Gujarat State Fertilizers and Chemicals Limited University

Vadodara, Gujarat

Topic: limits in two variable

MATHS ASSIGNMENT

ON

BOOK REVIEW

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SCHOOL OF TECHNOLOGY

B.TECH COMPUTER SCIENCE AND ENGINEERING

STUDENT DETAILS: -

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NAME :- A Course in Multivariable Calculus and Analysis

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About the author:

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- **Balmohan V. Limaye** is Professor Emeritus in the Department of Mathematics at the Indian Institute of Technology (IIT) Bombay. He is the author of several research monographs and textbooks, including *Linear Functional Analysis for Scientists and Engineers* (Springer, 2016). He worked at IIT Bombay for more than 40 years and has twice received the Award for Excellence in Teaching from IIT Bombay. His research interests include Banach algebras, approximation theory, numerical-functional analysis, and linear algebra.

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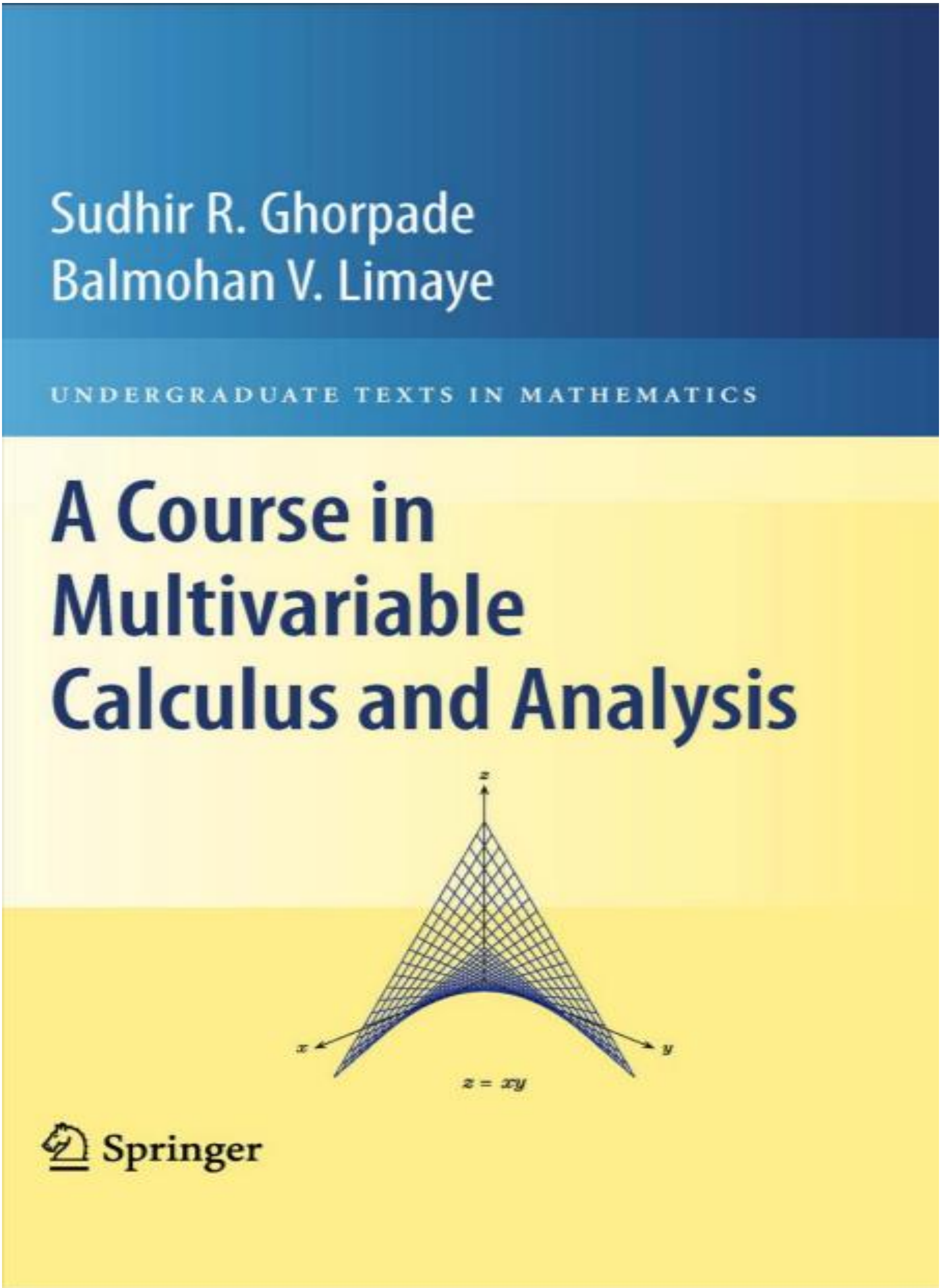
About this book:

Calculus of real-valued functions of several real variables, also known as multivariable calculus, is a rich and fascinating subject. On the one hand, it seeks to extend eminently useful and immensely successful notions in one-variable calculus such as limit, continuity, derivative, and integral to “higher dimensions.” On the other hand, the fact that there is much more room to move n about in the n-space R than on the real line R brings to the fore deeper geometric and topological notions that play a significant role in the study of functions of two or more variables.

In this book **courses in multivariable calculus**, the author had explained all concepts at an undergraduate level and even at an advanced level with the unenviable task of conveying the multifarious and multifaceted aspects of multivariable calculus to a student in the span of just about a semester or two. Ambitious courses and teachers would try to give some idea of the general Stokes’s theorem for differential forms on manifolds as a grand generalization of the fundamental theorem of calculus and prove the change of variables formula in all its glory.

The author had tried to do justice to important results such as the implicit function theorem, which really has no counterpart in one-variable calculus. Most courses would require the student to develop a passing acquaintance with the theorems of Green, Gauss, and Stokes, never mind the tricky questions about orientability, simple connectedness, etc.

In this book, there are a total of 7 chapters with a size of 486 pages.



Index of this book is given below:

Contents

Note: From this book, we are going to review a particular topic named limit and exactly going to focus on limits in two variables, which is in the 2nd chapter of this book.

CHAPTER-10:

Sequences, Continuity, and Limits

- This chapter contains 43-82 pages of this book.
- It contains 3 sessions.

- Session 1 (sequence) covers topics like:
  - Sequences in R^2
  - Subsequences and Cauchy Sequence
  - closure, boundary, and interior

- Session 2 (continuity) covers the topics like:
  - Continuity; Composition of Continuous Function
  - Piecing Continuous Functions on Overlapping Subsets
  - Characterizations of Continuity
  - Continuity and Boundedness
  - Continuity and Monotonicity
  - Continuity, Bounded Variation and Bounded Bivariation
  - Continuity and Convexity
  - Continuity and Intermediate Value Property
  - Uniform Continuity
  - Implicit Function Theory

❖ Session 3 (limits) covers the topics like:

[Note: here author had explained a particular theory and at the end of the theory its example are given to solve and practice that particular theory.]

– Limits in 2 variables.

- If a limit of f as (x, y) tends to (x0, y0) exists, then it is unique. With this in view, if f(x, y) → ℓ as (x, y) → (x0, y0), then we may refer to ℓ as the limit of f(x, y) as (x, y) tends to (x0, y0)

  and write

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell.}$$

- Example 2.47 is given to solve which is related to the above concept.
  - (i) Consider f: R2 → R defined by f(0, 0) := 1 and f(x, y) := sin(XY) for (X, Y) ∈ R2 \ {(0, 0)}. Then the limit of f as (x, y) tends to (0, 0) exists and is equal to 0. Indeed, if (Xn, Yn) is a sequence in R2 \ {(0, 0)} such that (Xn, Yn) → (0, 0), then XnYn → 0, and by the continuity of the sine function, sin(XnYn) → sin 0 = 0, that is, f(Xn, Yn) →0
  - (ii) Consider f : R2 → R defined by f(X, Y) = (X + Y if X 6= Y, 1 if X = Y. Then the limit of f as (X, Y) tends to (0, 0) does not exist. This can be seen by considering two sequences approaching (0, 0), one along the line Y = X and another staying away from this line. For example, if (Xn, Yn) := (1/n, 1/n) and (Un, Vn) := (−1/n, 1/n) for n ∈ N, then (Xn, Yn) and (Un, Vn) are sequences in R2 \{(0, 0)} converging to (0, 0), but f(Xn, Yn) → 1 and f(Un, Vn) → 0.

- (iii) Consider $f : R2 \setminus \{(0, 0)\} \to R$ given by $f(X, Y) = XY/(X2 + Y2)$ for $(X, Y) \in R2$, (X, Y) 6= (0, 0). Then the limit of f as (X, Y) tends to (0, 0) does not exist. This can also be seen by considering two sequences approaching (0, 0), along different lines through the origin. For example, if (Xn, Yn) := (1/n, 1/n) and (Un, Vn) := (1/n, 2/n) for $n \in N$, then (Xn, Yn) and (Un, Vn) are sequences in $R2 \setminus\{(0, 0)\}$ converging to (0, 0), but $f(Xn, Yn) \to$ 1 2 and $f(Un, Vn) \to 2$

– Limits and continuity.

- The concepts of continuity and limit are related in a similar way as in the case of functions of one variable.
- Let $D \subseteq R2$ and let $(X0, Y0) \in R2$ be an interior point of D, that is, $Sr(X0, Y0) \subseteq D$ for some $r > 0$.

  Let $f : D \to R$ be any function. Then f is continuous at (X0, Y0) if and only if the limit of f as (X, Y) tends to (X0, Y0) exists and is equal to f(X0, Y0).

  Proof. Assume that f is continuous at (X0, Y0). Let (Xn, Yn) be any sequence in D such that $(Xn, Yn) \to (X0, Y0)$. By the continuity of f at (X0, Y0), we see that $f(Xn, Yn) \to f(X0, Y0)$.

  It follows that the limit of f as (X, Y) tends to (X0, Y0) exists and is equal to f(X0, Y0).

  To prove the converse, assume that the limit of f as (X, Y) tends to (X0, Y0) exists and is equal to f(X0, Y0). Let (Xn, Yn) be any sequence in D such that $(Xn, Yn) \to (X0, Y0)$. If there is n0 $\in N$ such that (Xn, Yn) = (X0, Y0) for all $n \geq n0$, then it is clear that $f(Xn, Yn) \to f(X0, Y0)$.

  Otherwise, there are positive integers n1, n2, . . . such that $n1 < n2 < \cdots$ and $\{n \in N : (Xn, Yn)$ 6= $(X0, Y0)\} = \{nk : k \in N\}$. Now, (Xnk, Ynk) is a sequence in D\{(X0, Y0)} that converges to (X0, Y0), and therefore f(Xnk, Ynk) → f(x0, y0).

Since f(Xn, Yn) = f(X0, Y0) for all n ∈ N \ {nk : k ∈ N}, it follows that f(Xn, Yn) → f(X0, Y0). Hence f is continuous at (X0, Y0).

$$F(x, y) := \begin{cases} f(x, y) & \textit{if } (x, y) \in D \setminus \{(x_0, y_0)\}, \\ \ell & \textit{if } (x, y) = (x_0, y_0). \end{cases}$$

Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \textit{ exists and is equal to } \ell \iff F \textit{ is continuous at } (x_0, y_0).$$

- Example 2.50 is given to solve which is related to the above concept.
  - (i) In view of Proposition 2.48 and Example 2.16 (i), we see that every rational function has a limit wherever it is defined, that is, if p(X, Y) and q(X, Y) are polynomials in two variables and if (X0, Y0) ∈ R2 is such that q(X0, Y0) 6= 0, then

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} \frac{p(x, y)}{q(x, y)} = \frac{p(x_0, y_0)}{q(x_0, y_0)}.}$$

    On the other hand, if q(X0, Y0) = 0, then the limit of p(X, Y)/q(X, Y) may not exist, in general. For example, for any m, k ∈ N, the rational function f(X, Y) := Xm/Yk does not have a limit as (X, Y) tends to (0, 0). To see this, it suffices to approach (0, 0) along the parametric curve given by (X(t), Y(t)) = (αtk, βtm), t ∈ [−1, 1], where α, β are any nonzero constants. For example, if (Xn, Yn) := (1/nk, 1/nm) and (Un, Vn) := (2/nk, 1/nm) for n ∈ N, then (Xn, Yn) and (Un, Vn) are sequences in R2 \ {(0, 0)} converging to (0, 0), but f(Xn, Yn) → 1 and f(Un, Vn) →2^m

(ii) Consider f : R2 \ {(0, 0)} → R defined by f(X, Y) = X2Y/(X2 + Y2). Then we see that the limit of f(x, y) as (x, y) tends to (0, 0) exists and is equal to 0.

- Some other basic properties of limits of real-valued functions of two variables can be deduced from the corresponding properties of continuous functions

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \quad and \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = m.$$

- If there is δ > 0 with δ ≤ r such that f(X, Y) ≤ g(X, Y) for all (X, Y) in Sδ(X0, Y0)\{(X0, Y0)}, then ℓ ≤ m. Conversely, if ℓ < m, then there is δ > 0 such that δ ≤ r and f(X, Y) < g(X, Y) for all (X, Y) ∈ Sδ(X0, Y0)\ {(X0, Y0)}.

- If f(X, Y) ≥ 0 for all (X, Y) ∈ D, then ℓ ≥ 0 and for each k ∈ N, the limit of f 1/k : D → R as (X, Y) tends to (X0, Y0) exists, and is equal to ℓ1/k.

- (Cauchy Criterion for Limits of Functions)

Suppose D ⊆ R2 and (X0, Y0) ∈ R2 are such that D contains Sr(X0, Y0)\ {(X0, Y0)} for some r > 0. Let f : D → R be a function.
Then lim(X,Y)→(X0,Y0) f(X, Y) exists if and only if for every ϱ > 0, there is δ > 0 such that

$$(x,y),(u,v) \in D \cap \mathbb{S}_\delta(x_0,y_0) \setminus \{(x_0,y_0)\} \Longrightarrow |f(x,y) - f(u,v)| < \epsilon.$$

- For every ϱ > 0, there is δ > 0 such that

$$(x,y) \in D \cap \mathbb{S}_\delta(x_0,y_0) \;\; and \;\; (x,y) \neq (x_0,y_0) \Longrightarrow |f(x,y) - \ell| < \epsilon.$$

- [Sandwich Theorem]

  If ℓ = m and if there is h : D → R such that f(X, Y) ≤ h(X, Y) ≤ g(X, Y) for all (X, Y) ∈ D, then the limit of h as (X, Y) tends to (X0, Y0) exists and is equal to ℓ.

– Limits from a quadrant.

- An analog of the notion of left(-hand) or right(-hand) limits for functions of one variable is given by limits from any one of the four quadrants for functions of two variables. These may be defined as follows. Let D ⊆ R2 and (x0, y0) ∈ R2 be such that (x0, x0 + r) × (y0, y0 + r) ⊆ D for some r > 0. Given a function f : D → R, we say that a limit of f from the first quadrant as (x, y) tends to (x0, y0) exists if there is a real number ℓ such that whenever (xn, yn) is a sequence in D \ {(x0, y0)} satisfying (xn, yn) ≥ (x0, y0) for all n ∈ N and (xn, yn) → (x0, y0), we have f(xn, yn) → ℓ. It is easy to see that if such a limit exists, then it is unique. In this case, we write

$$f(x,y) \to \ell \text{ as } (x,y) \to (x_0^+, y_0^+) \quad \text{or} \quad \lim_{(x,y)\to(x_0^+,y_0^+)} f(x,y) = \ell.$$

  Similarly, we can define limits of f from the second, the third, and the fourth quadrants. Obvious analogs of the above notation are then used.

- straightforward analogue:

$$F_1(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D_1 \setminus \{(x_0,y_0)\}, \\ \ell & \text{if } (x,y) = (x_0,y_0). \end{cases}$$

Then

$$\lim_{(x,y)\to(x_0^+,y_0^+)} f(x,y) \text{ exists and is equal to } \ell \iff F_1 \text{ is continuous at } (x_0,y_0).$$

- Let D ⊆ R2 and (x0, y0) ∈ R2 be such that D contains Sr(x0, y0) \ {(x0, y0)} for some r > 0. Let f : D → R be a function and let ℓ ∈ R.

  Then lim(x,y)→(x0,y0) f(x, y) = ℓ if and only if lim(x,y)→(x+ 0 ,y+ 0) f(x, y), lim(x,y)→(x− 0 ,y+ 0) f(x, y), lim(x,y)→(x− 0 ,y− 0) f(x, y), and lim(x,y)→(x+ 0 ,y− 0) f(x, y) exist and are all equal to ℓ.

  If, in addition, (x0, y0) ∈ D, then f is continuous at (x0, y0) if and only if the limit of f from each of the four quadrants as (x, y) tends to (x0, y0) exists and they are all equal to f(x0, y0).

– Approaching Infinity.

- Let D ⊆ R2 be such that D contains a product of semi-infinite open intervals of the form (a, ∞) × (c, ∞), where a, c ∈ R. Given a function f : D → R, we 2.3 Limits 73 say that a limit of f as (x, y) tends to (∞, ∞) exists if there is a real number ℓ satisfying the following property:

$$\boxed{((x_n, y_n)) \text{ any sequence in } D \text{ with } x_n \to \infty \text{ and } y_n \to \infty \implies f(x_n, y_n) \to \ell.}$$

- Let D ⊆ R2 be such that D ⊇ (a, ∞) × (c, ∞) for some a, c ∈ R, and let f : D → R be a function. Then lim(x,y)→(∞,∞) f(x, y) exists if and only if there is ℓ ∈ R satisfying the following ϱ-(α, β) condition: For every ϱ > 0, there are α, β ∈ R such that:

$$\boxed{(x, y) \in D \ \textit{with} \ (x, y) \geq (\alpha, \beta) \implies |f(x, y) - \ell| < \epsilon.}$$

- Let D ⊆ R2 and (x0, y0) ∈ R2 be such that D contains Sr(x0, y0)\ {(x0, y0)} for some r > 0 and let f : D → R be any function. Then f(x, y) → ∞ as (x, y) → (x0, y0) if and only if the following α-δ condition holds: For every α ∈ R, there is δ > 0 such that:

$$\boxed{(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \ \textit{and} \ (x, y) \neq (x_0, y_0) \implies f(x, y) > \alpha.}$$

Likewise, f(x, y) → −∞ as (x, y) → (x0, y0) if and only if the following β-δ condition holds: For every β ∈ R, there is δ > 0 such that:

$$\boxed{(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \ \ and\ (x, y) \neq (x_0, y_0) \Longrightarrow f(x, y) < \beta.}$$

- Let a, b, c, d ∈ RU {−∞, ∞} with a < b and c < d be such that either a, c ∈ R or a = c = −∞, and either b, d ∈ R or b = d = ∞. Let f : (a, b) × (c, d) → R be a monotonically increasing function. Then (i) lim(x,y)→(b−, d−) f(x, y) exists if and only if f is bounded above; in this case, lim(x,y)→(b−, d−) f(x, y) = sup{f(x, y) : (x, y) ∈ (a, b) × (c, d)}. If f is not bounded above, then f(x, y) → ∞ as (x, y) → (b−, d− 2.3
- lim(x,y)→(a+, c+) f(x, y) exists if and only if f is bounded below; in this case, lim(x,y)→(a+, c+) f(x, y) = inf{f(x, y) : (x, y) ∈ (a, b) × (c, d)}. If f is not bounded below, then f(x, y) → −∞ as (x, y) → (a+, c+).

Note:

❖ After the theory, 38 questions are given by the author to practice this theory questions in terms of exercise.

<u>THANK YOU</u>