# MA 109 : Calculus-I
# D1 T4, Tutorial 2

Devansh Jain

IIT Bombay

2nd December 2020

# Questions to be Discussed

- Sheet 1
  - 7 - Proof using $\epsilon$ - $n_0$ definition (Optional, Discussed last time)
  - 10 - Proof of Even-odd convergence (Optional, Talked about last time)
  - 13 (ii) - Check Continuity of a function
  - 15 - Check differentiability of a function
  - 18 - Evaluate derivative of Multiplicative Cauchy functional equation
- Sheet 2
  - 3 - Intermediate Value Property (IVP) and Rolle's Theorem
  - 5 - Mean Value Theorem (MVT)

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

Let us choose $\epsilon = \dfrac{|L|}{2}$. Why is this a valid choice of $\epsilon$?)

By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.

$$\begin{aligned}
& |a_n - L| < \epsilon && \forall n \geq n_0 \\
\implies & ||a_n| - |L|| < \epsilon && \forall n \geq n_0 \\
\implies & -\epsilon < |a_n| - |L| < \epsilon && \forall n \geq n_0 \\
\implies & |L| - \epsilon < |a_n| && \forall n \geq n_0 \\
\implies & \frac{|L|}{2} < |a_n| && \forall n \geq n_0
\end{aligned}$$

10. To show:
$\{a_n\}_{n\geq 1}$ is convergent $\iff$ $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to the same limit.

*Proof.* ($\implies$) Let $b_n := a_{2n}$ and $c_n := a_{2n+1}$. We are given that $\lim\limits_{n\to\infty} a_n = L$. We must show that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$.
Let $\epsilon > 0$ be given. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for $n \geq n_0$.
Note that $2n > n$ and $2n + 1 > n$ for all $n \in \mathbb{N}$. Thus, we have that $|b_n - L| < \epsilon$ and $|c_n - L| < \epsilon$ for all $n \geq n_0$.
Thus, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$. ∎

$(\impliedby)$ Let $(b_n)$ and $(c_n)$ be as defined before. We are given that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$. We must show that $(a_n)$ converges.
Let $\epsilon > 0$ be given. By hypothesis, there exists $n_1,\ n_2 \in \mathbb{N}$ such that

$$|b_n - L| < \epsilon \text{ for all } n \geq n_1 \tag{1}$$

$$\text{and } |c_n - L| < \epsilon \text{ for all } n \geq n_2. \tag{2}$$

Choose $n_0 = \max\{2n_1,\ 2n_2 + 1\}$.
Let $n \geq n_0$ be even. Then, $n \geq 2n_1$ or $n/2 \geq n_1$ and $a_n = b_{n/2}$. By (1), we have it that $|a_n - L| < \epsilon$.
Similarly, let $n \geq n_0$ be odd. Then, $n \geq 2n_2 + 1$ or $(n-1)/2 \geq n_2$ and $a_n = c_{(n-1)/2}$. By (2), we have it that $|a_n - L| < \epsilon$.
Thus, we have shown that $|a_n - L| < \epsilon$ whenever $n \geq n_0$. This is precisely what it means for $(a_n)$ to converge to $L$. ■

13. (ii) The function is continuous everywhere.

*Proof.* For $x \neq 0$, it simply follows from the fact that product and composition of continuous functions is continuous.

To show continuity at $x = 0$ :

Let $(x_n)$ be any sequence of real numbers such that $x_n \to 0$. We must show that $f(x_n) \to 0$.

Let $\epsilon > 0$ be given.

Observe that $|f(x_n) - 0| = \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$.

Now, we shall use the fact $x_n \to 0$. By this hypothesis, there must exist $n_1 \in \mathbb{N}$ such that $|x_n| = |x_n - 0| < \epsilon \quad \forall n \geq n_1$.

Choosing $n_0 = n_1$, we have it that $|f(x_n) - 0| \leq |x_n| < \epsilon \quad \forall n \geq n_0$. ∎

15. For $x \neq 0$, it simply follows from the fact that product and composition of differentiable functions is differentiable.
To show differentiable at $x = 0$ and evaluating $f'(0)$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin(1/h)$$

$$|f'(0)| \leq \lim_{h \to 0} |h| = 0 \implies f'(0) = 0$$

We can compute $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$ and $f'(0) = 0$.
$f'$ is not continuous as limit at $x = 0$ is not defined (Why?).

18. Given: $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. (1)
Let $x = y = 0$. This gives us that $f(0) = (f(0))^2$.
Thus, $f(0) = 0$ or $f(0) = 1$.

Case 1. $f(0) = 0$.
Substitute $y = 0$ in (1). Thus, $f(x) = f(0)f(x) = 0$.
Therefore, $f$ is identically 0 which means it's differentiable everywhere with derivative 0.
Verify that $f'(c) = f'(0)f(c)$ does hold for all $x \in \mathbb{R}$. (We did not need to use the fact that $f$ is differentiable at 0, it followed from definition.)

Case 2. $f(0) = 1$.
As $f$ is differentiable at 0, we know that:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0) \implies \lim_{h\to 0} \frac{f(h) - 1}{h} = f'(0). \tag{2}$$

# Tutorial Sheet 1

Now, let us show that $f$ is differentiable everywhere.
Let $c \in \mathbb{R}$. We must show that the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Using (1), we can write the above expression as:

$$\lim_{h \to 0} \frac{f(c)f(h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c)(f(h) - 1)}{h} = f(c) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

By (2), we know that the above limit exists. Thus, we have it that $f$ is differentiable at $c$ for every $c \in \mathbb{R}$. Moreover, $f'(c) = f'(0)f(c)$.

**(Optional)** We have gotten that the derivative of $f$ is a scalar multiple of $f$. Use this to conclude.

3. Part 1. We will first show the existence of such an $x_0 \in (a, b)$.
*Proof.* $I := [a, b]$ is an interval and $f$ is continuous. Thus, $f$ has the intermediate value property on $I$. Thus, the range $J := f(I)$ must be an interval. As $f(a)$ and $f(b)$ are of different signs, 0 lies between them. As $f(a), f(b) \in J$ and $J$ is an interval, we have it that $0 \in J = f(I)$. Thus, $0 = f(x_0)$ for some $x_0 \in I = (a, b)$. ■

Part 2. Now we will show the uniqueness of $x_0$. Assume that there exists $x_1 \in (a, b)$ such that $f(x_1) = 0$. We may assume that $x_0 < x_1$.
Now, we know the following:
(i) $f$ is continuous on $[x_0, x_1]$,
(ii) $f$ is differentiable on $(x_0, x_1)$, and
(iii) $f(x_0) = f(x_1)$.
Thus, by Rolle's Theorem, there exists $x_2 \in (x_0, x_1)$ such that $f'(x_2) = 0$. But this contradicts the hypothesis that $f'(x) \neq 0$ for all $x \in (a, b)$. ■

5. To prove that $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
Case 1. $a = b$. Trivial.
Case 2. $a \neq b$. Without loss of generality, we can assume that $a < b$.
As $f(x) := \sin(x)$ is continuous and differentiable on $\mathbb{R}$, there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (By MVT)

Also, we know that $|f'(c)| = |\cos c| \leq 1$.

Thus, we have it that $\left|\dfrac{f(b) - f(a)}{b - a}\right| \leq 1$.

This is equivalent to what we wanted to prove. ■

## References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)