Math 345 Fall 2025

Homework #5
Due: December 8, 2025, 11:59 p.m.

Show work to justify your answers and write neatly (or type) for full credit. Your work must be uploaded onto Gradescope (accessed through Canvas).

1. Problem 5.6.1. Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lower eigenvalue of
$$\frac{d}{dx}\left[(1+x^2)\frac{d\phi}{dx}\right] + \lambda\phi = 0, \qquad \phi(0) = 1, \qquad \phi(1) = 0.$$
Specify your test function $u_T(x)$ and justify your choice. You may use an integration table, but include the formulas and show how you are applying them.

2. Problem 5.9.1. Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for
$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + \left[\lambda\sigma(x) + q(x)\right]\phi = 0.$$
if the boundary conditions are $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$.
HINT: When computing $\frac{d\phi}{dx}$, small terms can be set to zero. Use the fact that $\lambda \gg 1$.

3. Let f be a continuous function on $0 \le x \le \pi$. Prove that the following inequality holds:
$$\left|\int_0^\pi f(x)\sin x\,dx\right|^2 + \cdots + \left|\int_0^\pi f(x)\sin(nx)\,dx\right|^2 \le \frac{\pi}{2}\int_0^\pi |f(x)|^2\,dx.$$

4. Problem 7.3.5. Solve
$$\frac{\partial u}{\partial t} = k_1\frac{\partial^2 u}{\partial x^2} + k_2\frac{\partial^2 u}{\partial y^2}$$
on a rectangle ($0 < x < L$, $0 < y < H$) subject to the initial condition $u(x, y, 0) = \alpha(x, y)$ and boundary conditions $u(0, y, t) = 0$, $u(L, y, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, t) = 0$, and $\frac{\partial u}{\partial y}(x, H, t) = 0$.

5. Problem 7.5.4. Consider the heat equation in three dimensions with no sources but with nonconstant thermal properties
$$c\rho\frac{\partial u}{\partial t} = \nabla\cdot(K_0\nabla u),$$
where c, ρ, and K_0 are functions of x,y, and z. Assume that $u = 0$ on the boundary. Show that the time variable can be separated by assuming that
$$u(x, y, z, t) = \phi(x, y, z)h(t).$$
Show that $\phi(x, y, z)$ satisfies the eigenvalue problem
$$\nabla\cdot(p\nabla\phi) + \lambda\sigma(x, y, z)\phi = 0, \qquad \text{with } \phi = 0 \text{ on the boundary.}$$
What are $\sigma(x, y, z)$ and $p(x, y, z)$?