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MATH 345 Homework 2 Solutions

Problem 1.5.8
We can use the divergence theorem with $A = \nabla u$. If u satisfies Laplaces's equation in $\mathbb{R}^3$

$\nabla^2 u = \nabla \cdot (\nabla u) = 0$ in V
for any volume V bounded by the closed surface S, then:

$$\iint_S \nabla u \cdot \hat{n} dS = \iiint_V \nabla \cdot (\nabla u) dV = \iiint_V \nabla^2 u dV = 0$$

which proves the claim for every closed surface S

In a steady conduction with constant conductivity K and no internal sources, the heat flux is $q = -K\nabla u$.

$$\iint_S q \cdot \hat{n} dS = -K \iint_S \nabla u \cdot \hat{n} dS = 0$$

so the net heat flow throught any clsoed surface is 0. The heat leaving through some parts of S is exactly balanced by heat entering elsewhere. Equivalently, there are no sources/sinks of heat inside S at steady state.

Problem 2.2.5
(a)
If $L(u_p) = f$ and $L(u_1) = L(u_2) = 0$, then for any constant c_1, c_2,

$$L(u_p + c_1 u_1 + c_2 u_2) = L(u_p) + c_1 L(u_1) + c_2 L(u_2) = f + 0 + 0 = f$$

so $u = u_p + c_1 u_1 + c_2 u_2$ is another particular solution to $L(u) = f$
(b)
If $L(u) = f_1 + f_2$ and $u_{p,1}, u_{p,2}$ satisfy $L(u_{p,1}) = f_1$ and $L(u_{p,2}) = f_2$, then

$$L(u_{p,1} + u_{p,2}) = L(u_{p,1}) + L(u_{p,2}) = f_1 + f_2$$

so a particular solution for $f_1 + f_2$ is $u_p = u_{p,1} + u_{p,2}$
Problem 2.3.1
Let $u(r,t) = R(r)T(t)$

$$u_t = \frac{k}{r^2}\frac{\partial}{\partial r}(r^2 u_r)$$

$$R(r)T'(t) = \frac{k}{r^2}(\frac{d}{dr}(r^2 R'(r)))T(t)$$

$$\frac{1}{k}\frac{T'}{T} = \frac{1}{r^2 R}\frac{d}{dr}(r^2 R') = -\lambda$$

ODEs:

$$T'(t) + k\lambda T(t) = 0$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 R'(r)) + \lambda R(r) = 0 \iff r^2 R'' + 2rR' + \lambda r^2 R = 0$$

Problem 2.3.2

$$y'' + 4y = 0 \implies y(x) = A\cos(2x) + B\sin(2x)$$

(a)

$$y(0) = 3 \implies A = 3, y(1) = 5 \implies 3\cos(2) + B\sin(2) = 5$$

$\sin(2) \neq 0$

$$B = \frac{5 - 3\cos 2}{\sin 2}$$

$$y(x) = 3\cos(2x) + \frac{5 - 3\cos 2}{\sin 2}\sin(2x)$$

(b)

$$y(\pi) = 3\cos(2\pi) + B\sin(2\pi) = 3 \neq 5$$

thus, no solution

(c)

$$y(0) = 3 \implies A = 3$$

$$y(\pi) = 3\cos(2\pi) + B\sin(2\pi) = 3 \text{ for any } B$$

$$y(x) = 3\cos(2x) + B\sin(2x), B \in \mathbb{R}$$

thus, infinitely many solutions

Problem 2.3.7

We consider two cases.

Case 1: $n = m$.

$$\begin{aligned}
I &= \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\
&= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx \\
&= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L \\
&= \frac{1}{2}(L - 0) = \frac{L}{2}
\end{aligned}$$

Case 2: $n \neq m$.

$$\begin{aligned}
I &= \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx \\
&= \frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\right]_0^L \\
&= \frac{1}{2}\left(\frac{L}{(n-m)\pi}\sin((n-m)\pi) - \frac{L}{(n+m)\pi}\sin((n+m)\pi)\right) - 0 \\
&= 0
\end{aligned}$$

since $\sin(k\pi) = 0$ for any integer k.
Combining these results, we get the orthogonality condition:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Problem 2.4.4 (a)

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi'(L) = 0$$

Case 1: $\lambda < 0$. Let $\lambda = -k^2$ for $k > 0$.

$$\phi(x) = A\cosh(kx) + B\sinh(kx)$$
$$\phi(0) = A = 0 \implies \phi(x) = B\sinh(kx)$$
$$\phi'(x) = Bk\cosh(kx) \implies \phi'(L) = Bk\cosh(kL) = 0$$

Since $k, L > 0$, $\cosh(kL) \neq 0$, so $B = 0$. This gives the trivial solution.
Case 2: $\lambda = 0$.

$$\phi(x) = Ax + B$$
$$\phi(0) = B = 0 \implies \phi(x) = Ax$$
$$\phi'(x) = A \implies \phi'(L) = A = 0$$

This gives the trivial solution.
Case 3: $\lambda > 0$. Let $\lambda = k^2$ for $k > 0$.

$$\phi(x) = A\cos(kx) + B\sin(kx)$$
$$\phi(0) = A = 0 \implies \phi(x) = B\sin(kx)$$
$$\phi'(x) = Bk\cos(kx) \implies \phi'(L) = Bk\cos(kL) = 0$$

For a non-trivial solution, $B \neq 0$, so $\cos(kL) = 0$.

$$kL = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, 3, \ldots$$

The eigenvalues are:

$$\lambda_n = k_n^2 = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions are:

$$\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

(b)
Case 1: $n = m$.

$$\begin{aligned} I &= \int_0^L \sin^2\left(\frac{(2n-1)\pi x}{2L}\right) dx \\ &= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{(2n-1)\pi x}{L}\right)\right) dx \\ &= \frac{1}{2}\left[x - \frac{L}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi x}{L}\right)\right]_0^L \\ &= \frac{1}{2}\left(L - \frac{L}{(2n-1)\pi}\sin((2n-1)\pi)\right) - 0 \\ &= \frac{L}{2} \end{aligned}$$

since $\sin(k\pi) = 0$ for any integer k.
Case 2: $n \neq m$.

$$
\begin{aligned}
I &= \int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right) \sin\left(\frac{(2m-1)\pi x}{2L}\right) dx \\
&= \frac{1}{2} \int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m-1)\pi x}{L}\right)\right] dx \\
&= \frac{1}{2} \left[\frac{L}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{L}\right) - \frac{L}{(n+m-1)\pi} \sin\left(\frac{(n+m-1)\pi x}{L}\right)\right]_0^L \\
&= \frac{1}{2} \left(\frac{L}{(n-m)\pi} \sin((n-m)\pi) - \frac{L}{(n+m-1)\pi} \sin((n+m-1)\pi)\right) - 0 \\
&= 0
\end{aligned}
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