"Stylized facts" are general characteristics or patterns that we often observe in stock price data.

If this stock doesn't pay dividends between these two times, we can calculate something called the "one-period return." This is just the change in the stock price from yesterday to today divided by the price yesterday.

We can also calculate the "logarithmic return" or "continuously compounded return," which is a fancy way of measuring how the stock price changes over time. It's calculated by taking the natural logarithm of today's price divided by yesterday's price.

The "k-period log return" measures the change in the stock price over a longer period, not just one day. It's the sum of the individual daily returns over that period. In other words, if you want to know how the stock performed over a week (k=7), you add up the daily returns for those seven days.

Low-frequency returns, in the context of financial markets and time series analysis, refer to returns that are observed over longer time intervals or periods. These returns are typically calculated and analysed at intervals greater than a day, such as weekly, monthly, quarterly, or annually.

High-frequency, in the context of financial markets and trading, refers to the practice of analysing and trading assets over very short time intervals, such as seconds, minutes, or hours. It involves processing and reacting to a large volume of data and market information in real-time. High-frequency trading (HFT) is a form of trading that takes advantage of these short time intervals to make quick and frequent trades.

In simple words, this passage is talking about interesting things that happen when we look at the history of stock prices over time, especially when we're looking at returns (how much money you make or lose from owning stocks).

1. Non-Normality: The first thing is that these returns don't follow a nice, even pattern. They often have more extreme ups and downs than what we'd expect based on a simple, smooth curve.

2. Small Autocorrelations (relationship between a financial variable (in this case, price changes) and its past values): When we look at how today's return is related to yesterday's return, there isn't a strong connection. They don't move in lockstep (synchronized moving).

3. Strong Autocorrelations of Volatility: However, when we look at how uncertain or volatile these returns are (that's the "volatility"), we see a stronger connection. If things are uncertain one day, they tend to stay uncertain for a while.

4. Asymmetry in Returns: When stocks go up a lot, the response isn't as strong as when they go down a lot. It's like there's a bigger reaction to bad news than to good news.

5. Changes in Volatility: Lastly, when important things happen, like big financial news or company earnings reports, the level of uncertainty in the stock market can change a lot. Sometimes it's calm, and other times it's very jumpy.

So, in essence, the passage is explaining that when we look at how stocks behave over time, we see some interesting and not-so-predictable patterns in how they go up and down, how they react to good and bad news, and how they get more or less uncertain. These patterns are important for understanding the behaviour of financial markets.

In high-frequency financial data, such as stock or futures price movements, prices are quoted in specific discrete units or "ticks." Because of this discrete nature, the observed price changes can only take a limited number of distinct values, often including zero. This tendency for prices to cluster around certain values is known as "price clustering."

In addition to the discrete nature of transaction prices, they found other interesting patterns in the data. These included intraday periodicity and seasonality in trading intensity. "Intraday periodicity" and "seasonality" are patterns and cycles that occur within a single trading day in financial markets.

Intraday Periodicity: This refers to regular and recurring patterns that unfold within the same trading day. Intraday periodicity often occurs due to the structure of the trading day and the behavior of market participants. Some common examples of intraday periodicity include:

Opening and Closing Routines: There's often heightened activity and volatility at the beginning and end of the trading day. This is because investors and traders tend to react to overnight news, corporate announcements, or economic data at the opening, and then they make closing trades or adjustments at the end.

Lunchtime Lull: Around midday, there's frequently a decrease in trading activity and volume, often referred to as a "lunchtime lull." This can be attributed to traders taking breaks, reduced news flow during the lunch hour, and lower liquidity.

Seasonality: Seasonality refers to recurring patterns or trends that take place over longer periods, such as weeks, months, or even years. In the context of intraday trading, seasonality doesn't relate to calendar seasons but instead to regular market behaviour on specific days of the week or specific months. Some examples of seasonality in intraday trading include:

In stock exchanges, this seasonality is referred to as "U-shaped" because trading tends to be busiest at the opening and closing of trading hours, with a lull around lunchtime. This periodicity also affects the durations between transactions.

The Eurodollar market displayed a similar U-shaped daily pattern, but with differences, as the first half of the trading day was more active when European markets were open, and the second half was quieter after European markets closed. Furthermore, Niederho
er and Osborne identified negative lag 1 autocorrelation in price changes, meaning that there was a tendency for consecutive price changes to move in opposite directions. For example, if the price increased in the previous transaction, there is a tendency for it to decrease in the next transaction, and vice versa. This finding is significant because it suggests that price changes in the market are not purely random or independent from one trade to the next. Instead, there seems to be a certain degree of "mean reversion" or "reversal" in the short term. In other words, if the price has recently gone up, there's a tendency for it to correct or move downward in the next transaction, and if it has recently gone down, there's a tendency for it to recover or move upward in the next transaction.

Brownian motion models for speculative prices

The scientific analysis of stock price data traces back to the pioneering work of Regnault in 1863 and Bachelier in 1900. Both researchers utilized quote data from the Paris Exchange to investigate the mean deviation of price changes. What they discovered was that this deviation was proportional to the square root of the time interval, expressed as $E[|P_{t+1} - P_t|] \propto \sqrt{t}$, where Pt represents the price at time 't'. While Regnault's study was primarily empirical and suggested this square-root relationship, Bachelier, in his 1900 Ph.D. thesis, derived it from a heuristic version of the functional central limit theorem, ultimately leading to the concept of Brownian motion.

Bachelier, in his 1900 and 1938 writings, also introduced the idea that price changes (Di) could be modelled as independent symmetric random variables (are both statistically independent of each other and follow a symmetric probability distribution(A symmetric probability distribution is a type of probability distribution where the probabilities of events on one side of the distribution are mirrored or equal to the probabilities on the other side, relative to a central point within the distribution)). This model implied an equal probability that transaction prices would move up or down by 's' ticks. When the continuous-time Brownian motion 'Bt' was discretized at evenly spaced times, it resulted in a random walk with normal increments $N(0, \sigma^2)$.

Continuous-Time Brownian Motion ('Bt'): Brownian motion, often denoted as 'Bt,' is a continuous-time stochastic process that models the random and continuous movement of a particle or the price of a financial asset.

A random walk is a simple mathematical model often used to describe the behaviour of asset prices. In a random walk, the future value is determined by adding a random increment to the current value. The key characteristic is that the increments are typically assumed to be independent and identically distributed (i.i.d.).

When you discretize the continuous-time Brownian motion 'Bt,' the increments between consecutive data points in the discretized process are assumed to be normally distributed. The '$N(0, \sigma^2)$' notation indicates that these increments have a mean (expected value) of 0 and a variance of σ^2.

This means that the increments can be positive or negative, with a higher variance leading to larger fluctuations.

In 1973, Samuelson reviewed earlier empirical studies conducted by Cowles in 1933 and 1944, and Kendall in 1953. These studies supported the notion of a random walk model for 'speculative prices,'. Samuelson noted that, as measured by the absence of significant serial correlation, 18 common-stock price series from English exchanges appeared to resemble random walks.

However, Samuelson also raised concerns about Bachelier's model, arguing that while it was seminal, it led to impractical outcomes. He advocated for an alternative model based on geometric Brownian motion for stock prices, and, consequently, an independently and identically distributed (i.i.d.) model of log returns for low-frequency data.