FEDERAL UNIVERSITY OF TECHNOLOGY, MINNA

CENTRE FOR OPEN DISTANCE AND e-LEARNING CODeL

MAT 111

ALGEBRA AND NUMBER THEORY

Courseware Draft - Comments are welcome
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MAT 111: ALGEBRA AND NUMBER

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STUDY GUIDE

Introduction

MAT 111: Algebra and Number theory is a 3 credit unit course made of 10 units. It is a course compulsory for students in various disciplines, which is aimed at giving them a foundation which is not devoid of the knowledge of mathematical calculations. In this course we introduce topics like the set theory, mapping and function, number systems, inequalities among others.

Course Guide

The course guide introduces to you what you will learn in this course and how to make the best use of the material. It brings to your notice the general guidelines on how to navigate through the course and on the expected actions you have to take for you to complete this course successfully. Also, the guide will hint you on how to respond to your Self Assessment Question and Tutor-Marked Assignments.

What you will learn in this Course

The main purpose of this course is to introduce you to concepts relating to computer architecture and provide you with the indepth knowledge of the internal components of the computer hardware components and their basic functions. This we intend to achieve through the following:

Course Objectives

Certain objectives have been set out to ensure that the course achieves its aims. Apart from the course objectives, every unit of this course has set objectives. In the course of the study, you will need to confirm, at the end of each unit, if you have met the objectives set at the beginning of each unit. By the end of this course you should be able to:

1. Explain the meaning of a set

2. Perform basic operations on sets

3. Solve real- life problems using Venn diagram.

4. Use the concepts of mappings in solving problems;

5. Identify the important properties of mappings (functions);

6. Find product of mapping;

7.Find inverses of functions.

Working through this Course

In order to have a thorough understanding of the course units, you will need to read and understand the contents, practice the steps by designing and implementing a mini mini system for your department, and be committed to learning and implementing your knowledge.

This course is designed to cover approximately nineteen weeks, and it will require your devoted attention. You should do the exercises in the Tutor-Marked Assignments and submit to your tutors.

Course Materials

The major components of the course are:

1. Course Guide
2. Study Units
3. Text Books
4. Assignment File
5. Presentation Schedule

Study Units

There are 10 study units in this course. They are:

Recommended Texts

Odili G. A. (2000), Algebra for Colleges and Universities: An Integrated Approach. Anachuna Educational Books.

Akinnola V. J., Mohammed J., Aiyesimi Y. M., Akinwande N. I., and Ogunfiditimi E. O.(2005), College Algebra and Trigonometry. Y- Books.

Tuttuh-Adegun, Sivasubramniam S., Adekoge R. (2003) Further Mathematics Project 2. NPS Educational.

Tuttuh-Adegun, Sivasubramniam S., Adekoge R. (2008) Further Mathematics Project 1. NPS Educational.

Assignment File

The assignment file will be given to you in due course. In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain for these assignments will count towards the final mark for the course. Altogether, there are tutor marked assignments for this course.

Presentation Schedule

The presentation schedule included in this course guide provides you with important dates for completion of each tutor marked assignment. You should therefore endeavour to meet the deadlines.

Assessment

There are two aspects to the assessment of this course. First, there are tutor marked assignments; and second, the written examination. Therefore, you are expected to take note of the facts, information and problem solving gathered during the course. The tutor marked assignments must

be submitted to your tutor for formal assessment, in accordance to the deadline given. The work submitted will count for 40% of your total course mark.
At the end of the course, you will need to sit for a final written examination. This examination will account for 60% of your total score.

Tutor-Marked Assignments (TMAs)
There are TMAs in this course. You need to submit all the TMAs. The best 10 will therefore be counted. When you have completed each assignment, send them to your tutor as soon as possible and make certain that it gets to your tutor on or before the stipulated deadline. If for any reason you cannot complete your assignment on time, contact your tutor before the assignment is due to discuss the possibility of extension. Extension will not be granted after the deadline, unless on extraordinary cases.

Final Examination and Grading
The final examination for CPT 214 will be of last for a period of 2 hours and have a value of 60% of the total course grade. The examination will consist of questions which reflect the self assessment exercise and tutor marked assignments that you have previously encountered. Furthermore, all areas of the course will be examined. It would be better to use the time between finishing the last unit and sitting for the examination, to revise the entire course. You might find it useful to review your TMAs and comment on them before the examination. The final examination covers information from all parts of the course.

The following are practical strategies for working through this course
1. Read the course guide thoroughly
2. Organize a study schedule. Refer to the course overview for more details. Note the time you are expected to spend on each unit and how the assignment relates to the units. Important details, e.g. details of your tutorials and the date of the first day of the semester are available. You need to gather together all these information in one place such as a diary, a wall chart calendar or an organizer. Whatever method you choose, you should decide on and write in your own dates for working on each unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course works. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the table of content at the beginning of each unit. You will almost always need both the study unit you are working on and one of the materials recommended for further readings, on your desk at the same time.

6. Work through the unit, the content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit, you will be encouraged to read from your set books
7. Keep in mind that you will learn a lot by doing all your assignments carefully. They have been designed to help you meet the objectives of the course and will help you pass the examination.
8. Review the objectives of each study unit to confirm that you have achieved them.
If you are not certain about any of the objectives, review the study material and consult your tutor.
9. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you can keep yourself on schedule.
10. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor marked assignment form and also written on the assignment. Consult you tutor as soon as possible if you have any questions or problems.
11. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this course guide).

Tutors and Tutorials

There are 8 hours of tutorial provided in support of this course. You will be notified of the dates, time and location together with the name and phone number of your tutor as soon as you are allocated a tutorial group. Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter and provide assistance to you during the course. You must mail your tutor marked assignment to your tutor well before the due date. At least two working days are required for this purpose. They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail or discussion board if you need help. The following might be circumstances in which you would find help necessary: contact your tutor if:

• You do not understand any part of the study units or the assigned readings.
• You have difficulty with the self test or exercise.
• You have questions or problems with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.

You should endeavour to attend the tutorials. This is the only opportunity to have face to face contact with your tutor and ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit from the course tutorials, have some questions handy before attending them. You will learn a lot from participating actively in discussions.

GOODLUCK!

Table of Contents

COURSE DEVELOPMENT TEAM....................i

STUDY GUIDE.................... **Error! Bookmark not defined.**

UNIT 1: SET THEORY....................1

UNIT2: MAPPING AND FUNCTION....................9

UNIT3: NUMBER SYSTEMS....................18

UNIT4: INEQUALITY....................31

UNIT5: SURDS....................38

UNIT6: INDICES AND LOGARITHMS....................43

UNIT7: MATHEMATICAL INDUCTION....................48

UNIT8: BINOMIAL THEOREM....................51

UNIT9: MATRICES AND DETERMINANT....................55

UNIT10: QUADRATIC EQUATION....................66

UNIT 1: SET THEORY

Content

1.0 INTRODUCTION
2.0 OBJECTIVES
3.0 MAIN CONTENT
3.1 Sets
3.2 Subsets and Supersets
3.3 The Universal Set, the Unit Set and the Empty Set
3.4 Complementary Set
3.5 The Union of Sets
3.7 The Intersection of Sets
3.7 Venn Diagram
4.0 CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

INTRODUCTION

In this unit, you will study the relationship between the union and the intersection of the set. We shall use the Venn diagram to further explain the union, intersection and complement of the set. You will also solve real- life problems using Venn diagram.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Explain the meaning of a set

2. Perform basic operations on sets

3. Solve real- life problems using Venn diagram.

3.0 MAIN CONTENT

3.1 Sets

We are all familiar with the following collections:

1. A collection of books in a public library,

2. A collection of tools in a carpentary shop,

3. A collection of historical aircrafts in a museum.

Basically, any clearly defined collection of things, objects or numbers constitutes a set.

A collection of books in a public library is for instance, constitutes a set. Each member of a set is called an element of the set.

We shall use capital letters X, Y, Z etc to denote sets, while small letters *x, y, z*, etc will be used to denote the element of a set. This is purely conventional. When an element *x*, belongs to a set X, we write $x \in X$ and say that *x* is a member or element of X. if *x* is not a member or a element of X, we write $x \notin X$ and we say that *x* does not belong to X.

A set is completely specified in the following ways:

1. By listing all the members of the set;

2. By describing the elements of the set;

3. By enclosing within braces, any general element with a clearly defined property, associated with the set. The symbol { } denotes 'the set of'.

For instance:

A = {2, 3, 5, 7}, is the set consisting of prime numbers between 1 and 10, lists all the members of set A.

B is the set of all odd numbers between 1 and 10 describes the elements of the set B.

C = {all positive integer *x*: $1<x\leq 12$} is red the set of all positive integer *x* such that *x* is greater than1 but less than or equal to 12.

3.2 Subsets and Supersets

Consider two given sets A and B. If the set A consists of some or all elements of the set B, then A is said to be a subset of B. We denote this by set notation $\subseteq$. Thus $A\subseteq B$ means; A is a subset of B. if the set A is a subset of B which is not in , then the set A is called a proper subset of B. Thus $A\subset B$ means, A is a proper subset of B. B is considered a superset of A.

The set notation for superset is $\supset$. Thus B $\supset$A means B is a superset of A.

Example

If P = {2, 4, 6, 8,10}, Q = {4, 10}, R = {6, 8} and S = {2, 4, 6, 8, 10, 12}, then:

(i) $Q\subset P$ (ii) $N\subset P$ (iii) $S\supset P$

In sets, the order in which the elements are written is irrelevant. For instance {2, 4, 6, 8} is the same as {2, 6, 8, 4}.

3.3 The Universal Set, the Unit Set and the Empty Set

`The set which contains all the possible elements under consideration, is called the universal set.

The universal set is denoted ï K N.

A set which consists of only one element is called a unit set. For instance, A={*x*} is a unit set because if contains one element *x*.

When a set contains no element, it is called an empty set or a null set. A null set is denoted Î or {}.

Example

(i) P = {human beings with tails}= Î
(ii) Q = {all prime numbers divisible by 4}= Î
(iii) R = {all months of the year beginning with the letter b}= Î
(iv) S = {o} is a unit set.

3.4 Complementary Set

Given that P is a subset of the universal set, the elements of the universal set which are not in the set P constitute a set which is called the complementary set of P. The complement of P is denoted P'.

Note that $(P')' = P$, $\mu' =$ Î, Î ï

Examples

1. If ï = {all integers} and A = {all odd integers} then, A' = {all even integers}
2. If ï = {all letters of the alphabet} and X = {all consonants} then, X' = {all vowels}

3.5 **The Union of Sets**

The union of set A and B, is the set which consists of elements that are either n A or B or both. The set notation for the operation of union is $\cup$. Thus A union B is written A$\cup$B.

In set theoretical notation,

A$\cup$B = {x: $x \in A$, or $x \in B$, $x \in both$ A and B}.

Example

1.) Given that G= {h, e, a, p}, H = {l, a, k, e} then,
$G \cup H$ = {h, e, a, p, l, k}
2.) If X= {2, 4, 6, 8}, Y = {2, 5, 9, 11, 12}
Then $X \cup Y$ = {2, 4, 5, 6, 8, 9, 11, 12}

3.6 The Intersection of Sets

The intersection of two sets A and B; is the set which consists of elements that are in A as well as in B. The set notation for the operation of intersection is $\cap$. $A \cap B$ means; A intersection B.

In set theoretical $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example

1.) Let R = {all positive even integers less than or equal to 20}

Q = {all factors of 20}

R = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

Q = {1, 2, 4, 5, 10, 20}

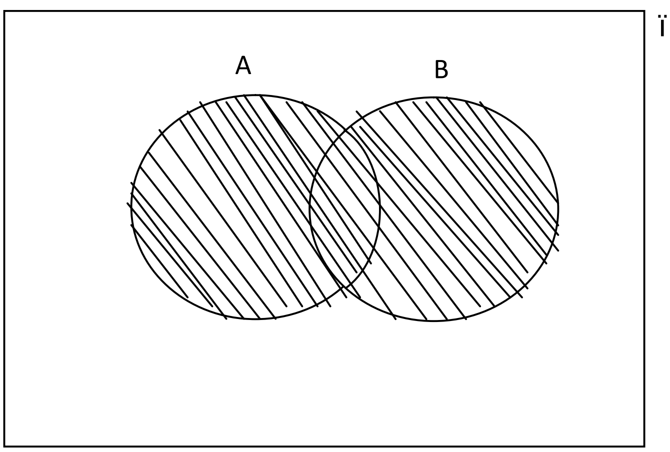
$\therefore R \cap Q$ = {2, 4, 10, 20}

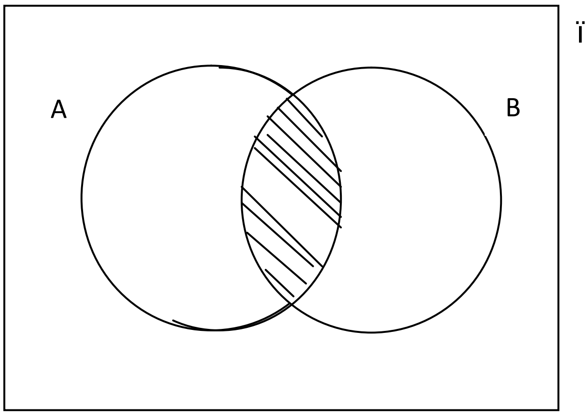
2.) If ï = {all the letters of the alphabet}
A = {f, a, k, e}
B = {s, p, e, a, k}
Then,
$A \cap B$ = {a, e, k}

3.7 Venn Diagram

Sets can be represented diagrammatically by closed figures. This method of set representation was developed by John Venn.

A Venn diagram is therefore a pictorial representation of sets. The operations of intersection, union and complementation of sets can easily be demonstrated by using Venn diagrams.





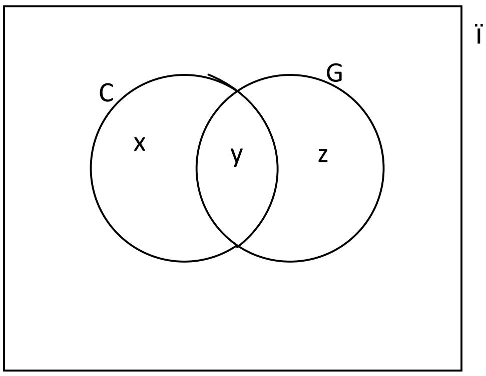
$A \cup B$

$A \cap B$

Example

1. In a class of 500 students, 270 offer Chemistry, 250 offer Geography, while 110 offer neither. How many students offer:
   (i) Both courses?
   (ii) Only Geography?

Solution



Let ï = the universal set ,C= the set of students offering Chemistry, G = the sets offering Geography. Then

$u = C \cap G' + C' \cap G + C \cap G + (C \cup G)'$

$n(C \cap G') = x = 270 - y$

$n(C' \cap G) = z = 250 - y$

$n(C \cap G) = y$

$n(C \cup G)' = 110$

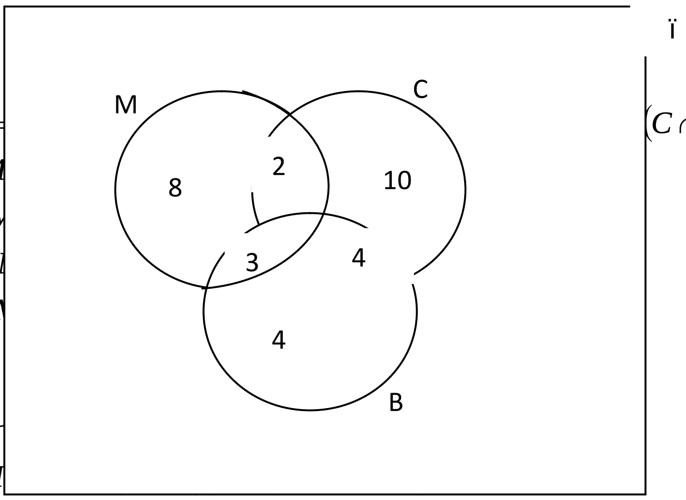
$500 = 270 - y + 250 - y + y + 110$

$y = 130$

2. All the 50 science students in Government Secondary School, Minna wereb asked their subject combinations. 18 of the students offered Further Mathematics, 21 offered chemistry while 16 offered Biology. 7 students offered Further Mathematics and Chemistry, 8 students offered Further Mathematics and Biology, 9 students offered Chemistry and Biology while 5 students offered the three subject combinations. Using Venn diagrams, find:
   (i) The number of students that offered Further Mathematics but offered neither Chemistry nor Biology;
   (ii) The number of students that offered Chemistry but offered neither Further Mathematics nor Biology;
   (iii) The number of students that offered Biology but offered neither Further Mathematics nor Chemistry;
   (iv) The number of students who did not offer any of the three combinations.

   Solution

Let ï = all the 50 Science students in GSS, Minna, M = students offering Further Mathematics, C = students offering Chemistry and B = students offering Biology.



$n(M)$ ... $(C \cap B) = 9, n(M \cap C \cap B = 5)$

$(i) n(M$

$(ii) n(M$

$(iii) n($

$(iv) n(M$

$n(M \cup$

$\therefore n(M$

Hence,

(v) The number of students that offered Further Mathematics but offered neither Chemistry nor Biology is 8;

(vi) The number of students that offered Chemistry but offered neither Further Mathematics nor Biology is 10;

(vii) The number of students that offered Biology but offered neither Further Mathematics nor Chemistry is 4;

(viii) The number of students who did not offer any of the three combinations is 14.

4.0 CONCLUSION

In this unit you have studied the basic operations of the set, the intersection, the union and the complement of the set. You have also learnt how to use Venn diagram to solve real life problems.

5.0 SUMMARY

In this unit you have studied:

A set is completely specified in the following ways:

1. By listing all the members of the set;

2. By describing the elements of the set;

3. By enclosing within braces, any general element with a clearly defined property, associated with the set. The symbol { } denotes 'the set of'.

(i) $(P')' = P$, $\mu' =$ î, î ï

(ii) A $\cup$ B = {x: $x \in A$, or $x \in B$, $x \in both$ A and B}.

(iii) A $\cap$ B = {x: $x \in A$ and $x \in B$ }.

6.0 TUTOR-MARKED ASSESSMENT

1. Given that ï {all the days in a week}, B = {all the days in the week whose letters begin with S}.
   (a) List all the elements of ï.
   (b) List the members of B.
   (c) List the members of B'.
2. ï {Natural numbers}
   F = {Factors of 20}

S = {all multiples of 3 less than or equal to 36}

Find:

(a) $F \cap S$ (b) $F \cup S$

3. A gamesmaster of a school called 50 students of the school, in Football and Volleyball. 30 students could play Football, while 20 students could play Volleyball. If 8 students could play both games, find:
   (a) The number of students that could play football but not volleyball;
   (b) The number of students that could play volleyball but not football;
   (c) The number of students that could not play any of the teo games.
4. In an examination for the final year students of a school, 60 offered History, 50, Economics and 48, Literature. 30 offered History and Economics, 16 offered History and Literature, 22 offered Econom and literature. If 10 candidates offered all the three subjects,
   (a) find :
      (i) the number that offered only History;
      (ii) the number that offered only Economics;
      (iii) the number that offered only Literature;
   (b) How many candidates sat for the examination, assuming that each candidate sat for at least one subject?

7.0 REFERENCES/FURTHER READING

Odili G. A. (2000), Algebra for Colleges and Universities: An Integrated Approach. Anachuna Educational Books.

Akinnola V. J., Mohammed J., Aiyesimi Y. M., Akinwande N. I., and Ogunfiditimi E. O.(2005), College Algebra and Trigonometry. Y- Books.

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UNIT2: MAPPING AND FUNCTION

1.0 INTRODUCTION
2.0 OBJECTIVES
3.0 CONTENTS

3.1 Mapping

3.2 Functions

3.3 One – One Mapping

3.4 Onto – Mapping
3.5 Composite Mapping
3.6 Inverse Function
4.0 CONCLUSION
5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READING

1.0 **INTRODUCTION**

In this unit you will study Mapping and Function. You will also see the important properties of Mapping.

2.0 **OBJECTIVES**

At the end of this unit you should be able to:

1. Use the concepts of mappings in solving problems;
2. Identify the important properties of mappings (functions);
3. Find product of mapping;
4. Find inverses of functions.

3.0 MAIN CONTENT

3.1 Mapping

You will recall that a set is a clearly defined collection of objects, things or numbers. Given two non-empty sets A and B, if there is a rule, which assigns an element $x \in A$ a unique element $y \in B$, then such a rule is called a mapping.

The set A is called the Domain of the mapping while the set B is called the Co-domain of the mapping.

If the rule which associates each element $x \in A$, a unique element $y \in B$ is denoted by f, the mapping between the set A and the set B can be represented by any of the following notations:

A $\xrightarrow{f} B$

$f : A \xrightarrow{f} B$

$f(x) = y$ where y is the unique element in B which corresponds to the element x in A. The element y in B is called the Image of $x \in A$.

A subset of the co-domain, which is a collection of all the images of the elements of the domain is called the Range.

If $f : A \to B$ is a mapping of the set A into the set B, then the range is usually denoted by f(A). f(A) is a subset of the co-domain, hence f(A) $\subset B$.

Example

(1) Given that P = {*w*, *x*, *y*, *z*} and Q = {a, b, f, e, g}. Let the mapping $f : P \to Q$ be defined by the diagram in fig. 3.1.

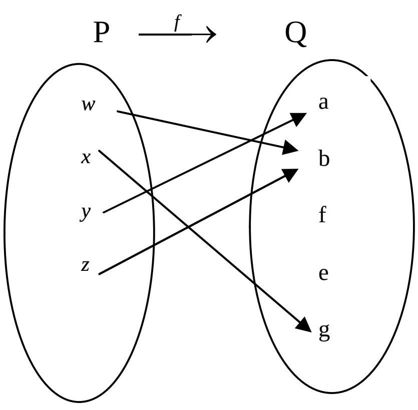

Fig. 3.1

The above representation is called an arrow diagram.

As $f(w)$ = b, $f(x)$ = g, $f(y)$ = a and $f(z)$ = b. b is the image of both w and z, g is the image of x while a is the image of y.

3.2 Functions

A function is a mapping whose co- domain is the set of numbers.

Consider the mapping in fig 3.3

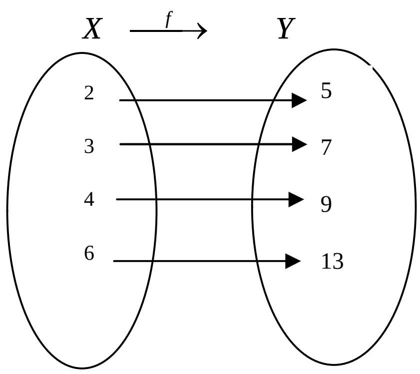

Fig. 3.3

Both the domain and the co-domain are sets of numbers. The mapping f is a function.

We observe that 5 is the f- image of 2 and we write this as $f(2) = 5$.

Similarly, we write $f(3) =7$, $f(4) = 9$ and $f(6) = 13$. Can you discover the rule which associates an elements from X, a unique image in the set Y? It appears that when 1 is added to twice an element in X, it produces the corresponding image in the set Y.

Hence, if $y \in Y$ is the f- image of $x \in X$, then we can write this as $f : x \to 2x + 1$ or $f(x) = 2x + 1$ or $y = 2x + 1$.

3.3 One – One Mapping

Let $f : X \to Y$ be the mapping that establishes the correspondence between the sets X and Y. The mapping f is called a One- one mapping, if different elements in the domain X contain have different images in the co- domain Y. Thus, $f : X \to Y$ is one – one mapping, if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ or $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

Examples

1. The mapping which associates each state in Nigeria with its Governor is a one- one mapping.
2. The mapping which assigns each University in West Africa with its Vice- Chancelloor is a one- one mapping.

5.1 Onto – Mapping

Let $f : X \to Y$ be a mapping from the set X to the set Y. The mapping f is called an Onto- mapping if every elements of the co-domain is an image of at least one element in the domain.

As every member of the co-domain is an image of at least one member of the domain, the range of an onto- mapping is equal to the co- domain.

Examples

1. Let R be a set of all real number and $f : R \to R$ a mapping defined by $f(x) = x^2 + 1$. Then f is not an onto- mapping because the range excludes negative numbers.

5.2 Composite Mapping

Let $f : X \to Z$ and $g : Z \to Y$ be two mappings such that the co- domain of f is the domain of g.

$$X \xrightarrow{f} Z \xrightarrow{g} Y$$

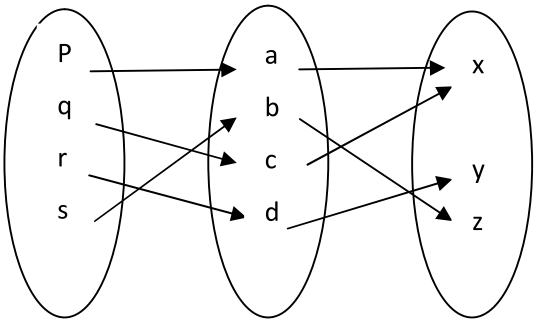

Fig. 3.4

The mapping f takes an element in X and produces an image in Z. the mapping g takes an element in Z and produces an image in Y. the mapping $g \circ f$ takes an element in X and produces an image in Y via Z.

From Fig. 3.4

$g \circ f(p) = g(a) = x$
$g \circ f(q) = g(c) = x$
$g \circ f(r) = g(d) = y$
$g \circ f(s) = g(b) = z$

The mapping $g \circ f$ is called a composite mapping. It is sometimes written as gf.

The composite mapping can also be represented vectorially as illustrated in Fig. 3.5.

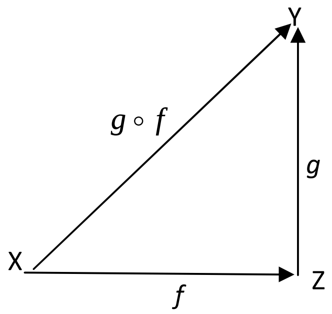

Fig. 3.5

Example

1. Let $f : X \to Y$ and $g : Y \to Z$ be mappings on the set of real numbers defined by:

$f(x) = x+1$ and $g(y) = (y+1)^2$, find $g \circ f$.

Solution

$$\begin{aligned} g \circ f &= gf(x) \\ &= g(x+1) \\ &= [(x+1)+1]^2 \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

2. Let f, g, and h be mappings on the set of real numbers defined by $f(x) = 2x+1$, $g(x) = x^2 - 3$ and $h(x) = 3x+2$. Show that $(f \circ g) \circ h = f \circ (g \circ h)$.
   Solution

$$\begin{aligned}(f \circ g) \circ h &= fg[3x+2] \\ &= f\left[(3x+2)^2 - 3\right] \\ &= f\left[9x^2 + 12x + 1\right] \\ &= 2(9x^2 + 12x + 1) + 1 \\ &= 18x^2 + 24x + 3\end{aligned}$$

$$\begin{aligned}f \circ (g \circ h) &= 2(g \circ h) + 1 \\ &= 2[g[h(x)]] + 1 \\ &= 2[g(3x+2)] + 1 \\ &= 2\left[(3x+2)^2 - 3\right] + 1 \\ &= 2\left[9x^2 = 12x + 1\right] + 1 \\ &= 18x^2 + 24x + 3\end{aligned}$$

$$\therefore (f \circ g) \circ h = f \circ (g \circ h)$$

5.3 Inverse Function

Consider the function $f : x \to 2x + 3$ on the set A = {-2, 1, 3} into the set B = {-1, 5, 9} illustrated in Fig. 3.6 below

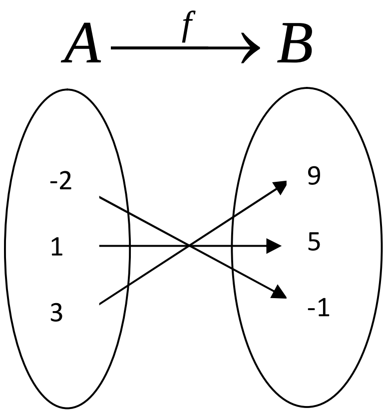

Fig. 3.6

The function *f* takes an element from the set A to produce a unique image in the set B. Alternatively, the function *f* can be defined as the set {(-2,-1), (1, 5), (3, 9)} i.e. *f* = {(-2,-1), (1, 5), (3, 9)}. Suppose we consider the relation, which reverses the operation of associating an element from the set A, a unique element in the set B. Let this relation be *g*. The relation *g* associates with every element in B a unique element in A.

The relation *g* can be defined as the set of the ordered pairs *g* = {(-2,-1), (1, 5), (3, 9)}

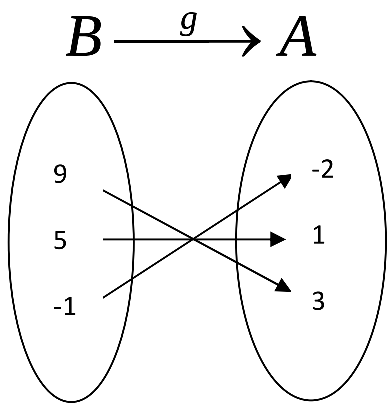

Fig. 3.7

The relation g is a mapping and it is called the inverse of the function f and it is usually denoted by f^{-1}.

Example

1. The function f over the set of real numbers is defined by $f(x) = \frac{1}{2}x - 3$.

   Find $f^{-1}(x)$.

   Solution

   Let $y = f(x) = \frac{1}{2}x - 3$

   $y + 3 = \frac{1}{2}x$

   $2(y+3) = x$

   $\therefore x = f^{-1}(y) = 2(y+3)$

   replacing y by x

   $f^{-1}(x) = 2(x+3)$

4.0 CONCLUSION

In this unit you have studied mapping, function, one- one mapping, onto mapping, composite mapping and inverse functions.

5.0 SUMMARY

1. Given two non-empty sets A and B, if there is a rule, which assigns an element $x \in A$ a unique element $y \in B$, then such a rule is called a mapping.

2. Given the mapping $f : A \to B$, the set A is called the Domain, while the set B is called the Co-domain.

3. A subset of the co-domain, which is a collection of all the images of the elements of the domain is called the Range.

4. The mapping $g \circ f$ is called a composite mapping. It is sometimes written as gf .

6.0 TUTOR-MARKED ASSIGNMENT

1. Determine the domain D of the mapping $g : x \to 2x^2$ -1, if R = {1, 7, 17}

is the range and g is defined on D.

2. $f : x \to 2x + 3$ is a mapping defined on the set R of real numbers. Determine the pre-images of : (a)1 (b) -1 (c) 7 (d) -5
3. Given the mapping $g : x \to \frac{3+x}{2x-1}$, determine:

(a) g(0) (b) g(1) (c) g(-1) (d) $g(\frac{1}{3})$ (e) $g(-\frac{1}{2})$

for what value of *x* is the mapping not defined? Hence or otherwise the largest domain of *g*.

4. Determine the inverse function in each of the following:

$(a)\, f(x) = 2x + 3 \quad (b)\, g(x) = (x-1)^{\frac{1}{3}} \quad (c)\, h(x) = 4x - 5 \quad (d)\, r(x) = \frac{x+3}{x+4}\, (x \neq -4)$

7.0 REFERENCES/FURTHER READING

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UNIT3: NUMBER SYSTEMS

Content

1.0 INTRODUCTION

2.0 OBJECTIVES

3.0 MAIN CONTENTS

3.1 Real Numbers

3.2 Operations with Real Numbers
3.3 Absolute Value of Real Numbers

3.4 Complex Number

4.0 CONCLUSION

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

Mathematics has its own language with numbers as the alphabet. The language is given structure with the aid of connective symbols, rules of operation, and a rigorous mode of thought (logic).

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Know different types of numbers

2. Perform operations with real numbers

3.0 MAIN CONTENT

3.1 Real Numbers

1. Natural numbers 1, 2, 3, 4, . . ., also called positive integers, are used in counting members of a set. The symbols varied with the times, e.g., the Romans used I, II, III, IV, ... The sum $a+b$ and product $a \bullet b$ or ab of any two natural numbers a and b is also a natural number. This is often expressed by saying that the set of natural numbers is closed under the operations of addition and multiplication, or satisfies the closure property with respect to these operations.

2. Negative integers and zero denoted by _1,_2,_3, . . . and 0, respectively, arose to permit solutions of equations such as $x+b=a$, where a and b are any natural numbers. This leads to the operation of subtraction, or inverse of addition, and we write $x=a-b$.
The set of positive and negative integers and zero is called the set of integers.

3. Rational numbers or fractions such as $\frac{2}{3}, -\frac{5}{4}, ...$ arose to permit solutions of equations such as $bx=a$ for all integers a and b, where $b\neq 0$. This leads to the operation of division, or inverse of multiplication, and we write $x=\frac{a}{b}$ or $a\div b$ where a is the numerator and b the denominator.
The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b=1$.

4. Irrational numbers such as $\sqrt{2}$ and π are numbers which are not rational, i.e., they cannot be expressed as $\frac{a}{b}$ (called the quotient of a and b), where a and b are integers and $b\neq 0$.
The set of rational and irrational numbers is called the set of real numbers.

3.2 Operations with Real Numbers

If a, b, c belong to the set R of real numbers, then:

1. $a+b$ and ab belong to R — Closure law
2. $a+b=b+b$ — Commutative law of addition
3. $a+(b+c)=(a+b)+c$ — Associative law of addition
4. $ab=ba$ — Commutative law of multiplication
5. $a(bc)=(ab)c$ — Associative law of multiplication
6. $a(b+c)=ab+ac$ — Distributive law
7. $a+0=0+a=a$, $1\bullet a=a\bullet 1=a$

0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication.

8. For any a there is a number x in R such that $x+a=0$.

x is called the inverse of a with respect to addition and is denoted by $-a$.

9. For any $a\neq 0$ there is a number x in R such that $ax=1$.

x is called the inverse of a with respect to multiplication and is denoted by a^{-1} 1 or $1/a$.

3.3 Absolute Value of Real Numbers

The absolute value of a number x is defined by:

$$|x|. = \begin{cases} x \text{ if } & x > 0 \\ 0 \text{ if } & x = 0 \\ -x \text{ if } & x < 0 \end{cases}$$

e.g., $|4| = 4, |-4| = 4$

The absolute value of a number is taken as the distance of the point representing the number on the real line from 0. This value is taken to be positive.

EXAMPLES

1. If $|x| = a \Rightarrow$ either

   $x = a$ or $x = -a$

2. If $|x| < a \Rightarrow$ either

   $x < a$ or $-x < a \Rightarrow x > -a$ these give

   $$-a < x < a$$

3. If $|x| \geq a \Rightarrow$ either

   $x \geq a$ or $-x \geq a \Rightarrow x \leq -a$ these gives

   $$x \leq a \text{ and } x \geq a$$

4. Find the range of x if $|x-3| = 5$

   Solution

   if $|x-3| = 5$

   either $x - 3 = 5 \Rightarrow x = 5 + 3 = 8$

   or $-(x-3) = 5 \Rightarrow -x + 3 = 5 \Rightarrow -x = 5 - 3 = 2 \Rightarrow x = -2$

   so $x = 5$ or $x = -2$

3.4 Complex Number

As noted in the study of the real number system, the solution of the quadratic equation

$$x^2 + 1 = 0 \qquad (3.1)$$

has no real solution, i.e. there is no real number which will satisfy the equation. This led to the introduction of the Complex Numbers.

From the above equation,

$$x^2 = -1$$

Or

$$x = \pm\sqrt{-1}$$

We define the complex number

$$i = \sqrt{-1} \qquad (3.2)$$

And also the solution of the quadratic equation (3.1) is then given by

$$x = \pm\sqrt{-1} = \pm i \qquad (3.3)$$

The powers of the complex number i are given by:

$$i = \sqrt{-1}\,,\ i^2 = \left(\sqrt{-1}\right)^2 = -1,\ i^3 = \left(i^2\right)(i) = -i\,,\ i^4 = \left(i^2\right)^2 = (-1)^2 = 1 \qquad (3.4)$$

The complex number system is the thus defined by:

V T E|Ʊ UĐ (3.5)

For the complex number $z = x + iy$, x is called the real part while y is the imaginary part with notations: $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$

3.4.1 The Argand Diagram

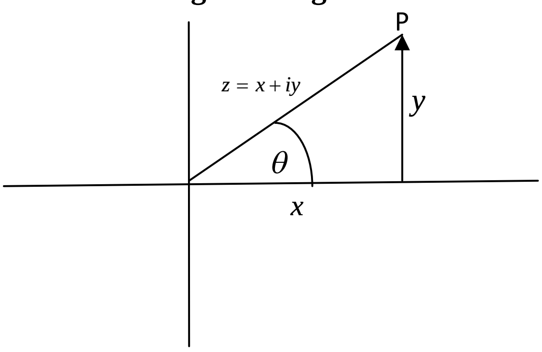

Figure 3.1

In the Argand diagram, the complex number $z = x + iy$ is uniquely represented by a point (x, y) on the Euclidean such that

The horizontal distance $OB = x$, which is the real part of z; also the vertical distance $AB = y$, which is the imaginary part of z.

The distance OA is called the modulus (or magnitude) of z and is denoted by:

$r = \text{mod} = |z|$

The angle $\theta =< AOB$ is called the argument of z denoted $\theta = \arg z$

From the diagram we have that

$$r^2 = |z|^2 = x^2 + y^2$$

And

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

3.4.2 Algebra of the Complex Number System

1. **Addition:**

   Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be Complex numbers, the addition of z_1 and z_2 is defined, and we have that

   $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

   The subtraction $z_1 - z_2$ is similarly defined.

2. **Multiplication:**

   The multiplication of z_1 and z_2 is defined, and we have that

   $$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)\cdot(x_2 + iy_2) = x_1 \cdot (x_2 + iy_2) + i(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

3. **Conjugate:**

   The conjugate of the Complex number

   $z = x + iy$ is given by $\overline{z} = x - iy$

   Note that

   $$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) = x(x - iy) + iy(x - iy) \\ &= x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 \end{aligned}$$

   i.e. $z\overline{z} = |z|^2$

4. **Rationalization:**

   $$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{z}\cdot\frac{\overline{z}}{\overline{z}} \\ &= \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} \end{aligned}$$

5. **Quotient or Division:**

   The quotient $\frac{z_1}{z_2}$ is rationalized as follow

   $$\frac{z_1}{z_2} = \frac{z_1}{z_2}\cdot\frac{\overline{z_1}}{\overline{z_2}} = \frac{z\overline{z_2}}{|z_2|^2}$$

   From figure (3.1), note that

   $x = r\cos\theta,\ y = r\sin\theta$

   And so

$z = x + iy \ = r(\cos\theta + i\sin\theta)$

Also

$r = \dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

The angle $2n\pi + \theta n = 0,1,2,...$ will satisfy the argument of z.

3.4.3 Examples

1. If $x + iy = \dfrac{3-i}{2+i}$, find x and y.

   Solution

   Rationalizing:

   $$x + iy = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-2i+i^2}{4-2i+2i-i^2} = \frac{6-3i-2i-1}{4-2i+2i+1} = \frac{5-5i}{5} = 1-i$$

   $x = 1$, $y = -1$

2. If z_1 and z_2 are given by:

   $z_1 = \dfrac{(2+i)^2(1-i)}{3i-1}$, $z_2 = \dfrac{i-3}{2+i}$; find:

   (i) $z_1 + z_2$

   (ii) Modulus of z_1

   (iii) Argument of z_2

   Solution

   $$(2+i)^2 = (2+i)(2+i) = 2(2+i) + i(2+i) = 4 + 2i + 2i + i^2 = 3 + 4i$$
   $$(2+i)^2(1-i) = (3+4i)(1-i) = 3(1-i) + 4(1-i) = 3 - 3i + 4i - 4i^2 = 3 + i + 4 = 7 + i$$

   $$z_1 = \frac{(2+i)^2(1-i)}{3i-1} = \frac{7+i}{3i-1} = \frac{(7+i)(-3i-1)}{(3i-1)(-3i-1)} = \frac{-21i-7+3-i}{10} = \frac{-22i-4}{10}$$
   $$z_1 = \frac{-4}{10} - i\frac{22}{10} = -\frac{2}{5} - i\frac{11}{5}$$
   $$z_2 = \frac{i-3}{2+i} = \frac{(i-3)(2-i)}{(2+i)(2-i)} = \frac{2i+1-6+3i}{5} = \frac{5i-5}{5} = i-1$$

   (i) $z_1 + z_2 = -\dfrac{2}{5} - i\dfrac{11}{5} + i - 1 = -\dfrac{(7+6i)}{5}$

   (ii) modulus of z_1

$$r_1 = |z_1| = \sqrt{\left(-\frac{2}{5}\right)^2 + \left(-\frac{11}{5}\right)^2} = \sqrt{\frac{4+121}{25}} = \sqrt{\frac{125}{25}} = \sqrt{5}$$

(iii)Argument of z_2

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$$

3.4.4 De 'Moivre's Theorem

The De'Moivre's theorem states that

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

For n= 1,2,3,…

Proof

Proof is by induction:

Let the statement $P(n)$ be :

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

For n = 1,

LHS= $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ = RHS

So $P(k)$ is true, i.e.

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

Let $n = k+1,$

LHS

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^{k} \\
&= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta) \\
&= \cos\theta(\cos k\theta + i\sin k\theta) = i\sin\theta(\cos k\theta + i\sin k\theta) \\
&= \cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta + i^{2}\sin\theta\sin k\theta \\
&= (\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta) \\
&= \cos(\theta + \mathrm{k}\theta) + i\sin(\theta + \mathrm{k}\theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta
\end{aligned}
$$

i.e. $P(k+1)$ is true.

Remark

In general the De'Moivre's theorem is true for when *n* is negative,

If $n = -p$

$$(\cos\theta + i\sin\theta)^{-p} = \cos(-p\theta) + i\sin(-p\theta)$$

But

$$\cos(-p\theta) = \cos p\theta\,,\ \sin(-p\theta) = -\sin p\theta$$

So

$$(\cos\theta + i\sin\theta)^{-p} = \cos p\theta - i\sin p\theta$$

3.4.5 Trigonometric Expansion Using De'Moivre's Theorem

If $z = \cos\theta + i\sin\theta$, the conjugate $\bar{z} = \cos\theta - i\sin\theta$

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^{2}}$$

$$|z|^{2} = \cos^{2}\theta + \sin^{2}\theta = 1$$

And so

$$z^{-1} = \frac{1}{z} = \bar{z}$$

From above,

$$z+\bar{z}=z+\frac{1}{z}=(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)$$
$$=2\cos\theta$$

Also,

$$z-\bar{z}=z-\frac{1}{z}=(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)$$
$$=2i\sin\theta$$

Since

$$z^m=(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$$

and

$$\frac{1}{z^m}=z^{-m}=(\cos\theta+i\sin\theta)^{-m}=\cos m\theta-i\sin m\theta$$

Recursively we deduce that

$$z^m+\frac{1}{z^m}=2\cos m\theta$$

and

$$z^m-\frac{1}{z^m}=2i\sin m\theta$$

Examples

1. Express $\cos^8\theta$ in multiple angles

   Solution

   Let $z=\cos\theta+i\sin\theta$

   Then

$$\left(z+\frac{1}{z}\right)^8 = (2\cos\theta)^8$$

i.e.

$$2^8\cos^8\theta = \left(z+\frac{1}{z}\right)^8$$

$$= z^8 + 8z^7\cdot\frac{1}{z} + 28z^6\cdot\frac{1}{z^2} + 56z^5\cdot\frac{1}{z^3} + 70z^4\cdot\frac{1}{z^4} + 56z^3\cdot\frac{1}{z^5} + 28z^2\cdot\frac{1}{z^6} + 8z\cdot\frac{1}{z^7} + \cdot\frac{1}{z^8}$$

$$= \left(z^8+\frac{1}{z^8}\right) + 8\left(z^6+\frac{1}{z^6}\right) + 28\left(z^4+\frac{1}{z^4}\right) + 56\left(z^2+\frac{1}{z^2}\right) + 70$$

$$= 2\cos 8\theta + 16\cos 6\theta + 56\cos 4\theta + 112\cos 2\theta + 70$$

$$= 2[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

and so

$$2^8\cos^8\theta = 2[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

and hence

$$\cos^8\theta = 2^{-7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$

2. Express $\sin^9\theta$ in multiple angles

Solution

Let $z = \cos\theta + i\sin\theta$

Then

$$\left(z-\frac{1}{z}\right)^9 = (2i\sin\theta)^9 = (2i)^9\sin^9\theta = 2^9 i\sin^9\theta$$

i.e.

$$2^9 i\sin^9\theta = \left(z-\frac{1}{z}\right)^9$$

$$= z^9 - 9z^8\cdot\frac{1}{z} + 36z^7\cdot\frac{1}{z^2} - 84z^6\cdot\frac{1}{z^3} + 126z^5\cdot\frac{1}{z^4} - 126z^4\cdot\frac{1}{z^5} + 84z^3\cdot\frac{1}{z^6} - 36z^2\cdot\frac{1}{z^7} + 9z\cdot\frac{1}{z^8} - \cdot\frac{1}{z^9}$$

$$= \left(z^9-\frac{1}{z^9}\right) - 9\left(z^7-\frac{1}{z^7}\right) + 36\left(z^5-\frac{1}{z^5}\right) - 84\left(z^3-\frac{1}{z^3}\right) + 126\left(z-\frac{1}{z}\right)$$

$$= 2i[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$$

and so

$$2^9 i\sin^9\theta = 2i[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$$

and hence

$$\sin^9\theta = 2^{-8}[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$$

4.0 CONCLUSION

In this unit you have studied the operations with real numbers, and absolute value of real numbers. We have also studied the complex number, algebra of the complex number, the De'Moivre's theorem and trigonometric expansion using De'Moivre's.

5.0 SUMMARY

In this unit you have studied:

If a, b, c belong to the set R of real numbers, then:

1. $a+b$ and ab belong to R	Closure law
2. $a+b=b+b$	Commutative law of addition
3. $a+(b+c)=(a+b)+c$	Associative law of addition
4. $ab=ba$	Commutative law of multiplication
5. $a(bc)=(ab)c$	Associative law of multiplication
6. $a(b+c)=ab+ac$	Distributive law
7. $a+0=0+a=a$, $1 \bullet a = a \bullet 1 = a$	

The absolute value of a number x is defined by:

$$|x|. = \begin{cases} x \text{ if } & x>0 \\ 0 \text{ if } & x=0 \\ -x \text{ if } & x<0 \end{cases}$$

$$x = \pm\sqrt{-1} \;\; ; i = \sqrt{-1}$$
$$x = \pm\sqrt{-1} = \pm i$$

The powers of the complex number i are given by:

$$i = \sqrt{-1}, i^2 = \left(\sqrt{-1}\right)^2 = -1, \; i^3 = \left(i^2\right)(i) = -i, i^4 = \left(i^2\right)^2 = (-1)^2 = 1 \qquad (3.4)$$

The complex number system is the thus defined by:

V T E|Ư, UĐ (3.5)

For the complex number $z = x + iy$, x is called the real part while y is the imaginary part with notations: $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$

$$r^2 = |z|^2 = x^2 + y^2$$

And

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The conjugate of the Complex number

$z = x + iy$ is given by $\bar{z} = x - iy$

Note that

$$z\bar{z} = (x+iy)(x-iy) = x(x-iy) + iy(x-iy)$$
$$= x^2 - ixy + ixy - i^2y^2 = x^2 + y^2$$

i.e. $z\bar{z} = |z|^2$

$$z^{-1} = \frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}}$$
$$= \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

The quotient $\frac{z_1}{z_2}$ is rationalized as follow

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\cdot\frac{\overline{z_1}}{\overline{z_2}} = \frac{z\overline{z_2}}{|z_2|^2}$$

If $z = \cos\theta + i\sin\theta$, the conjugate $\bar{z} = \cos\theta - i\sin\theta$

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$|z|^2 = \cos^2\theta + \sin^2\theta = 1$$

$$z^{-1} = \frac{1}{z} = \bar{z}$$

$$z + \bar{z} = z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$
$$= 2\cos\theta$$

$$z - \bar{z} = z - \frac{1}{z} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$$
$$= 2i\sin\theta$$

6.0 TUTOR-MARKED ASSIGNMENT

1. If $$x + iy = \frac{a+ib}{c+id}$$

show that
$$x - iy = \frac{a - ib}{c - id}$$

2. Determine the modulus and argument of $z_1 + z_2$ if
$$z_1 = \frac{(2+i)(1-i)}{3i-1}, \quad z_2 = \frac{(i-3)(2-i)}{2+i}$$
3. Express $\cos^5 x$ in multiple angles.
4. Express $\sin^6 x$ in multiple angles.

7.0 REFERENCES/FURTHER READING

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UNIT4: INEQUALITY

Content

1.0 INTRODUCTION
2.0 OBJECTIVES
3.0 MAIN CONTENT
3.1 Real Numbers

3.2 Operations with Real Numbers
3.3 Absolute Value of Real Numbers

3.4 Complex Number
4.0 CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

In this unit you shall study use of number line in inequality. We shall study the inequality of one variable, two variable, quadratic inequalities and simultaneous inequality.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Solve inequalities in one variable graphically and analytically;
2. Solve inequalities in two variables graphically;
3. Apply solution of inequalities to problems.

3.0 MAIN CONTENT

3.1 Basic Rules of Inequalities

1. (a) If $a > b$ and $c > 0$ then $a + c > b + c$
 (b) If $a < b$ and $c > 0$ then $a + c < b + c$

We can add the same constant to both sides of an inequality:

2. (a) If $a > b$ and $c > 0$ then $a - c > b - c$
(b) If $a < b$ and $c > 0$ then $a - c < b - c$
We can subtract the san constant from both side of an inequality.

3. (a) If $a > b$ and $c > 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$
 (b) If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

If both side of inequality are multiplied or divided by the same positive number, then the sense of the inequality remains unchanged.

4. (a) If $a > b$ and $c < 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$
(b) If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

If both side of an inequality are multiplied or divided by the same negative number, the sense of the inequality is reversed.

5. (a) If $a > b$ and $c < d$ then $a - c > b - d$
(b) If $a < b$ and $c > d$ then $a - c < b - d$

6. (a) If $a > b$ and $n > 0$ then $a^n > b^n$
 (b) If $a > b$ and $n < 0$ then $a^n < b^n$

7. (a) If $a < b$ and $n > 0$ then $a^n < b^n$
 (b) If $a < b$ and $n < 0$ then $a^n > b^n$

3.2 Linear Inequalities in one Variable

Most of the basic rules for solving a linear inequality in one variable are similar to those for solving a linear equation in one variable with exception of the rules on multiplication and division by negative number which reversed the sense of the inequality.

Example

1. (a) $2x-3<x+7$ (b) $4x+\frac{1}{3}\geq\frac{2}{3}x-2$ (c) $\frac{1}{3}x-2\leq\frac{2}{5}x+\frac{1}{4}$

Solution

(a) $2x-3<x+7$

$2x<10$

$x<5$

(b) $4x+\frac{1}{3}\geq\frac{2}{3}x-2$

$4x\geq\frac{2}{3}x-\frac{7}{3}$

$\frac{10}{3}x\geq-\frac{7}{3}$

$10x\geq-7$

$x\geq-\frac{7}{10}$

(c) $\frac{1}{3}x-2\leq\frac{2}{5}x+\frac{1}{4}$

$\frac{1}{3}x-\frac{2}{5}x\leq2+\frac{1}{4}$

$-\frac{1}{2}x\leq\frac{9}{4}$

$-4x\leq135$

$x\geq-135$

3.3 Graphical Representation

The solutions set of linear inequalities in one variable are represented graphically as intervals on the real number line.

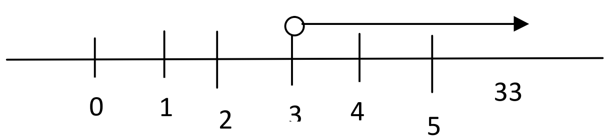
Example

1. Represent each of the following inequalities graphically:

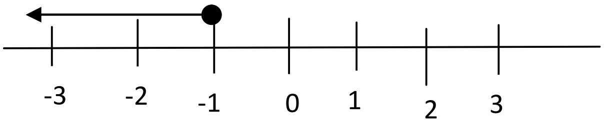
(a) $x>3$ (b) $x\leq-1$ (c) $-2<x<3$ (d) $-1\leq x\leq4$

Solution

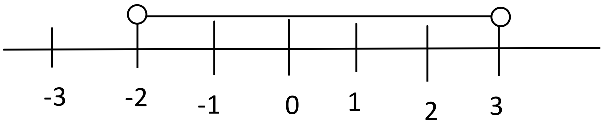
(a) $x>3$



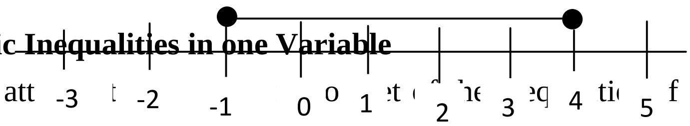
(b) $x \le -1$



(c) $-2 < x < 3$



(d) $-1 \le x \le 4$



3.4 Quadratic Inequalities in one Variable

We shall att[illegible] f the form $ax^2 + bx + c \ge 0$ or $ax^2 + bx + c \le 0$.

If $a.b > 0$ then either

(i) $a > 0$ and $b > 0$ or

(ii) $a < 0$ and $b < 0$

If $a.b < 0$ then either

(i) $a < 0$ and $b < 0$ or

(ii) $a > 0$ and $b > 0$

These facts can be used to solve quadratic inequalities in one variable.

Examples

(i) Find the solution set of $x^2 + x + 6 > 0$

Solution

If $x^2 + x + 6 > 0$

Then $(x-2)(x+3) > 0$

Either $x - 2 > 0$ and $x + 3 > 0$

$\therefore x > 2$ and $x > -3$

These can be combined as $x > 2$

Or

$x - 2 < 0$ and $x + 3 < 0$

$\therefore x < 2$ and $x < -3$

These can be combined as $x < -3$

Hence the solution set is

$\{x : x < -3\} \cup \{x : x > 2\}$

(2) Find the solution set of the inequality $x^2 + 3x - 4 \le 0$.

Solution

If $x^2 + 3x - 4 \le 0$ then

$(x-1)(x+4) \le 0$

Either $x - 1 \le 0$ and $x + 4 \ge 0$

$\therefore x \le 1$ and $x \ge -4$.

These can be combined as: $-4 \le x \le 1$

Or

$x - 1 \ge 0$ and $x + 4 \le 0$

$\therefore x \ge 1$ and $x \le -4$

But x cannot be greater than1 and at the same time be less than -4, hence the solution set is $\{x : -4 \le x \le 1\}$

3.5 Inequalities in Two Variables

A straight line has the general equation $ax + by + c = 0$, where a, b, and c are real numbers.

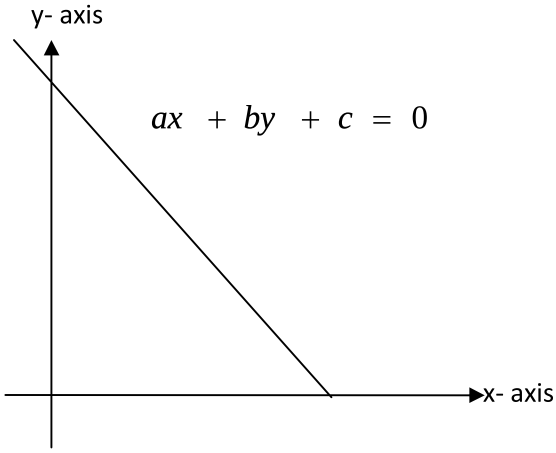
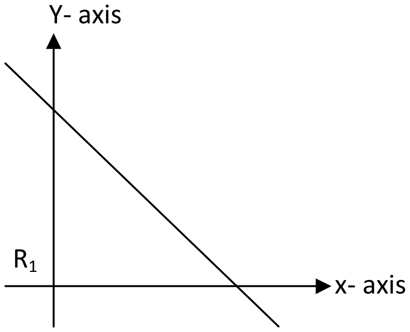
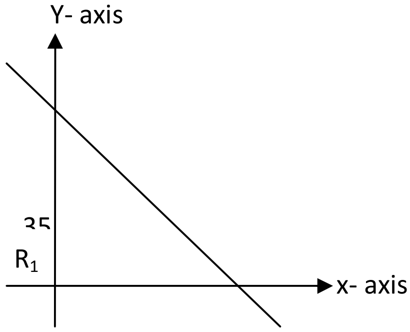


Fig. 3.1

The line $ax + by + c = 0$ partitions the $x - y$ plane into two regions.





(a) (b)

Fig. 3.2

Example

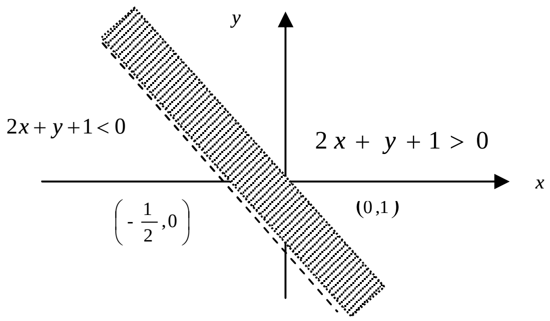
1. Solve graphically the regions represented by the inequalities:
   (a) $2x + y + 1 > 0$ and $2x + y + 1 < 0$
   (b) $2y - x - 3 > 0$ and $2y - x - 3 < 0$

Solution

(a) Consider the line $2x + y + 1 = 0$

The intercepts on the axes are $\left(-\frac{1}{2}, 0\right)$ and $(0, -1)$ if we substitute the coordinates of the origin $(0, 0)$ into the inequality, we have $1 > 0$ which is true.

Therefore every point on the same side of the origin will make the relation $2x + y + 1 > 0$ true.



The region on the same side of the shaded part is represented by $2x + y + 1 > 0$. The other opposite side is represented by the inequality $2x + y + 1 < 0$.

(b) Consider the line
$2y - x - 3 = 0$

The intercepts are $(-3,0)$ and $\left(0, \frac{3}{2}\right)$. Substitute the coordinates of the origin $(0, 0)$ into the inequality, we have $-3 < 0$ which is true.

Therefore every point on the same side of the origin will make the relation $2y - x - 3 < 0$ true.

$\left(0, \frac{3}{2}\right)$

The region on the same side of the shaded part is $2y - x - 3 < 0$ while the region on the side is $2y - x - 3 > 0$.

4.0 CONCLUSION

In this unit you have studied how to solve; inequality graphically and analytically, quadratic inequalities and inequalities of two variables and how to apply inequalities to solve real life problems.

5.0 SUMMARY

1. Most of the basic rules for solving a linear inequality in one variable are similar to those for solving a linear equation in one variable with exception of the rules on multiplication and division by negative number which reversed the sense of the inequality.
2. The solutions set of linear inequalities in one variable are represented graphically as intervals on the real number line.
3. The line $ax + by + c = 0$ partitions the $x - y$ plane into two regions R_1 and R_2.

6.0 TUTOR-MARKED ASSIGNMENT

1. Represent the solution set of each of the following inequalities graphically:

   (a) $2x + 1 \le x + 5$ (b) $\frac{1}{3}x + 2 \ge \frac{1}{2}x + 1$ (c) $x^2 + 6x + 8 < 0$ (d) $4x^2 + x - 3 < 0$

2. Show graphically the regions represented by the inequalities:

   (a) $2x + 3y + 6 > 0$ (b) $4y - 3x + 5 \le 0$ (c) $2y + 3x + 3 < 0$ (d) $x - 4y + 7 \ge 0$

UNIT5: SURDS

Content

1.0 INTRODUCTION

2.0 OBJECTIVES

3.0 MAIN CONTENT

3.1 Rule of Surds

3.2 Basic Forms and Similar Surds
3.3 Conjugate Surds
3.4 Rationalizing the Denominator

4.0 CONCLUSION

5.0 SUMMARY

6.0 TUTOR-MARKED ASSIGNMENT

7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

In this unit you will study addition, subtraction, multiplication and division of surds. We shall also study the conjugate and rationalization of surds.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Reduce surds to it basic form;

2. Evaluate the problems of surds;

3. Rationalize and take the conjugate of surds.

3.0 MAIN CONTENT

3.1 Rule of Surds

1. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

2. $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Examples

1. $\sqrt{3 \times 17} = \sqrt{3} \times \sqrt{17}$
2. $\sqrt{5} \times \sqrt{7} = \sqrt{5 \times 7} = \sqrt{35}$

Note that $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$

3.5 Basic Forms and Similar Surds

If the rational number under the square root sign contains a factor which is a square of a number, the surd can be reduced to a simpler form. Such a simpler form is called its basic form.

These are surds which are multiples of the same surds. $3\sqrt{5}$, $2\sqrt{5}$, $6\sqrt{5}$ and $7\sqrt{5}$ are similar surds because each of the surds is a multiple of $\sqrt{5}$. Two similar surds can be added together or subtracted from each other.

Examples

1. Simplify the following:
   (a) $\sqrt{20} + \sqrt{45} + \sqrt{125} - 2\sqrt{80}$
   (b) $3\sqrt{2} - \sqrt{32} + \sqrt{50} + \sqrt{98}$

Solution

(a) $\sqrt{20} + \sqrt{45} + \sqrt{125} - 2\sqrt{80}$

$= 2\sqrt{5} + 3\sqrt{5} + 5\sqrt{5} - 8\sqrt{5}$

$= 10\sqrt{5} - 8\sqrt{5}$

$= 2\sqrt{5}$

(b) $3\sqrt{2} - \sqrt{32} + \sqrt{50} + \sqrt{98}$

$= 3\sqrt{2} - 4\sqrt{2} + 5\sqrt{2} + 7\sqrt{2}$

$= 15\sqrt{2} - 4\sqrt{2}$

$= 11\sqrt{2}$

3.6 Conjugate Surds

Two surds are said to be conjugate of each other if their product gives rise to a rati
From our knowledge of difference of two square we know that $(a+b)(a-b) = a^2 -$

Similarly,

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a-b$$

While $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are not rational, their product a-b is rational. Hence, $\sqrt{a}+$
$\sqrt{a}-\sqrt{b}$ are conjugate of each other.

Example

(1) Write down the conjugate of each of the following:

(a) $\sqrt{3} - \sqrt{2}$ (b) $2\sqrt{5} + 3\sqrt{3}$

Solution

(a) The conjugate of $\sqrt{3} - \sqrt{2}$ is $\sqrt{3} + \sqrt{2}$

(b) The conjugate of $2\sqrt{5} + 3\sqrt{3}$ is $2\sqrt{5} - 3\sqrt{3}$

(2) Simplify the following:

(a) $\left(3\sqrt{2}-1\right)\left(3\sqrt{2}+1\right)$ (b) $\left(7\sqrt{2}+\sqrt{3}\right)\left(7\sqrt{2}-\sqrt{3}\right)$

Solution

(a) $\left(3\sqrt{2}-1\right)\left(3\sqrt{2}+1\right) = \left(3\sqrt{2}\right)^2 - 1 = 18 - 1 = 17$

(b) $\sqrt{\ }$ $\sqrt{\ }$

Examples

(1) Simplify (a) $\frac{5}{\sqrt{2}}$ (b) $\frac{3+\sqrt{2}}{\sqrt{5}+\sqrt{3}}$

Solution

(a) $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

(b) $\frac{3+\sqrt{2}}{\sqrt{5}+\sqrt{3}} = \frac{3+\sqrt{2}}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{(3+\sqrt{2})(\sqrt{5}-\sqrt{3})}{5-3}$

$= \frac{1}{2}\left(3\sqrt{5} - 3\sqrt{3} + \sqrt{10} - \sqrt{6}\right)$

4.0 CONCLUSION

In this unit you have studied how to reduce surds to it basic form and evaluate the problems of surds. You also studied rationalization and how to take the conjugate of surds.

5.0 SUMMARY

You have studied:

- $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are conjugate of each other.

6.0 Tutor-Marked Assignment

(1) Express each of the following in basic form:

(a) $\sqrt{200}$ (b) $\sqrt{162}$ (c) $\sqrt{648}$

(2) Simplify each of the following:

(a) $\left(2\sqrt{2} + 3\sqrt{5}\right)^2$ (b) $\left(2\sqrt{6} - 1\right)\left(\sqrt{3} - \sqrt{2}\right)$

(3) Express $\frac{7\sqrt{2} + 3\sqrt{3}}{4\sqrt{2} - 2\sqrt{3}}$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers.

(4) Express $\sqrt{32} + \frac{6}{\sqrt{2}}$ as a single surd and hence find the value of $\frac{7}{\sqrt{2}}\left(\sqrt{32} + \frac{6}{\sqrt{2}}\right)$

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UNIT6: INDICES AND LOGARITHMS

Content

1.0 INTRODUCTION
2.0 OBJECTIVES
3.0 MAIN CONTENT
3.1 Laws of Indices
3.2 Exponential Equations
3.3 Laws of Logarithms
4.0 CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

In this unit you will study the laws of indices and how you will apply the laws to solve problems.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Use the laws of indices in calculations and simplification;

2. Use the relationship between indices and logarithms to solve problems.

3.0 MAIN CONTENT

3.1 Laws of Indices

(a) $a^m \times a^n = a^{m+n}$

e.g., $3^2 \times 3^5 = 3^{2+5} = 3^7$

(b) $a^m \div a^n = a^{m-n}$

e.g., $2^5 \div 2^3 = 2^{5-3} = 2^2$

(c) $\left(a^m\right)^n - a^{m\times n} = a^{mn}$

e.g., $\left(4^3\right)^2 = 4^{3\times 2} = 4^6$

(d) Zero index

$$a^m \div a^m = \frac{a^m}{a^m} = 1$$

Also $a^m \div a^m = a^{m-m} = a^0$

$\therefore a^0 = 1$

(e) Negative index

$$a^{-m} = \frac{1}{a^m}$$

e.g., $a^3 \div a^5 = a^{3-5} = a^{-2}$

also $a^3 \div a^5 = \dfrac{a^3}{a^5} = \dfrac{1}{a^2}$

$$\therefore a^{-2} = \frac{1}{a^2}$$

(f) Fractional index

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\times n} = a$$

$$\left(\sqrt{a}\right)^n = a$$

$$\therefore a^{\frac{1}{n}} = \sqrt[n]{a}$$

e.g., $\left(a^{\frac{1}{2}}\right)^2 = a^{\frac{1}{2}\times 2} = a$

Also $\left(\sqrt{a}\right)^2 = a$

$\therefore a^{\frac{1}{2}} = \sqrt{a}$

Similarly,

$\left(a^{\frac{1}{3}}\right)^3 = a^{\frac{1}{3}\times 3} = a$

$\left(\sqrt[3]{a}\right)^3 = a$

$\therefore a^{\frac{1}{3}} = \sqrt[3]{a}$

3.2 Exponential Equations

Examples

Solve the following equations

(1) $2^x = 0.125$ (2) $5^{2x+4} \times 5^{x+1} - 125 = 0$

Solution

(1) $2^x = 0.125 \Rightarrow 2^x = \frac{1}{8} \Rightarrow 2^x = \frac{1}{2^3} \Rightarrow 2^x = 2^{-3}$

$\therefore x = -3$

(2) $5^{2x+4} \times 5^{x+1} - 125 = 0 \Rightarrow 5^{2x+4+x+1} = 5^3$

$2x + 4 + x + 1 = 3$

$3x = 3 - 5 \Rightarrow 3x = -2$

$\therefore x = \frac{-2}{3}$

3.4 Laws of Logarithms

(a) $\log_a x + \log_a y = \log_a (x \times y) = \log_a xy$

(b) $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

(c) $\log_a x^n = n \log_a x$

(d) $\log_a a = 1$

(e) $\log_a 1 = 0$

Examples

(1) Simplify each of the following:

(a) $\log_3 27 + 2\log_3 9 - \log_3 54$ (b) $\frac{1}{2}\log_4 8 + \log_4 32 - \log_4 2$

Solution

(a) $\log_3 27 + 2\log_3 9 - \log_3 54$

$= \log_3(27 \times 81) - \log_3(27 \times 2)$

$= \log_3 27 + \log_3 81 - \log_3 27 - \log_3 2$

$= 4\log_3 3 - \log_3 2$

$= 4 - \log_3 2$

(b) $\frac{1}{2}\log_4 8 + \log_4 32 - \log_4 2$

$= \frac{1}{2}\log_4 8 + \log_4 32 - \log_4 2$

$= \frac{3}{4}\log_2 2 + \frac{5}{2}\log_2 2 - \frac{1}{2}\log_2 2$

$= \frac{3}{4} + \frac{5}{2} - \frac{1}{2}$

$= \frac{11}{4}$

4.0 CONCLUSION

In this unit you have studied how to Use the laws of indices in calculations and simplification. You have studied how to solve exponential equations. You have also studied the relationship between indices and logarithms to solve problems.

5.0 SUMMARY

In this unit you studied:

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $\left(a^m\right)^n - a^{m \times n} = a^{m^n}$
4. $a^0 = 1$

5. $a^{-m} = \dfrac{1}{a^m}$
6. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
7. $\log_a x + \log_a y = \log_a (x \times y) = \log_a xy$
8. $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$
9. $\log_a x^n = n \log_a x$
10. $\log_a a = 1$
11. $\log_a 1 = 0$

6.0 TUTOR-MARKED ASSIGNMENT

(1) Evaluate the following:

(a) $32^{\frac{2}{5}}$ (b) $25^{1.5}$ (c) $(0.00001)^2$

(2) Solve the following exponential equations:

(a) $\dfrac{4}{2^x} = 64^x$ (b) $3^{2x} - 4(3^{x+1}) + 27 = 0$ (c) $10^x = \dfrac{1}{0.001}$

(3) Simplify each of the following:

(a) $\log_9 3 + \log_9 243 - 2\log_9 3$ (b) $\log_5 12.5 + \log_5 5$ (c) $\log_2 \sqrt{8} + \log_3 \sqrt{3}$

(4) solve the following equations

(a) $\log_{10}(10 + 9x) - \log_{10}(11 - x) = 2$ (b) $\log_2 (x - 3) + \log_2 x = 2$

7.0 REFERENCES/FURTHER READING

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UNIT7: MATHEMATICAL INDUCTION

Content

1.0 INTRODUCTION
2.0 OBJECTIVES
3.0 MAIN CONTENT
3.1 Definition
3.2 Examples
4.0 CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

2.0 OBJECTIVES

At the end of this unit you should be able to apply the principles of Mathematical Induction.

3.0 MAIN CONTENT

3.1 Definition

The principle of Mathematical Induction is to establish the truth of a mathematical statement P(n) for all positive integer (natural numbers) n.

The principle states that that:

(i) P(n) is a mathematical statement and

(ii) P(1) is true; (i.e. P(n) is true for n=1) and

(iii) P(k+1) is true for any arbitrary natural number k; (i.ei when we assume that P(k) is true, we can by this assumption establish that P(k+1) is also true),

Then P(n) is true for all natural numbers n = 1, 2, 3,…

3.2 Examples

(1) Prove the statement P(n): $1^3+2^3+\ldots+n^3 = (1 + 2 +\ldots+ n)^2$ by mathematical induction.

Solution

Let P(n) be the statement $1^3+2^3+\ldots+n^3=(1+2+\ldots+n)^2$

When n = 1,

The LHS = 1^3= 1

The RHS = 1^2 = 1

So P(n) is true

Suppose P(k) is true, then

$1^3 + 2^3 + ... + k^3 = (1 + 2 + ... + k)^2$ (i)

Now for the statement $p(k+1)$,

The LHS will be given by: $1^3 + 2^3 + ... + k^3 + (k+1)^3 = (1 + 2 + ... + k)^2 + (k+1)^3$

Using the RHS of (i)

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{2}(k^2 + 4k + 4) = \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$= (1 + 2 + ... + k + (k+1))^2$$

Which gives the RHS of $p(k+1)$,

Hence $p(k+1)$ is true, so it follows that $p(n)$ is true for all *n*.

(2) $p(n): \sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1)$

Solution

When $n = 1$,

The LHS = $1^2 = 1$

The RHS = $\frac{1}{6}(1)(2)(3) = 1$

So $p(1)$ is true, then

$\sum_{r=1}^{k} r^2 = \frac{1}{6} k(k+1)(2k+1)$ (i)

Now for the statement $p(k+1)$,

The LHS will be given by:

$$\sum_{r=1}^{k+1} r^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{1}{6}(k+1)(k+2)(2k+3)$$

Which gives the RHS of $p(k+1)$

Hence $p(k+1)$ is true, so it follows that $p(n)$ is true for all *n*.

4.0 CONCLUSION

In this unit you have studied how to apply the principles of Mathematical induction.

5.0 SUMMARY

The principle states that that:

(i) P(n) is a mathematical statement and

(ii) P(1) is true; (i.e. P(n) is true for n=1) and

(iii) P(k+1) is true for any arbitrary natural number k; (i.ei when we assume that P(k) is true, we can by this assumption establish that P(k+1) is also true),

Then P(n) is true for all natural numbers n = 1, 2, 3,…

6.0 TUTOR MARKED ASSIGNMENT

Prove all the mathematical statements for all natural number n using mathematical induction:

1. $2^n = n^2$
2. $1+3+5+...+(2n-1) = n^2$
3. $4^n \geq 3n^2 + 1$

7.0 REFERENCES/FURTHER READING

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UNIT8: BINOMIAL THEOREM

1.0 INTRODUCTION

2.0 OBJECTIVES
3.0 MAIN CONTENT
3.1 Pascal Triangle
3.2The Binomial Expansion Formula

4.0 CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

In this unit you will study the pascal triangle and binomial expansion formula and how to solve problems using the formula.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Use pascal triangle to solve problems.

2. Use binomial expansion formula to solve problems.

3.0 MAIN CONTENT

3.1 Pascal Triangle

The binomial coefficients appear as the entries of Pascal's triangle where each entry is the sum of the two above it.
In elementary algebra, the **binomial theorem** describes the algebraic expansion of powers of a binomial. According to the theorem, it is possible to expand the power $(x + y)^n$ into a sum involving terms of the form ax^by^c, where the exponents b and c are nonnegative integers with $b + c = n$, and the coefficient a of each term is a specific positive integer depending on n and b. When an exponent is zero, the corresponding power is usually omitted from the term.

This formula and the triangular arrangement of the binomial coefficients are often attributed to Blaise Pascal, who described them in the 17th century, but they were known to many mathematicians who preceded him.

Consider the expansions of each of the following:

$$(x+y)^0;\ (x+y)^1;\ (x+y)^2;\ (x+y)^3;\ (x+y)^4;\ (x+y)^5;\ (x+y)^6;\ (x+y)^7$$

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x+1y$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$
$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$
$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

The coefficients of x and y can be displayed in an array as:

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

The array of coefficients displayed above, is called Pascal's Triangle, and it is used to in determining the coefficients of the terms of the powers of a binomial expression.

Notice that

1. the powers of x go down until it reaches 0 ($x^0 = 1$),starting value is n (the n in $(x+y)^n$.)
2. the powers of y go up from 0 ($y^0 = 1$) until it reaches n (also the n in $(x+y)^n$.)
3. the nth row of the Pascal's Triangle will be the coefficients of the expanded binomial. (Note that the top is row 0.)
4. for each line, the number of products (i.e. the sum of the coefficients) is equal to 2^n.
5. for each line, the number of product groups is equal to $n+1$.

Examples

7.0 Using Pascal’s triangle, expand and simplify completely: $(2x+3y)^4$.

Solution

$$(2x+3y)^4 = (2x)^4 + 4(2x)^3(3y) + 6(2x)^2(3y)^2 + 4(2x)(3y)^3 + (3y)^4$$
$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

8.0 Using Pascal’s triangle, expand and simplify completely: $(x-2y)^5$.

Solution

$$(x-2y)^5 = x^5 + 5x^4(-2y) + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + (-2y)^5$$
$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

9.0 Using Pascal’s triangle, simplify, correct to 5 decimal places, $(1.01)^4$.

Solution

We can write $(1.01)^4 = (1+0.01)^4$

$$(1+0.01)^4 = 1 + 4(0.01) + 6(0.01)^2 + 4(0.01)^3 + (0.01)^4$$
$$= 1 + 0.04 + 0.0006 + 0.000004 + 0.00000001$$
$$= 1.04060401$$
$$= 1.04060(5\text{d.p.})$$

The Binomial Expansion Formula

According to the theorem, it is possible to expand any power of *x* + *y* into a sum of the form

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x$$

where each $\binom{n}{k}$ is a specific positive integer known as binomial coefficient. This formula is also referred to as the **Binomial Formula** or the **Binomial Identity**. Using summation notation, it can be written as

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

The final expression follows from the previous one by the symmetry of *x* and *y* in the first expression, and by comparison it follows that the sequence of binomial coefficients in the formula is symmetrical.

A variant of the binomial formula is obtained by substituting 1 for *y*, so that it involves only a single variable. In this form, the formula reads

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n,$$

or equivalently

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Where,

$$\binom{n}{k} = {}^nC_k = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)(n-3)\ldots(n-k+1)(n-k)!}{(n-k)!k!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

Example

1. (a) Using the binomial theorem, expand $(1+2x)^5$, simplifying all the terms.

   (b) use your expansion to calculate the value of 1.02^5, correct to six significant figures.

   Solution

   (a)

$$(1+2x)^5 = 1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + {}^5C_3(2x)^3 + {}^5C_4(2x)^4 + {}^5C_5(2x)^5$$

$$= 1 + 5(2x) + \frac{5.4}{1.2}.4x^2 + \frac{5.4.3}{1.2.3}.8x^3 + \frac{5.4.3.2}{1.2.3.4}.16x^4 + 32x^5$$

$$= 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$$

   (b) $(1.05)^5 = (1+0.002)^5$

   Put $(1+0.02)^5 = 1+2x$

   $2x = 0.02$

   $x = 0.01$

   Hence,

$$(1.05)^5 = 1 + 10(0.01) = 40(0.01)^2 + 80(0.01)^3 + 80(0.01)^4 + 32(0.01)^5$$

$$= 1 + 0.1 + 0.004 + 0.00008 + 0.00000008$$

$$= 1.10408(6 \text{ s.f.})$$

4.0 CONCLUSION

In this unit you have studied Pascal's triangle and Binomial expansion formula. We have also use both Pascal's triangle and Binomial expansion formula to solve problems of binomial theorem.

5.0 SUMMARY

In this unit you studied:

The binomial expansion formula for $(x+y)^n$ is

$$(x+y)^n = x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_k x^{n-k} y^k + \ldots y^n$$

Where

$${}^nC_k = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)(n-3)\ldots(n-k+1)(n-k)!}{(n-k)!k!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

6.0 TUTOR-MARKED ASSIGNMENT

1. Use the binomial theorem to write down and simplify all the terms of the expansion of $\left(1-\frac{1}{4}x\right)^5$.
2. Obtain the first three terms of the expansion of $\left(3+\frac{1}{3}x\right)^9$.
3. Obtain the expansion of $(1+x)^4 + (1-x)^4$.
   Use your expansion to evaluate $(1.01)^4 + (0.99)^4$, correct to 4 decimal places.

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The horizontal lines in a matrix are called **rows** and the vertical lines are called **columns**. A matrix with m rows and n columns is called an m-by-n matrix (or $m \times n$ matrix) and m and n are called its **dimensions**.

The places in the matrix where the numbers are, are called *entries*. The entry of a matrix A that lies in the row number i and column number j is called the i,j entry of A. This is written as $A[i,j]$ or a_{ij}.

We write $A := (a_{ij})_{m \times n}$ to define an $m \times n$ matrix A with each entry in the matrix called a_{ij} for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

$$A = \begin{bmatrix} a1,1 & a1,2 & a1,3 & \ldots \\ a2,1 & a2,2 & a2,3 & \ldots \\ a3,1 & a3,2 & a3,3 & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{bmatrix}$$

Example

The matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 7 \\ 4 & 9 & 2 \\ 6 & 1 & 5 \end{bmatrix}$$

is a 4×3 matrix. This matrix has m=4 rows, and n=3 columns.

The element $A[2,3]$ or a_{23} is 7.

3.1.2 Operations of Matrices

Addition

The sum of two matrices is the matrix, which (*i*,*j*)-th entry is equal to the sum of the (*i*,*j*)-th entries of two matrices:

$$\begin{bmatrix}1 & 3 & 2\\ 1 & 0 & 0\\ 1 & 2 & 2\end{bmatrix} + \begin{bmatrix}0 & 0 & 5\\ 7 & 5 & 0\\ 2 & 1 & 1\end{bmatrix} = \begin{bmatrix}1+0 & 3+0 & 2+5\\ 1+7 & 0+5 & 0+0\\ 1+2 & 2+1 & 2+1\end{bmatrix} = \begin{bmatrix}1 & 3 & 7\\ 8 & 5 & 0\\ 3 & 3 & 3\end{bmatrix}$$

The two matrices have the same dimensions. Here $A + B = B + A$ is true.

Multiplication of two matrices

The multiplication of two matrices is a bit more complicated:

$$\begin{bmatrix}a1 & a2\\ a3 & a4\end{bmatrix} \cdot \begin{bmatrix}b1 & b2\\ b3 & b4\end{bmatrix} = \begin{bmatrix}(a1 \cdot b1 + a2 \cdot b3) & (a1 \cdot b2 + a2 \cdot b4)\\ (a3 \cdot b1 + a4 \cdot b3) & (a3 \cdot b2 + a4 \cdot b4)\end{bmatrix}$$

So with Numbers:

$$\begin{bmatrix}3 & 5\\ 1 & 4\end{bmatrix} \cdot \begin{bmatrix}2 & 3\\ 5 & 0\end{bmatrix} = \begin{bmatrix}(3 \cdot 2 + 5 \cdot 5) & (3 \cdot 3 + 5 \cdot 0)\\ (1 \cdot 2 + 4 \cdot 5) & (1 \cdot 3 + 4 \cdot 0)\end{bmatrix} = \begin{bmatrix}31 & 9\\ 22 & 3\end{bmatrix}$$

1. two matrices can be multiplied with each other even if they have different dimensions, as long as the number of columns in the first matrix is equal to the number of rows in the second matrix.
2. the result of the multiplication, called the product, is another matrix with the same number of rows as the first matrix and the same number of columns as the second matrix.
3. the multiplication of matrices is not commutative, this means, in general that $A \cdot B \neq B \cdot A$
4. the multiplication of matrices is associative, this means $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

3.1.2 Special matrices

There are some matrices that are special.

Square matrix

A square matrix has the same number of rows as columns, so m=n.

An example of a square matrix is

$$\begin{bmatrix} 5 & -2 & 4 \\ 0 & 9 & 1 \\ -7 & 6 & 8 \end{bmatrix}$$

This matrix has 3 rows and 3 columns: m=n=3.

Identity

Every square dimension set of a matrix has a special counterpart called an "identity matrix". The identity matrix has nothing but zeroes except on the main diagonal, where there are all ones. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix. There is exactly **one** identity matrix for each square dimension set. An identity matrix is special because when multiplying any matrix by the identity matrix, the result is always the original matrix with no change.

Inverse matrix

An inverse matrix is a matrix that, when multiplied by another matrix, equals the identity matrix. For example:

$$\begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 7 & -8 \\ -6 & 7 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 7 & 8 \\ 6 & 7 \end{bmatrix}$.

One column matrix

A matrix, that has many rows, but only one column, is called a column vector.

Transpose of a Matrix

The *transpose* of an m-by-n matrix A is the n-by-m matrix A^T formed by turning rows into columns and columns into rows, i.e. $A_{i,j} = A^T_{j,i} \forall i, j$. An example is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

- A square matrix whose transpose is equal to itself is called a *symmetric matrix*; that is, A is symmetric if $A^{\mathrm{T}} = A$.. An example is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$$

- A square matrix whose transpose is equal to its negative is called *skew-symmetric matrix*; that is, A is skew-symmetric if $A^{\mathrm{T}} = -A$.. An example is

$$\begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix}$$

3.2 Determinants

The determinant takes a square matrix and returns a number. To understand what the number means, take each column of the matrix and draw it as a vector. The parallelogram drawn by those vectors has an area, which is the determinant. For all 2x2 matrices, the formula is very simple:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

For 3x3 matrices the formula is more complicated:

$$\det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right) = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

There are no simple formulas for the determinants of larger matrices, and many computer programmers study how to get computers to quickly find large determinants.

Properties of determinants

There are three rules that all determinants follow. These are:

1. The determinant of an identity matrix is 1
2. If two rows or two columns of the matrix are exchanged, then the determinant is multiplied by -1. Mathematicians call this *alternating*.
3. If all the numbers in one row or column are multiplied by another number *n*, then the determinant is multiplied by *n*. Also, if a matrix *M* has a column *v* that is the sum of two column matrices v_1 and v_2, then the determinant of *M* is the sum of the determinants of *M* with v_1 in place of *v* and *M* with v_2 in place of *v*. These two conditions are called *multi-linearity*.

Solution of Systems of Linear Equations

1. Solve the system of simultaneous linear equations

   $3x + 2y + z = 0$

   $x - y + 3z = 0$

   $2x + 3y - 2z = 0$

   Solution

   The matrix equation is

   $$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 3 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   So the augmented matrix is

   $$\begin{pmatrix} 3 & 2 & 1 & : & 0 \\ 1 & -1 & 3 & : & 0 \\ 2 & 3 & -2 & : & 0 \end{pmatrix}$$

   The elementary row operations that reduced this matrix are as follows:

   $$\begin{pmatrix} 3 & 2 & 1 & : & 0 \\ 1 & -1 & 3 & : & 0 \\ 2 & 3 & -2 & : & 0 \end{pmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 1 & -1 & 3 & : & 0 \\ 3 & 2 & 1 & : & 0 \\ 2 & 3 & -2 & : & 0 \end{pmatrix}$$

   $R_2 \to R_2 - 3R_1$

   $R_3 \to R_3 - 2R_1$

   $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & -1 & 3 & : & 0 \\ 0 & 5 & -8 & : & 0 \\ 0 & 5 & -8 & : & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 3 & : & 0 \\ 0 & 5 & -8 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

$$R_1 \to R_1 + \frac{1}{5}R_2$$

$$R_2 \to \frac{1}{5}R_2$$

$$\begin{pmatrix} 1 & 0 & \frac{7}{5} & : & 0 \\ 0 & 1 & -\frac{8}{5} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

This implies the system is equivalent to

$$x + \frac{7}{5}z = 0$$

$$y - \frac{8}{5}z = 0$$

It is thus possible to solve for x and y in terms of the unknown z. thus, if we let z =1, we get

$$x = \begin{pmatrix} -\frac{7}{5} \\ \frac{8}{5} \\ 1 \end{pmatrix}$$

Hence, any solution is of the form

$$c\begin{pmatrix} -\frac{7}{5} \\ \frac{8}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{5}c \\ \frac{8}{5}c \\ c \end{pmatrix}$$

For example, if c = 5, then we get the solution

$\begin{pmatrix} -7 \\ 8 \\ 5 \end{pmatrix}$ and if c = -5 we get the solution $\begin{pmatrix} 7 \\ -8 \\ -5 \end{pmatrix}$

if c = 0 we obtain the trivial solution

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2. Solve the following system of linear equations

   $x + 3y + 2z = -13$

   $2x - 6y + 3z = 32$

   $3x - 4y - z = 12$

   Solution

   The solution requires the values of four determinants: the denominator, the determinant of the coefficient matrix A.

   $$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -6 & 3 \\ 3 & -4 & -1 \end{vmatrix} = 1(6+12) - 3(-2-9) + 2(-8+18)$$

   $$= 18 + 33 + 20 = 71$$

The denominator of x

$$|A_x| = \begin{vmatrix} -13 & 3 & 2 \\ 32 & -6 & 3 \\ 12 & -4 & -1 \end{vmatrix} = -13(6+12) - 3(-32-36) + 2(-128+72)$$

$$= 234 + 204 - 112 \ = -142$$

The denominator of y

$$|A_y| = \begin{vmatrix} 1 & -13 & 2 \\ 2 & 32 & 3 \\ 3 & 12 & -1 \end{vmatrix} = -1(-32-36) - (-13)(-2-9) + 2(24-96)$$

$$= -68 - 143 - 144 \ = -355$$

The denominator of z

$$|A_z| = \begin{vmatrix} 1 & 3 & =13 \\ 2 & -6 & 32 \\ 3 & -4 & 12 \end{vmatrix} = 1(-72+128) - 3(24-96) + (-13)(-8+18)$$

$$= 56 + 216 - 130 \ = 142$$

Then

$$x = \frac{|A_x|}{|A|} = -\frac{142}{71} = -2$$

$$y = \frac{|A_y|}{|A|} = -\frac{355}{71} = -5$$

$$z = \frac{|A_z|}{|A|} = \frac{142}{71} = 2$$

4.0 CONCLUSION

In this unit you have studied matrices, determinant, basic operations of matrices and determinant. We also studied the solutions of systems of linear equations using determinant.

5.0 SUMMARY

In this unit you studied:

1. Addition and subtraction of matrices.
2. Multiplication of matrices.
3. Transpose of a matrices $A = A^T$
4. Inverse of a matrices is equal to identity matrix.
5. A square matrix has the same number of rows as columns, so m=n.

6. The determinant takes a square matrix and returns a number.
7. For all 2x2 matrices, the formula is very simple:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

8. For 3x3 matrices the formula is more complicated:

$$\det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right) = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

6.0 TUTOR-MARKED ASSIGNMENT

1. Let $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 2 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -2 & 5 \\ 2 & -2 & 4 & 7 \end{bmatrix}$. Determine $A + B$.

2. Let $A = \begin{bmatrix} 3 & 4 & -1 \\ 7 & -8 & 2 \\ 9 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 & -1 \\ 3 & -1 & 8 \\ 4 & 9 & 0 \end{bmatrix}$. Determine $A - B$.

3. Compute AB if $A = \begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 6 & 4 & 1 \\ 3 & 4 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ 2 & 1 \\ -1 & 4 \end{bmatrix}$.
4. Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 0 \end{bmatrix}$ Compute A^T.
5. Evaluate $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 4 & 3 \\ 0 & 0 & 0 \end{vmatrix}$

6. Solve the systems of equations

$$x + y + z = -2$$
$$2x + 6y - z = 3$$
$$3x + 2y + 3z = -4$$

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UNIT10: QUADRATIC EQUATION

1.0 INTRODUCTION

2.0 OBJECTIVES
3.0CONTENTS
3.1 Definition

3.2 Solutions of Quadratic Equations

3.3 The Discriminant of a Quadratic Equation

3.4 The Sum and Product of the Roots

3.5 Examples

4.0CONCLUSION
5.0 SUMMARY
6.0 TUTOR-MARKED ASSIGNMENT
7.0 REFERENCES/FURTHER READING

1.0 INTRODUCTION

In this unit you will study the quadratic equation. How to solve problems of quadratic equation.

2.0 OBJECTIVES

At the end of this unit you should be able to:

1. Define quadratic equation.

2. solve problems quadratic equations.

3.0 MAIN CONTENT

3.1 Definition

A quadratic expression in one independent variable x is given by:

$$ax^2 + bx + c \qquad (3.1)$$

Where a, b, c are constants with $a \neq 0,$ otherwise the expression becomes a linear one.

A quadratic equation is given by:

$$ax^2 + bx + c = 0 \qquad (3.2)$$

3.2 Solutions of Quadratic Equations

A quadratic equation with real or complex coefficients has two solutions, called *roots*. These two solutions may or may not be distinct, and they may or may not be ***real***.

Having

$$ax^2 + bx + c = 0,$$

the roots are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the symbol "±" indicates that both

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions of the quadratic equation.

Derivations of the quadratic formula

By completing the square

The quadratic formula can be derived by the method of completing the square,[19] so as to make use of the algebraic identity:

$$x^2 + 2xh + h^2 = (x + h)^2.$$

Dividing the quadratic equation

$$ax^2 + bx + c = 0$$

by *a* (which is allowed because *a* is non-zero), gives:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

The quadratic equation is now in a form to which the method of completing the square can be applied. To "complete the square" is to add a constant to both sides of the equation such that the left hand side becomes a complete square:

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\frac{b}{a}\right)^2 = -\frac{c}{a} + \left(\frac{1}{2}\frac{b}{a}\right)^2,$$

which produces

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

The right side can be written as a single fraction, with common denominator $4a^2$. This gives

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Taking the square root of both sides yields

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Isolating *x*, gives

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

3.3 The Discriminant of a Quadratic Equation

In the above formula, the expression underneath the square root sign is called the ***discriminant*** of the quadratic equation, and is often represented using an upper case *D* or an upper case Greek delta, the initial of the Greek word **Δ**ΞΙΟΦΗΣΥΨLΙ."*Diakrínousa*, discriminant:

$$\Delta = b^2 - 4ac.$$

A quadratic equation with *real* coefficients can have either one or two distinct real roots, or two distinct complex roots. In this case the discriminant determines the number and nature of the roots. There are three cases:

1. If the discriminant is positive, then there are two distinct roots, both of which are real numbers:

$$\frac{-b+\sqrt{\Delta}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{\Delta}}{2a}$$

For quadratic equations with integer coefficients, if the discriminant is a perfect square, then the roots are rational numbers—in other cases they may be quadratic irrationals.

2. If the discriminant is zero, then there is exactly one distinct real root, sometimes called a double root:

$$-\frac{b}{2a}.$$

3. If the discriminant is negative, then there are *no* real roots. Rather, there are two distinct (non-real) complex roots, which are complex conjugates of each other:[3]

$$\frac{-b}{2a}+i\frac{\sqrt{-\Delta}}{2a}, \quad \text{and} \quad \frac{-b}{2a}-i\frac{\sqrt{-\Delta}}{2a},$$

where *i* is the imaginary unit.

Thus the roots are distinct if and only if the discriminant is non-zero, and the roots are real if and only if the discriminant is non-negative.

3.4 The Sum and Product of the Roots

Given the roots

$$\alpha=\frac{-b-\sqrt{b^2-4ac}}{2a}; \quad \beta=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

of the quadratic equation

$$ax^2+bx+c=0,$$

the sum of the roots is given by:

$$\alpha+\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b+\sqrt{b^2-4ac}}{2a}=-\frac{b}{a}$$

While the product of the roots is given by:

$$\alpha\beta = \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) = \frac{c}{a}$$

So in general

$$\alpha+\beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}$$

3.5 Examples

1. Given the two quadratic equations $ax^2+bx+c=0$ and $px^2+qx+r=0,$ find the conditions on the coefficients a, b, c, p, q, r for the two quadratic equations to have the same roots.

Solution:

Let the common roots be α and $\beta,$ then

$$\alpha+\beta = -\frac{b}{a} = -\frac{q}{p} \qquad \text{(i)}$$

$$\alpha\beta = \frac{c}{a} = \frac{r}{p} \qquad \text{(ii)}$$

From (i)

$$aq = bp \Rightarrow a = \frac{bp}{q} \qquad \text{(iii)}$$

From (ii)

$$ar = cp \Rightarrow a = \frac{cp}{r} \qquad \text{(iv)}$$

(iii) and (iv) imply that

$$a = \frac{bp}{q} = \frac{cp}{r} \qquad \text{(v)}$$

2. Solve the equation $z^4 - 3z^2 + 2 = 0$

Solution

Let $u = z^2$ (i)

The given equation is transformed to

$u^2 - 3u + 2 = 0$ (ii)

The roots of (ii) are given by

$$u = \frac{3 \pm 1}{2} = 1,2 \quad \text{(iii)}$$

And so,

$u = z^2 = 1,\ \ u = z^2 = 2$ (iv)

And $z = \pm 1,\ z = \pm\sqrt{2}.$

3. If the roots of the equation $\mathbf{a}x^2 + bx + c = 0$ are α and β find the equation whose roots are

$$\left(\frac{\alpha}{\beta} - \frac{1}{\beta}\right),\ \text{and} \left(\frac{\beta}{\alpha} - \frac{1}{\alpha}\right)$$

solution

if the roots of the equation $\mathbf{a}x^2 + bx + c = 0$ are α and β the

$(\alpha + \beta) = -\frac{b}{a},\ \alpha\beta = \frac{\mathrm{c}}{\mathrm{a}}$

if the required equation is given by $x^2 + Bx + D = 0$

the sum of the roots is given by

$$-B=\frac{\alpha}{\beta}-\frac{1}{\beta}+\frac{\beta}{\alpha}-\frac{1}{\alpha}=\frac{\alpha^2-\alpha+\beta^2-\beta}{\alpha\beta}$$

$$=\frac{\alpha^2+\beta^2-(\alpha+\beta)}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta-(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(-b/a\right)^2-2\left(c/a\right)-\left(-b/a\right)}{c/a}=\frac{b^2-2ac+ab}{ac}$$

And so

$$B=\frac{b^2-2ac+ab}{ac}$$

The product of the roots is given by

$$D=\left(\frac{\alpha}{\beta}-\frac{1}{\beta}\right)\left(\frac{\beta}{\alpha}-\frac{1}{\alpha}\right)=\frac{1}{\alpha\beta}(\alpha-1)(\beta-1)$$

$$=\frac{1}{\alpha\beta}[\alpha\beta-(\alpha+\beta)-1]=\frac{1}{c/a}\left[c/a-\left(-b/a\right)-1\right]=\frac{1}{c}(a+b+c)$$

And so

$$x^2+Bx+D=x^2-\frac{b^2-2ac+ab}{ac}x+\frac{1}{c}(a+b+c)$$

The required equation is thus given as

$$acx^2-\left(b^2-2ac+ab\right)x+a(a+b+c)=0$$

4.0 CONCLUSION

In this unit you have studied quadratic equation, the discriminant and sum and product of the roots of quadratic equation. We also studied the solutions of quadratic equations.

5.0 SUMMARY

In this unit you studied:

1. The general form of quadratic equation $ax^2+bx+c=0$

2. The discriminant $\Delta = b^2 - 4ac$.
3. The sum and product of the roots of quadratic equation $\alpha + \beta = -\frac{b}{a}; \ \alpha\beta = \frac{c}{a}$

6.0 TUTOR- MARKED ASSIGNMENT

1. If α and β are the roots of the equation $2x^2 - x - 4 = 0$ determine the

   (i) $\alpha^3 + \beta^3$ (ii) equation whose roots are $\alpha - \beta/\alpha$ and $\beta - \alpha/\beta$

2. Find the value of γ for which the equation $x^2 - 5x - 6 + \gamma(3x + 2) = 0$ has real roots.

3. Solve the equation $u(u+1) + \frac{12}{u(u+1)} = 8$.

4. Solve the equation $x^4 + 2x^3 - x^2 + 2x + 1 = 0$

7.0 REFERENCES/FURTHER READING

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