MA 109 : Calculus-I
D1 T4, Tutorial 1

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Questions to be Discussed

- Sheet 1
  - 2 (iv) - Sandwich Theorem for finding limits
  - 3 (ii) - Checking convergence of a sequence
  - 5 (iii) - Monotonic Bounded sequences are convergent
  - 7 - Proof using ϵ - n_0 definition
  - 9 - Relation between product and convergence
  - 11 - Conditions for exchanging product and limits (Discussed in lecture)

Tutorial Sheet 1

2. (iv) $\lim_{n\to\infty} (n)^{1/n}$.

Define $h_n := n^{1/n} - 1$.
Then, $h_n \geq 0 \quad \forall n \in \mathbb{N}$. (Why?)
Observe the following for $n > 2$:

$$n = (1 + h_n)^n > 1 + nh_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 = \frac{n(n-1)}{2} h_n^2.$$

Thus, $h_n < \sqrt{\dfrac{2}{n-1}} \quad \forall n > 2$.

Using Sandwich Theorem, we get that $\lim_{n\to\infty} h_n = 0$ which gives us that $\lim_{n\to\infty} n^{1/n} = 1$.

You may use AM-GM on $(n-1)$ 1s and two $\sqrt{n}$ to get upper bound for $n^{1/n}$.

Tutorial Sheet 1

3. (ii) To show: $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$ is *not* convergent.

We will use the following two results: (a) Sum of convergent sequences is convergent. (b) The sequence $\{(-1)^n\}_{n\geq 1}$ is not convergent.
We now proceed as follows:

$$a_n := (-1)^n\left(\frac{1}{2}-\frac{1}{n}\right) = \frac{(-1)^n}{2}-\frac{(-1)^n}{n}.$$

It is easy to show that $b_n := \dfrac{(-1)^n}{n}$ is convergent. (Its absolute value will behave the same way as $1/n$.)
Now, for the sake of contradiction, let us assume that (a_n) converges. Then, by (a), we have it that $c_n := a_n + b_n = \dfrac{(-1)^n}{2}$ must be convergent.
However, (c_n) converging is equivalent to $\{(-1)^n\}_{n\geq 1}$ converging. (Why?)
However, by (b), we know that the above is false. Thus, we have arrived at a contradiction.

5. (iii) $a_1 = 2,\ a_{n+1} = 3 + \dfrac{a_n}{2} \quad \forall n \geq 1.$

Claim 1. $a_n < 6 \quad n \in \mathbb{N}.$

Proof. We shall prove this via induction. The base case $n = 1$ is immediate as $2 < 6$.
Assume that it holds for $n = k$.

$a_{k+1} = 3 + \dfrac{a_n}{2} < 3 + \dfrac{6}{2} = 6.$

By principle of mathematical induction, we have proven the claim. ■

Claim 2. $a_n < a_{n+1} \quad \forall n \in \mathbb{N}.$

Proof. $a_{n+1} - a_n = 3 - \dfrac{a_n}{2} = \dfrac{6 - a_n}{2} > 0 \implies a_{n+1} > a_n.$ ■

Thus, (a_n) is a monotonically increasing sequence that is bounded above. Therefore, it must converge. Use $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = L$ and then solve for L.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

Let us choose $\epsilon = \frac{|L|}{2}$. (Why is this a valid choice of ϵ?)

By hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ whenever $n \geq n_0$.

$$\begin{aligned}
& |a_n - L| < \epsilon && \forall n \geq n_0 \\
\implies & ||a_n| - |L|| < \epsilon && \forall n \geq n_0 \\
\implies & -\epsilon < |a_n| - |L| < \epsilon && \forall n \geq n_0 \\
\implies & |L| - \epsilon < |a_n| && \forall n \geq n_0 \\
\implies & \frac{|L|}{2} < |a_n| && \forall n \geq n_0
\end{aligned}$$

9. (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

Both are false.

The sequences, $a_n := 1 \quad \forall n \in \mathbb{N}$ and $b_n := (-1)^n \quad \forall n \in \mathbb{N}$ act as a counterexample for both the statements.

11. (i) We shall show that the statement is false with the help of a counterexample.
Let $a = -1,\ b = 1,\ c = 0$. Define f and g as follows:
$f(x) = x$ and $g(x) = \begin{cases} 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$.
It can be seen that $\lim_{x \to 0} f(x) = 0$ but $\lim_{x \to c} [f(x)g(x)] = \lim_{x \to 0} 1 = 1$.

(ii) We shall prove that the given statement is true.
We are given that g is bounded. Thus, $\exists M \in \mathbb{R}^+$ such that $|g(x)| \leq M \quad \forall x \in (a,\ b)$.
Let $\epsilon > 0$ be given. We want to show that there exists $\delta > 0$ such that
$|f(x)g(x) - 0| < \epsilon$ whenever $0 < |x - c| < \delta$.
Let $\epsilon_1 = \epsilon / M$. As $\lim_{x \to c} f(x) = 0$, there exists $\delta > 0$ such that
$0 < |x - c| < \delta \implies |f(x)| < \epsilon_1$.
Thus, whenever $0 < |x - c| < \delta$, we have it that
$|f(x)g(x) - 0| = |f(x)||g(x)| \leq |f(x)| \cdot M < \epsilon_1 \cdot M = \epsilon$. ∎

(iii) We shall prove that the given statement is true.
Let $\epsilon > 0$ be given.
Let $l := \lim_{x \to c} g(x)$.
Let $\epsilon_1 = \epsilon/(|l| + \epsilon)$.

By hypothesis, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1 \implies |g(x) - l| < \epsilon$.
Also, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2 \implies |f(x)| < \epsilon_1$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, whenever $0 < |x - c| < \delta$, we have that:
$|f(x)g(x)| = |f(x)g(x) - lf(x) + lf(x)| \leq |f(x)||(g(x) - l)| + |l||f(x)| <$
$|f(x)|\epsilon + |l||f(x)| = |f(x)|(\epsilon + |l|) < \epsilon_1(\epsilon + |l|) = \epsilon$.
Thus, we have it that $0 < |x - c| < \delta \implies |f(x)g(x) - 0| < \epsilon$. ■

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)