Oreofe Solarin

MATH 345 Homework 3 Solution
Problem 2.5.27

$$\psi(r,\theta) = c_1 ln(\frac{r}{a}) + U(r - \frac{a^2}{r})\sin\theta$$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U(1-\frac{a^2}{r^2})\cos\theta, u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{c_1}{r} - U(1+\frac{a^2}{r^2})\sin\theta$$

On the cylinder $r = a : u_r(a,\theta) = 0$ and

$$u_\theta(a,\theta) = -\frac{c_1}{a} - 2U\ \sin\theta$$

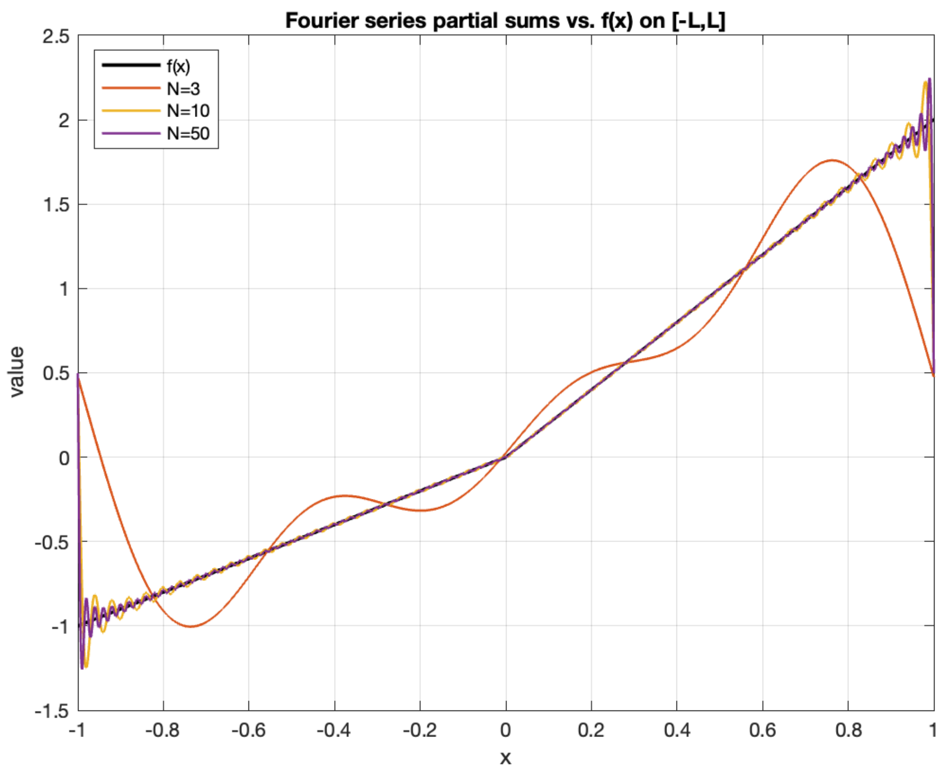
A stagination point on the cylinder need $u_\theta = 0$, so

$$\sin\theta = -\frac{c_1}{2aU}$$

This has a real solution $\iff |c_1| \le 2aU$
Since the circulation is $\tau = \int_0^{2\pi} u_\theta r\, d\theta = 2\pi c_1$
Thus, it needs to satisfy

$$|\tau| \le 4\pi aU$$

Problem 3.2.3 (e)



MATLAB code reference: 0.0.1

Problem 3.2.4

$$f(x) = e^{-x}, [-L, L]$$

We use

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\Big],$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Let $k = \dfrac{n\pi}{L}$.

$$\int e^{-x}\cos(kx)\,dx = \frac{e^{-x}\big(-\cos kx + k\sin kx\big)}{1+k^2}, \qquad \int e^{-x}\sin(kx)\,dx = \frac{e^{-x}\big(-\sin kx - k\cos kx\big)}{1+k^2}.$$

Our constant term value

$$a_0 = \frac{1}{L}\int_{-L}^{L} e^{-x}\,dx = \frac{1}{L}\left[-e^{-x}\right]_{-L}^{L} = \frac{e^L - e^{-L}}{L} = \frac{2\sinh L}{L}.$$

Cosine coefficients

$$\begin{aligned}
a_n &= \frac{1}{L}\left[\frac{e^{-x}(-\cos kx + k\sin kx)}{1+k^2}\right]_{-L}^{L} \\
&= \frac{1}{L(1+k^2)}\Big[(-\cos kL + k\sin kL)e^{-L} - (-\cos(-kL) + k\sin(-kL))e^{L}\Big] \\
&= \frac{1}{L(1+k^2)}\Big[(-\cos n\pi)e^{-L} - (-\cos n\pi)e^{L}\Big] \\
&= \frac{2\sinh L}{L}\cdot\frac{(-1)^n}{1+k^2}.
\end{aligned}$$

Since $1 + k^2 = 1 + (n\pi/L)^2 = (L^2 + n^2\pi^2)/L^2$

$$a_n = \frac{2L\,\sinh L\,(-1)^n}{L^2 + n^2\pi^2}.$$

Sine coefficients

$$\begin{aligned}
b_n &= \frac{1}{L}\left[\frac{e^{-x}(-\sin kx - k\cos kx)}{1+k^2}\right]_{-L}^{L} \\
&= \frac{1}{L(1+k^2)}\Big[(-\sin n\pi - k\cos n\pi)e^{-L} - (\sin n\pi - k\cos n\pi)e^{L}\Big] \\
&= \frac{1}{L(1+k^2)}\Big[-k(-1)^n e^{-L} + k(-1)^n e^{L}\Big] \\
&= \frac{2\sinh L}{L}\cdot\frac{k(-1)^n}{1+k^2}.
\end{aligned}$$

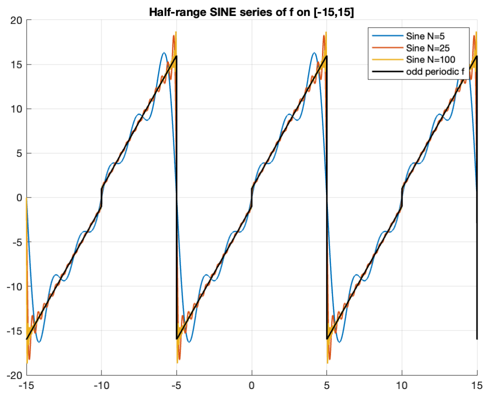
Thus

$$b_n = \frac{2n\pi\,\sinh L\,(-1)^n}{L^2 + n^2\pi^2}.$$

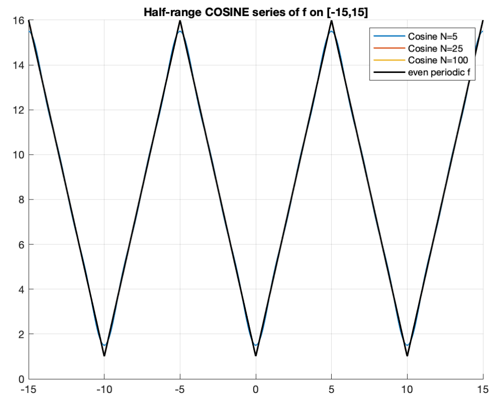
Final series

$$e^{-x} \sim \frac{\sinh L}{L} + \sum_{n=1}^{\infty}\left[\frac{2L\,\sinh L\,(-1)^n}{L^2+n^2\pi^2}\cos\frac{n\pi x}{L} + \frac{2n\pi\,\sinh L\,(-1)^n}{L^2+n^2\pi^2}\sin\frac{n\pi x}{L}\right] \quad (-L < x < L).$$

Problem 3.3.1



(a)



(b)

x	y	SineSeriesValue	CosineSeriesValue
5	-5	-16	16
9	-1	-4	4
22	2	7	7
101	1	4	4

Table 1: Values of x, y, and series evaluations

MATLAB code reference: 0.0.2
Problem 3.3.8 We can split a function $f(x)$ into even and odd parts:

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

For $f(x) = e^x$:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \cosh x + \sinh x$$

where $\cosh x$ is even and $\sinh x$ is odd
Problem 3.4.6
We can write the half-range cosine series

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

Differentiating term-by-term gives

$$e^x = \frac{d}{dx}e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\left(\frac{n\pi x}{L}\right),$$

Differentiating again yields (note no extra factor of x)

$$e^x = \frac{d^2}{dx^2}e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos\left(\frac{n\pi x}{L}\right),$$

It is *not* valid to conclude from these two series that $A_0 = 0$ and $A_n = 0$ by direct termwise comparison: the constant term and the other coefficients of the cosine series of e^x and those of $e^{x''}$ can differ by boundary terms

Since e^x satisfies $f'' = f$ on $(0, L)$, we can multiply the identity $f = f''$ by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate on $[0, L]$. Doing integration by parts twice gives, for $m \geq 1$,

$$\int_0^L f(x) \cos\Big(\frac{m\pi x}{L}\Big)\, dx = \big[f'(x)\cos\big(\tfrac{m\pi x}{L}\big)\big]_0^L - \Big(\frac{m\pi}{L}\Big)^2 \int_0^L f(x) \cos\Big(\frac{m\pi x}{L}\Big)\, dx.$$

Because $\sin(m\pi) = \sin 0 = 0$, the intermediate boundary terms vanish and then using orthogonality, $\int_0^L f \cos(\frac{m\pi x}{L})\, dx = \frac{L}{2} A_m$ for $m \geq 1$, and $f'(0) = 1$, $f'(L) = e^L$, we find

$$\Big(1 + \big(\tfrac{m\pi}{L}\big)^2\Big)\, \frac{L}{2} A_m = e^L(-1)^m - 1,$$

so

$$A_m = \frac{2}{L}\, \frac{(-1)^m e^L - 1}{1 + \big(\frac{m\pi}{L}\big)^2} \quad (m \geq 1)$$

For the constant term, integrate $f = f''$ on $[0, L]$:

$$\int_0^L f(x)\, dx = f'(L) - f'(0) = e^L - 1.$$

But $\int_0^L f = LA_0$, hence

$$A_0 = \frac{e^L - 1}{L}.$$

Problem 3.5.2

(a) Using the given Fourier sine series of x on $(0 < x < L)$,

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right),$$

we want to find the Fourier cosine series of x^2 on $(0 < x < L)$.
Write

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \qquad a_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx.$$

Let $k = \dfrac{n\pi}{L}$. Integrate by parts twice:

$$\begin{aligned} a_n &= \frac{2}{L}\left[\frac{x^2 \sin(kx)}{k}\right]_0^L - \frac{4}{Lk}\int_0^L x \sin(kx)\, dx \\ &= -\frac{4}{Lk}\left(\left[-\frac{x\cos(kx)}{k}\right]_0^L + \frac{1}{k}\left[\frac{\sin(kx)}{k}\right]_0^L\right) \\ &= -\frac{4}{Lk}\left(-\frac{L\cos(n\pi)}{k}\right) = \frac{4(-1)^n}{k^2} = \frac{4L^2}{\pi^2}\frac{(-1)^n}{n^2}. \end{aligned}$$

Also

$$\frac{a_0}{2} = \frac{1}{L}\int_0^L x^2\, dx = \frac{L^2}{3}.$$

Therefore

$$x^2 \sim \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L.$$

(b) From part (a), determine the Fourier sine series of x^3 on $(0 < x < L)$.
Write

$$x^3 \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right) dx.$$

With $k = \dfrac{n\pi}{L}$ and one integration by parts,

$$\begin{aligned} b_n &= \frac{2}{L}\left[-\frac{x^3\cos(kx)}{k}\right]_0^L + \frac{6}{Lk}\int_0^L x^2 \cos(kx)\, dx \\ &= -\frac{2L^3(-1)^n}{n\pi} + \frac{6}{Lk}\cdot\frac{L}{2}a_n = -\frac{2L^3(-1)^n}{n\pi} + \frac{3}{k}a_n. \end{aligned}$$

Using $a_n = \dfrac{4L^2(-1)^n}{n^2\pi^2}$ and $1/k = L/(n\pi)$,

$$b_n = -\frac{2L^3(-1)^n}{n\pi} + \frac{12L^3(-1)^n}{n^3\pi^3} = 2L^3(-1)^{n+1}\left(\frac{1}{n\pi} - \frac{6}{n^3\pi^3}\right).$$

Hence

$$x^3 \sim \sum_{n=1}^{\infty} 2L^3(-1)^{n+1}\left(\frac{1}{n\pi} - \frac{6}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L.$$

Appendix

MATLAB source listings

0.0.1 p323.m

```
% Fourier series sketch for:
% f(x) = x,   -L < x < 0
%       2x,   0 < x < L
clear; clc;

L = 1;
Ns = [3 50 100];
xx = linspace(-L, L, 4000);

% original piecewise f on (-L,L) (one period)
ftrue = (xx<0).*xx + (xx>=0).*2.*xx;

% Fourier coefficients (period 2L):
% f(x) ~ a0/2 + sum_{n>=1} [ a_n cos(n*pi*x/L) + b_n sin(n*pi*x/L) ]
% a0 = L/2
% a_n = L*((-1)^n - 1)/(n^2*pi^2)   (=> a_n = 0 for even n; = -2L/((2k-1)^2*pi^2) for odd n)
% b_n = -3*L*(-1)^n/(n*pi)

a0 = L/2;

figure; hold on;
plot(xx, ftrue, 'k', 'LineWidth', 1.8);  % original f

for N = Ns
    S = (a0/2)*ones(size(xx));
    for n = 1:N
        an = L*((-1)^n - 1)/(n^2*pi^2);
        bn = -3*L*(-1)^n/(n*pi);
        S = S + an*cos(n*pi*xx/L) + bn*sin(n*pi*xx/L);
    end
    plot(xx, S, 'LineWidth', 1.2, 'DisplayName', sprintf('N = %d', N));
end

grid on; box on;
xlabel('x'); ylabel('value');
title('Fourier series partial sums vs. f(x) on [-L,L]');
legend('f(x)', 'N=3','N=50','N=100', 'Location','northwest');
xlim([-L L]);
```

0.0.2 p331.m

```
%% Problem 3.3.1  f(x)=1+3x on 0<x<5
clear; clc; close all
L = 5;  T = 2*L; % period 10
xs = linspace(-15,15,6000);

% odd (sine) extension:
y = mod(xs+L,T) - L; % map to (-L, L]
f_odd  = (y>0).*(1+3*y) + (y<0).*(3*y-1);  % at y=0 or +/-L the series -> 0
f_even = 1 + 3*abs(y); % even (cosine) extension
```

```
% Half-range sine: f(x) ~ sum_{n>=1} b_n sin(n*pi*x/L) on (0,L)
% b_n = (2/(n*pi))*(1 - 16*(-1)^n)
bs = @(n) (2./(n*pi)).*(1 - 16*(-1).^n);

% Half-range cosine: f(x) ~ a0/2 + sum a_n cos(n*pi*x/L) on (0,L)
% a0 = (2/L)int_0^L (1+3x) dx = 17
% a_n = 30*((-1)^n - 1)/(n^2*pi^2) -> = -60/(n^2*pi^2) for n odd, 0 for n even
a0 = 17;
ac = @(n) 30*((-1).^n - 1)./( (n.^2)*(pi^2) );

% partial sums and plots
Ns = [5 25 100];

% Sine series
figure; hold on
for N = Ns
    S = zeros(size(xs));
    for n = 1:N
        S = S + bs(n)*sin(n*pi*xs/L);
    end
    plot(xs,S,'LineWidth',1.2,'DisplayName',sprintf('Sine N=%d',N));
end
plot(xs,f_odd,'k','LineWidth',1.5,'DisplayName','odd periodic f');
title('Half-range SINE series of f on [-15,15]'); grid on; legend show

% Cosine series
figure; hold on
for N = Ns
    C = (a0/2)*ones(size(xs));
    for n = 1:N
        C = C + ac(n)*cos(n*pi*xs/L);
    end
    plot(xs,C,'LineWidth',1.2,'DisplayName',sprintf('Cosine N=%d',N));
end
plot(xs,f_even,'k','LineWidth',1.5,'DisplayName','even periodic f');
title('Half-range COSINE series of f on [-15,15]'); grid on; legend show

%% (c) and (d)
pts = [5 9 22 101];

% helper that returns value of odd/even periodic extensions at points,
% with the convention: at y=0 or +/- L the odd series -> 0; even series -> 1+3*|+/- L| = 16
wrap = @(x) mod(x+L,T)-L;
ypts = wrap(pts);

sine_vals   = (ypts==0 | abs(ypts)==L).*0 + ... % endpoints -> 0
              (ypts>0).*(1+3*ypts) + (ypts<0).*(3*ypts-1);

cosine_vals = 1 + 3*abs(ypts); % endpoints -> 16

disp(table(pts.', ypts.', sine_vals.', cosine_vals.', ...
    'VariableNames', {'x','y_in_(-5,5]','SineSeriesValue','CosineSeriesValue'}));
```