MA 109 : Calculus-I
D1 T4, Tutorial 1

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Questions to be Discussed

- Sheet 1
  - 2 (iv) - Sandwich Theorem for finding limits
  - 3 (ii) - Checking convergence of a sequence
  - 5 (iii) - Monotonic Bounded sequences are convergent
  - 7 - Proof using ϵ - n_0 definition
  - 9 - Relation between product and convergence
  - 11 - Conditions for exchanging product and limits (Discussed in lecture)

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Where did we use that $h_n \geq 0$?

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However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\geq 1}$ converging. (Why?)
However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

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Thus, (a_n) is a monotonically increasing sequence that is bounded above. Therefore, it must converge. Using the same method as earlier gives this limit to be 6.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

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By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.

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(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

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Both are false.
The sequences, $a_n := 1 \quad \forall n \in \mathbb{N}$ and $b_n := (-1)^n \quad \forall n \in \mathbb{N}$ act as a counterexample for both the statements.

11. (i) We shall show that the statement is false with the help of a counterexample.
Let $a = -1,\ b = 1,\ c = 0$. Define f and g as follows:
$f(x) = x$ and $g(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$.
It can be seen that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to c} [f(x)g(x)] = \lim_{x \to 0} 1 = 1$.

(ii) We shall prove that the given statement is true.
We are given that g is bounded. Thus, $\exists M \in \mathbb{R}^+$ such that $|g(x)| \leq M \quad \forall x \in (a,\ b)$.
Let $\epsilon > 0$ be given. We want to show that there exists $\delta > 0$ such that
$|f(x)g(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.
Let $\epsilon_1 = \epsilon / M$. As $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that
$0 < |x - c| < \delta \implies |f(x)| < \epsilon_1$.
Thus, whenever $0 < |x - c| < \delta$, we have it that
$|f(x)g(x) - 0| = |f(x)||g(x)| \leq |f(x)| \cdot M < \epsilon_1 \cdot M = \epsilon$. ■

(iii) We shall prove that the given statement is true.
Let $\epsilon > 0$ be given.
Let $l := \lim_{x \to c} g(x)$.
Let $\epsilon_1 = \epsilon/(|l| + \epsilon)$.

By hypothesis, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1 \implies |g(x) - l| < \epsilon$.
Also, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2 \implies |f(x)| < \epsilon_1$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x - c| < \delta$, we have that:
$|f(x)g(x)| = |f(x)g(x) - lf(x) + lf(x)| \leq |f(x)||(g(x) - l)| + |l||f(x)| <$
$|f(x)|\epsilon + |l||f(x)| = |f(x)|(\epsilon + |l|) < \epsilon_1(\epsilon + |l|) = \epsilon.$
Thus, we have it that $0 < |x - c| < \delta \implies |f(x)g(x) - 0| < \epsilon$. $\blacksquare$

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)