MA 109 : Calculus-I
D1 T4, Tutorial 01

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Summary

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Expectations

What we expect from you, before you come to the tutorial is the following:

1. You have read the lecture slides that have been uploaded up to that tutorial.
2. You have *attempted* the questions that are to be discussed in the tutorials.

Sheet 1

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Let $n_0 = \left\lfloor \frac{10}{\epsilon} \right\rfloor + 1$. It is clear that $n_0 > \frac{10}{\epsilon}$.

Moreover, for any $n \geq n_0$, we will have $n > \frac{10}{\epsilon}$.

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Let $n_0 = \left\lfloor \frac{10}{\epsilon} \right\rfloor + 1$. It is clear that $n_0 > \frac{10}{\epsilon}$.

Moreover, for any $n \geq n_0$, we will have $n > \frac{10}{\epsilon}$.

Thus, we have shown that for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{10}{n}\right| < \epsilon$ for all $n \geq n_0$. $\therefore \lim_{n \to \infty} \frac{10}{n} = 0.$

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One such choice is $n_0 = \max\left\{1, \left\lfloor \frac{1}{3}\left(\frac{5}{\epsilon} - 1\right)\right\rfloor\right\} + 1$.

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One such choice is $n_0 = \max\left\{1, \left\lfloor \frac{1}{3}\left(\frac{5}{\epsilon} - 1\right)\right\rfloor\right\} + 1$.
Note: The choice of n_0 is not unique. Our choice of n_0 might not be the smallest but that is okay.

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Let $\epsilon > 0$ be given. We must show that there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{n^{2/3}\sin(n!)}{n+1} - 0\right| < \epsilon$ for all $n \geq n_0$.

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$\left|\dfrac{n^{2/3}\sin(n!)}{n+1} - 0\right| < \epsilon$ for all $n \geq n_0$.

$$\left|\frac{n^{2/3}\sin(n!)}{n+1} - 0\right| < \epsilon \iff \left|\frac{n^{2/3}\sin(n!)}{n+1}\right| < \epsilon \Longleftarrow \left|\frac{n^{2/3}}{n+1}\right| < \epsilon$$

Note the direction of implication of the red arrow. We have used the fact that $|\sin x| < 1$ for all real x.

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By our arrows of implication, it can be seen that for $n \geq n_0$, the desired inequality holds.

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Let $\epsilon>0$ be given. We must show that there exists $n_0\in\mathbb{N}$ such that
$\left|\frac{n^{2/3}\sin(n!)}{n+1}-0\right|<\epsilon$ for all $n\geq n_0$.

Thus, if we choose $n_0=\left\lfloor\frac{2}{\epsilon}\right\rfloor+1$, we have it that the desired inequality holds.

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Observe the following:

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Observe the following:

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$$=\frac{1}{n+1}+\frac{1}{n}<\frac{2}{n}$$

Thus, if we choose $n_0=\left\lfloor\frac{2}{\epsilon}\right\rfloor+1$, we have it that the desired inequality holds.

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Thus, the sequence given is bounded below by $n-1$, but by Archimedean property, we know that $n-1$ is not bounded above. Thus, our sequence is not bounded (above). As a result, it is not convergent. ■

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Now, for the sake of contradiction, let us assume that (a_n) converges. Then, by (a), we have it that $c_n := a_n + b_n = \dfrac{(-1)^n}{2}$ must be convergent.

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We now proceed as follows:
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However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\geq 1}$ converging. (Why?)
However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

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Choose $n_0 = n_1$, then, for any $n \geq n_0$, we have that $|b_n - L| = |a_{n+1} - L| < \epsilon$. (2)

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The last inequality is due to the following:
$n + 1 > n \geq n_0 = n_1$ and using (1).
Thus, by (2), we have shown that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_{n+1} = L$.

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$|b_n - |L|| = |\ |a_n| - |L|\ | \leq |a_n - L| < \epsilon$. (2)
The last inequality is due to the following:
$|\ |x| - |y|\ | \leq |x - y|$ for all $x,\ y \in \mathbb{R}$ and using (1).

6. Given $\lim_{n\to\infty} a_n = L$, we need to find $\lim_{n\to\infty} |a_n|$.
Like before, let us define $b_n := |a_n|$. It seems reasonable to guess that the limit must $|L|$, let us try to prove that.
Let $\epsilon > 0$ be given. As (a_n) is convergent, there exists $n_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_1$. (1)
Choose $n_0 = n_1$, then, for any $n \geq n_0$, we have that
$$|b_n - |L|| = |\,|a_n| - |L|\,| \leq |a_n - L| < \epsilon. \quad (2)$$
The last inequality is due to the following:
$|\,|x| - |y|\,| \leq |x - y|$ for all $x,\ y \in \mathbb{R}$ and using (1).
Thus, by (2), we have shown that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} |a_n| = L$.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

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Let $\epsilon > 0$ be given. This means that $\epsilon^2 > 0$.

Sheet 1

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By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| = a_n < \epsilon^2$ for all $n \geq n_0$.

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By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| = a_n < \epsilon^2$ for all $n \geq n_0$.
Thus, $|a_n^{1/2} - 0| = a_n^{1/2} < \epsilon$ for all $n \geq n_0$.

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At what place(s) did we use that $a_n \geq 0$?

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Hint for **optional:** Use the inequality $\left|\sqrt[n]{a} - \sqrt[n]{b}\right| \leq \sqrt[n]{|a-b|}$ for $n \in \mathbb{N}$.

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   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

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Both are false.
The sequences, $a_n := 1 \quad \forall n \in \mathbb{N}$ and $b_n := (-1)^n \quad \forall n \in \mathbb{N}$ act as a counterexample for both the statements.

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)