Mr. Sturm Lious Section

General form. A (regular) Sturm–Liouville eigenvalue problem on (a,b) has the form

$$\frac{d}{dx}\Big(p(x)\,\phi'(x)\Big) + \big(\lambda r(x) + q(x)\big)\,\phi(x) = 0, \tag{1}$$

with boundary conditions of the form

$$\alpha_1\phi(a) + \alpha_2\phi'(a) = 0,$$
$$\beta_1\phi(b) + \beta_2\phi'(b) = 0,$$

(where (α_1,α_2) and (β_1,β_2) are not both zero).
Typical choices:

- Dirichlet: $\phi(a)=0$, $\phi(b)=0$.
- Neumann: $\phi'(a)=0$, $\phi'(b)=0$.
- Robin: $\phi(a)-h_1\phi'(a)=0$, $\phi(b)-h_2\phi'(b)=0$, etc.

Assume

$$p(x)>0, \qquad r(x)>0 \quad \text{on } (a,b).$$

1.2 Weight, inner product, orthogonality

The *weight function* is

$$w(x)=r(x).$$

Define the inner product

$$\langle f,g\rangle = \int_a^b w(x)\,f(x)\,g(x)\,dx = \int_a^b r(x)f(x)g(x)\,dx.$$

If ϕ_m,ϕ_n are eigenfunctions corresponding to distinct eigenvalues $\lambda_m\neq\lambda_n$ of (1), then

$$\boxed{\int_a^b r(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0.}$$

That is, eigenfunctions for different eigenvalues are orthogonal with respect to the weight r.

1.3 Rayleigh quotient (for sign of eigenvalues)

Rewriting (1) as

$$-(p\phi')' - q\phi = \lambda r\phi,$$

multiplying by ϕ and integrating, then integrating by parts, gives for any nontrivial eigenfunction ϕ:

$$\lambda = \frac{\displaystyle\int_a^b \big[\,p(x)\,(\phi'(x))^2 - q(x)\,\phi(x)^2\,\big]\,dx}{\displaystyle\int_a^b r(x)\,\phi(x)^2\,dx}.$$

Typical sign arguments:

- If $p>0$, $r>0$, $q\le 0$, then $\lambda\ge 0$.
- If $q<0$ somewhere (and not identically zero), often $\lambda>0$.

2. Separation of variables: general templates

2.1 Heat-type equations with variable coefficients

General 1D heat equation:

$$r(x)\,u_t = \frac{\partial}{\partial x}\Big(p(x)\,u_x\Big) + q(x)\,u, \qquad a<x<b,\ t>0. \tag{2}$$

Here

$$r(x)>0, \quad p(x)>0.$$

Assume homogeneous boundary conditions that make the operator self-adjoint (e.g. Dirichlet/Neumann/Robin of SL type).

Separation of variables. Try $u(x,t)=\phi(x)T(t)$:

$$r\phi T' = (p\phi')'T + q\phi T.$$

Divide by $r\phi T$:

$$\frac{T'}{T} = \frac{(p\phi')'+q\phi}{r\phi} = -\lambda.$$

Hence

$$T'(t)+\lambda T(t)=0 \quad\Rightarrow\quad T(t)=e^{-\lambda t},$$

and the spatial SL problem is

$$(p\phi')' + q\phi + \lambda r\phi = 0 \qquad\Longleftrightarrow\qquad -(p\phi')' - q\phi = \lambda r\phi. \tag{3}$$

Let $\{\lambda_n,\phi_n(x)\}_{n\ge1}$ be the eigenpairs of (3) satisfying the BCs, with eigenfunctions orthogonal in the weight $r(x)$.

General solution. Expand the initial data $u(x,0)=f(x)$ in the eigenfunctions:

$$f(x)=\sum_{n=1}^{\infty} a_n\phi_n(x),$$

with coefficients

$$a_n = \frac{\displaystyle\int_a^b r(x)\,f(x)\,\phi_n(x)\,dx}{\displaystyle\int_a^b r(x)\,\phi_n(x)^2\,dx}. \tag{4}$$

Then the solution is

$$u(x,t)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\,\phi_n(x). \tag{5}$$

Long-time behavior. If all $\lambda_n>0$, then $u(x,t)\to 0$ as $t\to\infty$. If there is a zero eigenvalue $\lambda_0=0$ with eigenfunction ϕ_0, then

$$u(x,t) = a_0\phi_0(x) + \sum_{n\ge1} a_n e^{-\lambda_n t}\phi_n(x)$$

and

$$\lim_{t\to\infty} u(x,t) = a_0\phi_0(x).$$

Example: For Neumann (insulated) heat problems one typically has $\phi_0(x)=1$ and

$$a_0 = \frac{\displaystyle\int_a^b r(x)f(x)\,dx}{\displaystyle\int_a^b r(x)\,dx},$$

so the solution tends to the weighted spatial average of the initial data.

2.2 Wave-type equations with variable coefficients

General 1D wave equation:

$$r(x)\,u_{tt} = \frac{\partial}{\partial x}\Big(p(x)\,u_x\Big) + q(x)\,u, \qquad a<x<b,\ t>0. \tag{6}$$

Try $u(x,t)=\phi(x)G(t)$:

$$r\phi G'' = (p\phi')'G + q\phi G.$$

Divide by $r\phi G$:

$$\frac{G''}{G} = \frac{(p\phi')'+q\phi}{r\phi} = -\lambda.$$

Time ODE:

$$G''+\lambda G=0 \quad\Rightarrow\quad G(t)=A\cos\Big(\sqrt{\lambda}\,t\Big)+B\sin\Big(\sqrt{\lambda}\,t\Big).$$

Spatial eigenvalue problem (same SL operator as heat-type case):

$$(p\phi')' + q\phi + \lambda r\phi = 0. \tag{7}$$

Let $\{\lambda_n,\phi_n\}_{n\ge1}$ be eigenpairs. The general solution is

$$u(x,t)=\sum_{n=1}^{\infty}\Big(A_n\cos\Big(\sqrt{\lambda_n}\,t\Big)+B_n\sin\Big(\sqrt{\lambda_n}\,t\Big)\Big)\,\phi_n(x). \tag{8}$$

Given initial conditions

$$u(x,0)=f(x), \qquad u_t(x,0)=g(x),$$

we obtain

$$f(x)=\sum_{n=1}^{\infty}A_n\phi_n(x), \qquad g(x)=\sum_{n=1}^{\infty}B_n\sqrt{\lambda_n}\,\phi_n(x).$$

Using orthogonality in weight $r(x)$:

$$A_n = \frac{\displaystyle\int_a^b r(x)\,f(x)\,\phi_n(x)\,dx}{\displaystyle\int_a^b r(x)\,\phi_n(x)^2\,dx}, \tag{9}$$

$$B_n = \frac{\displaystyle\int_a^b r(x)\,g(x)\,\phi_n(x)\,dx}{\displaystyle\sqrt{\lambda_n}\int_a^b r(x)\,\phi_n(x)^2\,dx}. \tag{10}$$

Special case (as in 5.5.16):

- For $u_{tt}=c^2(x)u_{xx}$ with $u(0,t)=u(L,t)=0$ we have

$$p(x)=1, \quad q(x)=0, \quad r(x)=\frac{1}{c^2(x)}.$$

Orthogonality is with weight $1/c^2(x)$.

4. Equidimensional (Euler–Cauchy) ODEs

A typical equidimensional ODE:

$$x^2\phi'' + ax\phi' + b(x)\phi = 0, \qquad x > 0, \tag{11}$$

where $b(x)$ may be constant or contain eigenvalue parameters.

4.1 Power-law ansatz

Try $\phi(x) = x^m$:

$$\phi' = mx^{m-1}, \quad \phi'' = m(m-1)x^{m-2}.$$

Substitute into (11) (for constant b) to get the algebraic (indicial) equation:

$$m(m-1) + am + b = 0.$$

Solve for m:

- Two distinct real roots $m_1 \neq m_2$:
$$\phi(x) = C_1x^{m_1} + C_2x^{m_2}.$$
- Repeated root $m_1 = m_2 = m$:
$$\phi(x) = C_1x^m + C_2x^m \ln x.$$
- Complex roots $m = \alpha \pm i\beta$:
$$\phi(x) = x^\alpha\Big(C_1\cos(\beta \ln x) + C_2\sin(\beta \ln x)\Big).$$

Example pattern from the practice problems:

$$x^2\phi'' + 2x\phi' + (\lambda - 1)\phi = 0 \quad \Rightarrow \quad m^2 + m + (\lambda - 1) = 0.$$

If $\lambda > 5/4$ the roots are complex and the general real solution is

$$\phi(x) = x^{-1/2}\Big(A\cos(\mu \ln x) + B\sin(\mu \ln x)\Big),$$

with $\mu = \frac{1}{2}\sqrt{4\lambda - 5}$, etc.

5. Orthogonality proof pattern

Given an SL problem

$$(p\phi')' + (\lambda r + q)\phi = 0$$

with appropriate self-adjoint BCs, the orthogonality for $\lambda_m \neq \lambda_n$ is obtained by:

1. Write the equations for ϕ_m and ϕ_n:
$$(p\phi_m')' + (\lambda_m r + q)\phi_m = 0, \qquad (p\phi_n')' + (\lambda_n r + q)\phi_n = 0.$$
2. Multiply the first by ϕ_n, the second by ϕ_m, and subtract:
$$\phi_n(p\phi_m')' - \phi_m(p\phi_n')' + (\lambda_m - \lambda_n)r\phi_m\phi_n = 0.$$
3. Recognize a derivative:
$$\phi_n(p\phi_m')' - \phi_m(p\phi_n')' = \frac{d}{dx}\big(p(\phi_n\phi_m' - \phi_m\phi_n')\big).$$
4. Integrate over (a, b):
$$\big[p(\phi_n\phi_m' - \phi_m\phi_n')\big]_a^b + (\lambda_m - \lambda_n)\int_a^b r\phi_m\phi_n\,dx = 0.$$
5. Use the boundary conditions to show the boundary term is 0; conclude
$$(\lambda_m - \lambda_n)\int_a^b r(x)\phi_m(x)\phi_n(x)\,dx = 0,$$
hence orthogonality for $\lambda_m \neq \lambda_n$.

Diri

$$u(x,t) = \sum_{n=1}^{\infty}\Big[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\Big]\sin\frac{n\pi x}{L}, \qquad \omega_n = \frac{cn\pi}{L}.$$

$$A_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx,$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\,\sin\frac{n\pi x}{L}\,dx.$$

Neu

$$u(x,t) = A_0 + B_0t + \sum_{n=1}^{\infty}\Big[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\Big]\cos\frac{n\pi x}{L}, \qquad \omega_n = \frac{cn\pi}{L}.$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad B_0 = \frac{1}{L}\int_0^L g(x)\,dx.$$

$$A_n = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx,$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\,\cos\frac{n\pi x}{L}\,dx, \qquad n \geq 1.$$

Mixed BC: fixed at x=0, free at x=L

$$\phi_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right), \qquad \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \qquad \omega_n = c\sqrt{\lambda_n} = \frac{(2n+1)\pi c}{2L}.$$

$$u(x,t) = \sum_{n=0}^{\infty}\Big[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\Big]\sin\left(\frac{(2n+1)\pi x}{2L}\right).$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)\,dx,$$

$$B_n = \frac{2}{L\omega_n}\int_0^L g(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)\,dx = \frac{4}{c(2n+1)\pi}\int_0^L g(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)\,dx.$$

Mixed BC: free at x=0, fixed at x=L

$$\phi_n(x) = \cos\left(\frac{(2n+1)\pi x}{2L}\right), \qquad \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2, \qquad \omega_n = \frac{(2n+1)\pi c}{2L}.$$

$$u(x,t) = \sum_{n=0}^{\infty}\Big[A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\Big]\cos\left(\frac{(2n+1)\pi x}{2L}\right).$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)\,dx,$$

$$B_n = \frac{2}{L\omega_n}\int_0^L g(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)\,dx = \frac{4}{c(2n+1)\pi}\int_0^L g(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)\,dx.$$

1. Full Fourier Series on $[-L, L]$

A $2L$-periodic function $F(x)$ has Fourier series

$$F(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} F(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{n\pi x}{L}\,dx.$$

If F is piecewise C^1 (piecewise smooth, finitely many jumps) and $2L$-periodic, then:

$$\lim_{N\to\infty} S_N(x) = \begin{cases} F(x), & F \text{ continuous at } x, \\ \frac{1}{2}\big(F(x^-) + F(x^+)\big), & x \text{ a jump point,} \end{cases}$$

where $S_N(x)$ is the N-th partial sum of the Fourier series.

4. Sturm–Liouville Summary (Key Properties)

Consider a regular Sturm–Liouville problem

$$\frac{d}{dx}\big(p(x)y'(x)\big) + \big(\lambda\,w(x) - q(x)\big)y(x) = 0,$$

on $[a, b]$, with $p > 0$, $w > 0$, and appropriate self-adjoint (separated) boundary conditions. Then:

1. **Eigenvalues are real.**
   All eigenvalues λ_n are real numbers.
2. **Eigenvalues are discrete and ordered.**
   There is an infinite sequence
$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots, \qquad \lambda_n \to +\infty.$$
3. **Orthogonality w.r.t. the weight.**
   If y_m, y_n correspond to different eigenvalues $\lambda_m \neq \lambda_n$, then
$$\int_a^b w(x)\,y_m(x)\,y_n(x)\,dx = 0.$$
4. **One-dimensional eigenspaces.**
   For each λ_n, all eigenfunctions are multiples of a single eigenfunction (no independent "extra" eigenfunctions for the same eigenvalue under separated BCs).
5. **Zero (oscillation) count.**
   The n-th eigenfunction y_n has exactly $n - 1$ zeros in the open interval (a, b).

(Additionally, in applications one often uses that the eigenfunctions $\{y_n\}$ form a complete orthogonal set in $L^2_w[a, b]$, so any sufficiently nice function can be expanded in a Fourier series of eigenfunctions.)