Math 345 Fall 2025

Homework #1
Due: September 8, 2025, 11:59 p.m.

Show work to justify your answers and write neatly (or type) for full credit. Your work must be uploaded onto Gradescope (accessed through Canvas).

1. Problem 1.2.4. Derive the heat equation for a rod assuming constant thermal properties with variable cross-sectional area $A(x)$ assuming no sources by considering the total thermal energy between $x = a$ and $x = b$.

2. Problem 1.2.7. Suppose that the specific heat is a function of position and temperature, $c(x, u)$.

   (a) Show that the heat energy per unit mass necessary to raise the temperature of a thin slice of thickness Δx from 0° to $u(x,t)$ is not $c(x)u(x,t)$, but instead $\int_0^u c(x, \bar{u})\, d\bar{u}$.

   (b) Rederive the heat equation in this case. Show that

   $$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

   remains unchanged. Hint: Assume $\bar{u}$ is not a function of time. Apply Leibniz's rule.

3. Problem 1.4.2. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

   (a) $Q = 0$, $\frac{\partial u}{\partial x}(0) = 0$, $u(L) = T$.

   (b) $\frac{Q}{K_0} = 1$, $u(0) = T_1$, $u(L) = T_2$.

4. Problem 1.4.8. Determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

   $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad \frac{\partial u}{\partial x}(0,t) = 1, \qquad \frac{\partial u}{\partial x}(L,t) = \beta.$$

5. Problem 1.4.11. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, and $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

6. Problem 1.5.3. Note that we will start this problem in class. Consider the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

   (a) Since $r^2 = x^2 + y^2$, show that

   $$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}.$$

   (b) Show that $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$.

   (c) Using the chain rule, show that

   $$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}, and hence \nabla u = \hat{\mathbf{r}}\frac{\partial u}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial u}{\partial \theta}.$$