MATH 345 HW4

Q1

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho_0(x) \quad \cdots (I)$$

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} - T_0\frac{\partial^2 u}{\partial x^2} - Q(x,t)\,\rho_0(x) = 0$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} - Q(x,t) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} + Q(x,t) \quad \cdots (II)$$

$u_E(x)$ should satisfy $\frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} + Q(x,t) = 0$ $\quad u(0) = u(L) = 0$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\rho_0}{T_0}Q(x,t) = g\frac{\rho_0}{T_0}$$

$$\frac{\partial^2 u}{\partial x^2} = g\frac{\rho_0}{T_0} \Rightarrow u_E'(x) = g\frac{\rho_0}{T_0}x + c_1$$

$$u_E(x) = g\frac{\rho_0}{T_0}\frac{x^2}{2} + c_1 x + c_2$$

$u_E(0) = 0 \Rightarrow c_2 = 0$

$u_E(L) = 0 \Rightarrow g\frac{\rho_0}{T_0}\frac{L^2}{2} + c_1 L = 0$

$$c_1 = -g\frac{\rho_0 L}{2T_0}$$

$$u_E(x) = g\frac{\rho_0 x^2}{2T_0} - g\frac{\rho_0 L x}{2T_0} = g\frac{\rho_0}{2T_0}\left[x^2 - xL\right]$$

$$u_E(x) = g\frac{\rho_0}{2T_0}\left[x(x-L)\right]$$

(b)

wws: $v(x,t) = u(x,t) - u_E(x)$ satisifies $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$$v(x,t) = u(x,t) - u_E(x)$$

$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}$, since u_E doesn't depend on t

$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} + Q(x,t)$ from (1)

We have $v_{tt} = u_{tt} = \frac{T_0}{\rho_0}u_{xx} + Q$

Since $u_E(x)$ satisifies $\frac{T_0}{\rho_0}(u_E)_{xx} + Q = 0$

$$Q = -\frac{T_0}{\rho_0}(u_E)_{xx} = 0$$

$$v_{tt} = u_{tt} = \frac{T_0}{\rho_0}u_{xx} + Q = \frac{T_0}{\rho_0}u_{xx} - \frac{T_0}{\rho_0}(u_E)_{xx}$$

$$= \frac{T_0}{\rho_0}\left(u_{xx} - (u_E)_{xx}\right)$$

$$v_{tt} = c^2\left(u_{xx} - (u_E)_{xx}\right) = c^2(u - u_E)_{xx} = c\, v_{xx}$$

$$= c\,\frac{\partial^2 v}{\partial x^2}$$

Q2

(a) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$0 < x < L$

$u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = x(10-x)$

Dirichlet BC

$u(0,t) = u(L,t) = 0$

$u(x,t) = \phi(x) h(t)$

$$\phi_n(x) = \sin\left(\frac{n\pi}{L} x\right) \quad n = 1, 2, 3, \dots, \quad \lambda_n = \frac{n\pi}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right]$$

$$\omega_n = c\frac{n\pi}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow A_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\omega_n t) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} B_n \omega_n \cos(\omega_n t) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \omega_n \sin\left(\frac{n\pi x}{L}\right) = x(10-x)$$

$$B_n \omega_n = \frac{2}{2} \int_0^L x(10-x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{\omega_n 2} \int_0^L x(10-x) \sin\left(\frac{n\pi x}{2}\right) dx$$

let

$$I_n = \int_0^L x(10-x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$u = x(10-x) \qquad dv = \sin\left(\frac{n\pi x}{2}\right) dx$$

$$du = -x + (10-x) \qquad v = -\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)$$

$$= 10 - 2x$$

$$-\frac{L}{n\pi} x(10-x) \cos\left(\frac{n\pi x}{L}\right) + \int \frac{L}{n\pi} (10-2x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$K_n = \int_0^L (10-2x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$u = 10 - 2x \qquad dv = \cos\left(\frac{n\pi x}{2}\right) dx$$

$$du = -2 \qquad v = \frac{L}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$K_n = (10-2x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + 2\int \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{(10-2x)2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - 2\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$= \left[\frac{L(10-2L)}{n\pi}\underbrace{\sin(n\pi)}_{0} - 2\left(\frac{L}{n\pi}\right)^2\cos(n\pi)\right] - \left[-2\left(\frac{2}{n\pi}\right)^2\right]$$

$$= -2\left(\frac{L}{n\pi}\right)^2(-1)^n + 2\left(\frac{L}{n\pi}\right)^2 = -2\left(\frac{L}{n\pi}\right)^2\left[(-1)^n - 1\right]$$

$$K_n = -2\left(\frac{L}{n\pi}\right)^2\left[(-1)^n - 1\right]$$

$$I_n = \frac{L}{n\pi}(10-x)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{L}{n\pi}K_n$$

$$I_n = \frac{L}{n\pi}(10-L)\underbrace{\cos(n\pi)}_{(-1)^n} - \frac{L}{n\pi}(10) + \frac{2}{n\pi}K_n$$

$$I_n = \frac{L}{n\pi}\left[(10-L)(-1)^n - 10 + K_n\right]$$

$$I_n = \frac{L}{n\pi}\left[(10-L)(-1)^n - 10 - 2\left(\frac{L}{n\pi}\right)^2\left[(-1)^n - 1\right]\right]$$

$$B_n = \frac{2}{\omega_n L} I_n$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\omega_n t) \sin\left(\frac{n\pi x}{L}\right)$$

(b) $u(x,t) = R(x - ct) + S(x + ct)$

for some func. S & R

$a = x - ct, \quad b = x + ct$

$u(x,t) = U(a,b)$

$$u_x = U_a \frac{\partial a}{\partial x} + U_b \frac{\partial b}{\partial x} = U_a(1) + U_b(1) = U_a + U_b$$

$$u_t = U_a \frac{\partial a}{\partial t} + U_b \frac{\partial b}{\partial t} = U_a(-c) + U_b(c) = -cU_a + cU_b$$

$$u_{xx} = (U_a + U_b)_x = U_{aa} + 2U_{ab} + U_{bb}$$

$$u_{tt} = (-cU_a + cU_b)_t = c^2(U_{aa} - 2U_{ab} - U_{bb})$$

$a_x = 1, \quad b_x = 1, \quad a_t = -c, \quad b_t = c$

$u_{tt} = c^2 u_{xx}$

$\cancel{c^2}(U_{aa} - 2U_{ab} + U_{bb}) = \cancel{c^2}(U_{aa} + 2U_{ab} + U_{bb})$

$-2U_{ab} = 2U_{ab}$

$4U_{ab} = 0$

$U_{ab} = 0$

Solve $U_{ab} = 0$

$U_a(a, b) = R'(a)$

for some func R

integrate wrt a

$U(a, b) = R(a) + S(b)$

$u(x,t) = U(a,b) = R(a) + S(b) = R(x - ct) + S(x + ct)$

Q3

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L$$

$$E(t) = \int_0^L \left[\frac{1}{2}(u_t)^2 + \frac{c}{2}(u_x)^2\right] dx$$

$$\frac{dE}{dt} = \int_0^L (u_t u_{tt} + c^2 u_x u_{xt})\, dx$$

$$\frac{dE}{dt} = \int_0^L (c^2 u_t u_{xx} + c^2 u_x u_{xt})\, dx = c^2 \int_0^L (u_t u_{xx} + u_x u_{xt})\, dx$$

$$\frac{\partial}{\partial x}(u_t u_x) = u_{xt} u_x + u_t u_{xx}$$

$$\frac{dE}{dt} = c^2 \int_0^1 \frac{\partial}{\partial x}(u_t u_x)\, dx = c^2 \left[u_t u_x\right]_0^L$$

$$= c^2 u_x u_t \Big|_0^L$$

For a fixed-end string $\Rightarrow$ Energy conserved

$u(0,t) = u(L,t)$

$u(0,t) = u_t(L,t) = 0$

$$\frac{dE}{dt} = c^2 u_x u_t \Big|_0^L = 0$$

Q4

$$\frac{dE}{dt} = c^2 u_x u_t \Big|_0^L = c^2\left(u_x(L,t)\, u_t(L,t) - u_x(0,t)\, u_t(0,t)\right)$$

(a) $u(0,t)=0 \quad u(L,t)=0$ fixed ends

$u_t(0,t)=0 \quad , \quad u_t(L,t)=0$

$$\frac{dE}{dt} = c^2\left(u_x(L,t)\cdot 0 - u_x(L,t)\cdot 0\right) = 0$$

$E(t)$ is constant thus conserved

(b) $u_x(0,t)=0 \quad , \quad u(L,t)=0$ Left end free

Right end fixed

$u(L,t)=0 \Rightarrow u_t(L,t)=0$

$$\frac{dE}{dt} = c^2\left(u_x(L,t)\cdot 0 - 0\cdot u_t(0,t)\right) = 0$$

$E(t)$ is conserved

(c) $u(0,t)=0 \qquad u_x(L,t) = -\gamma u(L,t) \quad \gamma > 0$

From $u(0,t)=0$ we get $u_t(0,t)=0$

$$\frac{dE}{dt} = c^2 u_x(L,t)\, u_t(L,t) = c^2(-\gamma u(L,t))\, u_t(L,t)$$

$$= -\gamma c^2 u(L,t)\, u_t(L,t)$$

$$u(L,t)\, u_t(L,t) = \frac{1}{2}\frac{d}{dt}\left(u(L,t)^2\right)$$

$$\frac{dE}{dt} = -\frac{\gamma c^2}{2}\frac{d}{dt}\left(u(L,t)^2\right)$$

$$\frac{d}{dt}\left[E(t) + \frac{\gamma c^2}{2} u(L,t)^2\right] = 0$$

$$E(t) + \frac{\gamma c^2}{2} u(L,t)^2 = K$$

K is a constant

$E(t)$ by itself is not conserved. It can increase or decrease as energy is exchanged with a spring

(d) $$\frac{dE}{dt} = -\gamma^2 c^2 u(L,t)\, u_t(L,t)$$

$$E(t) + \frac{\gamma c^2}{2} u(L,t)^2 = K$$

K is a constant

$E(t)$ can increase over time drawing energy from the boundary. This is an unstable, energy-generating boundary. As $|u(L,t)|$ grows $E(t)$ must grow

Q5

$$u(x,t) = A\cos(kx - \omega t + \phi_0)$$

$$\Phi(x,t) = kx - \omega t + \phi_0$$

$$\Phi(x(t),t) = \text{constant}$$

$$\frac{d}{dt}\Phi(x(t),t) = k\frac{dx}{dt} - \omega = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

$$v = \frac{\omega}{k}$$

Q6

$$1+R=T$$

$$ik_{3_I}(1-R) = Tk_I\sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}$$

let $\alpha = \sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}$

$$ik_{3_I}(1-R) = Tk_I\alpha$$

$$ik_{3_I}(1-R) = Tk_I(1+R)$$

$$ik_{3_I} - ik_{3_I}R = k_I\alpha + k_I\alpha R$$

$$(-ik_{3_I} - k_I\alpha)R = k_I\alpha - ik_{3_I}$$

$$R = \frac{k_I\alpha - ik_{3I}}{-k_{3I} - k_I\alpha} = \frac{ik_{3I} - k_I\alpha}{ik_{3I} + k_I\alpha}$$

$$T = 1 + \frac{ik_{3I} - k_I\alpha}{ik_{3I} + k_I\alpha} = \frac{ik_{3I} + k_I\alpha + ik_{3I} - k_I\alpha}{ik_{3I} + k_I\alpha}$$

$$= \frac{2ik_{3I}}{ik_{3I} + k_I\alpha}$$

$(k_{3I} + \,), a$

$$R = \frac{i k_{3I} - k_1 \sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}}{i k_{3I} + k_I \sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}}$$

$$T = \frac{2 i k_{3I}}{i k_{3I} + k_I \sqrt{\sin^2\theta_I - \frac{c_+^2}{c_-^2}}}$$

Q7

$\phi'' + \lambda\phi = 0 \qquad -L < x < L$

$\phi(L) = \phi(-L) \qquad \phi'(L) = \phi'(-L)$

Case $\lambda = 0$

$\phi'' = 0 \qquad \phi(x) = Ax + B$

$\phi(L) = \phi(-L)$

$AL + B = -AL + B \implies A = 0$

so $\phi(x) = B$ Constant

so $\lambda_0 = 0$ with $\phi_0(x) = 1$

Case $\lambda > 0$

let $\lambda = u^2 > 0$

$$\phi(x) = A\cos(ux) + B\sin(ux)$$

$\phi(L) = \phi(-L)$

$A\cos(uL) + B\sin(uL) = A\cos(-uL) + B\sin(-uL)$

$= A\cos(uL) - B\sin(uL)$

So $2B\sin(\mu L) = 0$

$\phi'(L) = \phi'(-L)$, we get $2A\mu\sin(\mu L) = 0$

$\sin(\mu L) = 0 \qquad \mu L = n\pi \quad n = 1, 2, \cdots$

$$\lambda_n = \mu^2 = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$\phi_n(x) = A\cos\left(\frac{n\pi x}{L}\right) + B\sin\left(\frac{n\pi x}{L}\right)$$

(1) True. Eigenvalues are real

(2) True. $\lambda_1 = 0$, $\lambda_2 = \left(\frac{\pi}{L}\right)^2$, $\lambda_3 = \left(\frac{2\pi}{L}\right)^2, \cdots$

(3) False. i.e. $\lambda_2 = \left(\frac{\pi}{L}\right)^2$

$$\phi(x) = A\cos\left(\frac{\pi}{L}x\right) + B\sin\left(\frac{\pi}{L}x\right)$$

eigen functions in the form of above, depending on A, B has far more than one zero in $(-L, L)$ since $\sin(\pi/L\, x)$ has $2 \cdot 1 - 1 = 1$ interior zero? actually more. cos has 2, sin have 1 interior zero, higher n gives more zeros, but they do not match exactly "n-1"

(4) False.

The space of each λ_n $n \geq 1$ is 2-D.

If we pick only one eigenfunction for each λ_n, we miss the rest of that eigenspace. For example, Choosing only cosines for each n leaves out all sine functions; then many periodic functions (odd ones) cannot be expanded in that set alone.

(5) True. $\int_{-L}^{L} \phi_m(x)\, \phi_n(x)\, dx = 0$ $\quad \lambda_m \neq \lambda_n$