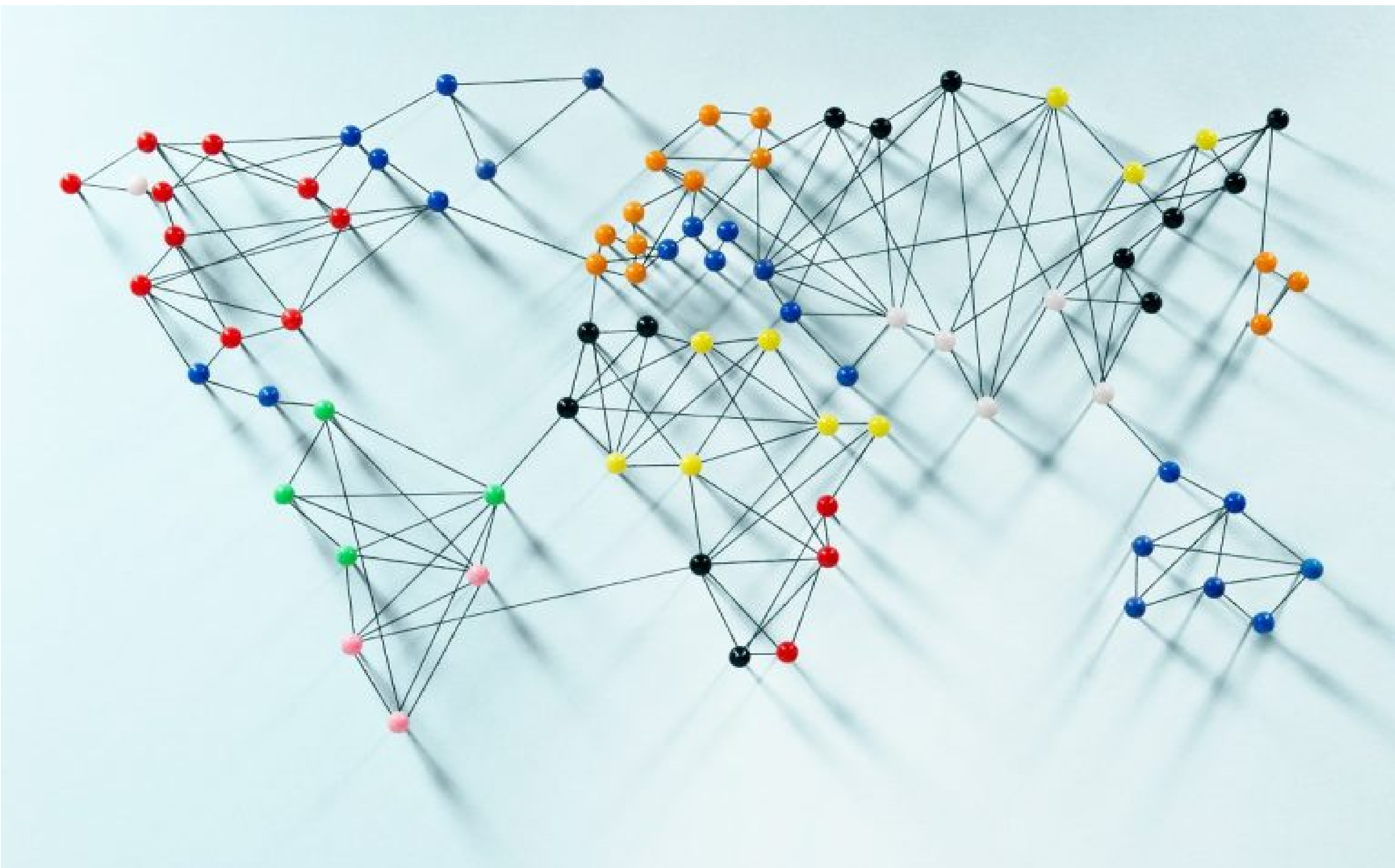
Graph Theory and Algorithms

An Introduction to the world of Graphs



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Preface

Before there were computers, there were algorithms. But now that there are computers, there are even more algorithms, and algorithms lie at the heart of computing
In the domain of mathematics and computer science, graph theory is the study of graphs that concerns the relationship among edges and vertices.
In recent years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from electrical engineering and geography to sociology and architecture. At the same time it has also emerged as a worthwhile mathematical discipline in its own right.

About the report

This work is written as a part of the Summer Of Science, 2020 project organized by the MnP Club, IIT Bombay. (Mentor: Tathagat Verma)
I have followed the course MIT 6.042J/18.062J and MIT 6.006 offered by MIT. The report consists mainly of my notes for these lectures.
All content here is highly inspired by the courses and at several points might simply be a paraphrasing of the course content. Nevertheless, I have tried to compress the contents of the course, skipping a few of the details, while still preserving sufficient depth.

Prerequisites for understanding the content:

- You should have some programming experience. In particular, you should understand recursive procedures and simple data structures such as arrays and linked lists.
- You should have some facility with mathematical proofs, and especially proofs by mathematical induction.

Though it may be hard to believe for a report of this size, space constraints prevented me from including many interesting algorithms and problems.

About the project and its future

Even after completion of SoS 2020, I intend to learn further in this field.
I would be moving on to MIT 6.046J/18.410J (Design and Analysis of Algorithms) and MIT 6.854J/18.415J (Advanced Algorithms - a Graduate level course).
More courses in this context are MIT 6.045J/18.400J, MIT 18.404J/6.840J, MIT 6.079/6.975, MIT 18.315.

MIT 6.042J/18.062J (Fall 2010) : Mathematics for Computer Science

This course is available on OpenCourseWare MIT (ocw.mit.edu) as well as on Youtube channel MIT OpenCourseWare.

Course Instructor(s)

Prof. Tom Leighton
Dr. Marten van Dijk

Course Description

This course covers elementary discrete mathematics for computer science and engineering. It emphasizes mathematical definitions and proofs as well as applicable methods. Topics include formal logic notation, proof methods; induction, well-ordering; sets, relations; elementary graph theory; integer congruences; asymptotic notation and growth of functions; permutations and combinations, counting principles; discrete probability. Further selected topics may also be covered, such as recursive definition and structural induction; state machines and invariants; recurrences; generating functions.

In this report, I have included notes of Lectures 6 thru 11 which cover 'Mathematical basics of Graph Theory'.
More detailed notes of the course are present here

Lecture 6: Graph Theory and Coloring

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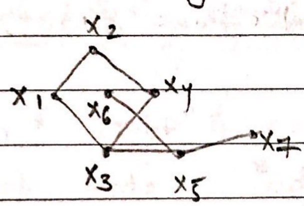
L-6 Graph are ~~very~~ incredibly useful structures in computer science. They come up in all sorts of applications, scheduling, optimization, communications, the design and analysis of algorithms.

⇒ Informally, a graph is just a bunch of dots and lines connecting the dots



⇒ A graph G is a pair of sets (V, E) where
- V is a non-empty set of items called vertices or nodes
- E is a set of 2-item subsets of V called edges. (can be empty set). ⇒

$V=\{x_1,x_2,x_3\}$
$E=\phi$

$V=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$

$E=\{ (x_1,x_2), (x_2,x_4), (x_4,x_3), (x_3,x_1)$
$(x_5,x_6), (x_3,x_5), (x_5,x_7) \}$

equivalent to $x_1 — x_2$

Lecture 6: Graph Theory and Coloring

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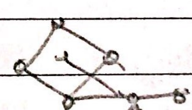
⇒ Two nodes x_i & x_j are adjacent if they're connected by an edge i.e. $x_i - x_j \in E$

⇒ An edge $e = (x_i, x_j)$ is said to be incident to its end-points x_i & x_j.

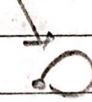
⇒ The number of edges incident to a node is called the degree of the node.

~~deg(x)~~.



$\deg(x_5) = 3$

⇒ Graph is simple if it has no loops or multiples edges. (multi-edge).



$|V| \rightarrow$ cardinality notation
i.e. no. of elements in set V.

Men-Women partners ratio

Lecture 6: Graph Theory and Coloring

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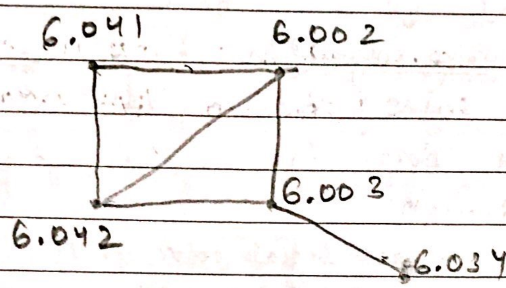


Slots
5-7
7-9
9-11
11-1
1-3

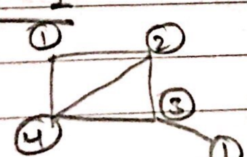
Graph-coloring problem

Given a graph G and K colors, assign a color to each node so adjacent nodes get different colors.

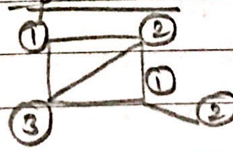
Minimum value of K for which such a coloring exists is the Chromatic Number of G $\chi(G)$

→ Color ⇒ Slots
Nodes ⇒ Courses

Option 1



Option 2



$\chi(G)=3$

2 colors can't work because of triangle formation

Lecture 6: Graph Theory and Coloring

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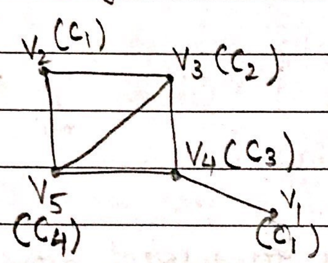
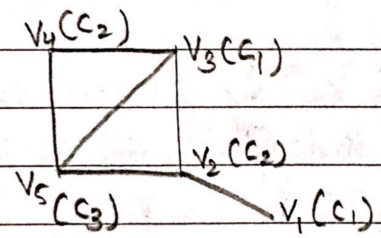
Basic graph coloring algorithm

1. Order the nodes $V_1, \dots, V_n$
2. Order the colors $C_1, \dots$
3. For $i = 1, 2, \dots, n$

∴ Assign the lowest legal color to V_i



We know this is not best case.



So, if you change the ordering, you may get a better answer.

Now this algorithm is an example of what's known as greedy algorithm.
You just go one step after the next, taking the best you can do at each step. You never go back and try to make things better.

Lecture 6: Graph Theory and Coloring

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Thm. If every node in G has degree $\leq d$, then this basic algorithm uses atmost $d+1$ colors for G.

Proof By induction,

Induction Hypothesis:

//Never induct on d, induct on 'n' or 'e'
node

Predicate
$P(n)$ = "If every node in n-node graph G with max. degree d then this basic algorithm uses atmost $d+1$ colors for G."

Base case:

$n = 1 \Rightarrow |E| = 0 \Rightarrow d = 0 \Rightarrow$ one color

$\therefore P(1)$ is true

Ind. step: Assume $P(n)$ is true

Let G be any $n+1$ node graph.

Let d be the max. degree of nodes in G

Order the nodes: $V_1, \ldots, V_n, V_{n+1}$

Let G' be the graph after removing V_{n+1}

max. degree of nodes in $G \leq d$

$\therefore$ Basic algorithm uses atmost $d+1$ colors for G'.

Lecture 6: Graph Theory and Coloring

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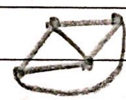
In G, color nodes $v_1 \dots v_n$ same as G'.
v_{n+1} has atmost d neighbours therefore atleast one color is not present in v_{n+1}'s neighbours.
$\therefore$ Give v_{n+1} that color.

$\therefore$ $P(n+1)$ is true

$K_n \rightarrow$ n-node complete graph (or clique)



Degree $= n-1$
$\chi(K_n) = n$

$d = n/2$
$\chi(G) = 2$

Bipartite graph

Star graph
$d = n-1$
$\chi(G) = 2$

The Algorithm may be much better than the theorem

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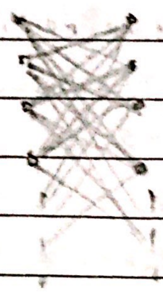


Good ordering (2 Colors)

$(C_1)\ v_1 \bullet \quad \bullet\ v_5\ (C_2)$
$(C_1)\ v_2 \bullet \quad \bullet\ v_6\ (C_2)$
$(C_1)\ v_3 \bullet \quad \bullet\ v_7\ (C_2)$
$(C_1)\ v_4 \bullet \quad \bullet\ v_8\ (C_2)$
$\vdots$

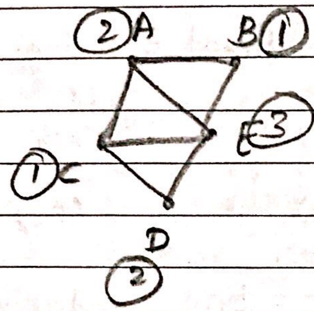
Bad ordering ($n/2$ colors)

$(C_1)\ v_1 \bullet \quad \bullet\ v_2\ (C_1)$
$(C_2)\ v_3 \bullet \quad \bullet\ v_4\ (C_2)$
$(C_3)\ v_5 \bullet \quad \bullet\ v_6\ (C_3)$
$(C_4)\ v_7 \bullet \quad \bullet\ v_8\ (C_5)$
$\vdots$

A graph $G = (V, E)$ is said to be bipartite if V can be split into V_L, V_R so that all the edges connect V_L to V_R.

Lecture 7: Matching Problems

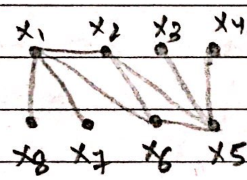
resource allocation problems (load balancing traffic on internet)



$\chi(G) = 3$

Matching algorithm - usage: online dating agencies, assignments problems (matching interns to hospitals on Match day)

L-7 ⇒ Given a graph $G = (V, E)$, a matching is a subgraph of G where every node has degree 1.

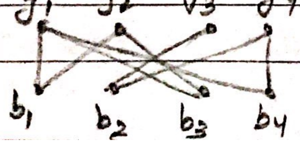


$\{x_1 - x_6, x_2 - x_5\}$ is a matching of size 2.

$\{x_1 - x_7, x_2 - x_6, x_3 - x_5\}$ is a matching of size 3.

Size 4 matching is not possible in this graph.

⇒ A matching is perfect if it has size $\frac{|V|}{2}$



$\{g_1 - b_1, g_2 - b_3, g_3 - b_2, g_4 - b_4\}$ is a matching

Lecture 7: Matching Problems

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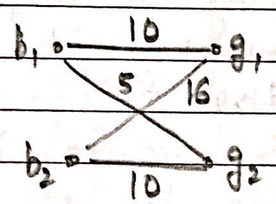
Weights?

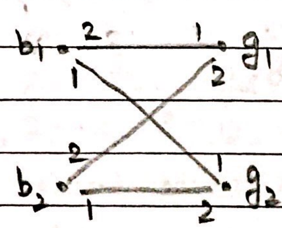
⇒ The weight of a matching (M) is the sum of the weights on the edge of M.

⇒ A minimum weight matching for graph G is a perfect matching for G with the minimum weight.



Min. weight matching
$\{b_1 - g_1, b_2 - g_2\}$
weight = 20

Preference list - similar to weights



If $b_1 - g_1$, $b_2 - g_2$ then $b_1 - g_2$ will be rogue couple.

Given a matching M, x&y form rogue couple if they prefer each other to their mates in M.

A matching is stable if there aren't any rogue couple.

Lecture 7: Matching Problems

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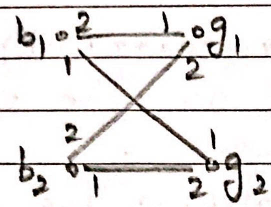
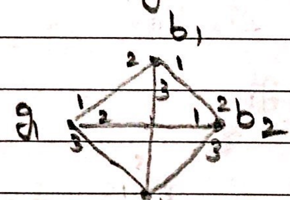


$b_1 - g_2$ & $b_2 - g_1$ is stable matching

If b-g matching is only allowed then there always exist a stable perfect matching.

If b-b & g-g is allowed, it may not always possible to find stable perfect matching.



g_2 (Preference doesn't matter)

Theorem: There is no stable perfect matching

Proof: By contradiction,

If there is a stable matching, g_2 would be paired to someone

Without loss of generality (by symmetry) (the Δ is symmetric)

$\{g_2 - b_2, g_1 - b_1\} \Rightarrow b_1, b_2$ form rougue couple.

Lecture 7: Matching Problems

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Stable Marriage Algorithm (David Gale - Lloyd Shapley Algo) 1962

- N boys & N girls
- each boy has his own ranked list of all the girls. Same for each girl
- to find stable perfect matching.

Eg:- 5 boys & 5 girls

Boys	Preference	Girls	Preference
1 ♂	CBEAD	A	35214
2 ♂	ABECD	B	52143
3 ♂	DCBAE	C	43512
4 ♂	ACDBE	D	12345
5 ♂	ABDEC	E	23415

Using greedy algorithm,

1 – C, 2 – A, 3 – D, 4 – B, 5 – E

Rogue couple : 4 – C

Using mating algorithm,

serenaders

Boys	Girls	Day 1	Day 2	Day 3	Day 4	
1 ~~C~~BEAD	A	24 5	5	5	5	A – 5
2 ~~A~~BECD	B		2	21	2	B – 2
3 DCBAE	C	1	14	4	4	C – 4
4 ~~A~~CDBE	D	3	3	3	3	D – 3
5 ABDEC	E				1	E – 1

Lecture 7: Matching Problems

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Need to show:

1. Algorithm terminates (Matching returned)
2. Everyone gets matched (Perfect matching)
3. No rogue couple (Stable matching)
4. It runs quickly
5. Fairness

Theorem 1 Algo terminates in less than N^2+1 days

Proof By contradiction.
Suppose, algo doesn't terminate in N^2+1 days.

Claim: If we don't terminate on a day, then atleast one girl had more than one boy and so she rejected atleast one boy who crosses her name.

If the algo doesn't terminate in N^2+1 days then atleast N^2+1 cross-outs have happened.

As there are N^2 names on N lists, there are atmost N^2 cross-outs. $(< N^2+1)$
Thus, not possible.

Lecture 7: Matching Problems

(35)

Predicate P = "If a girl G ever rejected a boy B, then G has a suitor who she prefers to B".

Lemma 1 : P is an invarient for the algo

Proof: ~~Base case:~~ By induction on number of days.

Base case: Day 0 (Vacuously true)

(No one rejected yet)

Ind step: P holds at the end of Day d.

Case 1 G rejects B on day d+1.

Then there was a better boy

⇒ P holds on Day d+1

Case 2 G rejected B before day d+1

P on day d ⇒ G had a better boy on day d.

⇒ P holds on Day d+1.

Theorem 2 Everyone is married in Algo

Proof By contradiction

Assume that some boy B is not married, then he was rejected by every girl which

⇒ every girl by lemma 1 has a better suitor

⇒ every girl is married ⇒ every boy is married

⇒ B is married.

Lecture 7: Matching Problems

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Theorem 3 Algo produces a ~~stable~~ matching (No rogue couple)

Proof

Let B & G be any pair that are not married.

To prove: B & G aren't rogue.

Case 1 G rejected B
⇒ G has a better suitor than B (lemma 1)
⇒ G married someone whom she prefers over B.
⇒ G is not rogue with B.

Case 2 G didn't reject B
⇒ B never serenaded G
⇒ G is lower in order than B's spouse.
⇒ B is not rogue with G

∴ No rogue couples.

Lecture 7: Matching Problems

Let S be the set of all stable matching
$S \neq \phi$ as algo produces one stable matching.

For each person P, we define the realm of possibility for P to be $\{Q \mid \exists M \in S \mid \{P, Q\} \in M\}$

A person's optimal mate is his/her favorite from the realm of possibility.

A person's pessimal mate is his/her least favorite from the realm of possibility.

Theorem 4 Algo marries every boy with his optimal mate.

Proof —

Theorem 5 Algo marries every girl with her pessimal mate.

Proof —

TMA → The Marriage Algorithm

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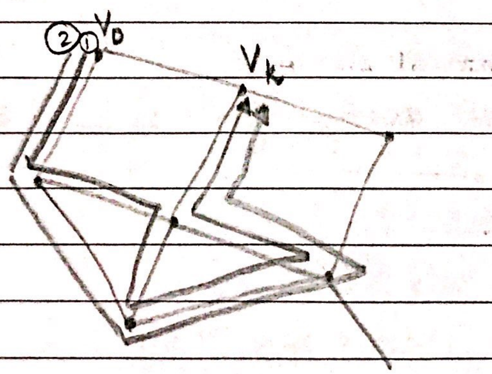
L-8
- Walks & Paths
- Connectivity
- Cycles & Closed walk
- Spanning Tree (ST)
- Min-weight spanning tree (MST)

Walks & Paths

⇒ A walk is a sequence of vertices that are connected by edges

Eg:- $v_0 - v_1 - \cdots - v_k$
Start ... end

It has k-edges & length = k



① → walk, not path
② → walk, and path

⇒ A path is a walk where all v_i's are distinct

Lecture 8: Graph Theory II: Minimum Spanning Trees

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Lemma 1 If $\exists$ walk from u to v, then $\exists$ path from u to v.

Proof $\exists$ walk u to v

By well-ordering principle: walk of minimal length.

$u = v_0 - v_1 - \cdots - v_k = v$

We prove that this walk (of minimal length) is a path.

Case $k=0$ No edge length $=0$

Case $k=1$ Just one edge length $=1$

Case $k \geq 2$

Suppose this walk is not a path

then $\exists_{i \neq j}$ $v_i = v_j$

$u = v_0 - \cdots - v_i = v_j - \cdots - v_k$

Now this a shorter walk

thus contradicting our assumption. ∎

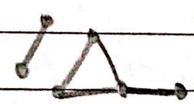
Lecture 8: Graph Theory II: Minimum Spanning Trees

Connectivity

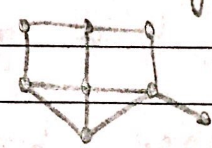
⇒ u and v are connected if there is a path from u to v.

⇒ A graph $G=(V,E)$ is connected if every pair of vertices $(v_1, v_2) \in V \times V$ are connected.

Not connected graph



Connected graph



Cycles & Closed walks

~~(also known as tours)~~

⇒ A closed walk is a walk starts and ends at exactly same vertex.

$$v_0 - v_1 - \cdots - v_k = v_0$$

a closed walk with ... and

⇒ If $k \geq 3$ if all $v_0, \ldots, v_{k-1}$ are distinct then it is called a cycle

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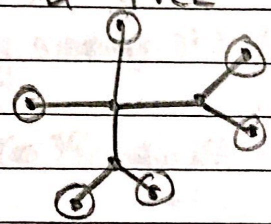
Trees

⇒ A connected and acyclic graph is called a tree.

⇒ A leaf is a node with degree = 1 in a tree



Lemma2 Any connected subgraph of a tree is a tree.

Proof By contradiction,
if the connected subgraph is not a tree then it must have a cycle.
As the subgraph is a part of a graph, the graph must have a cycle.
But the graph is a tree.
We get a contradiction. ∎

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Lemma3 A tree with n vertices has $n-1$ edges.

Proof By Induction

~~P(n)~~ Ind. Hypo. : $P(n)$ = "There are $n-1$ edges in an n-vertex tree"

Base case : $P(1)$ is true. (0)

Ind. step: Assume that $P(n)$ is true

Let T be an $(n+1)$-vertex tree,
Let v be a leaf of the tree
Delete v ; this creates a connected subgraph which is also a tree (Lemma 2)
By $P(n)$: this subgraph has $n-1$ edges.
Re-attach v $\therefore$ T has $(n-1)+1 = n$ edges
(degree of $v = 1$)

$P(n+1)$ is true

$\blacksquare$

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Spanning Trees

→ A spanning tree (ST) of a connected graph ~~G = (V, E)~~ is a subgraph that is a tree with same vertices as the graph.

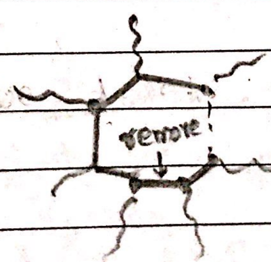
Theorem Every connected graph has a ST.

Proof By contradiction.

Assume a connected graph G with no ST.

Let T be a connected subgraph of G and has same vertices as G and has minimum number of edges.

As T is not a spanning tree, it must have atleast one cycle.



if we remove one edge of the cycle, we still have a connected subgraph.

Lecture 8: Graph Theory II: Minimum Spanning Trees

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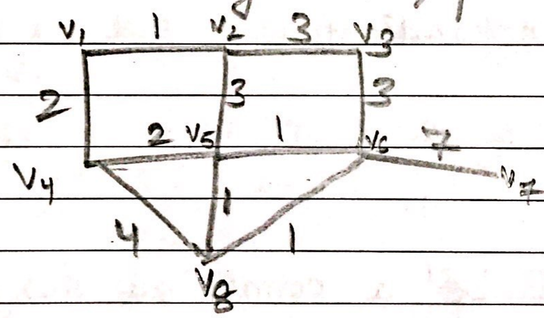
Subgraph we obtain is smaller than T (w.r.t. number of edges) thus contradicting our minimum condition on T.

Minimum-weighted spanning tree



$ST_1 = \{ v_1 - v_4,\ v_4 - v_5,\ v_5 - v_2,\ v_5 - v_6,\ v_6 - v_3,\ v_6 - v_7,\ v_5 - v_8 \}$

Weight of $ST_1 = 2 + 2 + 3 + 1 + 3 + 7 + 1 = 19$

$ST_2 = \{ v_1 - v_2,\ v_1 - v_4,\ v_4 - v_5,\ v_5 - v_6,\ v_6 - v_3,\ v_6 - v_7,\ v_5 - v_8 \}$

Weight of $ST_2 = 1 + 2 + 2 + 1 + 3 + 7 + 1 = 17$

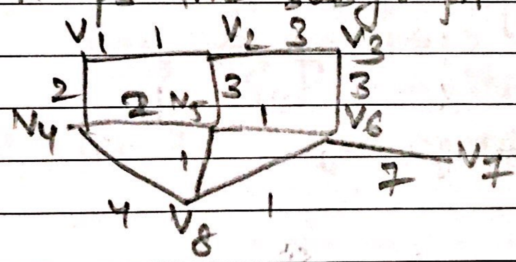
Lecture 8: Graph Theory II: Minimum Spanning Trees

⇒ The minimum spanning tree (MST) of an edge-weighted graph is defined as the ST of G with the smallest possible sum of edge-weights.

Algo

- Grow a subgraph one edge at a time, such that at each step:

⇒ Add the minimum weighted edge keeps the subgraph acyclic.



+1 ① $V_6 - V_8$

+1 ② $V_1 - V_2$

+1 ③ $V_5 - V_6$ ⇒ $V_5 - V_8$ ⊗

+2 ④ $V_1 - V_4$

+2 ⑤ $V_4 - V_5$ ⇒ $V_4 - V_8$ ⊗, $V_2 - V_5$ ⊗

+3 ⑥ $V_2 - V_3$ ⇒ $V_3 - V_6$ ⊗

+7 ⑦ $V_6 - V_7$

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Lecture 8: Graph Theory II: Minimum Spanning Trees

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Theorem For any connected weighted graph G, the algo produces a MST.

Lemma Let S consists of first k-edges by the algo, then $\exists$ MST $T=(V,E)$ for G such that $S \subseteq E$

Proof (Assumed lemma as true)

Graph $G=(V,E)$

$|V|=n$

1) Suppose $< n-1$ edges are picked, then $\exists$ edge in $E-S$ that can be added without creating a cycle.

2) Once $n-1$ edges are picked, we know S is an MST.

Proof By Induction:

$P(m) = \forall G\ \forall S$ consisting of first m selected edges $\exists$ MST of G $T=(V,E)$ such that $S \subseteq E$.

Base Case: $m=0 \Rightarrow S=\phi \subseteq E$ for any MST $T=(V,E)$

Ind. Step: Assume $P(m)$ holds

Let e denote the $(m+1)$th selected edge.

Let S denote the first m selected edges.

Lecture 8: Graph Theory II: Minimum Spanning Trees

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Let $T^* = (V, E^*)$ be MST of G such that $S \subseteq E^*$

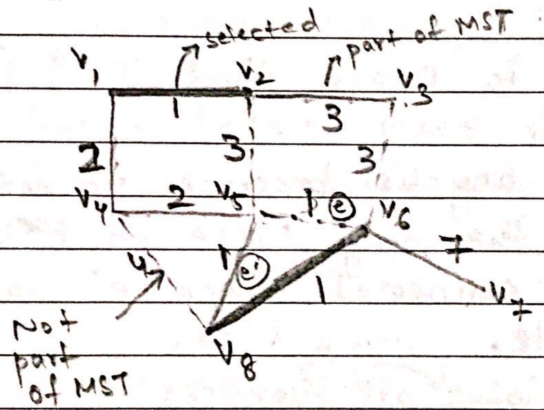
Case 1: $e \in E^* \Rightarrow S \cup \{e\} \subseteq E^* \Rightarrow P(m+1)$ holds

Case 2 $e \notin E^*$

$S = \{v_1 - v_2, v_6 - v_8\}$

$E^* = \{v_1 - v_2, v_2 - v_3, v_1 - v_4, v_4 - v_5, v_5 - v_8, v_6 - v_8, v_6 - v_7\}$

$e = v_5 - v_6$



~~From the algo,~~

Algo $\rightarrow S \cup \{e\}$ has no cycle

(This is how we choose $(m+1)^{th}$ edge)

T^* is a tree $\Rightarrow (V, E^* \cup \{e\})$ must contain a cycle. (Proof in book)
$= (V, E^*)$

this cycle has edge $e' \in E^* - S$

Lecture 8: Graph Theory II: Minimum Spanning Trees

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Algo could have selected e or e'
$\Rightarrow$ weight of $e \leq$ weight of e'.

Swap e and e' in T^*:

Let $T^{**} = (V, E^{**})$: $E^{**} = (E^* - \{e'\}) \cup \{e\}$

We need to prove that T^{**} is an MST.

- T^{**} is acyclic because we removed e' from the only cycle in $E^* \cup \{e\}$.
- T^{**} is connected since e' was on a cycle.
- T^{**} contains all vertices of G

$\Rightarrow$ T^{**} is ST of G.

weight of T^{**} = weighted sum of E^{**}
= Weighted sum of E^* − weight of e' + weight of e
= weight ~~sum of~~ of T^* − weight of e' + weight of e

As weight of $e \leq$ weight of e'
weight of $T^{**} \leq$ weight of T^*
And as T^* is MST, equality holds & T^{**} is also MST. $\square$

Lecture 9: Communication Networks

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L-9 → L-10

Communication Networks

- great application of graph theory
- how do you route packets through networks

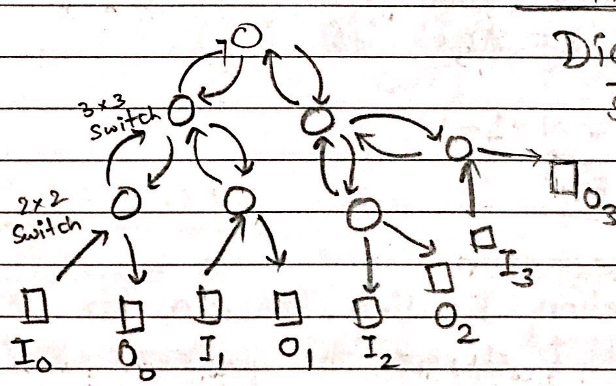
$N \times N$ Networks $(N = 2^k)$.

Complete Binary Tree

○ = switch (direct packets through a network)

□ = terminal (source and destination of data)

4 X 4 Network



~~Diameter = 6~~

~~Switch count = 7~~

$\pi(i) = i$

↳ Congestion = 1

$\pi(i) = N-1-i$

↳ Congestion = 4

Max Congestion = 4

N X N Network

Diameter $= 2(1 + \log_2 N)$

Max. Switch size $= 3 \times 3$

Switch count $= 2N - 1$

Max Congestion $= N$

Lecture 9: Communication Networks

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Latency: Time that is required for a packet to travel from an input to an output

Diameter of a network: Length of the shortest path b/w Input and Output that are farthest apart (Maximum length of shortest path

A permutation is a function $\pi : \{0 \dots N-1\} \to \{0 \dots N-1\}$ such that no two numbers are mapped to the same value
$(\pi(i) = \pi(j)$ iff $i = j)$

Eg:- $\pi(i) = N-1-i$
$\pi(i) = i$

Permutation Routing Problem for π
- For each i direct the packet at I_i to $O_{\pi(i)}$
  path taken is denoted by $P_{i,(\pi(i))}$

The congestion of paths $P_{0,\pi(0)}, \dots, P_{N-1,\pi(N-1)}$ is equal to the largest number of paths that pass through a single switch.

Max congestion = max possible value of congestion. (over permutations with best path)

Lecture 9: Communication Networks

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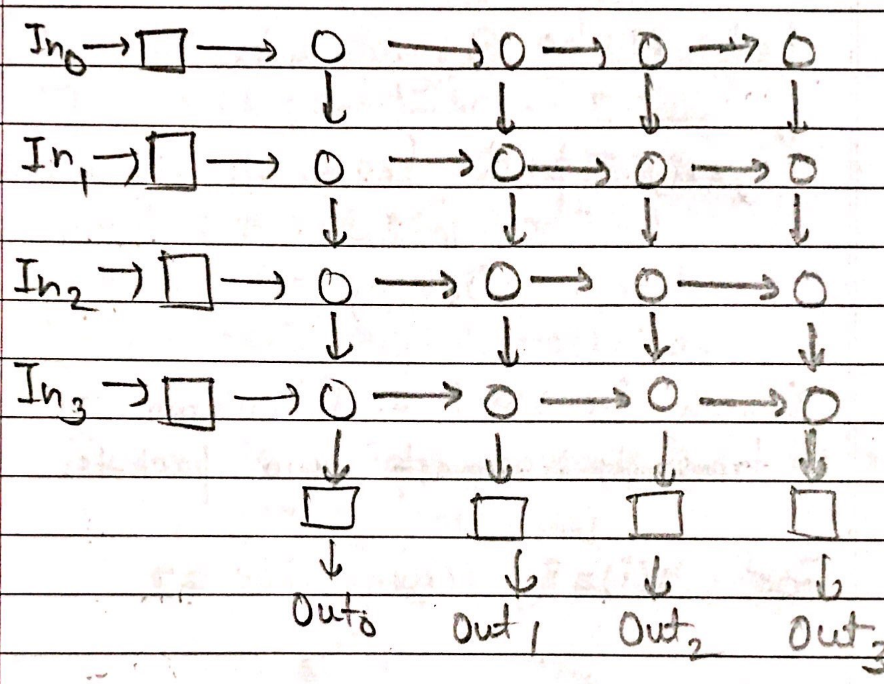
2D Arrays (Grid structure)



4x4 Network	**NXN Network**
Diameter = 8	Diameter = 2N
Max. switch size = 2x2	Max. switch size = 2x2
No. of switches = 16	No. of switches = N^2
Max. congestion = 2	Max. congestion = 2

Lecture 9: Communication Networks

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Theorem Congestion of an N-input array is 2.

Proof Let π be a permutation.

$P_{i,\pi(i)}$ = path from In_i rightward to column $\pi(i)$ and downward to $Out_{\pi(i)}$.

Switch in row i and column $\pi(i)$ transmits atmost two packets.

For $\pi(i)=i$, congestion $=2$

∎

Lecture 9: Communication Networks

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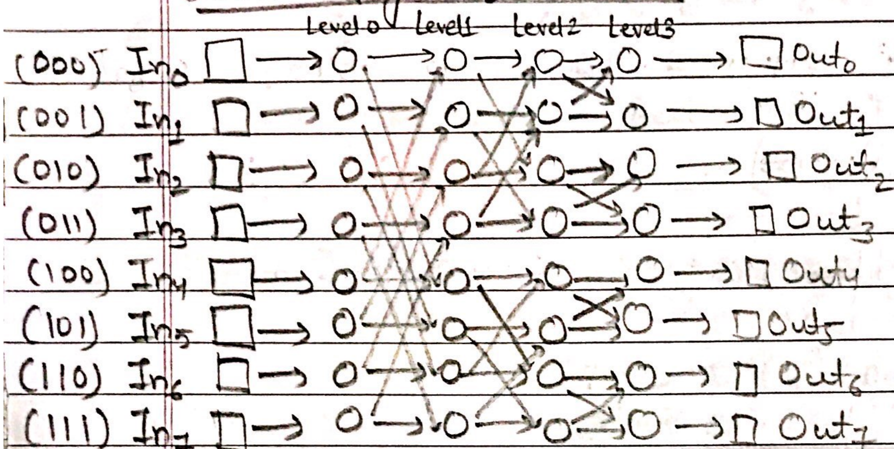
Butterfly ~~Structure~~ Network



Switch is uniquely identified its row & col

$$((\overbrace{b_1 \cdots b_{\log N}}^{\text{binary rep. of input}}), \overset{\text{level}}{\ell})$$

$$((b_1 \cdots \overline{b_{\ell+1}} \cdots b_{\log N}), \ell+1) \qquad ((b_1 \cdots b_{\ell+1} \cdots b_{\log N}), \ell+1)$$

complement of $(\ell+1)^{th}$ bit.

Routing

$$((x_1 \cdots x_{\log N}), 0) \longrightarrow ((y_1 x_2 \cdots x_{\log N}), 1)$$

$$((y_1 y_2 x_3 \cdots x_{\log N}), 2)$$

$$((y_1 \cdots y_\ell x_{\ell+1} \cdots x_{\log N}), \ell)$$

$$((y_1 \cdots y_{\log N}), \log N)$$

Lecture 9: Communication Networks

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In3 $\longrightarrow$ Out5

Eg:- $011 \longrightarrow 101$

$$
\begin{array}{l}
\text{In3} \rightarrow (011, 0) \\
\qquad\quad \downarrow \\
\qquad (111, 1) \\
\qquad\quad \downarrow \\
\qquad (101, 2) \\
\qquad\quad \downarrow \\
\text{Out5} \leftarrow (101, 3)
\end{array}
$$

<u>N×N Network</u>

Diameter = $2 + \log N$

Max. Switch size = 2×2

No. of Switches = $N(1 + \log N)$

Congestion = $\sqrt{N}$ $(N \in 2^{2n})$

$= \sqrt{N/2}$ $(N \in 2^{2n+1})$

Lecture 9: Communication Networks

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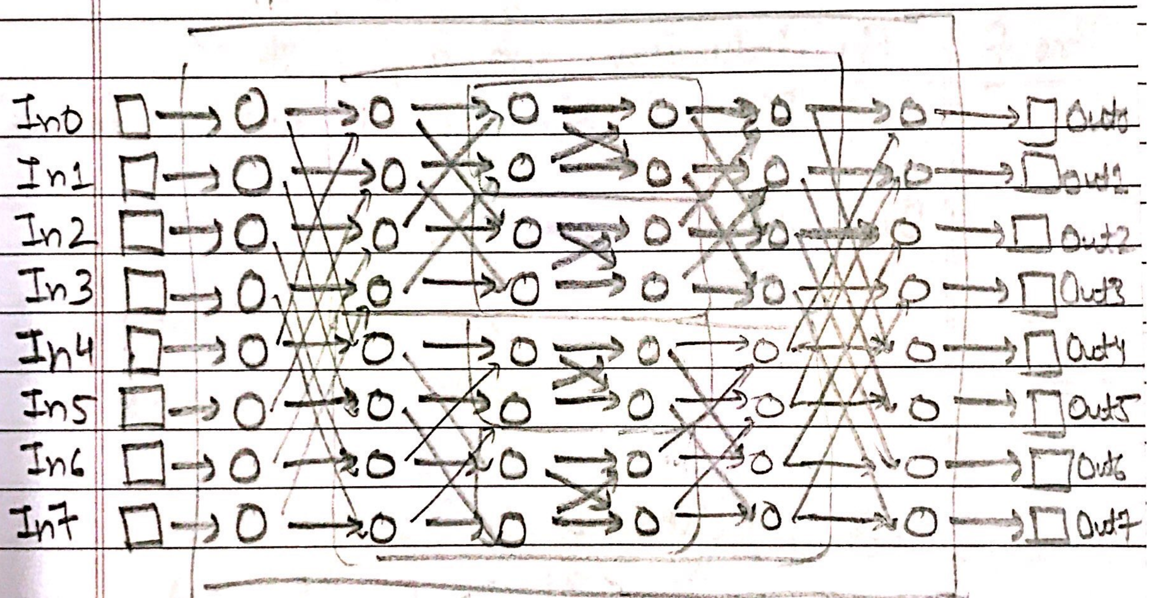
Benes

1960s : Benes, a Bell Labs researcher



NxN Network

Diameter = 1+ 2 log N

Max switch size = 2x2

No of switches = 2N log N

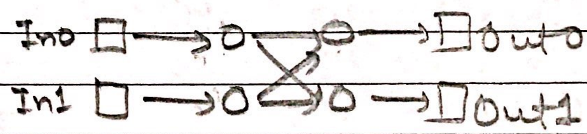
Congestion = 1

Lecture 9: Communication Networks

Theorem The Congestion of N-input Benes network is 1, when $N = 2^a$ for some $a \geq 1$

Proof By induction on a,
$P(a) =$ "This theorem is true for a".

Base case: $a = 1$ $N = 2$



$\pi(0) = 0$	$\pi(0) = 1$
$\pi(1) = 1$	$\pi(1) = 0$
Congestion = 1	Congestion = 1
(Just straight)	(Just cross-over)

Ind. Step: Assume $P(a)$ is true

Example: $\pi(0) = 1$ $\pi(4) = 3$
$\pi(1) = 5$ $\pi(5) = 6$
$\pi(2) = 4$ $\pi(6) = 0$
$\pi(3) = 7$ $\pi(7) = 2$

Constraint graph

If two packets must pass through different subnetworks then there is an edge b/w them.

Lecture 10: Graph Theory III

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L-10 → L-9

Euler tours

Directed graphs

- Definitions
- # walks (No. of)
- strong connectivity
- DAGs

Tournament Graphs

Euler tour

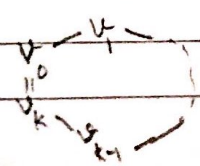
- he lived in Konigsberg
- seven bridges (birth of graph theory)

An Euler tour is a walk that traverses every edge exactly once and starts and finishes at same vertex.

Theorem A connected graph has an Euler tour iff every vertex has even degree.

Proof ($\Rightarrow$)

Assume $G = (V, E)$ has an Euler tour



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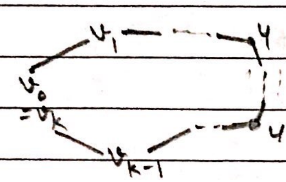
Since every edge in E is traversed once;
clearly $\deg(u) = (\#$ times u appears in this tour $v_0 \dots v_k) \times 2$



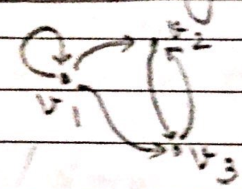
($\Leftarrow$)
For $G=(V,E)$, assume $\deg(v)$ is even $\forall v \in V$
Let $W: v_0 - v_1 - \dots\dots - v_k$ be the longest walk that traverses no edge more than once.

① $v_k - u$ not in $W \Rightarrow v_0 - v_1 \dots - v_k - u$ is longer than $W \Rightarrow$ Contradicts
$\therefore$ All the edges (in E) incident of v_k are traversed in W.

② $v_k \neq v_0 \Rightarrow v_k$ has odd degree in walk W
As we have showed that all edges (in E) of v_k are in W, so $\deg(v_k)$ is odd in G
$\Rightarrow$ Contradicts as $\deg(v)$ is even $\forall v \in V$.

Lecture 10: Graph Theory III

Directed graph (digraphs)



$$\underset{\text{(Tail)}}{v_2} \longrightarrow \underset{\text{(Head)}}{v_3}$$

indegree $(v_2) = 2$
outdegree $(v_2) = 1$

Theorem Let $G = (V, E)$ be an n-node graph with $V = \{v_1, \dots, v_n\}$. Let $A = \{a_{ij}\}$ denote the adjacency matrix for G that is

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \to v_j \text{ is an edge.} \\ 0 & \text{otherwise} \end{cases}$$

Let $P_{ij}^{(k)}$ = # directed walks of length k from v_i to v_j.

then $A^k = \{P_{ij}^{(k)}\}$

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Proof $a_{ij}^{(k)}$ denote the $(i,j)^{th}$ entry in A^k

By induction:

Predicate → $P(k)$ = "Theorem is true for k"
= " $\forall_{i,j} \quad a_{ij}^{(k)} = p_{ij}^{(k)}$ "

Base case $\quad k=1$

Edge $v_i \to v_j$: $p_{ij}^{(1)} = 1 = a_{ij}^{(1)}$

No edge $v_i \to v_j$: $p_{ij}^{(1)} = 0 = a_{ij}^{(1)}$

$\therefore \quad \forall_{i,j} \quad a_{ij}^{(1)} = p_{ij}^{(1)}$

Ind. step

Assume $P(k)$ holds.

$$p_{ij}^{(k+1)} = \sum_{h:\, v_h \to v_j \text{ is an edge in } E} p_{ih}^{(k)} = \sum_{h=1}^{n} p_{ih}^{(k)} \cdot a_{hj}$$

$v_i \overset{k}{\rightsquigarrow} v_h \xrightarrow{1} v_j$

Ind. step →
$$= \sum_{h=1}^{n} a_{ih}^{(k)} \cdot a_{hj}$$

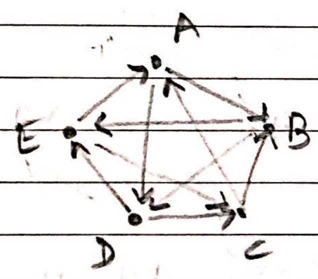
Matrix multiplication →
$$= a_{ij}^{(k+1)} \Rightarrow P(k+1) \text{ holds}$$

Lecture 10: Graph Theory III

⇒ A digraph $G=(V,E)$ is called strongly connected if $\forall\, u,v \in V$ $\exists$ directed path from u to v in G.

⇒ A digraph is called directed acyclic graph (DAG) if it does not contain any directed cycles.

<u>Tournament Graph</u>



either u beats v: $u \to v$
or v beats u: $v \to u$

→ E beats A, C and loses to B, D.

⇒ A directed Hamiltonian path is a directed walk that visits every vertex exactly once.

Theorem Every tournament graph contains a directed Hamiltonian path.

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Proof By Induction (on number of nodes)

$P(n)$ = "Every tournament graph on n-nodes contains a directed Ham. path."

Base case: $P(1)$ holds

Ind. step: Assume $P(n)$ holds

Consider a tournament graph on $(n+1)$-nodes

Take out a node v. We are left with a tournament graph on n-nodes.

By $P(n)$, $v_1 \to v_2 \to \cdots \to v_n$

Approach ① If $v \to v_1$ is an edge, then $P(n+1)$ holds.

(if $v_1 \to v$ and $v \to v_n$) then

Claim: $\exists$ i such that $v_i \to v$ & $v \to v_{i+1}$

Proof by contradiction

If no such i exists and we know $v_1 \to v$ then $v_i \to v$ $\forall i$ (by induction)

But $v \to v_n$ contradicts this.

~~There is also proof by strong induction.~~ See it.

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Approach ②

Case 1 $v \to v_1$

Case 2 $v_1 \to v$

Smallest i such that $v \to v_i$

$\Rightarrow v_k \to v \;\; \forall \; k < i$

$\Rightarrow v_{i-1} \to v$

$$v_1 \to \cdots \to v_{i-1} \to v_i \to \cdots \to v_n$$

$$v_{i-1} \to v \to v_i$$

A new directed Hamiltonian path is form

$\therefore P(n+1)$ holds

Lecture 10: Graph Theory III

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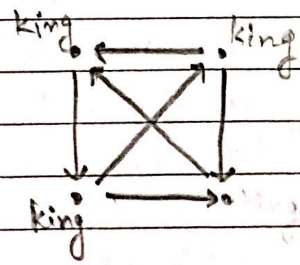
King Chicken Problem

either chicken u pecks chicken v : $u \to v$
or chicken v pecks chicken u : $v \to u$

u "virtually" pecks v if
- $u \to v$, or
- $\exists_w \; u \to w \to v$

A chicken that virtually pecks every other chicken is called king chicken.



Theorem The chicken with highest outdegree is definitely a king.

Proof By contradiction,

Let u have highest outdegree.
Suppose u is not king

Lecture 11: Relations, Partial Orders, and Scheduling

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$\Rightarrow \exists_v : v \longrightarrow u$, and $\forall_w : u \to w \Rightarrow v \to w$

(outdegree of v) $\geq$ (Outdegree of u) + 1

So, our assumption is false. ■

L-11
- Relations
  - properties
- Equivalent Relations
- Partial Orders
  - Hasse diagram
  - Total order
  - Topological Sort
- Parallel Task Scheduling
  - Dilworth's Lemma

Relations

→ A relation from a set A to a set B is a subset $R \subseteq A \times B$

Ex :- $R = \{ (a,b) :$ student a is taking class $b \}$

$(a,b) \in R$: aRb , $a \sim_R b$

Relational Symbol

Lecture 11: Relations, Partial Orders, and Scheduling

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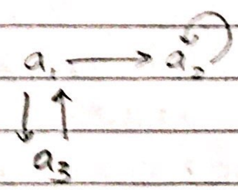
⇒ Relation on A is a subset $R \subseteq A \times A$

Eg:- $A = \mathbb{Z}$: xRy iff $x \equiv y \pmod 5$

$A = \mathbb{N}$: xRy iff $x \mid y$

$A = \mathbb{N}$: xRy iff $x \leq y$

Set A together with R is a directed graph $G = (V, E)$ with $V = A$, $E = R$.



Properties

A relation R on A is

* relfexive if xRx $\forall x \in A$
* symmetric if $xRy \Rightarrow yRx$ $\forall x, y \in R$
* anti-symmetric if $xRy \wedge yRx \Rightarrow x = y$ (&)
* transitive if $xRy \wedge yRz \Rightarrow xRz$

Lecture 11: Relations, Partial Orders, and Scheduling

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Ex	Reflexive	Symmetric	Anti-Symm	Transitive	
$x \equiv y \pmod 5$	✓	✓	× (2, 7)	✓	→ Equivalence
$x \mid y$	✓	× (1, 2)	✓	✓	Partial order
$x \leq y$	✓	× (1, 2)	✓	✓	

⇒ An equivalence relation is reflexive, symmetric, transitive.

Ex: $A = \mathbb{Z}$: xRy iff $x = y$ $\quad n \in \mathbb{N}$

$A = \mathbb{Z}$: xRy iff $x \equiv y \pmod n$

⇒ The equivalence class of $x \in A$ is the set of all elements in A related to x by R : denoted by $[x]$

$$[x] = \{y : xRy\}$$

Ex: $A = \mathbb{Z}$: xRy iff $x \equiv y \pmod 5$

$[7] = \{\ldots, -3, 2, 7, 12, 17, \ldots\}$

$[7] = [2] = [12] = [17] = \ldots$

Lecture 11: Relations, Partial Orders, and Scheduling

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A partition of A is a collection of disjoint nonempty sets $A_1, \ldots, A_n \subseteq A$ whose union is A.

Ex: $A = \mathbb{Z}$: xRy iff $x \equiv y \pmod 5$

$A_0 = \{\ldots -5, 0, 5, \ldots\}$
$A_1 = \{\ldots -4, 1, 6, \ldots\}$
$A_2 = \{\ldots -3, 2, 7, \ldots\}$
$A_3 = \{\ldots -2, 3, 8, \ldots\}$
$A_4 = \{\ldots -1, 4, 9, \ldots\}$

Theorem The equivalence classes of equivalence relation on a set A form a partition of A.

Partial Orders

A relation is a (weak) partial order if it is reflexive, antisymmetric and transitive.

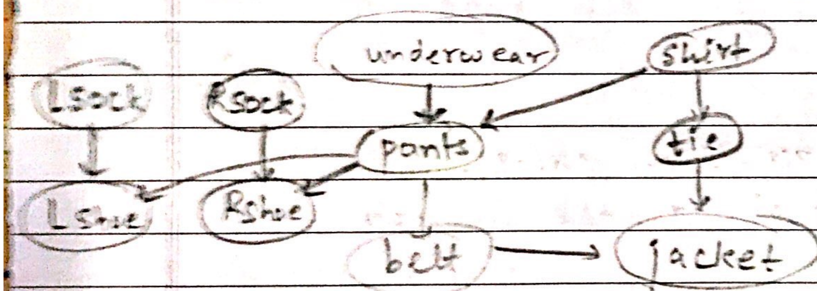
A relation is a (strong) partial order if it is irreflexive, antisymmetric and transitive. (See book).

Lecture 11: Relations, Partial Orders, and Scheduling

A partial order relation is denoted by $\preceq$ instead of R.

$(A, \preceq)$ is actually called partial ordered set. or poset

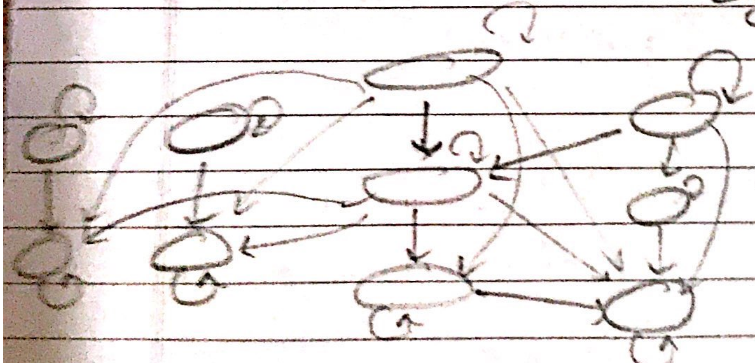
A poset is a directed graph with vertex set A and edge set $\preceq$.



(Hesse diagram)

Transitive $\Rightarrow$ No directed cycles

Antisymmetric $\Rightarrow$ Only one directed edges b/w two vertices



(Poset)

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A Hasse diagram for a poset $(A, \preceq)$ is a directed graph with vertex set A and edge set $\preceq$ minus:
* all self-loops, and
* all edges implied by transitivity

Theorem A poset has no directed cycles other than self loops.

Proof By contradiction

Suppose $\exists$ $n (\geq 2)$ distinct elements $a_1, \ldots, a_n$ such that

$a_1 \preceq a_2 \preceq a_3 \ldots \preceq a_n \preceq a_1 \Rightarrow a_n \preceq a_1$ — (I)

(induction)

By transitivity, $a_1 \preceq a_2 \preceq \ldots \preceq a_n$

$\Rightarrow a_1 \preceq a_n$ — (I)

anti-symmetry

$a_1 = a_n$

Contradicts distinct element assumption. $\blacksquare$

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Conclusion of the theorem

So, deleting self-loops from a poset, makes a directed acyclic graph (DAG)

→ a and b are incomparable if neither $a \preceq b$ nor $b \preceq a$

→ a and b are comparable if $a \preceq b$ or $b \preceq a$.

→ A total order is a partial order in which every pair of elements is comparable.

Hesse diagram of a total order is a straight line.

A total order consistent with a partial order is called a Topological sort.

A topological of a poset $(A, \preceq)$ is a total order $(A, \preceq_T)$ such that

$$\preceq \subseteq \preceq_T$$

$$(x \preceq y \Rightarrow x \preceq_T y)$$

Lecture 11: Relations, Partial Orders, and Scheduling

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Theorem Every finite poset has a topological sort.

Proof ⇒ $x \in A$ is minimal if there doesn't exist $y \in A \,;\, y \neq x$ such that $y \preceq x$.

⇒ $x \in A$ is maximal if $\neg\exists$ $y \in A, y \neq x$ s.t. $x \preceq y$.

not there exists

Lemma Every finite poset has a minimum element.

⇒ A chain is a sequence of distinct elements such that $a_1 \preceq a_2 \preceq \dots \preceq a_t$ (length of chain is t).

Proof Let $C = (a_1 \preceq a_2 \dots \preceq a_n)$ be the max length chain.

Case1: $a \notin \{a_1, \dots, a_n\}$

if $a \preceq a_1$ then C is not the longest chain.

So, $\neg\, a \preceq a_1$

Part of Lecture 13: Asymptotics

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Case 2: $a \in \{a_1, \ldots, a_n\}$

if $a \preceq a_1$ then we have a directed cycle which contradicts Theorem ___.

So, $\neg\, a \preceq a_1$

$\therefore \forall\, a \in A \quad \neg\, a \preceq a_1$

So, by definition, a_1 is the minimum element.

Add Parallel Task Scheduling & Dilworth's Lemma.

L-13 (45:30)

Asymptotic Notation

tilde $f(x) \sim g(x)$ if $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 0$

oh, big-oh $f(x) = O(g(x))$ if $\lim\limits_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$ (finite)

Multiple usage :

Formal math way

$f(x) \leq O(g(x))$; $f(x)$ is $O(g(x))$; $f(x) \in O(g(x))$

$O(g(x))$ is a set of all func. that grow slowly than $g(x)$

Part of Lecture 13: Asymptotics

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Theorem Let $f(x) = x$, $g(x) = x^2$
Then $f(x) \in O(g(x))$

Proof $\lim_{x \to \infty} \frac{x}{x^2} = 0$ which is finite

Theorem $x^2 \notin O(x)$

Proof $\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x$ is infinite.

Q. Is $x^2 \in O(10^6 x)$ true?
A. No.

Q. Is $10^6 x^2 \in O(x^2)$ true?
A. Yes

Q. Is $x^2 + 100x + 10^7 = O(x^2)$ true?
A. Yes

Theorem $x^{10} \in O(e^x)$

Proof $\lim_{x \to \infty} \frac{x^{10}}{e^x} = 0$ which is finite.
(L'Hopital's rule)

Part of Lecture 13: Asymptotics

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Theorem $4^x \notin O(2^x)$

Proof $\lim_{x \to \infty} \frac{4^x}{2^x} = \lim_{x \to \infty} 2^x \longrightarrow \infty$

Q. Is $10 \in O(1)$ true?

A. Yes

$f(x) \geq O(g(x))$ is meaningless.

omega $f(x) = \Omega(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| > 0$

Theorem $f(x) = O(g(x))$ iff $g(x) = \Omega(f(x))$

$f(x) \leq O(g(x))$ iff $g(x) \geq \Omega(f(x))$

$x^2 = \Omega(x)$

$2^x = \Omega(x^2)$

$\frac{x}{100} = \Omega(100x + 25)$

Part of Lecture 13: Asymptotics

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theta $f(x) = \Theta(g(x))$ if $0 < \lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$

<u>Theorem</u> $f(x) = \Theta(g(x))$ iff $f(x) = O(g(x))$ & $f(x) = \Omega(g(x))$

$10x^3 - 20x + 1 \in \Theta(x^3)$

$\frac{x}{\ln(x)} \notin \Theta(x)$ | $\frac{x}{\ln(x)} \in O(x)$

$T(n) = \Theta(n^2)$ means T grows quadratically in n.

$O \Rightarrow \leq$
$\Omega \Rightarrow \geq$
$\Theta \Rightarrow =$
$o \Rightarrow <$ ($\leq$ not $=$)
$\omega \Rightarrow >$ ($\geq$ not $=$)

Part of Lecture 13: Asymptotics

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little oh $f(x) = o(g(x))$ iff $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0$

little omega $f(x) = \omega(g(x))$ iff $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = \infty$

$\frac{x}{\ln(x)} \in o(x)$

$\frac{x}{100} \notin o(x)$ $\frac{x}{100} \in \Theta(x)$

$x^2 \in \omega(0)$

what not do

example.

MIT 6.006 (Fall 2011) : Introduction to Algorithms

This course is available on OpenCourseWare MIT (ocw.mit.edu) as well as on Youtube channel MIT OpenCourseWare.

Course Instructor(s)

Prof. Erik Demaine
Prof. Srini Devadas

Course Description

This course provides an introduction to mathematical modeling of computational problems. It covers the common algorithms, algorithmic paradigms, and data structures used to solve these problems. The course emphasizes the relationship between algorithms and programming, and introduces basic performance measures and analysis techniques for these problems.

In this report, I have included my notes of Lectures 13 thru 18 which cover 'Basic algorithms of Graph Theory'.
More detailed notes of the course are present here

Lecture 13: Breadth-First Search (BFS)

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L-13 **Graph I : BFS**

- applications of graph search
- graph representation
- BFS → Breadth-first search

Graph search

"explore" a graph

Eg:- (s) ⟶ (t) Find path

Recall: graph $G = (V, E)$

V → set of vertices

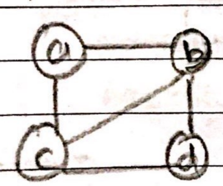
E → set of edges

Undirected Graph → $e = \{v, w\}$ ← unordered pairs

OR

Directed Graph → $e = (v, w)$ ← ordered pairs

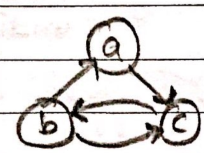
Every undirected graph is a directed graph $\{u, v\} \to (u, v), (v, u)$



UNDIRECTED

$V = \{a, b, c, d\}$

$E = \{ \{a,b\}, \{b,c\}, \{c,d\}, \{a,c\}, \{b,d\} \}$



DIRECTED

$V = \{a, b, c\}$

$E = \{(a,c), (b,c), (c,b), (b,a)\}$

Lecture 13: Breadth-First Search (BFS)

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Applications :
- web crawling
- social networking
- network broadcast
- garbage collection
- model checking
- check mathematical conjecture
- solving puzzles & games

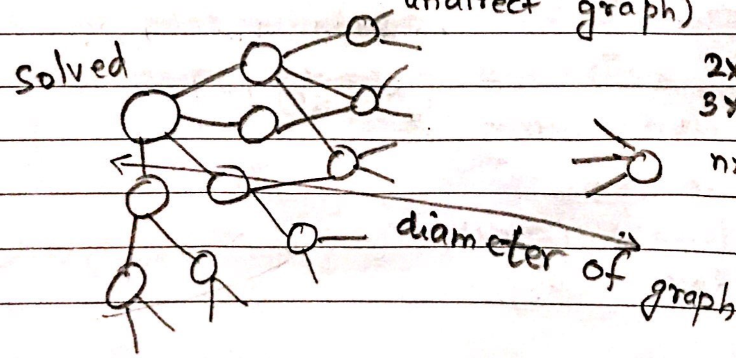
Pocket Cube : $2 \times 2 \times 2$
- configuration graph
  - vertex for each possible state of cube

    $\#\text{ vertices} = 8! \cdot 3^8 = 264,539,520$

    24 symmetries

    $1/3^{rd}$ of states can only be reached without breaking it apart
  - edge for each possible move

    (as every move is undoable, undirect graph)



$2\times2\times2$: 11

$3\times3\times3$: 20

$n\times n\times n$: $\Theta\left(\frac{n^2}{\lg n}\right)$

Lecture 13: Breadth-First Search (BFS)

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Graph representation:

Adjacency Lists

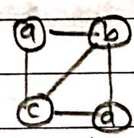
array Adj of size $|V|$

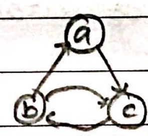
- each element is a pointer to a linked list

for each vertex $u \in V$,

Adj[u] stores u's neighbours

$\rightarrow \{ u \in V \mid (u, v) \in E \}$





Adj[a] = {b,c}
Adj[b] = {a, c, d}
Adj[c] = {a, b, d}
Adj[d] = {b, c}

Adj[a] = {c}
Adj[b] = {a, c}
Adj[c] = {b}

Object-oriented pattern

Vertex $v \longrightarrow$ v.neighbours = Adj[v]

The reason why object-oriented is not used is because you can simply have adjacency list of sub-graph in the direct method.
If you are dealing with only one graph, object-oriented method is more cleaner.

Lecture 13: Breadth-First Search (BFS)

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Implicit representation

- Adj(u) is a function
- v.neighbours() is a method

Uses less space

This would be helpful for puzzles like Rubik's cube.

Fun fact: 7x7x7 has configurations more than number of atoms in the known universe.

Space used by the basic rep. is $\Theta(|V|+|E|)$

Breadth-first Search

- Visit all the nodes reachable from given $s \in V$
- $O(V+E)$ time
- look at nodes reachable in 0 moves, 1 move, 2move, ...

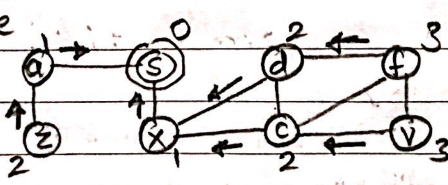
  $\{s\}$ Adj[s]
- careful to avoid duplicates

Lecture 13: Breadth-First Search (BFS)


```
BFS(s, Adj)
  level = {s : 0}
  parent = {s : None}
  i = 1
  frontier = [s]        ← level i-1
  while frontier:
      next = []         ← level i
      for u in frontier:
          for v in Adj[u]:
              if v not in level:
                  level[v] = i
                  parent[v] = u
                  next.append(v)
      frontier = next
      i += 1
```

UNDIRECTED Graph

Example



(←) parent direction

i=1 next = [a, x]

i=2 next = [z, d, c]

i=3 next = [f, v]

i=4 next = []

Lecture 13: Breadth-First Search (BFS)

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Shortest paths

$v \leftarrow parent[v] \leftarrow parent[parent[v]]$

$\uparrow$

$\vdots$

$\uparrow$

s

is the shortest path from s to v.
of length level[v].

Each vertex is checked once
Each edge is considered once (or twice (UDG))

$$\sum_{\forall v \in V} |Adj[v]| = \begin{cases} 2 \cdot |E| & (UDG) \\ |E| & (DG) \end{cases}$$

R-13 (Halloween Day)
Revised BFS
and operations on Byte (8 bit) Vs Word (16 bit)

Lecture 14: Depth-First Search (DFS), Topological Sort

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L-14

Graph II : DFS
- depth-first search
- edge classification
- cycle detection
- topological sort

Depth-first search (DFS)
- recurisevly explore graph, backtracking as necessary
- careful not to repeat

```
parent = { s: None }
DFS-Visit(V, Adj, s):
    for v in Adj[s]:
        if v not in parent:
            parent[v] = s
            DFS-Visit(V, Adj, v)
DFS (V, Adj):
    parent = {}
    for s in V:
        if s not in parent:
            parent[s] = None
            DFS-Visit(V, Adj, s)
```

Lecture 14: Depth-First Search (DFS), Topological Sort

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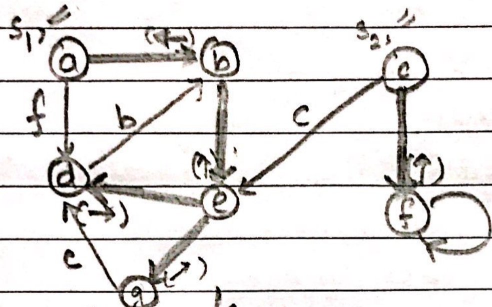

(Not strongly connected, can get from 'b' to 'c')

($\leftarrow$) parent direction

Running time

$O(V+E)$ (Linear time)

- visit each vertex once in DFS alone $O(V)$
- DFS-Visit(..., .., v) called atmost once per vertex v.
  - ↳ pay $|Adj[v]|$

$$O\left(\sum_{v\in V}|Adj[V]|\right) = O(E)$$

$\rightarrow O(V+E)$

Edge classification

- Tree edges (parent pointers) (Bold)
  visit new vertex via that edge.
- forward edges (f)
  goes from a node to a descendant in the tree
- backward edges (b)
  goes from a node to an ancestor in the tree
- Cross edges (or others) (c)
  between two non-ancestor-related subtrees

Lecture 14: Depth-First Search (DFS), Topological Sort

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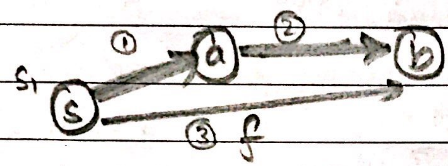
For undirected graphs,
forward and cross edges don't exist.



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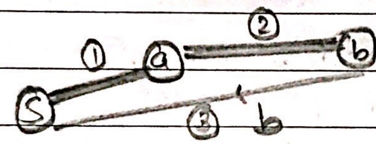

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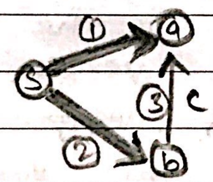

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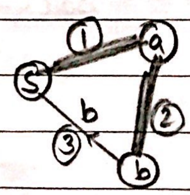

UNDIR.

Useful for cycle detection
and topological sort.

Lecture 14: Depth-First Search (DFS), Topological Sort

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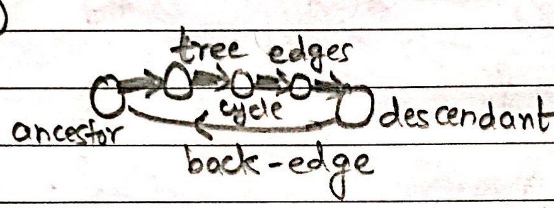
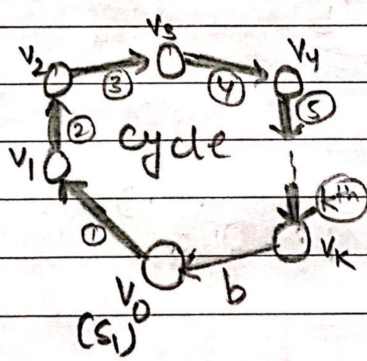
Cycle detection

Thm. G has a cycle $\Leftrightarrow$ DFS has a back edge

Proof. ($\Leftarrow$)



($\Rightarrow$)



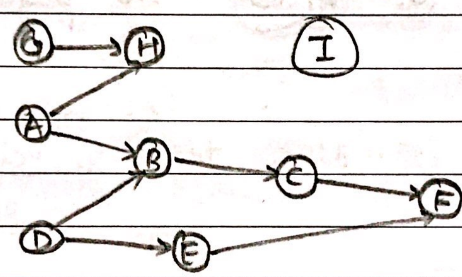
Assuming v_0 is the first vertex in cycle visited by DFS. then $v_k \to v_0$ is back-edge

Lecture 14: Depth-First Search (DFS), Topological Sort

1.05

Job Scheduling

given directed acyclic graph (DAG)
order vertices such that all edges point from lower vertices to higher order



Topological sort (Sorting vertices):
run DFS
output reverse of finishing times of vertices.

Correctness

for any edge $e = (u, v)$
v finishes before u finishes. (To show)

Case 1: u starts before v

$u \to v$ $\Rightarrow$ visit v before u finishes

visit u → start u → visit v → start v → finish v → finish u

Lecture 14: Depth-First Search (DFS), Topological Sort

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Case 2: v starts before u

$u \to v$ cycle # contradiction

else

$u \to v$ No path from v to u then v finishes first (Actually even before u starts)

R-14 Procedurally analysed example of appling DFS on a direct graph and an undirected graph.

Lecture 15: Single-Source Shortest Paths Problem

(107)

L-15

Shortest Path I : Intro

- Weighted graphs
- General approach
- Negative edges
- Optimal substructure

$G(V, E, W)$

vertices → V, edges → E, weight → W $w: E \to \mathbb{R}$

Two algos

- Dijkstra (after Edsger Dijkstra)
  - non-negative weight edges.
  - $O(V \lg V + E) \approx O(E)$ (almost $\because E = O(V^2)$
- Bellman-Ford
  - +/- weight edges
  - $O(VE)$

Lecture 15: Single-Source Shortest Paths Problem

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Path $p = \langle v_0, v_1, \ldots, v_k \rangle$ (sequence)

$(v_i, v_{i+1}) \in E \quad \forall \; 0 \leq i < k$

$$w(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$$

find p with minimum weight.

$A \to B$	1 paths
$A \to C$	2 paths
$A \to E$	4 paths
$A \to G$	8 paths

Paths increase exponentially

Weighted graphs

$V_0 \xrightarrow{p} V_k$ (V_0) is a path from V_0 to V_0 of weight 0.

Shortest path weight from u to v as

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\to} v\} & \exists \text{ any such path} \\ \infty & \text{otherwise} \end{cases}$$

Lecture 15: Single-Source Shortest Paths Problem

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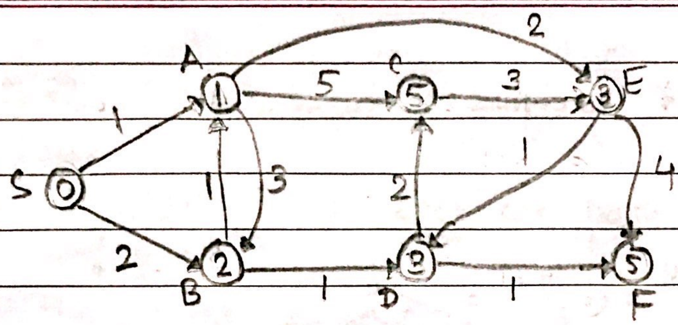


$d(u)$: current weight
$\pi(u)$: predecessor

	S	A	B	C	D	E	F
	0	∞	∞	∞	∞	∞	∞
From S	0	1	2	∞	∞	∞	∞
From A	0	1	2 (<4)	6	∞	3	∞
From C	0	1	2	6	∞	3 (<9)	∞
From E	0	1	2	6	4	3	7
From D	0	1	2	6	4	3	5 (<7)
From F	0	1	2	6	4	3	5
From B	0	1 (<3)	2	6	3 (<4)	3	5
From D	0	1	2	5 (<6)	4	3	5
From C	0	1	2	5	4	3	5

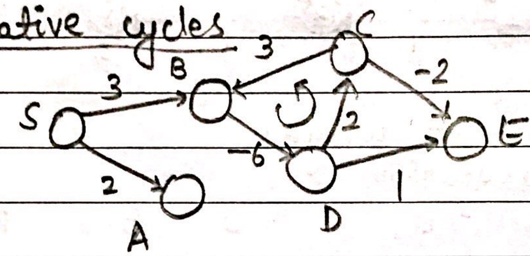
$\delta(S,A) = 1$ $\quad \delta(S,B) = 2$ $\quad \delta(S,C) = 5$

$\delta(S,D) = 4$ $\quad \delta(S,E) = 3$ $\quad \delta(S,F) = 5$

Lecture 15: Single-Source Shortest Paths Problem

Negative weights : reverse tolls, social networking (likes & dislikes)

Negative cycles



$\delta(s,s) = 0 \qquad \delta(s,A) = 2$

$\delta(s,B) = \delta(s,c) = \delta(s,D) = \delta(s,E) = -\infty$

Bellman-Ford detects negative cycles and thus, terminates in finite time.

General structure (no negative cycles)

Initialize for $u \in V$ $\quad d[u] \leftarrow \infty \quad \pi[u] \leftarrow NIL$

$d[s] \leftarrow 0$

Repeat select edge (u,v) [Somehow]

"Relax" edge (u,v) : if $d[v] > d[u] + w(u,v)$

$d[v] = d[u] + w(u,v)$

$\pi[v] \leftarrow u$

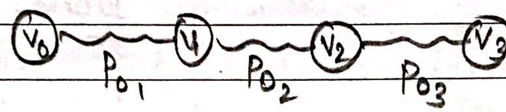
Until all edges have $d[v] \leq d[u] + w(u,v)$

(Tutorial) Recitation 15: Shortest Paths

(11)

Optimal substructure

- Subpaths of a shortest path are shortest paths



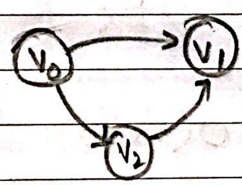
$\{P_{0_1}, P_{0_2}, P_{0_3}\} = SP$

then $\{P_{0_1}\} = SP$

$\{P_{0_2}\} = SP$

$\{P_{0_3}\} = SP$

Triangle inequality (used in L-16)

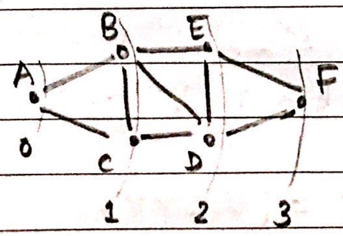


$$\delta(v_0, v_1) \leq \delta(v_0, v_2) + \delta(v_2, v_1)$$

R-15

Shortest path

Thinking of something simple.



BFS

So, now converting a weighted graph to a non-weighted graph

(Tutorial) Recitation 15: Shortest Paths

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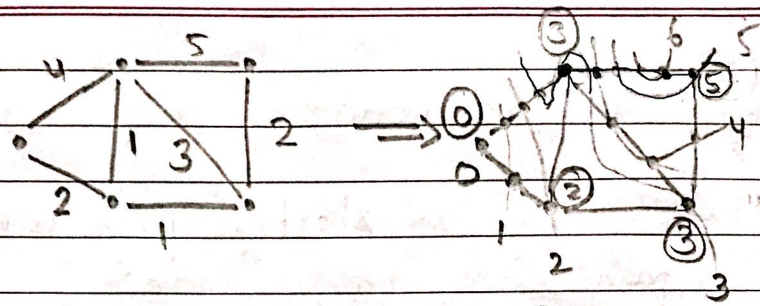
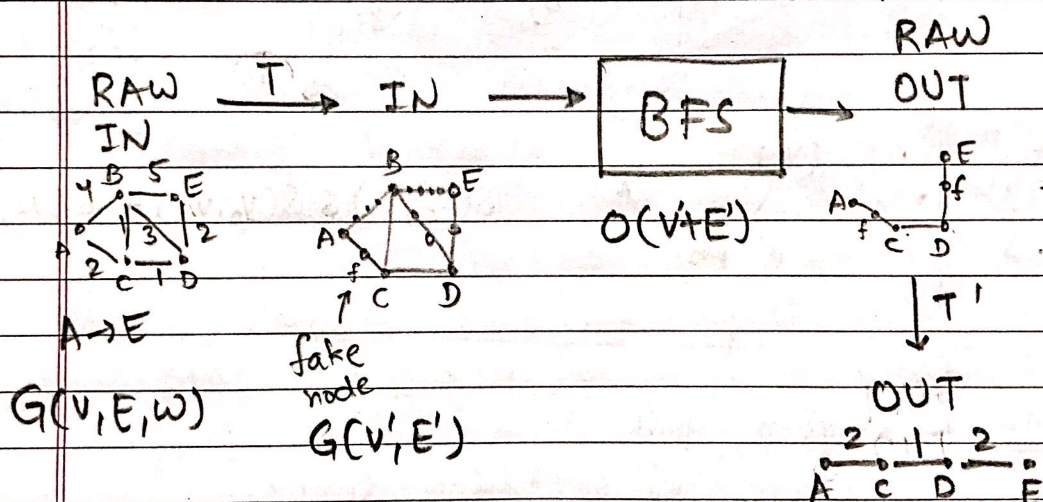


Done.



To find out running time,
we need to know V', E'.

$E' = O(\omega E)$ $\quad V' = (\omega E + V)$

$O(V' + E') = O(\omega E + V)$

For small paths, this is a nice algo.

(Tutorial) Recitation 15: Shortest Paths

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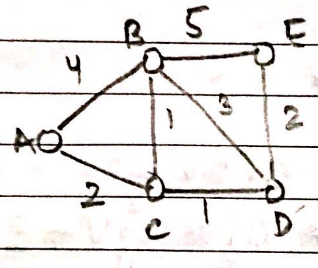


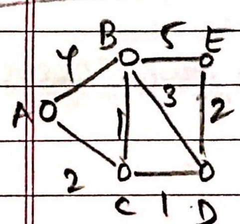
Task

Find the shortest path with odd no. of edges.

Thought 1: Need a transform which stores the data of (# edges travelled % 2) (even or odd)

Thought 2: If you reach X in even #edges then you can reach Y in odd #edges, this way.

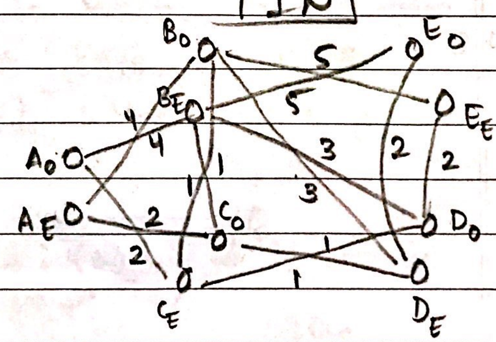
RAW IN



(A → D)

$\xrightarrow{T}$

IN



(Messed-up graph!)

↓

Dijkstra

$(A_E \to C_O \to B_E \to D_O)$ ← Dijkstra

RAW OUT

$(A \to C \to B \to D) \xleftarrow{T'} (A_E \to C_O \to B_E \to D_O)$

2 1 3

OUT

(Tutorial) Recitation 15: Shortest Paths

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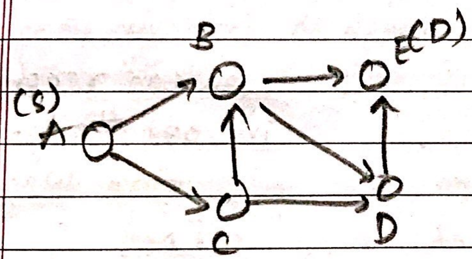
$$E' = 2E \qquad V' = 2V$$

$$O(V'\log V' + E') = O(2V\log 2V + 2E)$$
$$= O(V\log V + E)$$

A hard one (took me 20 min to understand)



Given DIR. Graph

Now each edge (u, v) has two costs: f_c, t_c.

$$u \xrightarrow[t_c(s)\,:\ \text{time cost}]{f_c\,:\ \text{fuel cost}} v$$

depends on start time.

We are given that we can reach any destination in a day and $t_c((u,v), s)$ has resolution of minutes.

$\Rightarrow$ We want least time consuming path from source to destination, if many then one with least fuel consumption

(Tutorial) Recitation 15: Shortest Paths

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In previous ques, we had two states: even & odd # edges.

Here, every vertex will have 1440 [60x24] states (M=1440)

Transform (T)

$v \xrightarrow{(\times M)} (v, t)$

$(u, v) \xrightarrow{(\times M)} \big((u,t), (v, t + t_c((u,v), t)) \big)$

fuel cost $= f_c$

Also, say we are ~~on~~ vertex u at time t. if we were to wait for one minute then we might have a smaller t_c.

So, new edges $((u,t), (u,t+1))$ are added. fuel cost $= 0$.

(You can think these as extension of $(u,u) \xrightarrow{\times M} ((u,t)), (u,t+1))$)

Now, we use Dijkstra's algo to find required path from (source, 0) to (dest., i), incrementing i till we get a finite soln.

(Tutorial) Recitation 15: Shortest Paths

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$i=0 \quad \infty$
$i=1 \quad \infty$ No such path
$i=2 \quad \infty$
$i=3 \quad \infty$
$\vdots$
$i=k-1 \quad \infty$
$i=k \quad P \longrightarrow$ eco-friendly fastest path.
$i=k+1$
$\vdots$

Using this we can get the shortest path taking a certain amount of time. (Feature)

Complexity of Dijkstra
$= O(V \lg V + E)$ (To be proved in L-16)

Here, $V' = V \cdot M$
$E' = E \cdot M + V \cdot M$

Complexity of algo
$= O(V' \lg V' + E')$
$= O(V \cdot M \lg(V \cdot M) + E \cdot M + V \cdot M)$
$= O(V \lg V + E)$
(though, constant is much worse than Dijkstra)

Lecture 16: Dijkstra

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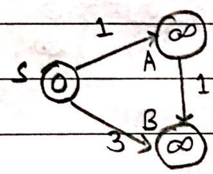
L-16

Shortest Paths II : Dijkstra

- Review
- Shortest paths - DAGs
- SPs - graphs w/o -ve edges
- Dijkstra's algo

Review



$d[v]$: length of current S.P. from source (s) to v.

$d[s] = 0$, $d[A] = \infty$, $d[B] = \infty$

$\delta(s, v)$: length of a shortest path from s to v.

$\pi(v)$: predecessor of v in the S.P. from s to v.

$d[s]=0$, $d[A]=1$, $d[B]=3$

$\pi[A]=s$, $\pi[B]=s$

$d[s]=0$, $d[A]=1$, $d[B]=2$

$\pi[A]=s$, $\pi[B]=A$

S.P. from S to B : $B \leftarrow \underbrace{\pi[B]}_{=A} \leftarrow \underbrace{\pi[\pi[B]]}_{=S} \leftarrow \dots \leftarrow S$

$\Rightarrow B \xleftarrow{1} A \xleftarrow{1} S$

$d[B]=2$ $\quad w(p)=2$

Lecture 16: Dijkstra

Relax(u, v, w) : ← weight

if $d[v] > d[u] + w(u,v)$

$d[v] = d[u] + w(u,v)$

$\pi[v] = u$

Lemma: The relaxation operation maintains the invariant that $d[v] \geq \delta(s,v)\ \forall\ v \in V$.

Proof By Induction on # steps.

Base case: $n=0$ $\quad d[v] = \begin{cases} 0 & v = s \\ \infty & \text{otherwise} \end{cases}$

satisfies

Inductive step:

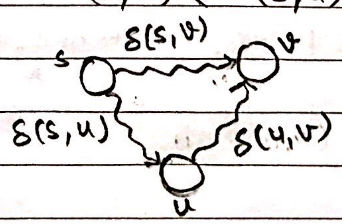
For $n=k$ steps, the lemma satisfies.

in $(k+1)^{th}$ step, we Relax (u,v,w).

We know that $d[u] \geq \delta(s,u)$

By triangle-inequality

$\delta(s,v) \leq \delta(s,u) + \delta(u,v)$



$\delta(s,v) \leq d[u] + \delta(u,v)$

$\delta(u,v) \leq w(u,v)$

$\delta(s,v) \leq d[u] + w(u,v)$

$\therefore\ \delta(s,v) \leq d[v]$ $\quad$ Proved.

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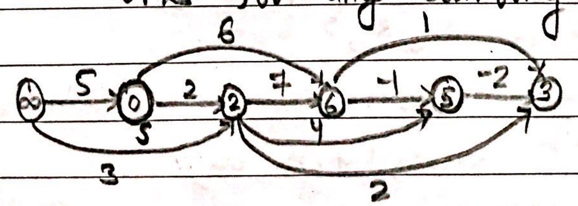
DAGs (Can't have (-ve) cycles)

1. Topologically sort the DAG.
   Path from u to v implies that u is before v in the ordering.
2. One pass over vertices in topologically sorted order relaxing each edge that leaves that vertex.

O(V+E) time

- This works for any starting vertex.



Step 0	∞	0	∞	∞	∞	∞
Step 1	∞	0	2	6	∞	∞
Step 2	∞	0	2	6(<9)	6	4
Step 3	∞	0	2	6	5(<6)	4(<7)
Step 4	∞	0	2	6	5	3(<4)
Step 5	∞	0	2	6	5	3

Lecture 16: Dijkstra

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Demo of Dijkstra
Mechanically validating Dijkstra
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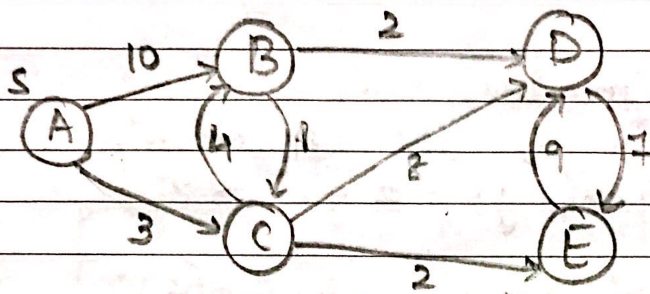
Dijkstra is a greedy algo.
It is easy to understand but not so easy to prove its correctness.

Dijkstra (G, ω, s):
$d[s]=0$ Initialize (G, s) $S \leftarrow \phi$ $Q = V[G]$
while $Q \neq \phi$
$u \leftarrow$ extract-min(Q)
$S \leftarrow S \cup \{u\}$
for each vertex $v \in Adj[u]$
Relax (u, v, ω)

Q is a priority-queue of $d[v]$ $\forall v \in V$.
extract-min(Q) returns vertex not yet evaluted and has minimum d[ ].

Proof in CLRS Textbook (Introduction to Algorithms, 3rd ed.)
(Reference Book 1)

Lecture 16: Dijkstra



d[A]	0	0	0	0	0	0
d[B]	∞	10	7	7	7	7
d[C]	∞	3	3	3	3	3
d[D]	∞	∞	11	11	7	7
d[E]	∞	∞	5	5	5	5
S	{ }	{A}	{A,C}	{A,C,E}	{A,C,E,B}	{A,C,E,B,D}
Q	{A,B,C,D,E}	{B,C,D,E}	{B,D,E}	{B,D}	{D}	{ }

A C E B D

Complexity

$\Theta(V)$ inserts into the priority-queue Q

$\Theta(V)$ extract-min ops.

$\Theta(E)$ Decrease-key / update ops.

Arrays: $\Theta(1)$ for insert, update

$\Theta(V)$ for extract-min

$\rightarrow \Theta(V + V^2 + E) = \Theta(V^2)$ $\quad (E = O(V^2))$

Lecture 16: Dijkstra

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Binary min heap:

$\Theta(\lg V)$ for extract-min, update

$\Rightarrow \Theta(V \lg V + E \lg V)$

Fibonacci heap (not 6.006):

$\Theta(\lg V)$ for extract-min

$\Theta(1)$ ammortized for update

$\Rightarrow \Theta(V \lg V + E)$

Theoretically,

complexity is $\Theta(E \lg V)$ (worst-case)

but practically, $\Theta(V \lg V + E)$ (ammortized).

(Tutorial) Recitation 16: Rubik's Cube, StarCraft Zero

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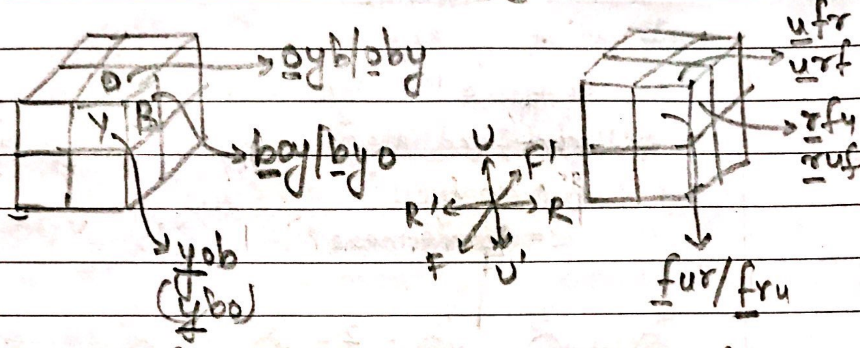
R-16 2×2×2 Rubik's cube



Plastic faces Wireframe

8 cublets

24 faces

Configuration is an array of 24

yob	boy	oyb	...	...
0	1	2	3 ...	23
(fur)	(ruf)	(urf)		

Total configurations: 24!

Six types of moves: $f_c, f_{cc}, r_c, r_{cc}, u_c, u_{cc}$

Finding Starcraft game strategy (Using Dijkstra)

Lecture 17: Bellman-Ford

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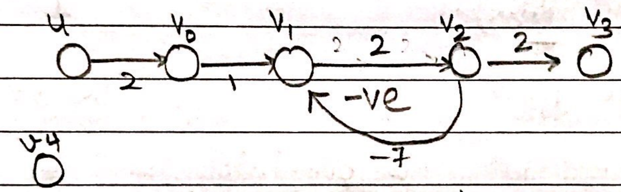
L-17 Shortest Paths III : Bellman-Ford

- Negative cycles
- Generic S.P. Algorithm
- Bellman-Ford Algo
  - Analysis
  - Correctness



$\delta(u, v_1)$, $\delta(u, v_2)$, $\delta(u, v_3)$ not defined $(-\infty)$

$\delta(u, v_0) = 2$

$\delta(u, v_4) = \infty$

This is what is required.

Generic S.P. Algo

Initialize for $v \in V$ $\quad d[v] \leftarrow \infty \quad \pi[v] \leftarrow$ None

$d[s] = 0$

Main loop : Repeat → select edge [somehow]

→ Relax (u, v, W)

until you can't relax anymore.

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Problems

1. Complexity could be exponential time (even for +ve edge weights)
2. Will not terminate if there is a negative weight cycles. reachable from the source

If we assume no cycles, i.e, DAGs, we have a $O(V+E)$ time algo (using topo. sort)
If we assume no -ve weights, we have a $O(V\lg V + E)$ time algo (Dijkstra)

```
Bellman-Ford (G, w, s)
    Initialize()
    for i=1 to i=|V|-1
        for each edge (u,v) ∈ E              O(VE)
            Relax (u, v, w)
    for each edge (u,v) ∈ E                   O(E) Check
        if d[v] > d[u] + w(u,v)
            then report -ve cycle exists.
```

```
Relax(u,v,w):
    if d[v] > d[u] + w(u,v):
        d[v] = d[u] + w(u,v)
        π[v] = u
```

Complexity : $O(VE)$

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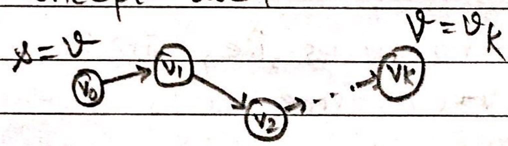
Thm. If $G = (V, E)$ contains no -ve weight cycle then after B-F excutes, $d[v] = \delta(s, v)\ \forall v \in V$

Corollary If a value $d[v]$ fails to converge after $|V|-1$ passes, there exists a -ve weight cycle reachable from s.

Concept used



$v_0, v_1, \dots, v_k$ is a path from s to v

$\Rightarrow\ k \le |V| - 1$

If $k > |V| - 1$ then it is a walk with atleast a cycle.

Proof Let $v \in V$

$p = \langle v_0, v_1, \dots, v_k \rangle \quad v_0 = s \quad$ to $\quad v_k = v$

This path p is a shortest path with min # edges.

No -ve weight cycle $\Rightarrow$ p has no cycles $\Rightarrow k \le |V| - 1$

After one pass thru E, we have $d[v_1] = \delta(s, v_1)$ because we will relax edge (v_0, v_1)

(If not, then p is not a shortest path.)

Lecture 17: Bellman-Ford

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After 2 passes, $d[v_2] = \delta(s, v_2)$ because in 2^{nd} pass, we will relax (v_1, v_2) as well.

After k passes, $d[v_k] = \delta(s, v_k)$

$\because k \leq |V| - 1$

$|V|-1$ passes $\Rightarrow$ all reachable vertices have $d[v] = \delta(s, v)$

Proof (Corollary) After $|V| - 1$ passes, we find an edge that can be relaxed.

$\Rightarrow$ current shortest path from S to some vertex V is not simple.

$\Rightarrow$ Have a repeated vertex

$\Rightarrow$ A repeated cycle

$\Rightarrow$ As the path weight decreases this cycle has -ve weight.

Lecture 17: Bellman-Ford

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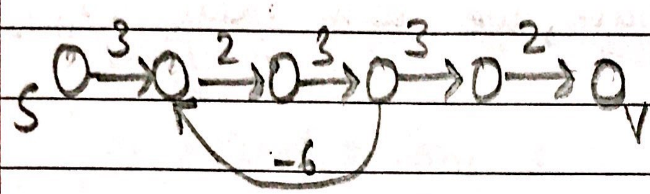
Shortest Simple Path



Shortest simple path is 13.

Bellman-Ford can't solve this.
Turns out this is a NP-hard problem.
i.e. we don't know an algo that is better than exponential time to solve this problem.

Lecture 18: Speeding up Dijkstra

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L-18 **Shortest Paths IV : Speeding up Dijkstra**

- Single source, single target
- Bi-directional search
- Goal-directed search - potentials, landmarks

Unlike previous lectures, we're going to be talking about optimizations that don't change the worst-case, or asymptotic, complexity, but improve empirical, real-life performance. (Performance in average case)

```
1. Dijkstra(G, W, s):
2.    Initialize()  ←  d[s]=0 , d[u≠s] = ∞
3.    Q ← V[G]
4.    while Q ≠ ϕ
5.       do  u ← extract-min(Q)
6.       for each vertex V ∈ Adj[u]
7.           do  Relax(u, V, w)
```

If we are asked to find shortest path from s to t. then we can replace 4. by while($Q = \phi$ && $u \neq t$)

This optimizes the run-time. Worst-case run-time still stays same but average improves.

Lecture 18: Speeding up Dijkstra

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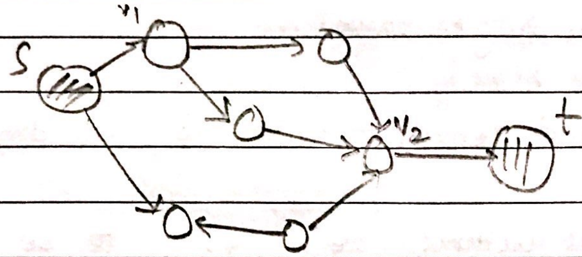
Bi-directional search

$\pi_f[v_1] = s$
$\pi_b[v_2] = t$



- Alternate forward search from s
  backward search from t
  (following edges backward)

$d_f[s] = 0$	$d_b[t] = 0$
$d_f[u \neq s] = \infty$	$d_b[u \neq t] = \infty$
Distances for forward search	Distances for backward search
π_f : normal parent	π_b : reverse parent

Priority Queues: Q_f (forward)
Q_b (backward).

Termination condition

- some vertex u has been processed both in the forward search & backward search i.e. deleted from Q_f & Q_b.

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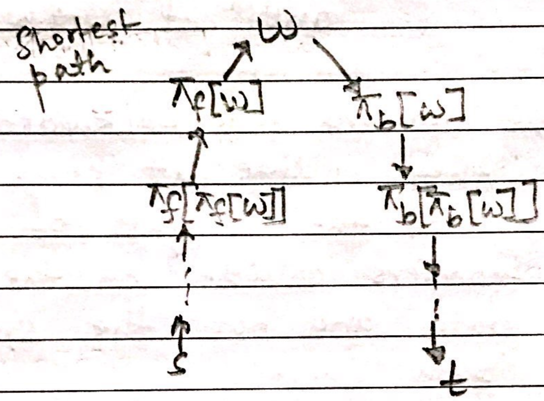
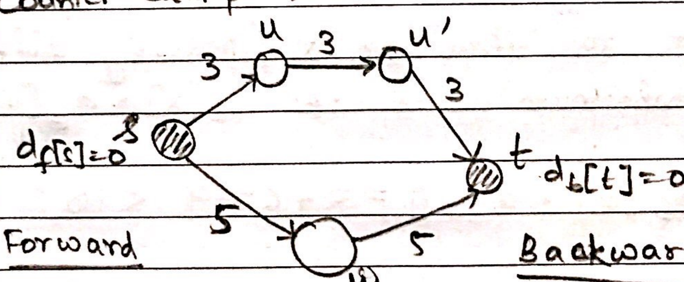
Finding the shortest path

Claim: If w was processed first from both Q_f & Q_b.



This claim is actually false.

Counter-example:



Forward

from s ① $d_f[u]=3$

$d_f[w]=5$

$Q_f=\{\underline{u}, u', w, t\}$

from u ② $d_f[u']=6$

$Q_f=\{u', \underline{w}, t\}$

from w ③ $d_f[t]=10$

$Q_f=\{u', t\}$

Backward

from t ① $d_b[u']=3$

$d_b[w]=5$

$Q_b=\{u, \underline{u'}, w, s\}$

from u' ② $d_b[u]=6$

$Q_b=\{u, \underline{w}, s\}$

from w ③ $d_f[s]=10$

$Q_b=\{u, s\}$

Lecture 18: Speeding up Dijkstra

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w has been processed from Q_b & Q_f

$\pi_b[w] = t \quad \pi_f[w] = s$

But, the shortest path is $s \to u \to u' \to t$ and not $s \to w \to t$.

When the forward and backward frontiers collide, they collide at some vertex, regardless of the weights of the edges.

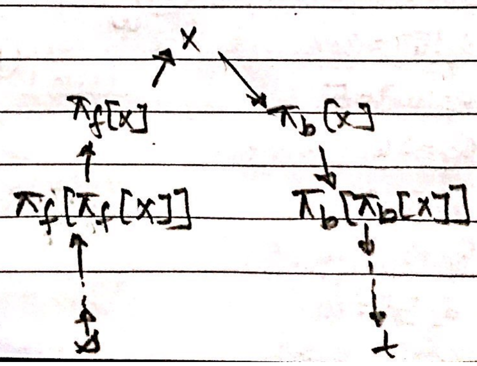
So, the frontiers collide on the shortest length path and not the shortest weight path (here).

So, how do we find the shortest weight path.

We want to find an x (possibly diff from w) with minimum value of $d_f[x] + d_b[x]$

(Here, $d_f[u] + d_b[u] = 3 + 6 = 9 < 10$)

Shortest path:



Lecture 18: Speeding up Dijkstra

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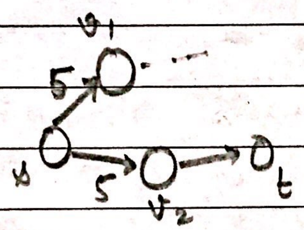
Goal-directed search

- modify edge weights with potential functions.

$$\bar{\omega}(u, v) = \omega(u, v) - \lambda(u) + \lambda(v)$$



we wish to increase the potential value of (s, v_1) (like going upwards) so that Dijkstra's algo chooses v_2 over v_1.

$$\bar{\omega}(p) = \omega(p) - \lambda_t(s) + \lambda_t(t)$$

Landmark $\ell \in V$

precompute $\delta(u, \ell)$ $\forall$ $u \in V$

$$\lambda_t(u) = \delta(u, \ell) - \delta(t, \ell)$$

References

Site References:

- https://www.geeksforgeeks.org/fundamentals-of-algorithms/
  (For Basics of algorithms)
- https://www.geeksforgeeks.org/graph-data-structure-and-algorithms/
  (For concepts of graph algorithms)
- http://cp-algorithms.com/
  (Problems on graphs algorithms)
- https://www.youtube.com/watch?v=09_LlHjoEiY&t=7s
  (Summary of graph algorithms)

Book References:

- Introduction to Algorithms - Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein - Third Edition (Textbook for MIT 6.006)
- Introduction to Graph Theory (2nd Edition) - Douglas B. West (For Mathematical concepts of Graph Theory)
- GRAPH THEORY - Keijo Ruohonen (For Mathematical concepts of Graph Theory)
- Algorithm Design: Parallel and Sequential - Xiuquan Lv (Algorithm Design)
- Competitive Programmer's Handbook - Antti Laaksonen (For competitive programming)
- NUS CS3233 - Competitive Programming (For competitive programming)

MIT Courses (which are related):

- MIT 6.042J/18.062J - Mathematics for Computer science
- MIT 6.006 - Introduction to Algorithms
- MIT 6.046J/18.410J - Design and Analysis of Algorithms
- MIT 6.854J/18.415J - Advanced Algorithms
- MIT 6.045J/18.400J - Automata, Computability, and Complexity
- MIT 18.404J/6.840J - Theory of Computation
- MIT 6.079/6.975 - Introduction to Convex Optimization
- MIT 18.315 - Combinatorial Theory: Introduction to Graph Theory, Extremal and Enumerative Combinatorics

I would be studying these courses as well and my notes I make would be available on Github (ID: devansh-dvj)