MA 109 : Calculus-I
D1 T4, Tutorial 3

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Questions to be Discussed

- Sheet 2
  - 8 (ii), (iii) Finding Functions with given Conditions
  - 10 (i) Sketching curve with given properties
  - 11 Sketching curve with given properties
- Sheet 3
  - 1 (ii) Taylor Series for $\arctan x$
  - 2 Taylor Series for a polynomial
  - 4 Convergence of Maclaurin Series of e^x
  - 5 Integration using Taylor Series

Tutorial Sheet 2

8. (ii). $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f'(1) = 2$

$f''(x) > 0$ for all $x \in \mathbb{R}$

We know such a curve!!!

Can try quadratic $\implies$ Let $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$ $\quad\quad$ $f''(x) = 2a$

$f''(x) > 0 \implies a > 0$
$f'(0) = 1 \implies b = 1$
$f'(1) = 2 \implies 2a + b = 2$

One such function is: $f(x) = \frac{x^2}{2} + x$

Tutorial Sheet 2

8. (iii). $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f(x) < 100$ for all $x > 0$

Idea. Derivative at 0 is positive.
f'' non negative, means derivative not decreasing, always positive after 0.
f must be strictly increasing with derivative > 1, which means,
as x goes to $x + 1$, $f(x)$ goes to something $> f(x) + 1$.
(That's what derivative tells us, right?)
But $f(x)$ is bounded above for positive x. Contradiction!!!

Such a function cannot exist!
Let's write a formal argument.

8. (iii). **Proof.**

$f''(x) \geq 0 \quad \forall x \in \mathbb{R} \implies f'$ not decreasing, so $\forall x > 0, f'(x) \geq 1$

To prove: f must exceed 100 at some point.

Can write, using MVT $\dfrac{f(x) - f(0)}{x - 0} \geq f'(c) \geq 1 \because c \geq 0$

Thus, $f(x) \geq x + f(0)$

Just take $x = 101 - f(0) \implies f(x) \geq 101$.

10. (i). Graphing the polynomial $f(x) = 2x^3 + 2x^2 - 2x - 1$.

$f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$

$f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$; so $f(x)$ is strictly increasing here.

and $f'(x) < 0$ in (-1, 1/3) so $f(x)$ is strictly decreasing here.

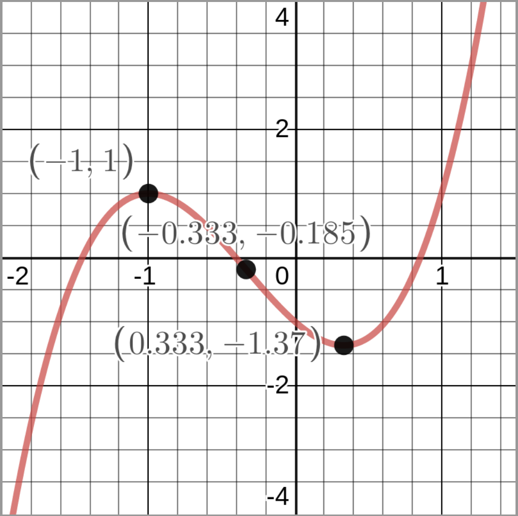
Thus, $x = -1$ is a local maximum, and $x = \frac{1}{3}$ is a local minimum.

$f''(x) = 12x + 4$

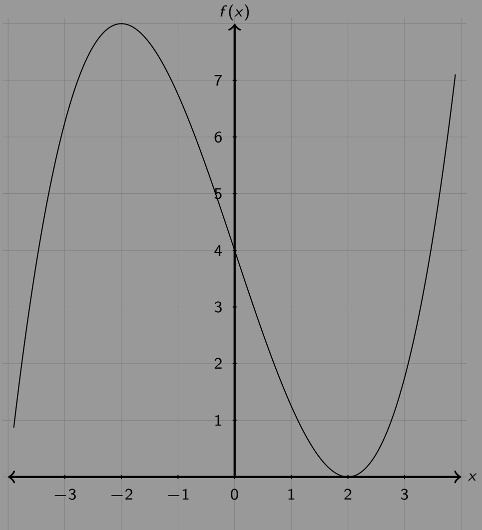
Thus, $f(x)$ is concave in $(-\infty, 1/3)$ and convex in $(-1/3, \infty)$,

with a point of inflection at $x = \dfrac{-1}{3}$.

10. Graph.



11.



11. (contd.)
I have actually graphed a polynomial that satisfies the given properties.
Can you come up with it?
Is there a unique such polynomial? (We discussed this)
What's the minimum degree of such a polynomial?
Is there a unique polynomial with that degree? (This you should think about)
Suppose you have two distinct polynomials f and g that satisfy the given conditions.
Can you come up with a distinct third polynomial such that it satisfies the conditions as well?

1. (ii). Taylor Series for arctan x at x=0.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = g(x) \text{ (say)}$$

$$g(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

(This is only true for $|x| < 1$, so we restrict ourselves to that domain.)

$$g^{(2k+1)}(0) = 0 \qquad g^{(2k)}(0) = (-1)^k(2k)! \; ; \qquad k \geq 0 \quad \text{(Do verify this!)}$$

but these are n^{th} derivatives of $(arctan(x))'$, so are $(n+1)^{th}$ derivatives of $(arctan(x))$.

Thus the series for arctan x , using the formula $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$, is :

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{(Do verify this!)}$$

2. Taylor Series of $f(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$

Can already guess!!! $f(x) = x^3 - 3x^2 + 3x - 1 \implies f(1) = 0$

$f'(x) = 3x^2 - 6x + 3 \implies f'(1) = 0$

$f''(x) = 6x - 6 \implies f''(1) = 0$

$f'''(x) = 6 \implies f'''(1) = 6$

$f^{(n)}(x) = 0$ for all $n > 3$.

The Taylor series is $P(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

Here, only the third derivative is non-zero! Only one term in the Taylor Series!

$$P(x) = \frac{f'''(0)}{3!}(x-1)^3 = \frac{6}{6}(x-1)^3 = (x-1)^3$$

4. Proving convergence of the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

We'll follow steps given in the question, i.e. prove Cauchy.

Let us denote the partial sums of of the given series by $s_m(x)$.
We should show that for every $\epsilon > 0$, there is a $N \in \mathbb{N}$, such that for all $m, n > N$, $|s_m(x) - s_n(x)| < \epsilon$.

It can be shown that,

for $n > N_0 = \lceil 2x \rceil + 1 > 2x$, $\quad \frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}$ (Iteratively, $\frac{x^{n+k}}{(n+k)!} < \frac{1}{2^k} \cdot \frac{x^n}{n!}$)

Observe that (assuming W.L.O.G. $m > n > N_0$),

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \le \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{m-n}} \right) \le \frac{|x^n|}{n!}$$

4. (contd.)

We have, $|s_m(x) - s_n(x)| \leq \dfrac{|x^n|}{n!}$ for all $m > n$

Also, by induction, for all $k > 0$, we have, $\dfrac{|x|^{N_0+k}}{(N_0+k)!} < \dfrac{1}{2^k} \cdot \dfrac{|x|^{N_0}}{N_0!}$

For a given x and ϵ, for $N_0 = \lceil 2x \rceil + 1$ choose a k such that $\dfrac{1}{2^k} \cdot \dfrac{|x|^{N_0}}{N_0!} < \epsilon$.

Choose $N = N_0 + k$.

Hence this sequence of partial sums of the series $a_n = s_n(x)$ is cauchy for every $x \in \mathbb{R}$ and thus the series is convergent.

5. To integrate: $\int \frac{e^x}{x} dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies$$

$$\int \frac{e^x}{x} dx = \int \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!} dx = \int \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \right) dx = \int \frac{1}{x} dx + \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} dx$$

$$= \log(x) + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!} dx = \log(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{x^n}{n} = \log(x) + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Tutorial slides prepared by Krushnakant Bhattad for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)