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MATH 345 Homework 1 Solutions

Prelimary Def: Thermal Energy (T.E.), Heat Energy (H.E.), Heat Flux (H.F.), Thermal Energy Density (T.E.D.), Heat Source (H.S.), Specific Heat (S.H.)

Problem 1.2.4

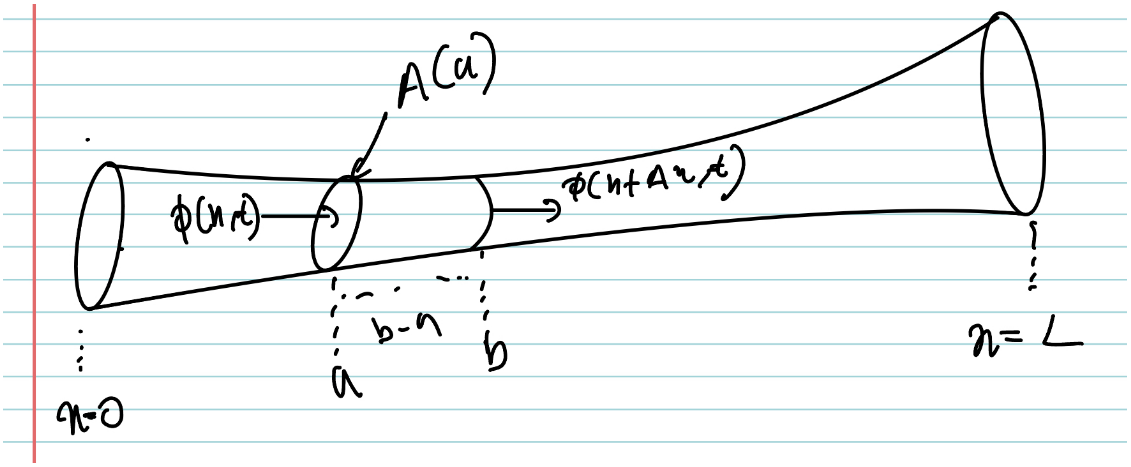

Figure 1: Rod

We define our surface area of the rod as $A(x)$.
T.E.D.:

$$e(x,t)$$

H.E.:

$$e(x,t)A(x)\cdot(b-a)$$

where the volume of a slice is $A(x)\cdot(b-a)$ and we define $(b-a)=\Delta x$.

$$\frac{\partial}{\partial t}\Big[e(x,t)\cdot A(x)\Delta x\Big]\approx\phi(x,t)A(x)-\phi(x+\Delta x,t)A(x+\Delta x)$$

(L: time rate of change of energy in slice. R: inflow at x minus outflow at $x+\Delta x$ plus sources.)

In the limit as $\Delta x\to 0$, divide by Δx and we use

$$\phi(x+\Delta x,t)A(x+\Delta x)=\phi(x,t)A(x)+\partial_x[\phi(x,t)A(x)]\,\Delta x+o(\Delta x)$$

which yields

$$\frac{\partial}{\partial t}[e(x,t)A(x)]=-\frac{\partial}{\partial x}[\phi(x,t)A(x)]$$

$$e(x,t)=\rho c\,T(x,t),\qquad \phi(x,t)=-k\,\frac{\partial T}{\partial x}(x,t).$$

$$\frac{\partial}{\partial t}\Big[\rho c\,A(x)\,T(x,t)\Big]=-\frac{\partial}{\partial x}\Big[-k\,A(x)\,T_x(x,t)\Big].$$

Since $A=A(x)$ is time-independent and k,ρ,c are constants,

$$\rho c\,A(x)\,T_t(x,t)\;=\;\frac{\partial}{\partial x}\Big(k\,A(x)\,T_x(x,t)\Big).$$

Hence the heat eq. for a rod with $A(x)$ and $Q=0$:

$$(\rho c)\,A(x)\,T_t \;=\; \big(k\,A(x)\,T_x\big)_x \qquad\Longleftrightarrow\qquad T_t \;=\; \alpha\,\frac{1}{A(x)}\,\frac{\partial}{\partial x}\Big(A(x)\,T_x\Big),\quad \alpha=\frac{k}{\rho c}.$$

One way we can confirm we have the right answer is to check if $A(x)\equiv A_0$, then $(kA_0T_x)_x = kA_0T_{xx}$ and we get $T_t=\alpha\,T_{xx}$.

Problem 1.2.7

Suppose that S.H. is $c=c(x,u)$

(a)

For a small temperature to raise $u\to u+du$ at fixed x requires per unit mass heat

$$dq = c(x,u)\,du.$$

$$q(x,t)=\int_0^{u(x,t)} c\big(x,\bar u\big)\,d\bar u,$$

(b)

T.E. per unit volume is

$$e(x,t)=\rho(x)\,q(x,t)=\rho(x)\int_0^{u(x,t)} c\big(x,\bar u\big)\,d\bar u.$$

Conservation of thermal energy

$$\frac{\partial e}{\partial t} \;=\; -\,\frac{\partial \phi}{\partial x}\;+\;Q,$$

Leibniz rule Since $\bar u$ has no explicit t–dependence,

$$\frac{\partial}{\partial t}\left[\int_0^{u(x,t)} c\big(x,\bar u\big)\,d\bar u\right] = c\big(x,u(x,t)\big)\,u_t(x,t)$$

Hence

$$\frac{\partial e}{\partial t} = \rho(x)\,c\big(x,u\big)\,u_t(x,t)$$

$$\rho(x)\,c\big(x,u\big)\,u_t \;=\; -\,\phi_x \;+\; Q$$

With Fourier's law

$$\phi(x,t) = -\,K_0(x)\,u_x \qquad\Longrightarrow\qquad -\phi_x = \partial_x\big(K_0(x)\,u_x\big)$$

$$\rho(x)\,c\big(x,u\big)\,u_t = \frac{\partial}{\partial x}\Big(K_0(x)\,u_x\Big) + Q(x,t)$$

If K_0,ρ are constants and $Q=0$ this is

$$\big(\rho\,c(x,u)\big)\,u_t = K_0\,u_{xx}, \qquad\qquad u_t = \frac{K_0}{\rho}\,\frac{u_{xx}}{c(x,u)}$$

Problem 1.4.2
Equilibrium $\Rightarrow u_t = 0$. With constant properties and Fourier's law $\phi = -K_0 u_x$

$$\frac{\partial e}{\partial t} = 0 = -\phi_x + Q \implies (K_0 u_x)_x + Q = 0 \quad \Rightarrow \quad K_0\, u_{xx} + Q = 0$$

(a)
$Q = 0,\ u_x(0) = 0,\ u(L) = T.$

$$u_{xx} = 0 \ \Rightarrow\ u(x) = ax + b, \quad u_x = a.$$

From $u_x(0) = 0$ we get $a = 0$, hence $u(x) \equiv b$. Using $u(L) = T$ gives

$$u(x) \equiv T$$

(b)
$\dfrac{Q}{K_0} = 1 \ \Rightarrow\ Q = K_0,\ u(0) = T_1,\ u(L) = T_2.$

$$K_0 u_{xx} + K_0 = 0 \ \Rightarrow\ u_{xx} = -1.$$

$$u_x(x) = \int u_{xx}(x)\,dx = \int (-1)\,dx = -x + C_1$$

$$u(x) = \int u_x(x)\,dx = \int (-x + C_1)\,dx = -\frac{x^2}{2} + C_1 x + C_2$$

Apply BCs:

$$u(0) = T_1 \Rightarrow C_2 = T_1, \qquad u(L) = T_2 \Rightarrow -\frac{L^2}{2} + C_1 L + T_1 = T_2 \Rightarrow C_1 = \frac{T_2 - T_1}{L} + \frac{L}{2}$$

Therefore

$$u(x) = -\frac{x^2}{2} + \Big(\frac{T_2 - T_1}{L} + \frac{L}{2}\Big)x + T_1 \ = \ T_1 + \frac{T_2 - T_1}{L}\,x + \frac{x(L-x)}{2}$$

$$u(x) = T_1 + \frac{T_2 - T_1}{L}\,x + \frac{x(L-x)}{2}$$

Problem 1.4.8

$$u_{xx} = 0 \quad \Longrightarrow \quad u(x) = ax + b,\ u_x(x) \equiv a$$

BCs give

$$u_x(0,t) = 1 \Rightarrow a = 1, \qquad u_x(L,t) = \beta \Rightarrow a = \beta$$

Consistency $\Rightarrow \beta = 1$

If $\beta = 1$, the family of steady states is

$$u_{\mathrm{eq}}(x) = x + C, \ \ C \in \mathbb{R}$$

Physical explanation. Integrate the PDE over $(0, L)$:

$$\frac{d}{dt}\int_0^L u\,dx = \int_0^L u_{xx}\,dx = u_x(L,t) - u_x(0,t) = \beta - 1.$$

Thus:

$$\beta \neq 1 \ \Rightarrow \ \text{net heat flux } \neq 0 \ \Rightarrow \ \int_0^L u\,dx \text{ drifts linearly in time } \Rightarrow \ \text{no equilibrium.}$$

$$\beta = 1 \ \Rightarrow \ \text{net flux } = 0 \ \Rightarrow \ \text{equilibrium exists.}$$

Which C? When $\beta = 1$, the spatial mean is conserved:

$$\int_0^L u(x,t)\,dx = \int_0^L f(x)\,dx.$$

$$\int_0^L (x + C)\,dx = \int_0^L f(x)\,dx \quad \Rightarrow \quad C = \frac{1}{L}\int_0^L f(x)\,dx - \frac{L}{2}.$$

So the steady state compatible with the initial data is

$$u_{\mathrm{eq}}(x) = x + \Big(\frac{1}{L}\int_0^L f(x)\,dx - \frac{L}{2}\Big) \quad \text{exists iff } \beta = 1.$$

Problem 1.4.11

Let

$$E(t) \;=\; \int_0^L u(x,t)\,dx$$

$$E'(t) \;=\; \int_0^L u_t\,dx = \int_0^L \big(u_{xx}+4\big)\,dx = \Big[u_x\Big]_0^L + 4L = u_x(L,t) - u_x(0,t) + 4L = 6 - 5 + 4L = 1 + 4L.$$

$$E(t) = E(0) + (1+4L)\,t \;=\; \int_0^L f(x)\,dx \;+\; (1+4L)\,t\,.$$

With constant density ρ, S.H. c, and cross-sectional area A, total T.E. is

$$\mathcal{H}(t) = \rho\,c\,A\,E(t) = \rho c A\left(\int_0^L f(x)\,dx + (1+4L)\,t\right).$$

Problem 1.5.3. $x = r\cos\theta,\ y = r\sin\theta.$

(a) Since $r^2 = x^2 + y^2$,

$$2r\,r_x = 2x \Rightarrow r_x = \frac{x}{r} = \cos\theta, \qquad 2r\,r_y = 2y \Rightarrow r_y = \frac{y}{r} = \sin\theta.$$

Also $\theta = \arctan(y/x)$, so

$$\theta_x = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}, \qquad \theta_y = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.$$

(b) Position vector $\vec{r} = x\,\hat{\vec{i}} + y\,\hat{\vec{j}} = r(\cos\theta\,\hat{\vec{i}} + \sin\theta\,\hat{\vec{j}})$, so

$$\hat{\vec{r}} = \cos\theta\,\hat{\vec{i}} + \sin\theta\,\hat{\vec{j}}\,, \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\vec{i}} + \cos\theta\,\hat{\vec{j}}$$

(Obtain $\hat{\boldsymbol{\theta}}$ by $+90°$ rotation or by $\partial_\theta\hat{\vec{r}}$.)

(c) Chain rule:

$$\partial_x = r_x\,\partial_r + \theta_x\,\partial_\theta = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta, \quad \partial_y = r_y\,\partial_r + \theta_y\,\partial_\theta = \sin\theta\,\partial_r + \frac{\cos\theta}{r}\,\partial_\theta.$$

Hence

$$\nabla = \hat{\vec{i}}\,\partial_x + \hat{\vec{j}}\,\partial_y = (\cos\theta\,\hat{\vec{i}} + \sin\theta\,\hat{\vec{j}})\,\partial_r + (-\sin\theta\,\hat{\vec{i}} + \cos\theta\,\hat{\vec{j}})\,\frac{1}{r}\partial_\theta = \boxed{\hat{\vec{r}}\,\partial_r + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\partial_\theta}.$$

Applying to u,

$$\nabla u = \hat{\vec{r}}\,u_r + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\,u_\theta$$