2. (iv)

Binomial Method

$h_n := n^{1/n} - 1 \qquad n \in \mathbb{N}$

$n \geq 1 \to n^{1/n} \geq 1 \to h_n \geq 0 \quad \forall n \in \mathbb{N}$

$$n = (1+h_n)^n \geq 1 + nh_n + \frac{n(n-1)}{2} h_n^2 \geq \frac{n(n-1)}{2} h_n^2 \qquad \forall n \geq 2 \qquad (h_n \geq 0)$$

$$h_n \leq \sqrt{\frac{2}{n-1}}$$

For every $\varepsilon > 0$, $\exists N_0$ such that $\forall n \geq N_0$

$$\left|\sqrt{\frac{2}{n-1}} - 0\right| < \varepsilon \Rightarrow N_0 = \left\lceil \frac{2}{\varepsilon^2} + 1 \right\rceil$$

$$\therefore \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$$

Using Sandwich Theorem,

$$0 \leq h_n \leq \sqrt{\frac{2}{n-1}} \qquad \forall n \geq 2$$

$$\lim_{n\to\infty} h_n = 0$$

$$\therefore \lim_{n\to\infty} n^{1/n} = 0 + 1 = 1 //$$

AM-GM Method

$n \geq 1 \Rightarrow n^{1/n} \geq 1$

$$\frac{\underbrace{1+1+\cdots+1}_{n-2 \text{ times}}+\sqrt{n}+\sqrt{n}}{n} \geq n^{1/n} \qquad \text{(AM-GM inequality)}$$

$$\frac{n-2+2\sqrt{n}}{n} \geq n^{1/n}$$

$$\lim_{n\to\infty} \frac{n-2+2\sqrt{n}}{n} = 1 - \left(2 \lim_{n\to\infty} \frac{1}{n}\right) + \left(2 \lim_{n\to\infty} \frac{1}{\sqrt{n}}\right)$$

$$= 1$$

Using Sandwich Theorem,

$$1 \leq n^{1/n} \leq \frac{n-2+2\sqrt{n}}{n}$$

$$\lim_{n\to\infty} n^{1/n} = 1 //$$

3. (ii) $a_n := (-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)$

$b_n := (-1)^n \left(\frac{1}{n}\right)$

For sake of contradiction,
let us assume a_n is convergent

b_n is convergent $\leftarrow$ $|b_n| < \varepsilon \quad \forall\, n \geq N = \lceil 1/\varepsilon \rceil$ for every $\varepsilon > 0$

$\therefore\ a_n + b_n = \frac{(-1)^n}{2}$ is convergent

But $\frac{(-1)^n}{2}$ oscillates between $\frac{1}{2}$ and $\frac{-1}{2}$ as $n \to \infty$
contradicting its convergence.

$\therefore$ Our assumption was incorrect.

$a_n = (-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)$ is a non-convergent sequence.

5. (iii) $a_1 = 2$, $a_{n+1} = 3 + \frac{a_n}{2} \quad \forall\, n \geq 1$

<u>Claim :</u> $a_n < 6 \quad \forall\, n \in \mathbb{N}$

Proof by induction: Base Case : $n = 1$, $a_1 < 6$ holds

Inductive step : Assume $a_n < 6$ for $n = k$.

Induction : $a_{k+1} = 3 + \frac{a_k}{2} < 3 + 3 < 6$

$\therefore$ By induction, $a_n < 6 \quad \forall\, n \geq 1$

<u>Claim:</u> $a_n < a_{n+1} \quad \forall\, n \in \mathbb{N}$

$a_{n+1} - a_n = 3 - \frac{a_n}{2} > 0$ (as $a_m < 6$)

$\therefore\ \{a_n\}$ is monotonically increasing and bounded above by 6.
(and below by 2)

$\therefore\ \lim_{n\to\infty} a_n = L$ exists

$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 3 + \frac{a_n}{2} = 3 + \frac{L}{2}$

$\Rightarrow L = 6$

$\therefore\ \lim_{n\to\infty} a_n = 6$ //