MA 109 : Calculus-I
D1 T4, Tutorial 3

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Questions to be Discussed

- Sheet 2
  - 8 (ii), (iii) Finding Functions with given Conditions
  - 10 (i) Sketching curve with given properties
  - 11 Sketching curve with given properties
- Sheet 3
  - 1 (ii) Taylor Series for $\arctan x$
  - 2 Taylor Series for a polynomial
  - 4 Convergence of Maclaurin Series of e^x
  - 5 Integration using Taylor Series

8. (ii). $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f'(1) = 2$

$f''(x) > 0$ for all $x \in \mathbb{R}$

We know such a curve!!!

Can try quadratic $\implies$ Let $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$ $\quad$ $f''(x) = 2a$

$f''(x) > 0 \implies a > 0$

$f'(0) = 1 \implies b = 1$

$f'(1) = 2 \implies 2a + b = 2$

One such function is: $f(x) = \frac{x^2}{2} + x$

8. (iii). $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f(x) < 100$ for all $x > 0$

Idea. Derivative at 0 is positive.
f'' non negative, means derivative not decreasing, always positive after 0.
f must be strictly increasing.
But f is bounded above. Contradiction!!!
Such a function cannot exist!
Let's write a formal argument.

8. (iii). **Proof.**
$f''(x) \geq 0 \quad \forall x \in \mathbb{R} \implies f'$ not decreasing, so $\forall x > 0, f'(x) \geq 1$
To prove: f must exceed 100.

Can't use integration here :(
MVT to the rescue!!

Can write $\dfrac{f(x) - f(0)}{x - 0} = f'(c) \geq 1 \because c \geq 0$
Thus, $f(x) \geq x + f(0)$
Just take $x = 101 - f(0) \implies f(x) \geq 101$.
Done!!!

1. (ii). Taylor Series for arctan x.
Can do by integration, as given in paragraph after question 4.

2. Taylor Series of $f(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$
Can already guess!!! $f(x) = x^3 - 3x^2 + 3x - 1 \implies f(1) = 0$

$f'(x) = 3x^2 - 6x + 3 \implies f'(1) = 0$

$f''(x) = 6x - 6 \implies f''(1) = 0$

$f'''(x) = 6 \implies f'''(1) = 6$

$f^{(n)}(x) = 0$ for all $n > 3$. Thus, the Taylor series is

$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ Here, only the third derivative is non-zero! Only one term in the Taylor Series! $P(x) = \frac{f'''(0)}{3!}(x-1)^3 = \frac{6}{6}(x-1)^3 = (x-1)^3$

4. Proving convergence of the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

We'll follow steps given in the question. Choose $N > 2x$. We see that for all $n > N$, $\frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}$ Let us denote the partial sums of of the given series by $s_m(x)$. We should show that for every $\epsilon > 0$, there is a $N \in \mathbb{N}$, such that for all $m, n > N$, $|s_m(x) - s_n(x)| < \epsilon$. For this, observe that (assuming WLOG $m > n$), $|s_m(x) - s_n(x)| =$

$$\left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{m-n}} \right) \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \ldots \right) \leq \frac{|x^n|}{n!}$$

4 contd.

We have, $|s_m(x) - s_n(x)| \leq 2 \cdot \dfrac{|x^n|}{n!}$ for all $m > n$

We also have,

for $n > N > 2x$, $\dfrac{x^{n+1}}{(n+1)!} < \dfrac{1}{2} \cdot \dfrac{x^n}{n!} < \dfrac{x^n}{n!} < \dfrac{x^N}{N!}$

5. To integrate: $\displaystyle\int \frac{e^x}{x}dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies$$

$$\int \frac{e^x}{x}dx = \int \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{n-1}dx = \sum_{n=0}^{\infty} \frac{1}{n!}(n-1)x^n = \sum_{n=0}^{\infty} \frac{(n-1)x^n}{n!}$$

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Krushnakant Bhattad for MA 109 (Autumn 2020)
Solutions to tutorial problems for MA 105 (Autumn 2019)