Exam 1 Cheatsheet (Methods & Templates)

Conservation & Physical Meaning

Heat eq (1D rod): $u_t = \alpha u_{xx}$, $\alpha = K/(\rho c)$.
Flux: $\phi = -Ku_x$. **Energy density**: $e = \rho c\, u$.
Integral conservation: $\dfrac{d}{dt}\displaystyle\int_0^L u\,dx = \dfrac{1}{\rho c K}\int_0^L Q\,dx$.
Equilibrium: $\Delta u = 0$. With *pure Neumann*, steady state exists iff net flux $= 0$; solution unique up to constant (fix mean or value at a point).
Divergence Thm (2D/3D): $\iint_\Omega \nabla\cdot F\,dA = \int_{\partial\Omega} F\cdot n\,ds$, $\iiint_\Omega \nabla\cdot F\,dV = \iint_{\partial\Omega} F\cdot n\,dS$.

Boundary Conditions (heat)

Dirichlet: fixed temperature $u=$ given. **Neumann**: insulated/flux given, u_x given.
Robin (Newton cooling): $-Ku_x = h(u - u_\infty)$.
Periodic: $u(0,t) = u(L,t)$, $u_x(0,t) = u_x(L,t)$.

Orthogonality you *don't* have on your sheet

Half-integer cosine/sine on $[0,H]$:

$$\int_0^H \cos\frac{(n+\frac12)\pi y}{H}\cos\frac{(m+\frac12)\pi y}{H}\,dy = \frac{H}{2}\delta_{mn},$$

$$\int_0^H \sin\frac{(n+\frac12)\pi y}{H}\sin\frac{(m+\frac12)\pi y}{H}\,dy = \frac{H}{2}\delta_{mn}.$$

Fourier coeffs for these bases:

$$C_n = \frac{2}{H}\int_0^H g(y)\cos\frac{(n+\frac12)\pi y}{H}\,dy, \quad S_n = \frac{2}{H}\int_0^H g(y)\sin\frac{(n+\frac12)\pi y}{H}\,dy.$$

Heat Equation Templates (1D)

Domain $0<x<L$, $u_t = \alpha u_{xx}$.

D–N (Dirichlet at $x=0$, Neumann at $x=L$):

$$u(0,t)=0, \quad u_x(L,t)=0, \quad u(x,0)=f(x).$$

Use eigenfunctions $\sin\frac{(n+\frac12)\pi x}{L}$.

$$u(x,t) = \sum_{n=0}^\infty b_n\, e^{-\alpha\left((n+\frac12)\pi/L\right)^2 t}\sin\frac{(n+\frac12)\pi x}{L},$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(n+\frac12)\pi x}{L}\,dx.$$

(Satisfies $u(0,t)=0$ since $\sin 0 = 0$ and $u_x(L,t)=0$ since $\cos\big((n+\frac12)\pi\big)=0$.)

N–D (Neumann at $x=0$, Dirichlet at $x=L$):

$$u_x(0,t)=0, \quad u(L,t)=0, \quad u(x,0)=f(x).$$

Use eigenfunctions $\cos\frac{(n+\frac12)\pi x}{L}$.

$$u(x,t) = \sum_{n=0}^\infty a_n\, e^{-\alpha\left((n+\frac12)\pi/L\right)^2 t}\cos\frac{(n+\frac12)\pi x}{L},$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(n+\frac12)\pi x}{L}\,dx.$$

(Satisfies $u_x(0,t)=0$ since $\sin 0 = 0$, and $u(L,t)=0$ since $\cos\big((n+\frac12)\pi\big)=0$.)

Periodic (for reference):

$$u(0,t)=u(L,t), \quad u_x(0,t)=u_x(L,t), \quad u(x,t) = A_0 + \sum_{n\neq 0} A_n e^{-\alpha(2\pi n/L)^2 t} e^{i2\pi n x/L}.$$

Real form: $u(x,t) = a_0 + \sum_{n\ge1} e^{-\alpha(2\pi n/L)^2 t}\big[a_n\cos(2\pi n x/L) + b_n\sin(2\pi n x/L)\big]$.

Laplace on Rectangles: Ready Forms

Let $\Delta u = 0$ on $0<x<L$, $0<y<H$.
(D at $y=0,H$; $u(0,y)=g$, $u(L,y)=0$):

$$u = \sum_{n\ge1}\frac{G_n}{\sinh(\alpha_n L)}\sinh(\alpha_n(L-x))\sin(\alpha_n y), \ \alpha_n = \tfrac{n\pi}{H}, \ G_n = \tfrac{2}{H}\int_0^H g\sin(\alpha_n y)\,dy.$$

(N at $y=0$; D at $y=H$; $u(0,y)=g$, $u(L,y)=0$):

$$u = \sum_{n\ge0}\frac{G_n}{\sinh(\alpha_n L)}\sinh(\alpha_n(L-x))\cos(\alpha_n y), \ \ \alpha_n = \tfrac{(n+\frac12)\pi}{H}.$$

(N at $x=0,L$; N at $y=H$; $u(x,0)=F$):

$$u(x,y) = A_0 + \sum_{n\ge1} A_n\cos\frac{n\pi x}{L}\,\frac{\cosh\big(\frac{n\pi}{L}(H-y)\big)}{\cosh\big(\frac{n\pi}{L}H\big)},$$

$A_0 = \frac{1}{L}\int_0^L F$, $A_n = \frac{2}{L}\int_0^L F\cos\frac{n\pi x}{L}\,dx$.
Trick: write $\sinh(\alpha(L-x))$ or $\cosh(\alpha(L-x))$ to auto-satisfy a boundary at $x = L$ (Dirichlet or Neumann).

Neumann Compatibility (use this!)

For steady Laplace with Neumann data:

$$\int_{\partial\Omega} \partial_n u\,ds = \iint_\Omega \Delta u\,dA = 0.$$

Example: if $u_x(0,y) = u_x(L,y) = u_y(x,0) = 0$ and $u_y(x,H) = f(x)$, then *must have* $\int_0^L f(x)\,dx = 0$. Unique up to additive constant.

Polar Geometry: Practical Forms

Exterior circle ($r \geq a$), finite at ∞, Dirichlet at $r = a$:

$$u(r,\theta) = A_0 + \sum_{n\geq 1}\left(\frac{a}{r}\right)^n \big(A_n\cos n\theta + B_n\sin n\theta\big),$$

A_0, A_n, B_n are the 2π Fourier coefficients of $u(a,\theta)$. If $u \to 0$ at ∞, set $A_0 = 0$.
Quarter-disk $(0 \leq \theta \leq \pi/2)$ with $u_\theta(r,0) = 0$, $u(r,\frac{\pi}{2}) = 0$, $u(1,\theta) = f(\theta)$:

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n\, r^{2n+1}\cos\big((2n+1)\theta\big), \qquad A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\varphi)\cos\big((2n+1)\varphi\big)\,d\varphi.$$

Fourier Tips You'll Use

Step on half-interval: $F(x) = \mathbf{1}_{[0,L/2)}(x)$ has cosine coeffs

$$A_n = \frac{2}{n\pi}\sin\frac{n\pi}{2} \Rightarrow A_{2k} = 0,\ A_{2k+1} = \frac{2(-1)^k}{(2k+1)\pi}.$$

Even/odd extensions (to use pure cos/sin on $[0,L]$):
Even $\Rightarrow$ cosine series; Odd $\Rightarrow$ sine series.

Divergence Theorem (2D/3D Full Forms)

General 2D (Cartesian):

$$\boxed{\iint_\Omega\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)dA = \oint_{\partial\Omega}(F_x n_x + F_y n_y)\,ds = \oint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}\,ds}$$

Rectangular region $\Omega = [0,L]\times[0,H]$:

$$\iint_\Omega\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)dA = \int_0^H \big[F_x(L,y) - F_x(0,y)\big]\,dy + \int_0^L \big[F_y(x,H) - F_y(x,0)\big]\,dx.$$

For heat flux $\mathbf{F} = -K\nabla u$, this equals total outward heat flow through all 4 sides.
Polar coordinates (r,θ): If $\mathbf{F} = F_r\,\mathbf{e}_r + F_\theta\,\mathbf{e}_\theta$,

$$\boxed{\iint_\Omega\left[\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right]r\,dr\,d\theta = \oint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}\,ds}$$

For a circular region $r = a$:

$$\iint_\Omega\left[\frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right]r\,dr\,d\theta = \int_0^{2\pi} F_r(a,\theta)\,a\,d\theta.$$

For $\mathbf{F} = -K\nabla u$, this gives total heat leaving the circle of radius a.
3D version (for reference):

$$\iiint_\Omega\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dV = \iint_{\partial\Omega}(F_x n_x + F_y n_y + F_z n_z)\,dS.$$

Proof Tools (one-liners)

Dirichlet uniqueness for $\Delta u = g$: difference is harmonic $\Rightarrow$ max principle $\Rightarrow$ zero.
Mean value property: harmonic $u(x_0)$ equals average of u on any circle/sphere centered at x_0 inside domain (quick sanity check).
Strong max principle: interior max/min $\Rightarrow$ u constant.

Checklists

1D Heat:

- Choose sin/cos basis.
- Compute Fourier coeffs of $u(x,0)$.
- Multiply by $e^{-\alpha\lambda_n t}$.
- Add constant mode (Neumann).

Polar:

- Periodicity $\Rightarrow$ angular $\cos n\theta, \sin n\theta$.
- Radial Euler $\Rightarrow$ keep r^n (interior) or r^{-n} (exterior).
- Coeffs are Fourier coeffs of boundary data.