MA 109 : Calculus-I
D1 T4, Tutorial 1

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Questions to be Discussed

- Sheet 1
  - 2 (iv) - Sandwich Theorem for finding limits
  - 3 (ii) - Checking convergence of a sequence
  - 5 (iii) - Monotonic Bounded sequences are convergent
  - 7 - Proof using ϵ - n_0 definition
  - 9 - Relation between product and convergence
  - 11 - Conditions for exchanging product and limits
  - 13 (iii) - Checking continuity of a function

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Where did we use that $h_n \geq 0$?

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However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\geq 1}$ converging. (Why?)
However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

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Thus, (a_n) is a monotonically increasing sequence that is bounded above. Therefore, it must converge. Using the same method as earlier gives this limit to be 6.

7. If $\lim_{n \to \infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

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By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.

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9. (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
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Both are false.
The sequences, $a_n := 1 \quad \forall n \in \mathbb{N}$ and $b_n := (-1)^n \quad \forall n \in \mathbb{N}$ act as a counterexample for both the statements.

11. (i) We shall show that the statement is false with the help of a counterexample.
Let $a = -1,\ b = 1,\ c = 0$. Define f and g as follows:

$$f(x) = x \text{ and } g(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}.$$

It can be seen that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to c} [f(x)g(x)] = \lim_{x \to 0} 1 = 1$.

(ii) We shall prove that the given statement is true.
We are given that g is bounded. Thus, $\exists M \in \mathbb{R}^+$ such that $|g(x)| \leq M \quad \forall x \in (a,\ b)$.
Let $\epsilon > 0$ be given. We want to show that there exists $\delta > 0$ such that
$|f(x)g(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.
Let $\epsilon_1 = \epsilon/M$. As $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that
$0 < |x - c| < \delta \implies |f(x)| < \epsilon_1$.
Thus, whenever $0 < |x - c| < \delta$, we have it that
$|f(x)g(x) - 0| = |f(x)||g(x)| \leq |f(x)| \cdot M < \epsilon_1 \cdot M = \epsilon$. ∎

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(iii) We shall prove that the given statement is true.
Let $\epsilon > 0$ be given.
Let $l := \lim_{x \to c} g(x)$.
Let $\epsilon_1 = \epsilon/(|l| + \epsilon)$.

By hypothesis, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1 \implies |g(x) - l| < \epsilon$.
Also, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2 \implies |f(x)| < \epsilon_1$.

Let $\delta = \min\{\delta_1,\ \delta_2\}$. Then, whenever $0 < |x - c| < \delta$, we have that:
$|f(x)g(x)| = |f(x)g(x) - lf(x) + lf(x)| \leq |f(x)||(g(x) - l)| + |l||f(x)| <$
$|f(x)|\epsilon + |l||f(x)| = |f(x)|(\epsilon + |l|) < \epsilon_1(\epsilon + |l|) = \epsilon$.
Thus, we have it that $0 < |x - c| < \delta \implies |f(x)g(x) - 0| < \epsilon$. ∎

13. (iii) The function can be rewritten as: $f(x) = \begin{cases} x & \text{if} \quad 1 \leq x < 2 \\ 1 & \text{if} \quad x = 2 \\ \sqrt{6-x} & \text{if} \quad 2 < x \leq 3 \end{cases}$

We claim that the function is continuous on $[1,2) \cup (2,3]$ and discontinuous at 2.
Given $x \in [1,2)$ and any sequence (x_n) in the domain such that $x_n \to x$, there must exist $n \in n_0$ such that $x_n \in [1,2) \quad \forall n \geq n_0$. Thus, $f(x_n) = x_n \quad \forall n \geq n_0$. It can now be easily shown that $f(x_n) \to x = f(x)$. (We have essentially used the continuity of the function $x \mapsto x$.) Thus, f is continuous on $[1,2)$.
Similarly, we can argue that f is continuous on $(2,3]$. Again, this will follow from the fact that the function $x \mapsto \sqrt{6-x}$ is continuous on its domain.

Now, we show that f is discontinuous at 2. Consider the sequence $x_n := 2 - 1/n$. It is clear that $x_n \to 2$.
Observe that $1 \leq x_n < 2$. Thus, $f(x_n) = 2 - 1/n$.
This gives us that $f(x_n) \to 2 \neq f(2)$. ■

References

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Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)