vecto~~ 2S-sparse vector x, for which
$\|\Phi x\|_{l_2}^2 = 0$ [See left inequality in definition, & the fact that δ_{2s} is 'smallest']

Let Φ_i be the i^{th} column of Φ & x_i be the i^{th} element of x.

Thus, $\Phi x = \sum_{i=0}^{n} \Phi_i x_i$

As x has $(n-2S)$ elements zero, we can reduce the above summation to the 2S non-zero elements of x.

we have, $\Phi x = \sum_{p=0}^{2S} \Phi_{i_p} x_{i_p}$.

(where i_p are the indices of non-zero elements of x.)

As $\|\Phi x\|_{l_2}^2 = 0$, $\Phi x = 0$. Thus, we have,

$$\sum_{p=0}^{2S} \Phi_{i_p} x_{i_p} = 0.$$

follows from defn of l_2 norm.

$\because$ x_{i_p} are scalars not all zero, the above is a linear combination of 2S-columns of Φ summing to 0.

$\therefore$ There maybe a set of 2S columns of Φ that are linearly dependent

2. First we note that, as x^* is a solution to the optimization problem constrained on $\|y - \Phi x^*\| \leq \epsilon$, we do have $\|\Phi x^* - y\|_{l_2} \leq \epsilon$. — (1).

Next, $y = \Phi x + z$, where z is noise with $\|z\|_{l_2} \leq \epsilon$.

$\therefore$ We have, $\|y - \Phi x\|_{l_2} \leq \epsilon$ — (2).

For vectors x & y, the triangle inequality for l_2 norm states that,

$$\|x+y\|_{l_2} \leq \|x\|_{l_2} + \|y\|_{l_2}. \quad \text{— (3)}.$$

Consider, $\|\Phi(x^* - x)\|_{l_2}$

$$= \|\Phi x^* - y + y - \Phi x\|_{l_2}$$

$$\leq \|\Phi x^* - y\|_{l_2} + \|y - \Phi x\|_{l_2} \quad \text{from (3)}$$

$$\leq \epsilon + \epsilon \quad \text{from (1) \& (2)}$$

$$= 2\epsilon.$$

Thus, we have things as desired.

3. Let i_p for $p = 1, \dots, s$ represent the indices of the s non-zero elements of a vector.

Let q_j = max magnitude element of h_{T_j}
& r_j = min magnitude element of h_{T_j}.

First by definitions of h_{T_j}, we have, $q_j \leq r_{j-1}$

Next by definition of l_∞-norm, $\|h_{T_j}\|_{l_\infty} = q_j$.

a. $\|h_{T_j}\|_{l_2} = \sqrt{\sum_{p=1}^{s} (h_{T_j})_{i_p}^2}$

$\leq \sqrt{\sum_{p=1}^{s} q_j^2}$ (by def^n of q_j)

$= \sqrt{s \cdot q_j^2} = s^{1/2} q_j$ ($\because q_j \geq 0$ by def^n)

$= s^{1/2} \|h_{T_j}\|_{l_\infty}$

b. $q_j \leq r_{j-1}$

$\Rightarrow s \cdot q_j \leq s \cdot r_{j-1}$

$\Rightarrow s \cdot q_j \leq \sum_{p=1}^{s} |(h_{T_{j-1}})_{i_p}|$ (By def^n of r_j) every element is greater in magnitude than r_j

~~$\Rightarrow s \cdot q_j$~~

$\Rightarrow s \|h_{T_j}\|_{l_\infty} \leq \|h_{T_{j-1}}\|_{l_1}$

$\Rightarrow s^{1/2} \|h_{T_j}\|_{l_\infty} \leq s^{-1/2} \|h_{T_{j-1}}\|_{l_1}$

We've thus justified both inequalities.

4. $\| h_{T_2} \|_{\ell_2} \leq s^{-1/2} \| h_{T_1} \|_{\ell_1}$ & so on.

$\therefore \forall j \geq 2$, $\| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \| h_{T_{j-1}} \|_{\ell_1}$

Sum on both sides,

for $j \geq 2$, we have,

$$\sum_{j \geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \left(\sum_{j \geq 2} \| h_{T_{j-1}} \|_{\ell_1} \right)$$

Thus,

$$\sum_{j \geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \left(\| h_{T_1} \|_{\ell_1} + \| h_{T_2} \|_{\ell_1} + \dots \right)$$

Next, consider $h_{T_0^c}$.

$\| h_{T_0^c} \|_{\ell_1}$ is sum of absolute values of all elements of $h_{T_0^c}$.

~~Also, $h_{T_0^c} = h_{T_1} + h_{T_2} + h_{T_3} + \dots$~~

~~Where the RHS~~

Next, h_{T_1} is vector containing s-largest values from $h_{T_0^c}$

, h_{T_2} — " — from $h_{(T_0 \cup T_1)^c}$

& so on.

Thus, $\| h_{T_1} \|_{\ell_1} + \| h_{T_2} \|_{\ell_1} + \dots + \| h_{T_k} \|_{\ell_1}$ is the sum of absolute values of $s \times k$ largest elements of $h_{T_0^c}$.

Thus for any k,

$$\sum_{i=1}^{k} \| h_{T_i} \|_{\ell_1} \leq \| h_{T_0^c} \|_{\ell_1}$$

This directly translates into the right inequality & we're done.

5. We note that the triangle inequality can be expanded to multiple terms, by induction, using the idea that -

$$\|x+y+z\| = \|(x+y)+z\| \leq \|x+y\| + \|z\| \leq \|x\| + \|y\| + \|z\|$$

(All are l_2 norms).

Thus,

$$\left\|\sum_{j\geq 2} h_{T_j}\right\|_{l_2} \leq \sum_{j\geq 2} \|h_{T_j}\|_{l_2}$$

follows directly from the Δ-inequality.

As already proved, $\sum_{j\geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} \|h_{T_0^c}\|_{l_1}$ in part (4).

& we're done.

6. $|x| + |y| \geq |x+y|$ (Δ-inequality for $|\cdot|$)

Let $x = a$ & $y = -a-b$

$\therefore |a| + |-a-b| \geq |a+(-a-b)| = |-b| = |b|$

$\therefore |-a-b| \geq |b| - |a|$

$\therefore |a+b| \geq |b| - |a|$

This also gives, $|a+b| \geq |a| - |b|$.

$\therefore \forall i \in T_0, \ |x_i + h_i| \geq |x_i| - |h_i|$

$$\therefore \sum_{i\in T_0} |x_i + h_i| \geq \sum_{i\in T_0} |x_i| - \sum_{i\in T_0} |h_i| = \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1}$$

Similarly proceeding using $\forall i \in T_0^c$, $|x_i + h_i| \geq |h_i| - |x_i|$ we get $\sum_{i\in T_0^c} |x_i + h_i| \geq \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1}$.

Combining above two inequalities, we're done

7. In (6) we proved that,

$$\|x\|_{l_1} \geq \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \|h_{T_0^c}\|_{l_1} - \|x_{T_0^c}\|_{l_1}$$

$$\therefore \|h_{T_0^c}\|_{l_1} \leq \|h_{T_0}\|_{l_1} + \|x_{T_0^c}\|_{l_1} + \|x\|_{l_1} - \|x_{T_0}\|_{l_1}$$

$$\|x\|_{l_1} = \sum_i |x_i| = \sum_{i \in T_0} |x_i| + \sum_{i \in T_0^c} |x_i|$$

$$= \|x_{T_0}\|_{l_1} + \|x_{T_0^c}\|_{l_1}$$

Substituting this in the inequality as $\|x\|_{l_1} - \|x_{T_0}\|_{l_1} = \|x_{T_0^c}\|_{l_1}$

we get,

$$\|h_{T_0^c}\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}$$

as desired.

8. For two vectors $u, v \in \mathbb{R}^s$, the Cauchy-Schwartz inequality states that

$$|\langle u, v\rangle| \leq \|u\|_{l_2}\|v\|_{l_2}.$$

Usual inner product, & l_2-norm.

i.e.,
$$\sum_{i=1}^{s} u_i v_i \leq \sqrt{\sum_{i=1}^{s} u_i^2}\sqrt{\sum_{i=1}^{s} v_i^2}$$

Let $x \in \mathbb{R}^s$

Let $u_i = |x_i|$ & $v_i = 1$, $\forall i \in \{1, 2, \ldots s\}$

Then we have,
$$\sum_{i=1}^{s} |x_i| \leq s^{1/2}\sqrt{\sum_{i=1}^{n} x_i^2}$$

i.e. $\|x\|_{l_1} \leq s^{1/2}\|x\|_{l_2}$.

$\therefore$ We have, $\|h_{T_0}\|_{l_1} \leq s^{1/2}\|h_{T_0}\|_{l_2}$.

We can thus write,

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq s^{-1/2}\|h_{T_0^c}\|_{l_1}$$
$$\leq s^{-1/2}\left(\|h_{T_0}\|_{l_1} + 2\|x_{T_0^c}\|_{l_1}\right)$$

Already proved

$$= s^{-1/2}\|h_{T_0}\|_{l_1} + 2\left(s^{-1/2}\|x_{T_0^c}\|_{l_1}\right)$$
$$\leq s^{-1/2}\left(s^{1/2}\|h_{T_0}\|_{l_2}\right) + 2\left(s^{-1/2}\|x - x_s\|_{l_1}\right)$$
$$= \|h_{T_0}\|_{l_2} + 2e_0$$
, as desired.

9. $h' = h_{T_0 \cup T_1}$ by def^n is a 2S-sparse vector.

As Φ obeys RIP of order 2S, we have,

$$\| \Phi h' \|_{\ell_2}^2 \leq (1 + \delta_{2s}) \| h' \|_{\ell_2}^2$$

i.e. $\| \Phi h' \|_{\ell_2} \leq \sqrt{1 + \delta_{2s}} \; \| h' \|_{\ell_2}$ — (1).

We've already proved that $\| \Phi (x^* - x) \| \leq 2\epsilon$

i.e. $\| \Phi h \| \leq 2\epsilon$. — (2)

By cauchy schwartz inequality $|\langle u, v \rangle| \leq \|u\| \, \|v\|$ we have,

$$| \langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle | \leq \| \Phi_{h_{T_0 \cup T_1}} \| \, \| \Phi h \| \quad \text{— (3).}$$

Combine (1), (2) & (3) appropriately to get desired result.

10. Lemma 2.1 states that,

$$|\langle \Phi x, \Phi x' \rangle| \leq \delta_{S+S'} \|x\|_{l_2} \|x'\|_{l_2}$$

where x, x' are supported on $T, T' \subseteq \{1, 2, \ldots n\}$ with $|T| \leq S$ & $|T'| \leq S'$.
(using parallelogram identity of inner product)

h_{T_0} & h_{T_j} for all j, by definition are s-sparse,

i.e. supported on $T_0, T_j \subseteq \{1, 2, \ldots n\}$ ~~ust~~ with $|T_0| \leq s$ & $|T_j| \leq s$. $(= \Rightarrow \leq)$

Thus using lemma 2.1, we can conclude,

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_0}\|_{l_2} \|h_{T_j}\|_{l_2}.$$

We could use the result because by definition, T_j is included in T_0^c. Hence, h_{T_0} & h_{T_j} are supported on disjoint s-sized subsets of $[1:n]$. Likewise for $j \geq 2$ & T_1 in place of T_0.

11. We can write $h_{T_0 \cup T_1} = h_{T_0} + h_{T_1}$ as T_0 & T_1 are disjoint by definition.

For $x, y \geq 0$,
We know that $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$
As $(x-y)^2 \geq 0$, we have, $x + y \leq \sqrt{2}\sqrt{x^2+y^2}$
If $x = \|h_{T_0}\|_{l_2}$, $y = \|h_{T_1}\|_{l_2}$

Then $\sqrt{x^2+y^2} = \|h_{T_0 \cup T_1}\|_{l_2}$ $(\because T_0, T_1$ are disjoint$)$

Thus we can write,

$$\|h_{T_0}\|_{l_2} + \|h_{T_1}\|_{l_2} \leq \sqrt{2}\, \|h_{T_0 \cup T_1}\|_{l_2}$$

v 2 We have, $\| \Phi h_{T_0 \cup T_1} \|_{\ell_2}^2 = \langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle - \langle \Phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle$.

(1) Firstly we already have, $\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle \leq 2\epsilon \sqrt{1+\delta_{2s}} \, \| h_{T_0 \cup T_1} \|_{\ell_2}$

Next, we have, & for $j \geq 2$

$$\begin{cases} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_0}\|_{\ell_2} \|h_{T_j}\|_{\ell_2} \\ |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_1}\|_{\ell_2} \|h_{T_j}\|_{\ell_2}. \end{cases}$$

By linearity of inner product in first argument, & $h_{T_0 \cup T_1} = h_{T_0} + h_{T_1}$ we get

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_j}\|_{\ell_2} \left(\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \right)$$

$$\leq \delta_{2s} \|h_{T_j}\|_{\ell_2} \left(\sqrt{2} \, \|h_{T_0 \cup T_1}\|_{\ell_2} \right) \quad \{\text{Proved in Q11}\}$$

$$= \|h_{T_j}\|_{\ell_2} \left(\sqrt{2} \, \delta_{2s} \|h_{T_0 \cup T_1}\|_{\ell_2} \right)$$

By linearity of inner product in 2^{nd} argument, summing the above inequality over $j \geq 2$, we have,

(2) $$\langle \Phi h_{T_0 \cup T_1}, \sum \Phi h_{T_j} \rangle \leq \left(\sqrt{2} \, \delta_{2s} \|h_{T_0 \cup T_1}\|_{\ell_2}^2 \right) \Big(\underbrace{\sum_{j \geq 2} \|h_{T_j}\|_{\ell_2}}_{(*)} \Big)$$

(1) & (2) both have RHS ≥ 0, & using the fact that $a - b \leq |a| + |b|$, we can write:

$$\| \Phi h_{T_0 \cup T_1} \|_{\ell_2}^2 \leq \|h_{T_0 \cup T_1}\|_{\ell_2} \left(2\epsilon \sqrt{1+\delta_{2s}} + \sqrt{2} \, \delta_{2s} (*) \right)$$

The left inequality in given question follows from the fact that $h_{T_0 \cup T_1}$ is a 2S-sparse vector, the definition of RIP & RIC & that Φ follows RIP of order 2S with RIC δ_{2s}.

<u>13.</u> In Q12 we proved that:

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_{l_2}^2 \leq \|h_{T_0 \cup T_1}\|_{l_2} \left(2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_{l_2}\right)$$

Let $\|h_{T_0 \cup T_1}\|_{l_2} = q \geq 0$.

We have,

$$q \leq \underbrace{\left(\frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\right)}_{\alpha} \epsilon + \underbrace{\left(\frac{\sqrt{2}\,\delta_{2s}}{1-\delta_{2s}}\right)}_{\rho} \left(\sum_{j \geq 2} \|h_{T_j}\|_{l_2}\right)$$

Also we proved in ~~Q~~ equation (10) in the paper that

$$\sum_{j \geq 2} \|h_{T_j}\|_{l_2} \leq s^{-1/2} \|h_{T_0^c}\|_{l_1}$$

Thus we have, $q \leq \alpha\epsilon + \rho\, s^{-1/2} \|h_{T_0^c}\|_{l_1}$

As desired.

14. First, we observe that $\|h_{T_0 \cup T_1}\|_{\ell_2} \geq \|h_{T_0}\|_{\ell_2}$
as the former has a
strictly greater no. of non-negative terms under the $\sqrt{\ }$
in addition to what latter has.
Next, in Q8 we proved that $\|h_{T_0}\|_{\ell_2} \geq S^{-1/2}\|h_{T_0}\|_{\ell_1}$
Using both we can write,

$$S^{-1/2}\|h_{T_0}\|_{\ell_1} \leq \|h_{T_0 \cup T_1}\|_{\ell_2}$$

Now consider,

$$S^{-1/2}\|h_{T_0^c}\|_{\ell_1} \leq S^{-1/2}\left(\|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}\right) \quad \left[\text{From (12) in the paper}\right]$$

$$= S^{-1/2}\|h_{T_0}\|_{\ell_1} + 2\underbrace{S^{-1/2}\|x - x_S\|_{\ell_1}}_{\text{defined to be } e_0}$$

$$\leq \|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0$$

Substituting this into the last eq^n of Q13 we get,

$$\|h_{T_0 \cup T_1}\|_{\ell_2} \leq \alpha\epsilon + \rho\|h_{T_0 \cup T_1}\|_{\ell_2} + 2\rho e_0 .$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_{\ell_2} \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0)$$

15. We've already proved in eq(13) in the paper that,

$$\|h_{(T_0 \cup T_1)^c}\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0 \quad — (1).$$

Also we know, $\|h_{T_0}\|_{l_2} \leq \|h_{T_0 \cup T_1}\|_{l_2}$ — (2).

Now, $\|h\|_{l_2}$

$= \|h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}\|_{l_2}$ (disjoint & exhaustive)

$\leq \|h_{T_0 \cup T_1}\|_{l_2} + \|h_{(T_0 \cup T_1)^c}\|_{l_2}$ (Δ-inequality for l_2 norm)

$\leq 2\|h_{T_0 \cup T_1}\|_{l_2} + 2e_0$ (Use (1) & (2))

$\leq 2\left[(1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0) + e_0(1-\rho)(1-\rho)^{-1}\right]$

$= 2\left[(1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0 + e_0 - \rho e_0)\right]$

$= 2(1-\rho)^{-1}(\alpha\epsilon + (1+\rho)e_0)$

16.

In Q13 we proved that, $\|h_{T_0 \cup T_1}\|_{\ell_2} \leq \alpha\epsilon + \rho s^{-1/2} \|h_{T_0^c}\|_{\ell_1}$

In noiseless case, $\epsilon = 0$, so we have,

$$s^{1/2} \|h_{T_0 \cup T_1}\|_{\ell_2} \leq \rho \|h_{T_0^c}\|_{\ell_1}$$

We've proved in Q8 that,

$$\|h_{T_0}\|_{\ell_1} \leq s^{1/2} \|h_{T_0}\|_{\ell_2}$$

$$\therefore \|h_{T_0}\|_{\ell_1} \leq s^{1/2} \|h_{T_0 \cup T_1}\|_{\ell_2}$$

$$\leq \rho \|h_{T_0^c}\|_{\ell_1}.$$

In eq^n (12) in the paper, we proved that

$$\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$$

$$\therefore \|h_{T_0^c}\|_{\ell_1} \leq \rho \|h_{T_0^c}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$$

$$\therefore \|h_{T_0^c}\|_{\ell_1} \leq 2(1-\rho)^{-1} \|x_{T_0^c}\|_{\ell_1}$$

Now, $\|h\|_{\ell_1} = \sum_i |h_i| = \sum_{i \in T_0} |h_i| + \sum_{i \in T_0^c} |h_i|$

$$= \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1}$$

$$\leq \rho \|h_{T_0^c}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1}$$

$$= \|h_{T_0^c}\|_{\ell_1} (1+\rho)$$

use this here.

$$\leq 2(1+\rho)(1-\rho)^{-1} \|x_{T_0^c}\|_{\ell_1}$$

Thus we're done.

Question 2

Instructions for running the code:

- After extracting submitted file, look for a directory named `q2`, and `cd` (change directory) to it.
- Files `q2a.m`, `q2b.m`, `q2c.m` and `q2d.m` contains the code for (a), (b), (c) and (d) respectively which use the ISTA function defined in file `ista.m`.
- Run the files `q2a.m`, `q2b.m`, `q2c.m` and `q2d.m`. The results can be found in `./results/`

ISTA Algorithm

```
function theta = ista(y, A, lambda, alpha, iter)
    theta = zeros(size(A, 2), 1);
    thres = lambda/(2*alpha);
    for i=1:iter
        theta = soft(theta + (A'*(y - A*theta))/alpha, thres);
    end
end

function y = soft(x,T)
    y = sign(x).*(max(0, abs(x)-T));
end
```

Function `ista` takes y, A, λ in the LASSO function, α used in Majorizer function and the number of iterations.

Initialization for (a)

```
% Setting seed
rng(0);

% Reading
orig = cast(imread("data/barbara256.png"),'double');
H = size(orig, 1);
W = size(orig, 2);
% figure; imshow(cast(orig, 'uint8'));

% Adding Gaussian Noise
noise_img = orig + 2*randn(H, W);
% figure; imshow(cast(noise_img, 'uint8'));
```

Initialization for (b) and (c)

```
% Setting seed
rng(0);

% Reading
orig = cast(imread("data/barbara256.png"),'double');
```

```
H = size(orig, 1);
W = size(orig, 2);
% figure; imshow(cast(orig, 'uint8'));
```

Setting Φ and Ψ for (a)

```
% Calculating phi, psi and thus, A.
phi = diag(ones(64,1));
psi = kron(dctmtx(8)', dctmtx(8)');
A = phi*psi;
```

Setting Φ and Ψ for (b)

```
% Calculating phi, psi and thus, A.
phi = randn(32, 64);
psi = kron(dctmtx(8)', dctmtx(8)');
A = phi*psi;
```

Setting Φ and Ψ for (c)

```
% Calculating phi, psi and thus, A.
phi = randn(32, 64);
psi = haarmtx(64);
A = phi*psi;
```

Function `haarmtx(N)` is defined at the end of file `q2c.m` which creates the Haar wavelet transformation matrix of size NxN.

Setting alpha, lambda and number of iterations

```
alpha = floor(eigs(A'*A,1)) + 1;
lambda = 1;
iter = 100;
```

Initializing reconstructed image and averaging matrix

```
recon_img = zeros(H, W, 'double');
avg_mat = zeros(H, W, 'double');
```

Iterating over all possible 8x8 patches in the image

```
for i=1:H-7
    for j=1:W-7
        y = phi * reshape(orig(i:i+7,j:j+7), [8*8 1]);
        theta = ista(y, A, lambda, alpha, iter);
        recon_img(i:i+7,j:j+7) = recon_img(i:i+7,j:j+7) + reshape(psi * theta, [8 8]);
        avg_mat(i:i+7,j:j+7) = avg_mat(i:i+7,j:j+7) + ones(8,8);
        i, j % Prints the coordinates, to check for speed and debugging
```

```
    end
end
```

Normalize the reconstructed image

```
recon_img(:,:) = recon_img(:,:)./avg_mat(:,:);
recon_img(recon_img < 0) = 0;
recon_img(recon_img > 255) = 255;
```

Save the image and calculate RMSE

```
figure; imshow(cast([recon_img(:,:), orig(:,:)], 'uint8'));
imwrite(cast([recon_img(:,:), orig(:,:)], 'uint8'), 'results/q2b.png');
fprintf('RMSE : %f\n', norm(recon_img(:,:) - orig(:,:), 'fro')/norm(orig(:,:), 'fro'));
```

Reporting Relative Mean Squared Error

- (a)
  RMSE = 0.013523
- (b)
  RMSE = 0.062871
- (c)
  RMSE = 0.063416

Initialization for (d)

```
% Setting the seed
rng(100);

% Reading
orig = zeros(100,1);
ind = randi(100, [10 1]);
orig(ind) = randi(10, [10 1]);
```

Calculating the matrix A for (d)

```
A = zeros(100, 100);
A(1:4, 1) = [10 3 2 1];
A(1:5, 2) = [6 4 3 2 1];
A(1:6, 3) = [3 3 4 3 2 1];
for i=4:97
     A(i-3:i+3,i) = [1 2 3 4 3 2 1];
end
A(95:100, 98) = [1 2 3 4 3 3];
A(96:100, 99) = [1 2 3 4 6];
A(97:100, 100) = [1 2 3 10];
A = A/16;
```

Reconstructing the signal

```
alpha = floor(eigs(A'*A,1)) + 1;
iter = 10000;
lambda = 0.01;

y = A*orig + 0.0005*norm(orig)*randn(100,1); % 5% noise wasn't giving proper reconstruction
recon = ista(y, A, lambda, alpha, iter);
fprintf('RMSE : %f\n', norm(recon - orig)/norm(orig));
```

The reconstruction error in (d) is pretty high compared to the error calculated in the first three parts.
The ISTA algorithm works perfectly fine, the possible reason for this is that A doesn't follow RIP and thus the reconstruction wasn't nice.
However, reducing the noise with changing the lambda gave better reconstruction with RMSE as low as 0.019029.

Question 3

(a)

As the signal is purely S-sparse, we can write as follows if we ignore the elements that we know will be zero:

$\boldsymbol{y} = \boldsymbol{\Phi_S x_S} + \boldsymbol{\eta}$

If we pre-multiply this by the pseudo-inverse, we have:

$((\boldsymbol{\Phi_S^T \Phi_S})^{-1}\boldsymbol{\Phi_S^T})\boldsymbol{y} = \boldsymbol{x_S} + ((\boldsymbol{\Phi_S^T \Phi_S})^{-1}\boldsymbol{\Phi_S^T})\boldsymbol{\eta}$

Using a method like ML-estimate or least squares, we could estimate the oracular solution, as the following, where other elements are zero and the vector of non-zero elements is given by :

$\boldsymbol{\tilde{x}_S} = (\boldsymbol{\Phi_S^T \Phi_S})^{-1}\boldsymbol{\Phi_S^T}\,\boldsymbol{y}$

(b)

(Unless specified otherwise, the norms are ℓ_2 norms)

Now, let $\boldsymbol{\Phi_S^\dagger} \triangleq (\boldsymbol{\Phi_S^T \Phi_S})^{-1}\boldsymbol{\Phi_S^T}$

The original $\boldsymbol{x_S}$, is however, $\boldsymbol{\Phi_S^\dagger}(\boldsymbol{y} - \boldsymbol{\eta})$ while, $\boldsymbol{\tilde{x}_S} = \boldsymbol{\Phi_S^\dagger}\,\boldsymbol{y}$

Thus we have $||\boldsymbol{x} - \boldsymbol{\tilde{x}}|| = ||\boldsymbol{x_S} - \boldsymbol{\tilde{x}_S}|| = ||\boldsymbol{\Phi_S^\dagger \eta}||$

By definition of the matrix norm (or the largest singular value), we have the following:

$||\boldsymbol{\Phi_S^\dagger}||_2 = \sup_{\boldsymbol{x}} \dfrac{||\boldsymbol{\Phi_S^\dagger x}||_2}{||\boldsymbol{x}||_2}$

Thus it immediately follows that:

$||\boldsymbol{\Phi_S^\dagger}||_2 \geq \dfrac{||\boldsymbol{\Phi_S^\dagger \eta}||_2}{||\boldsymbol{\eta}||_2}$ which we can also write as: $||\boldsymbol{\Phi_S^\dagger}||_2 ||\boldsymbol{\eta}||_2 \geq ||\boldsymbol{\Phi_S^\dagger \eta}||_2$

This brings us to our desired inequality:

$\|\boldsymbol{\tilde{x}} - \boldsymbol{x}\|_2 = \|\boldsymbol{\Phi_S^\dagger \eta}\|_2 \leq \|\boldsymbol{\Phi_S^\dagger}\|_2 \|\boldsymbol{\eta}\|_2$

(c)

$k = |S|$ and δ_{2k} is the RIC of $\boldsymbol{\Phi}$ of order $2k$

Thus for all $2k$-sparse vectors $\boldsymbol{q}$ we have:

$$(1 - \delta_{2k})||\boldsymbol{q}||_2^2 \leq ||\boldsymbol{\Phi q}||_2^2 \leq (1 + \delta_{2k})||\boldsymbol{q}||_2^2$$

That also guarantees us that the above will hold for all k-sparse vectors with $\boldsymbol{S}$ as a support.
Each vector $\boldsymbol{q}_S$ in $\mathbb{R}^{k\times 1}$ can be expressed as a k-sparse vector with $\boldsymbol{S}$ as a support, as $\boldsymbol{q}$ in $\mathbb{R}^{n\times 1}$, with the same ℓ_2 norm.

Also, for all such vectors , $\boldsymbol{\Phi q} = \boldsymbol{\Phi}_S \boldsymbol{q}_S$

Thus, for all vectors $\boldsymbol{q}_S \in \mathbb{R}^{k\times 1}$, we have:

$$(1-\delta_{2k})||\boldsymbol{q_S}||_2^2 \leq ||\boldsymbol{\Phi_S q_S}||_2^2 \leq (1+\delta_{2k})||\boldsymbol{q_S}||_2^2$$

$$\sqrt{(1-\delta_{2k})} \leq \frac{||\boldsymbol{\Phi_S q_S}||_2}{||\boldsymbol{q_S}||_2} \leq \sqrt{(1+\delta_{2k})}$$

The term in the middle tells us that the singular values of $\boldsymbol{\Phi_S}$ lie in the range $\sqrt{(1-\delta_{2k})}$ to $\sqrt{(1+\delta_{2k})}$.
If $\boldsymbol{\Phi_S} = USV^T$ is the singular-value decomposition of $\boldsymbol{\Phi_S}$,
then the SVD of $\boldsymbol{\Phi_S^\dagger}$ is $\boldsymbol{\Phi_S^\dagger} = (V^T)^{-1}S^{-1}U^{-1}$.

That is, each of the singular values of $\boldsymbol{\Phi_S^\dagger}$ is just the multiplicative inverse of the singular values of $\boldsymbol{\Phi_S}$
If $d \in [a,b]$ then, $\frac{1}{d} \in \left[\frac{1}{b}, \frac{1}{a}\right]$.
Thus the singular values of $\boldsymbol{\Phi_S^\dagger}$ (and by consequence, the largest singular value) lie in the range $\frac{1}{\sqrt{1+\delta_{2k}}}$ and $\frac{1}{\sqrt{1-\delta_{2k}}}$.

(d)

Let $\boldsymbol{x_t}$ be the answer given by theorem 3, $\boldsymbol{x}$ be the actual solution and $\tilde{\boldsymbol{x}}$ be the oracular solution. The error bound given by Theorem 3 is:
$||\boldsymbol{x_t} - \boldsymbol{x}||_2 \leq \frac{c_0}{\sqrt{|\boldsymbol{S}|}}||\boldsymbol{x} - \tilde{\boldsymbol{x}}||_{\ell_1} + c_1\epsilon$
For a purely sparse $\boldsymbol{x}$, the ℓ_1 norm term will be 0, and what is left will be $c_1\epsilon$.
Here the bound is:
$\frac{\epsilon}{\sqrt{1+\delta_{2k}}} \leq \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \leq \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$
Since both bounds are independent of n, evidently, the Theorem 3 bound is only a constant factor worse than the oracular solution.
For example, when $\delta_{2k} = 0.25$, we get
$0.89\epsilon \leq \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_2 \leq 1.16\epsilon$
while theorem 3 error bound is about 6ϵ.

Question 4

[In this answer, all norms are ℓ_2 norms unless stated otherwise]
For integer $k = 1, 2, \ldots, n$, the restricted isometry constant(RIC) δ_k of a matrix $\boldsymbol{A}$ of size $m \times n$ is the smallest number such that for any k-sparse vector $\boldsymbol{\theta}$, we have:

$$(1-\delta_k)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_k)||\boldsymbol{\theta}||^2$$

Let $s < t$.
Let Q_s be the set of all s-sparse vectors, and Q_t be the set of all t-sparse vectors. As $s < t$, any s-sparse vector is also a t-sparse vector. Hence, we have $Q_s \subset Q_t$.
δ_s is the smallest number such that for all $\boldsymbol{\theta} \in Q_s$,

$$(1-\delta_s)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_s)||\boldsymbol{\theta}||^2$$

And δ_t is the smallest number such that for all $\boldsymbol{\theta} \in Q_t$,

$$(1-\delta_t)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_t)||\boldsymbol{\theta}||^2$$

We have to prove that $\delta_s \leq \delta_t$.
On the contrary, let's assume $\delta_s > \delta_t$. As $Q_s \subset Q_t$, for all $\boldsymbol{\theta} \in Q_s \subset Q_t$, we have

$$(1-\delta_t)||\boldsymbol{\theta}||^2 \leq ||\boldsymbol{A\theta}||^2 \leq (1+\delta_t)||\boldsymbol{\theta}||^2$$

As δ_s was defined to be the smallest number satisfying above, it cannot be the case that $\delta_s > \delta_t$.
Hence, it must be the case that $\delta_s \leq \delta_t$.

Question 5

(1)

Title of the paper - "Low-Cost and High-Throughput Testing of COVID-19 Viruses and Antibodies via Compressed Sensing: System Concepts and Computational Experiments"
Link - `https://arxiv.org/abs/2004.05759`

(2)

The paper solves the Basis Pursuit problem (ℓ_1 norm optimization):

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} ||\boldsymbol{x}||_1, \text{ s.t. } ||\boldsymbol{Ax} - \boldsymbol{y}||_2 \leq \epsilon, \mathbf{x} \geq \mathbf{0}$$

where $\epsilon > 0$ is a parameter tuned to noise magnitude, and where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the measurement matrix, $\boldsymbol{x}$ is the signal to be recovered, and $\boldsymbol{y} \in \mathbb{R}^m$ is the measurement vector.

(3)

Differences between the proposed approach and the Tapestry pooling approach

1. The actual measurement matrix A in the proposed approach is the element-wise multiplication of mixing matrix E (binary) and allocation matrix W (percentage of each sample in each pool). The actual measurement matrix A in the Tapestry pooling approach is just a binary matrix. Also, the method of construction of the measurement matrix is different.
2. Even though the paper mentions several algorithms to solve the system, it uses the Basis pursuit to find $\boldsymbol{x}$. The Tapestry paper uses algorithms like NN-OMP which is an approximation algorithm for the ℓ_0 norm optimization problem (P0 problem with $\boldsymbol{x} \geq \mathbf{0}$)
3. One of the major difference is that the proposed approach doesn't remove samples belonging to a pool which has tested negative. This is taken care in the Tapestry pooling approach uses COMP (Combinatorial Group-testing) to make the system of equations as small as possible.
4. While designing the sensing matrix A, the Tapestry pooling approach for the ease of pipetting, A is made sparse. While in the proposed approach the sensing matrix was a Bernoulli random matrix where each entry of the matrix is '0' with probability 0.5, and is '1' with probability 0.5.

Question 6

Fix a $\lambda > 0$ and consider the LASSO Problem:
$J(\boldsymbol{x}) = ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1$

Suppose $\boldsymbol{r}$ minimizes $J(\cdot)$.

We claim that, if $\epsilon = ||\boldsymbol{y} - \boldsymbol{\Phi r}||_2$ then, $\boldsymbol{r}$ is also a solution to:
P1: $\min_{\boldsymbol{x}} ||\boldsymbol{x}||_1$ s. t. $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$

To prove this, we do as follows:
Firstly, for all $\boldsymbol{x} \neq \boldsymbol{r}$, we will have: $J(\boldsymbol{x}) \geq J(\boldsymbol{r})$.

Consider all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$.
Due to both sides being positive, we also have- $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 \leq \epsilon^2$

which we can write as: $0 \leq \epsilon^2 - ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2$

We do know that following holds true for all $\boldsymbol{x}$, so it does for our constrain too:

$||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1 \geq ||\boldsymbol{y} - \boldsymbol{\Phi r}||_2^2 + \lambda||\boldsymbol{r}||_1$

$||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 + \lambda||\boldsymbol{x}||_1 \geq \epsilon^2 + \lambda||\boldsymbol{r}||_1$

Now using our constrain, we can write,

$\lambda(||\boldsymbol{x}||_1 - ||\boldsymbol{r}||_1) \geq \epsilon^2 - ||\boldsymbol{y} - \boldsymbol{\Phi x}||_2^2 \geq 0$

As $\lambda > 0$ by definition, we thus have $||\boldsymbol{x}||_1 - ||\boldsymbol{r}||_1 \geq 0$ for all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$.

That is, for all $\boldsymbol{x}$ for which $||\boldsymbol{y} - \boldsymbol{\Phi x}||_2 \leq \epsilon$, $||\boldsymbol{r}||_1 \leq ||\boldsymbol{x}||_1$

Thus, $\boldsymbol{r}$ is also a solution to P1 for the value of ϵ stated above. $\square$