Oreofe Solarin

MATH 345 Homework 5 Solutions

Question 1

In SL-form:

$$-\frac{d}{dx}\Big[(1+x^2)\phi'(x)\Big] = \lambda\phi(x)$$

and we have $\phi(0) = 0$ and $\phi(1) = 0$

The Rayleigh quotient for a given u is:

$$R[u] = \frac{\int_0^1 p(x)[u'(x)]^2 dx}{\int_0^1 [u(x)]^2 dx} = \frac{\int_0^1 (1+x^2)[u'(x)]^2 dx}{\int_0^1 u(x)^2 dx}$$

for such u that satisfies the boundary conditions, we have $\lambda_1 \leq R[u]$

For our test function $u_T(x)$, the first eigenfunction should be positive on $(0,1)$, satisfy $\phi(0) = 0, \phi(1) = 0$ and have 0 derivative at $x = 0$

since $u_T(0) = 0$, $u_T'(0) = 1 - 0 = 1$, $u_T(1) = 0$

$$u_T(x) = x(1-x)$$

$$[u_T(x)]^2 = [x(1-x)]^2 = x^2(1-x)^2 = x^2(1-2x+x^2) = x^2 - 2x^3 + x^4$$

$$u_T'(x) = x \cdot -1 + (1-x)\cdot 1 = 1 - 2x$$

$$[u_T'(x)]^2 = [1-2x]^2 = 1 - 2x - 2x + 4x^2 = 1 - 4x + 4x^2$$

Numerator:

$$\begin{aligned}
N &= \int_0^1 (1+x^2)[u_T'(x)]^2 dx = \int_0^1 (1+x^2)(1-4x+4x^2)dx \\
&= \int_0^1 1 - 4x + 4x^2 + x^2 - 4x^3 + 4x^4 dx = \int_0^1 1 - 4x + 5x^2 - 4x^3 + 4x^4 dx \\
&= \Big[x - 2x^2 + \frac{5}{3}x^3 - x^4 + \frac{4}{5}x^5\Big]_0^1 \\
&= 1 - 2 + \frac{5}{3} - 1 + \frac{4}{5} = \frac{7}{15}
\end{aligned}$$

Denominator:

$$\begin{aligned}
D &= \int_0^1 u_T(x)^2 dx = \int_0^1 x^2 - 2x^3 + x^4 dx \\
&= \Big[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\Big]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}
\end{aligned}$$

Our Rayleigh Q:

$$R[u_T] = \frac{N}{D} = \frac{7/15}{1/30} = 14$$

So reasonable upper bound for the smallest eigenvalue is $\boxed{\lambda_1 \leq 14}$ If we use higher order terms in our u_T function we can get more accurate estitmates

Question 2

We have:

$$\frac{d}{dx}\Big(p(x)\frac{d\phi}{dx}\Big) + [\lambda\sigma(x) + q(x)]\phi = 0$$

with BCs $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

for $\lambda \gg 1$, $\lambda\sigma(x)$ dominates so we can drop $q(x)$. and we now have $(p\phi')' + \lambda\sigma\phi \approx 0$

$$\frac{\phi'' p}{p} + \frac{p'\phi}{p} + \frac{\lambda\sigma\phi}{p} \approx 0$$

$$\phi'' + \frac{p'\phi}{p} + \frac{\lambda\sigma\phi}{p} \approx 0$$

using ansatz:

$$\phi(x) = A(x)\cos\theta(x)$$

$$\phi' = A'\cos\theta(x) - A\theta'\sin\theta(x)$$

$$\phi'' = (A'' - A(\theta')^2)\cos\theta - (2A'\theta' + A\theta'')\sin\theta$$

$$(A'' - A(\theta')^2)\cos\theta - (2A'\theta' + A\theta'')\sin\theta + \frac{\lambda\sigma}{p}A\cos\theta \approx 0$$

cos coefficient:

$$A\Big(-(\theta')^2 + \lambda\frac{\sigma}{p}\Big) + A''.$$

sin coefficient:

$$-(2A'\theta' + A\theta'')$$

For large λ, the terms $A(\theta')^2$ and $\lambda\frac{\sigma}{p}A$ are $O(\lambda)$, while A'' is $O(1)$; thus we can drop A'' at leading order:

$$-(\theta')^2 + \lambda\frac{\sigma(x)}{p(x)} \approx 0 \quad \Rightarrow \quad (\theta')^2 \approx \lambda\frac{\sigma(x)}{p(x)}$$

$$\theta'(x) \approx \sqrt{\lambda}\sqrt{\frac{\sigma(x)}{p(x)}}.$$

$$\theta(x) \approx \sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(s)}{p(s)}}ds + C$$

$$\phi(x) \approx A(x)\cos\Big(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma}{p}}ds + C\Big)$$

using the Neumann boundary conditions

we have $\phi'(0) = \phi'(L) = 0$

$$\phi'(x) \approx -A\theta'(x)\sin\theta(x)$$

so

at $x = 0$:

$$\phi'(0) \approx -A\theta'(0)\sin\theta(0) = 0 \quad \Rightarrow \quad \sin\theta(0) \approx 0.$$

we can take $\theta(0) = 0$ (so $\delta = 0$)
at $x = L$:

$$\phi'(L) \approx -A\theta'(L)\sin\theta(L) = 0 \quad \Rightarrow \quad \sin\theta(L) \approx 0 \quad \Rightarrow \quad \theta(L) \approx n\pi,\ n = 0, 1, 2, \ldots$$

but

$$\theta(L) \approx \sqrt{\lambda}\int_0^L \sqrt{\frac{\sigma(s)}{p(s)}}ds = \sqrt{\lambda}J.$$

so the quantization condition is:

$$\sqrt{\lambda_n}\,J \approx n\pi \quad \Rightarrow \quad \boxed{\lambda_n \approx \frac{n^2\pi^2}{J^2}}, \qquad J = \int_0^L \sqrt{\frac{\sigma(x)}{p(x)}}\,dx$$

and the corresponding eigenfunctions (to leading order) are

$$\boxed{\phi_n(x) \approx A\cos\left(\frac{n\pi}{J}\int_0^x \sqrt{\frac{\sigma(s)}{p(s)}}ds\right)}$$

Question 3
WWS, for continuous $f:[0,\pi]\to\mathbb{R}$ (or even $\mathbb{C}$),

$$\sum_{k=1}^{n}\left|\int_0^\pi f(x)\sin(kx)\,dx\right|^2 \le \frac{\pi}{2}\int_0^\pi |f(x)|^2\,dx$$

using orthogonality of the sin
the inner product:

$$\langle g,h\rangle = \int_0^\pi g(x)\overline{h(x)}\,dx.$$

For $s_k(x)=\sin(kx)$ we have
For $k\neq m$:

$$\int_0^\pi \sin(kx)\sin(mx)\,dx = 0,$$

For $k=m$:

$$\int_0^\pi \sin^2(kx)\,dx = \frac{\pi}{2}$$

So $s_1,\ldots,s_n$ is an orthogonal set and

$$|s_k|^2 = \langle s_k, s_k\rangle = \frac{\pi}{2}$$

For any constants $c_1,\ldots,c_n$ we consider

$$g(x)=\sum_{k=1}^{n} c_k s_k(x)$$

and since $|f-g|^2\ge 0$,

$$0\le |f-g|^2 = |f|^2 - 2\Re\sum_{k=1}^{n} c_k\langle f,s_k\rangle + \sum_{k=1}^{n}|c_k|^2|s_k|^2$$

using $|s_k|^2=\pi/2$, this is

$$0\le |f|^2 - 2\Re\sum_{k=1}^{n} c_k\langle f,s_k\rangle + \frac{\pi}{2}\sum_{k=1}^{n}|c_k|^2 \tag{*}$$

we choose

$$c_k = \frac{2}{\pi}\langle f,s_k\rangle = \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)\,dx$$

Plug into (*):
Middle term:

$$-2\Re\sum c_k\langle f,s_k\rangle = -2\sum\frac{2}{\pi}|\langle f,s_k\rangle|^2 = -\frac{4}{\pi}\sum_{k=1}^{n}|\langle f,s_k\rangle|^2$$

Last term:

$$\frac{\pi}{2}\sum|c_k|^2 = \frac{\pi}{2}\sum\left(\frac{2}{\pi}\right)^2|\langle f,s_k\rangle|^2 = \frac{2}{\pi}\sum_{k=1}^{n}|\langle f,s_k\rangle|^2$$

so

$$0\le |f|^2 - \frac{4}{\pi}\sum|\langle f,s_k\rangle|^2 + \frac{2}{\pi}\sum|\langle f,s_k\rangle|^2 = |f|^2 - \frac{2}{\pi}\sum_{k=1}^{n}|\langle f,s_k\rangle|^2$$

rearranging:

$$\sum_{k=1}^{n}|\langle f,s_k\rangle|^2 \le \frac{\pi}{2}|f|^2$$

Finally, $\langle f, s_k \rangle = \int_0^\pi f(x)\sin(kx)\,dx$ and $|f|^2 = \int_0^\pi |f(x)|^2 dx$, so this is exactly

$$\boxed{\sum_{k=1}^{n} \left| \int_0^\pi f(x)\sin(kx)\,dx \right|^2 \le \frac{\pi}{2} \int_0^\pi |f(x)|^2\,dx}$$

Question 4

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x < L; 0 < y < H,$$

with $u(0, y, t) = u(L, y, t) = 0$ (Dirichlet in (x)),
$u_y(x, 0, t) = u_y(x, H, t) = 0$ (Neumann in (y)),
and $u(x, y, 0) = \alpha(x, y)$

SoV:

$$u(x, y, t) = X(x)Y(y)T(t)$$
$$XYT' = k_1 X''YT + k_2 XY''T$$
$$\frac{T'}{T} = k_1 \frac{X''}{X} + k_2 \frac{Y''}{Y}$$
$$\frac{T'}{T} = -\lambda, \qquad k_1 \frac{X''}{X} + k_2 \frac{Y''}{Y} = -\lambda.$$
$$k_1 \frac{X''}{X} + k_2 \frac{Y''}{Y} = -\lambda \quad \Rightarrow \quad \frac{X''}{X} + \frac{k_2}{k_1} \frac{Y''}{Y} = -\frac{\lambda}{k_1}.$$

separating again with another constant $-\mu$:

$$\frac{X''}{X} = -\mu, \qquad \frac{k_2}{k_1} \frac{Y''}{Y} = -(\tfrac{\lambda}{k_1} - \mu)$$

ODEs:

$$X'' + \mu X = 0, \qquad Y'' + \nu Y = 0, \qquad T' + \lambda T = 0,$$

where: $\nu = \frac{k_1}{k_2} \left(\frac{\lambda}{k_1} - \mu \right)$

For x: $X'' + \mu X = 0,\ X(0) = X(L) = 0$

$$\mu_n = \left(\frac{n\pi}{L} \right)^2, \qquad X_n(x) = \sin \frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

For y: $Y'' + \nu Y = 0,\ Y'(0) = Y'(H) = 0$

We have a neumann eigenproblem:

$$\nu_m = \left(\frac{m\pi}{H} \right)^2, \qquad Y_m(y) = \cos \frac{m\pi y}{H}, \qquad m = 0, 1, 2, \ldots$$

For each pair ((n,m)),

$$\lambda_{nm} = k_1 \mu_n + k_2 \nu_m = k_1 \left(\frac{n\pi}{L} \right)^2 + k_2 \left(\frac{m\pi}{H} \right)^2$$

$$T_{nm}(t) = e^{-\lambda_{nm} t}$$

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} \times e^{-\left(k_1 (n\pi/L)^2 + k_2 (m\pi/H)^2\right)t} \sin \frac{n\pi x}{L} \times \cos \frac{m\pi y}{H}$$

At $t = 0$,

$$\alpha(x, y) = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \cos \frac{m\pi y}{H}$$

so α is expanded in the mixed sin-cos part
using orthogonality:

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}dx = \frac{L}{2}\delta_{np}, \int_0^H \cos\frac{m\pi y}{H}\cos\frac{q\pi y}{H}dy = \begin{cases} H, & m = q = 0, \\ \frac{H}{2}\delta_{mq}, & m, q \geq 1. \end{cases}$$

we get

for $m \geq 1$:

$$A_{nm} = \frac{4}{LH}\int_0^L \int_0^H \alpha(x,y)\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}dydx;$$

for $m = 0$:

$$A_{n0} = \frac{2}{LH}\int_0^L \int_0^H \alpha(x,y)\sin\frac{n\pi x}{L}dydx.$$

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm}e^{-\left(k_1\left(\frac{n\pi}{L}\right)^2+k_2\left(\frac{m\pi}{H}\right)^2\right)t}\sin\frac{n\pi x}{L}\cos\frac{m\pi y}{H}}$$

with A_{nm} determined from $\alpha(x,y)$ by the formulas above

Question 5

$$c(x,y,z)\rho(x,y,z)\frac{\partial u}{dt} = \nabla \cdot (K_0(x,y,z)\nabla u)$$

$$u(x,y,z,t) = \phi(x,y,z)h(t)$$

$$\phi h' = \frac{1}{c\rho}(\nabla \cdot (K_0(h(t)\nabla\phi))) = \frac{h}{c\rho}\nabla \cdot (K_0\nabla\phi)$$

since K_0 depends only on space (last part)

$$\frac{h'}{h} = \frac{1}{c\rho\phi}\nabla \cdot (K_o\nabla\phi)$$

equate to $-\lambda$

$$\frac{h'}{h} = \frac{1}{c\rho\phi}\nabla \cdot (K_o\nabla\phi) = -\lambda$$

$$\frac{h'}{h} = -\lambda \qquad \frac{1}{c\rho\phi}\nabla \cdot (K_o\nabla\phi) = -\lambda$$

from the first h:

$$h(t) = Ce^{-\lambda t}$$

$$\nabla \cdot (K_0\nabla\phi) = -\lambda c\rho\phi \iff \nabla \cdot (K_0\nabla\phi) + \lambda c\rho\phi = 0$$

because we have $u = 0$ at the bds for all t and $u = \phi h$, we must have $\phi = 0$ on the region $\partial\Omega$

we just found the eigenvalue eq.:

$$\nabla \cdot (K_0\nabla\phi) + \lambda c\rho\phi = 0$$

and it matches expected

$$\nabla \cdot (p\nabla\phi) + \lambda\sigma(x,y,z)\phi = 0$$

$$\boxed{p(x,y,z) = K_0(x,y,z) \qquad \sigma(x,y,z) = c(x,y,z)\rho(x,y,z)}$$