Supplemental Materials for "Minimizing post shock forecasting error using disparate information"

Jilei Lin[*], Ziyu Liu[†], and Daniel J. Eck[‡]

Department of Statistics, University of Illinois at Urbana-Champaign

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In the supplementary materials, we provide details for some procedures that are not discussed in the manuscript. Section 1 provides statistical evidence for approximate independence between the shock-effects nested in 2008 September time series in the analysis of Conoco Phillips stock. Section 2 details the algorithms of $\mathcal{B}_f$ and $\mathcal{B}_u$. Section 3 lists the tables for simulations under $\mathcal{M}_1$, whose results are discussed in Section 4 in the manuscript. Section 4 lists an example for the non-uniqueness of $\mathbf{W}^*$ when $2p < n$, where $\mathbf{W}^*$ is very likely to lie in the boundary of parameter space $\mathcal{W}$.

1 Supplementary materials for data analysis

The independence of the estimated September, 2008 shock-effects are further tested using likelihood ratio test (LRT) based on their estimated covariance matrix. The estimated covariance matrix is

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 4.836 & 0.418 & -0.552 \\ 0.418 & 4.269 & 0.161 \\ -0.552 & 0.161 & 4.170 \end{pmatrix}.$$

with degrees of freedoms 14. Using the LRT for independence between blocks of random variables [Marden, 2015, Section 10.2], the LRT test statistic is 0.367 with p-value of 0.545. Therefore, we do not reject the null hypothesis that the three estimated shock-effects are independent under a significance level of 5%.

*jileil2@ilinois.edu

†ziyuliu3@illinois.edu

‡dje13@illinois.edu

2 Bootstrap algorithms of $\mathcal{B}_f$ and $\mathcal{B}_u$

Algorithm 1 presents the algorithms for the fixed donor pool bootstrapping $\mathcal{B}_f$ and Algorithm 2 outlines the steps for the unfixed donor pool bootstrapping $\mathcal{B}_u$.

Algorithm 1: Fixed donor pool bootstrapping $\mathcal{B}_f$ for estimation of shock-effect estimators

Input: B – the number of parametric bootstraps
$\{(y_{i,t}, \mathbf{x}_{i,t})\colon i = 2, \ldots, n+1, t = 0, \ldots, T_i\}$ – the data
$\{T_i^*\colon i = 1, \ldots, n+1\}$ – the time point just before the shock
$\{\hat{\varepsilon}_{i,t}\colon t = 1, \ldots, T_i\}$ – the collection of residuals for $t = 1, \ldots, T_i$
$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i\colon i = 2, \ldots, n+1\}$ – the OLS estimates

Result: The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

for $b = 1 : B$ **do**
  for $i = 2, \ldots, n+1$ **do**
    Sample with replacement from $\{\hat{\varepsilon}_{i,t}\colon t = 1, \ldots, T_i\}$ to obtain $\{\hat{\varepsilon}_{i,t}^{(b)}\colon t = 1, \ldots, T_i\}$
    Define $y_{i,0}^{(b)} = y_{i,0}$
    for $t = 1, \ldots, T_i$ **do**
      Compute $y_{i,t}^{(b)} = \hat{\eta}_i + \hat{\alpha}_i 1(t = T_i^* + 1) + \hat{\phi}_i y_{i,t-1}^{(b)} + \theta_i' \mathbf{x}_{i,t} + \beta_i' \mathbf{x}_{i,t-1} + \hat{\varepsilon}_{i,t}^{(b)}$
    end
    Compute $\hat{\alpha}_i^{(b)}$ based on OLS estimation with $\{(y_{i,t}^{(b)}, \mathbf{x}_{i,t})\colon t = 0, \ldots, T_i\}$
  end
  Compute the bth shock-effect estimate $\hat{\alpha}_{\text{est}}^{(b)}$ for est $\in \{\text{adj}, \text{wadj}, \text{IVW}\}$
end
Compute the sample variance of $\{\hat{\alpha}_{\text{est}}^{(b)}\colon b = 1, \ldots, B\}$ for est $\in \{\text{adj}, \text{wadj}, \text{IVW}\}$

Algorithm 2: Unfixed donor pool bootstrapping $\mathcal{B}_u$ for estimation of shock-effect estimators

Input: B – the number of parametric bootstraps
$\{(y_{i,t}, \mathbf{x}_{i,t})\colon i = 2, \ldots, n+1, t = 0, \ldots, T_i\}$ – the data
$\{T_i^*\colon i = 1, \ldots, n+1\}$ – the time point just before the shock
$\{\hat{\varepsilon}_{i,t}\colon t = 1, \ldots, T_i\}$ – the collection of residuals for $t = 1, \ldots, T_i$
$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i\colon i = 2, \ldots, n+1\}$ – the OLS estimates

Result: The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

for $b = 1 : B$ **do**
  Sample n elements with replacement from $I = \{2, \ldots, n+1\}$ to form $I^{(b)}$, where elements of $I^{(b)}$ are not necessarily unique in terms of their indices
  for $i \in I^{(b)}$ **do**
    Sample with replacement from $\{\hat{\varepsilon}_{i,t}\colon t = 1, \ldots, T_i\}$ to obtain $\{\hat{\varepsilon}_{i,t}^{(b)}\colon t = 1, \ldots, T_i\}$
    Define $y_{i,0}^{(b)} = y_{i,0}$
    for $t = 1, \ldots, T_i$ **do**
      Compute $y_{i,t}^{(b)} = \hat{\eta}_i + \hat{\alpha}_i 1(t = T_i^* + 1) + \hat{\phi}_i y_{i,t-1}^{(b)} + \theta_i' \mathbf{x}_{i,t} + \beta_i' \mathbf{x}_{i,t-1} + \hat{\varepsilon}_{i,t}^{(b)}$
    end
    Compute $\hat{\alpha}_i^{(b)}$ based on OLS estimation with $\{(y_{i,t}^{(b)}, \mathbf{x}_{i,t})\colon t = 0, \ldots, T_i\}$
  end
  Compute the bth shock-effect estimate $\hat{\alpha}_{\text{est}}^{(b)}$ for est $\in \{\text{adj}, \text{wadj}, \text{IVW}\}$
end
Compute the sample variance of $\{\hat{\alpha}_{\text{est}}^{(b)}\colon b = 1, \ldots, B\}$ for est $\in \{\text{adj}, \text{wadj}, \text{IVW}\}$

3 Simulations for $\mathcal{M}_1$

In this section, we present the simulation results for $\mathcal{M}_1$. To make it comparable to $\mathcal{M}_2$, we set $\mu_\alpha = 50$ with other parameter setup the same as that of $\mathcal{M}_2$. The corresponding tables are attached as follows.

Table 1: 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_u$ with varying n and σ_α

		Guess			LOOCV with k random draws			Distance to y_{1,T_1^*+1}			
n	σ_α	$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
	5	1 (0)	1 (0)	1 (0)	0.97 (0.01)	0.97 (0.01)	0.97 (0.01)	50.44 (3.04)	14.42 (2.64)	14.58 (2.69)	14.37 (2.64)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.93 (0.02)	0.93 (0.02)	51.17 (3.43)	16.97 (2.88)	16.82 (2.91)	17.04 (2.87)
5	25	1 (0)	1 (0)	1 (0)	0.85 (0.02)	0.83 (0.03)	0.85 (0.02)	53.63 (5.49)	27.12 (4.56)	28.8 (4.26)	27.31 (4.54)
	50	0.97 (0.03)	0.97 (0.03)	0.9 (0.06)	0.61 (0.05)	0.63 (0.05)	0.63 (0.05)	63.4 (8.74)	47.94 (8.09)	52.66 (7.56)	48.28 (8.06)
	100	0.7 (0.09)	0.63 (0.09)	0.63 (0.09)	0.55 (0.05)	0.6 (0.04)	0.55 (0.05)	92.54 (15.54)	91.55 (15.74)	102.28 (14.95)	92.35 (15.7)
	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.93 (0.02)	51.41 (2.47)	11.95 (1.75)	12.26 (1.88)	12.02 (1.78)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.9 (0.02)	0.92 (0.02)	50.22 (3.12)	14.17 (2.41)	14.54 (2.46)	14.17 (2.47)
10	25	1 (0)	0.97 (0.03)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	47 (5.94)	28.21 (4.39)	26.37 (4.8)	28.37 (4.46)
	50	0.87 (0.06)	0.9 (0.06)	0.73 (0.08)	0.65 (0.04)	0.64 (0.04)	0.63 (0.04)	52.83 (9.71)	54.55 (8.17)	51.15 (8.89)	54.71 (8.26)
	100	0.77 (0.08)	0.73 (0.08)	0.57 (0.09)	0.47 (0.04)	0.53 (0.05)	0.47 (0.04)	85.79 (17.29)	108.57 (15.99)	106.48 (16.44)	108.85 (16.09)
	5	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	47.84 (2.91)	13.2 (1.81)	12.78 (1.63)	13.04 (1.82)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.93 (0.02)	0.95 (0.02)	48.16 (3.24)	14.63 (2)	14.14 (1.91)	14.58 (1.98)
15	25	1 (0)	1 (0)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	49.39 (4.94)	21.64 (3.28)	23.4 (3.27)	21.59 (3.29)
	50	0.9 (0.06)	0.9 (0.06)	0.83 (0.07)	0.57 (0.05)	0.62 (0.04)	0.55 (0.04)	56.69 (7.53)	38.83 (5.67)	45.4 (5.59)	39.2 (5.56)
	100	0.67 (0.09)	0.63 (0.09)	0.47 (0.09)	0.47 (0.04)	0.44 (0.04)	0.46 (0.04)	85 (12.02)	77.64 (10.39)	91.43 (10.65)	77.98 (10.23)
	5	1 (0)	1 (0)	1 (0)	0.98 (0.01)	0.97 (0.01)	0.98 (0.01)	47.68 (3.18)	12.86 (1.96)	12.6 (2.09)	12.75 (1.98)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	46.14 (3.34)	14.31 (1.86)	13.74 (2.1)	14.33 (1.85)
25	25	1 (0)	1 (0)	0.97 (0.03)	0.79 (0.03)	0.8 (0.03)	0.79 (0.03)	42.26 (4.45)	21.32 (2.55)	22.65 (2.87)	21.46 (2.52)
	50	0.93 (0.05)	0.9 (0.06)	0.87 (0.06)	0.62 (0.04)	0.61 (0.04)	0.63 (0.04)	44.43 (6.07)	38.19 (4.48)	43.57 (4.93)	38.6 (4.44)
	100	0.9 (0.06)	0.8 (0.07)	0.73 (0.08)	0.53 (0.04)	0.54 (0.04)	0.51 (0.04)	71.69 (9.1)	72.64 (9.64)	87.82 (9.85)	73.73 (9.57)

Table 2: 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_u$ with varying σ and σ_α

		Guess			LOOCV with k random draws			Distance to y_{1,T_1^*+1}			
σ	σ_α	$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	49.68 (1.63)	7.62 (1.21)	7.77 (1.28)	7.62 (1.24)
	10	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	48.5 (2.53)	11.95 (1.85)	11.49 (2)	12.03 (1.88)
5	25	1 (0)	1 (0)	0.93 (0.05)	0.85 (0.03)	0.85 (0.03)	0.85 (0.03)	44.99 (5.78)	27.75 (4.03)	26.06 (4.42)	27.85 (4.07)
	50	0.9 (0.06)	0.93 (0.05)	0.8 (0.07)	0.65 (0.03)	0.66 (0.03)	0.63 (0.04)	52.5 (9.41)	54.76 (7.92)	53.73 (8.17)	54.92 (7.97)
	100	0.73 (0.08)	0.73 (0.08)	0.57 (0.09)	0.48 (0.04)	0.51 (0.04)	0.47 (0.04)	86.04 (17.13)	110.35 (15.5)	109.21 (15.79)	110.42 (15.62)
	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.93 (0.02)	51.41 (2.47)	11.95 (1.75)	12.26 (1.88)	12.02 (1.78)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.9 (0.02)	0.92 (0.02)	50.22 (3.12)	14.17 (2.41)	14.54 (2.46)	14.17 (2.47)
10	25	1 (0)	0.97 (0.03)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	47 (5.94)	28.21 (4.39)	26.37 (4.8)	28.37 (4.46)
	50	0.87 (0.06)	0.9 (0.06)	0.73 (0.08)	0.65 (0.04)	0.64 (0.04)	0.63 (0.04)	52.83 (9.71)	54.55 (8.17)	51.15 (8.89)	54.71 (8.26)
	100	0.77 (0.08)	0.73 (0.08)	0.57 (0.09)	0.47 (0.04)	0.53 (0.05)	0.47 (0.04)	85.79 (17.29)	108.57 (15.99)	106.48 (16.44)	108.85 (16.09)
	5	0.97 (0.03)	0.93 (0.05)	0.9 (0.06)	0.77 (0.03)	0.79 (0.03)	0.77 (0.03)	56.81 (5.73)	28.42 (3.9)	28.84 (4.24)	28.31 (4)
	10	0.97 (0.03)	0.93 (0.05)	0.9 (0.06)	0.77 (0.03)	0.79 (0.03)	0.77 (0.03)	55.69 (5.99)	29.55 (4.17)	29.47 (4.56)	29.6 (4.26)
25	25	0.93 (0.05)	0.9 (0.06)	0.87 (0.06)	0.71 (0.04)	0.69 (0.04)	0.69 (0.04)	54.03 (7.4)	35.79 (6.05)	35.34 (6.29)	35.8 (6.21)
	50	0.8 (0.07)	0.73 (0.08)	0.67 (0.09)	0.59 (0.05)	0.59 (0.05)	0.59 (0.05)	58.47 (10.63)	57.13 (9.47)	53.81 (10.1)	57.46 (9.61)
	100	0.7 (0.09)	0.73 (0.08)	0.6 (0.09)	0.48 (0.05)	0.53 (0.04)	0.47 (0.04)	89.65 (17.56)	108.87 (16.85)	100.54 (18.43)	109.32 (17.01)
	5	0.77 (0.08)	0.73 (0.08)	0.7 (0.09)	0.48 (0.04)	0.49 (0.04)	0.48 (0.04)	74.3 (9.4)	56.62 (7.74)	57.8 (8.28)	56.64 (7.83)
	10	0.77 (0.08)	0.73 (0.08)	0.7 (0.09)	0.47 (0.05)	0.49 (0.04)	0.47 (0.04)	73.97 (9.36)	57.3 (7.84)	57.88 (8.45)	57.29 (7.97)
50	25	0.73 (0.08)	0.63 (0.09)	0.67 (0.09)	0.49 (0.04)	0.5 (0.04)	0.5 (0.04)	74.15 (9.79)	61.5 (8.75)	60.05 (9.58)	61.64 (8.95)
	50	0.67 (0.09)	0.6 (0.09)	0.57 (0.09)	0.5 (0.04)	0.55 (0.04)	0.49 (0.04)	77.11 (12.25)	72.31 (12.11)	71.3 (12.48)	72.47 (12.4)
	100	0.67 (0.09)	0.6 (0.09)	0.5 (0.09)	0.47 (0.05)	0.51 (0.04)	0.47 (0.04)	100.81 (18.65)	114.75 (18.98)	107.41 (20.26)	115.39 (19.27)
	5	0.57 (0.09)	0.5 (0.09)	0.53 (0.09)	0.51 (0.04)	0.49 (0.04)	0.53 (0.05)	120.39 (16.06)	113.53 (15.4)	116.64 (16.14)	113.57 (15.53)
	10	0.57 (0.09)	0.5 (0.09)	0.53 (0.09)	0.52 (0.05)	0.49 (0.04)	0.53 (0.05)	119.9 (16.04)	113.73 (15.47)	115.28 (16.51)	113.78 (15.63)
100	25	0.6 (0.09)	0.5 (0.09)	0.53 (0.09)	0.51 (0.04)	0.55 (0.04)	0.53 (0.04)	120.59 (15.93)	115.82 (15.92)	115.95 (17.06)	115.74 (16.23)
	50	0.6 (0.09)	0.47 (0.09)	0.53 (0.09)	0.53 (0.05)	0.58 (0.05)	0.53 (0.04)	122.37 (17.24)	123.37 (17.55)	120.3 (18.99)	123.69 (17.93)
	100	0.63 (0.09)	0.47 (0.09)	0.5 (0.09)	0.47 (0.05)	0.53 (0.05)	0.48 (0.04)	137.56 (21.94)	145.22 (24.23)	143.1 (24.73)	145.67 (24.77)

Table 3: 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_f$ with varying n and σ_α

		Guess			LOOCV with k random draws			Distance to y_{1,T_1^*+1}			
n	σ_α	$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.93 (0.02)	0.93 (0.02)	49.36 (2.51)	12.52 (2.39)	12.82 (2.25)	12.27 (2.39)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.89 (0.02)	49.62 (2.89)	15.15 (2.61)	14.75 (2.55)	14.93 (2.59)
5	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.79 (0.03)	0.81 (0.03)	50.39 (5.33)	28.01 (3.76)	28 (3.52)	27.66 (3.72)
	50	0.97 (0.03)	0.97 (0.03)	0.97 (0.03)	0.66 (0.03)	0.65 (0.04)	0.67 (0.03)	61.3 (8.41)	51.79 (6.68)	51.84 (6.45)	51.36 (6.62)
	100	0.93 (0.05)	0.9 (0.06)	0.9 (0.06)	0.53 (0.05)	0.45 (0.05)	0.55 (0.05)	102.54 (13.46)	100.42 (13.26)	99.48 (13.38)	99.81 (13.15)
	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.85 (2.66)	11.93 (1.97)	13.19 (2.07)	11.85 (2.01)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.9 (0.02)	53.22 (3)	13.55 (2.1)	14.51 (2.19)	13.43 (2.15)
10	25	1 (0)	1 (0)	1 (0)	0.75 (0.04)	0.79 (0.04)	0.76 (0.04)	54.31 (4.77)	23.71 (2.71)	21.37 (3.47)	23.58 (2.72)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.59 (0.04)	0.64 (0.04)	0.59 (0.04)	58.73 (7.88)	41.53 (5.24)	37.18 (6.44)	41.37 (5.15)
	100	1 (0)	0.97 (0.03)	0.73 (0.08)	0.48 (0.05)	0.48 (0.05)	0.48 (0.04)	82.13 (12.74)	77.4 (11.24)	72.81 (12.65)	77.07 (11.02)
	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.94 (0.02)	46.76 (2.5)	11.39 (1.4)	13.23 (1.68)	11.38 (1.39)
	10	1 (0)	1 (0)	1 (0)	0.92 (0.02)	0.91 (0.02)	0.92 (0.02)	46.37 (2.59)	11.65 (1.56)	13.88 (1.96)	11.66 (1.55)
15	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.81 (0.03)	0.81 (0.03)	45.21 (3.62)	17.31 (2.23)	21.19 (2.76)	17.16 (2.33)
	50	1 (0)	1 (0)	0.87 (0.06)	0.64 (0.05)	0.67 (0.04)	0.65 (0.04)	44.29 (6.1)	30.7 (4.15)	36.42 (4.91)	31.17 (4.18)
	100	0.9 (0.06)	0.87 (0.06)	0.7 (0.09)	0.55 (0.04)	0.52 (0.05)	0.56 (0.04)	57.73 (9.79)	61.28 (8.18)	71.42 (9.17)	62.33 (8.22)
	5	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	47.87 (3.13)	12.4 (2.09)	13.37 (1.95)	12.4 (2.09)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.93 (0.02)	0.95 (0.02)	46.81 (3.5)	14.31 (2.26)	15.8 (2.17)	14.23 (2.26)
25	25	1 (0)	1 (0)	0.93 (0.05)	0.74 (0.03)	0.77 (0.03)	0.74 (0.03)	45.32 (4.84)	23.24 (3.26)	25.45 (3.64)	23.12 (3.23)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.61 (0.03)	0.59 (0.03)	0.6 (0.03)	52.43 (6.48)	43.31 (5.03)	46.98 (6.12)	43.24 (4.93)
	100	1 (0)	0.97 (0.03)	0.67 (0.09)	0.53 (0.04)	0.53 (0.03)	0.53 (0.04)	85.99 (9.26)	83.52 (9.56)	90.95 (11.69)	83.53 (9.39)

Table 4: 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_f$ with varying σ and σ_α

		Guess			LOOCV with k random draws			Distance to y_{1,T_1^*+1}			
σ	σ_α	$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.6 (1.52)	6.91 (1.05)	7.35 (1.09)	6.85 (1.07)
	10	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.96 (2.07)	10.22 (1.19)	9.5 (1.47)	10.18 (1.2)
5	25	1 (0)	1 (0)	1 (0)	0.79 (0.03)	0.82 (0.03)	0.79 (0.03)	53.05 (4.27)	20.92 (2.62)	18.7 (3.21)	20.87 (2.57)
	50	1 (0)	1 (0)	0.87 (0.06)	0.59 (0.04)	0.66 (0.04)	0.58 (0.04)	57.88 (7.51)	38.89 (5.61)	36.51 (6.32)	38.74 (5.51)
	100	1 (0)	1 (0)	0.73 (0.08)	0.45 (0.04)	0.46 (0.05)	0.46 (0.04)	80.3 (12.8)	76.09 (11.54)	72.77 (12.71)	75.51 (11.37)
	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.85 (2.66)	11.93 (1.97)	13.19 (2.07)	11.85 (2.01)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.9 (0.02)	53.22 (3)	13.55 (2.1)	14.51 (2.19)	13.43 (2.15)
10	25	1 (0)	1 (0)	1 (0)	0.75 (0.04)	0.79 (0.04)	0.76 (0.04)	54.31 (4.77)	23.71 (2.71)	21.37 (3.47)	23.58 (2.72)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.59 (0.04)	0.64 (0.04)	0.59 (0.04)	58.73 (7.88)	41.53 (5.24)	37.18 (6.44)	41.37 (5.15)
	100	1 (0)	0.97 (0.03)	0.73 (0.08)	0.48 (0.05)	0.48 (0.05)	0.48 (0.04)	82.13 (12.74)	77.4 (11.24)	72.81 (12.65)	77.07 (11.02)
	5	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.77 (0.03)	0.75 (0.03)	0.76 (0.03)	58.95 (5.73)	28.82 (4.87)	30.78 (5.44)	29.02 (4.91)
	10	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.78 (0.04)	0.76 (0.04)	0.77 (0.04)	59.25 (5.89)	28.98 (4.93)	31.55 (5.31)	29.1 (4.99)
25	25	0.97 (0.03)	0.97 (0.03)	0.9 (0.06)	0.67 (0.04)	0.67 (0.04)	0.67 (0.04)	61.16 (6.63)	33.6 (5.22)	35.64 (5.47)	33.34 (5.33)
	50	0.9 (0.06)	0.83 (0.07)	0.77 (0.08)	0.55 (0.04)	0.59 (0.04)	0.54 (0.05)	66.95 (8.6)	49.6 (6.03)	46.13 (7.36)	49.25 (6.08)
	100	0.9 (0.06)	0.87 (0.06)	0.7 (0.09)	0.49 (0.05)	0.46 (0.05)	0.47 (0.05)	89.11 (12.91)	85.23 (10.31)	75.55 (13.03)	84.46 (10.25)
	5	0.77 (0.08)	0.6 (0.09)	0.67 (0.09)	0.62 (0.04)	0.57 (0.05)	0.63 (0.04)	77.73 (9.69)	58.31 (9.66)	61.13 (11.05)	59.11 (9.66)
	10	0.77 (0.08)	0.67 (0.09)	0.73 (0.08)	0.64 (0.04)	0.58 (0.05)	0.62 (0.04)	78.36 (9.65)	57.79 (9.73)	61.58 (10.86)	58.54 (9.74)
50	25	0.8 (0.07)	0.67 (0.09)	0.73 (0.08)	0.63 (0.04)	0.57 (0.04)	0.61 (0.05)	81.3 (9.66)	59.28 (9.82)	64.17 (10.5)	59.24 (9.98)
	50	0.87 (0.06)	0.7 (0.09)	0.67 (0.09)	0.51 (0.05)	0.51 (0.05)	0.53 (0.05)	86.76 (10.76)	67.58 (10.33)	70.73 (10.98)	67.06 (10.54)
	100	0.87 (0.06)	0.7 (0.09)	0.63 (0.09)	0.48 (0.05)	0.53 (0.04)	0.47 (0.05)	102.61 (14.87)	98.97 (12.06)	91.38 (14.79)	98.17 (12.16)
	5	0.63 (0.09)	0.53 (0.09)	0.5 (0.09)	0.55 (0.04)	0.54 (0.03)	0.53 (0.04)	123.93 (17.91)	117.54 (19.27)	122.82 (22.17)	119.19 (19.23)
	10	0.6 (0.09)	0.5 (0.09)	0.5 (0.09)	0.54 (0.04)	0.54 (0.03)	0.53 (0.04)	124.06 (17.94)	117.01 (19.28)	122.57 (22.05)	118.62 (19.25)
100	25	0.63 (0.09)	0.47 (0.09)	0.5 (0.09)	0.51 (0.04)	0.53 (0.04)	0.52 (0.04)	125.25 (18.06)	115.88 (19.47)	123.94 (21.52)	116.93 (19.56)
	50	0.67 (0.09)	0.43 (0.09)	0.53 (0.09)	0.51 (0.05)	0.51 (0.04)	0.52 (0.05)	130.4 (18.23)	119.1 (19.58)	128.78 (20.9)	119.01 (19.88)
	100	0.57 (0.09)	0.53 (0.09)	0.43 (0.09)	0.49 (0.04)	0.49 (0.04)	0.45 (0.04)	143.5 (20.29)	135.98 (20.53)	141.58 (21.93)	134.9 (20.94)

4 Simulation for the Boundary Case ($p < n$)

In this section, we briefly present the result for the non-uniqueness and the boundary case when $p < n$ with the example $p = 2$ under $\mathcal{B}_u$ and $\mathcal{M}_2$. Proposition 1 in the main text tells that there are infinitely many solutions of $\mathbf{W}^*$ in this setup. To make it comparable to the main results of Section 4.3 in the main text, we set $\mu_\alpha = 50$. It is because in our simulation, as p decreases, $\mathrm{E}(\alpha_1)$ will decreases as well. The result is attached as below. See discussion in Section 3.2 and Section 6 in the main text. The following table verifies the claim at the end of Section 3.2 that non-uniqueness is not a serious problem for inferential purposes and the point in the discussion that boundary problems for $\mathcal{B}_u$ do not compromise inference.

Table 5: 30 Monte Carlo simulations of $\mathcal{M}_2$ for $\mathcal{B}_u$ with varying n and σ_α ($p = 2$, boundary case)

		Guess			LOOCV with k random draws			Distance to y_{1,T_1^*+1}			
n	σ_α	$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.71 (1.69)	8.84 (1.23)	9.55 (1.31)	8.81 (1.26)
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	52.63 (1.87)	10.14 (1.4)	10.33 (1.39)	10.17 (1.42)
5	10	1 (0)	1 (0)	1 (0)	0.97 (0.01)	0.97 (0.01)	0.96 (0.02)	53.78 (2.42)	12.44 (1.84)	13.16 (1.83)	12.44 (1.86)
	25	0.9 (0.06)	1 (0)	0.9 (0.06)	0.75 (0.04)	0.79 (0.04)	0.75 (0.04)	57.62 (4.64)	22.34 (3.43)	25.19 (4.13)	22.33 (3.44)
	50	0.7 (0.09)	0.7 (0.09)	0.7 (0.09)	0.59 (0.05)	0.57 (0.04)	0.57 (0.05)	68.41 (7.84)	42.53 (5.97)	49.34 (8.12)	42.44 (6.01)
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	52.19 (2.18)	11.1 (1.32)	12.18 (1.81)	11.19 (1.32)
	5	1 (0)	1 (0)	1 (0)	0.97 (0.02)	0.96 (0.01)	0.97 (0.02)	52.33 (2.5)	12.51 (1.58)	14.19 (1.94)	12.61 (1.58)
10	10	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.5 (3.09)	14.69 (2.12)	17.46 (2.25)	14.76 (2.13)
	25	0.9 (0.06)	0.9 (0.06)	0.9 (0.06)	0.74 (0.03)	0.76 (0.03)	0.74 (0.03)	54.62 (4.81)	24.86 (3.7)	28.35 (3.96)	24.94 (3.73)
	50	0.7 (0.09)	0.73 (0.08)	0.73 (0.08)	0.59 (0.04)	0.55 (0.04)	0.57 (0.04)	62.89 (7.56)	44.32 (6.5)	49.12 (7.13)	44.44 (6.54)
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	1 (0)	0.99 (0.01)	53.83 (1.89)	9.12 (0.92)	9.95 (1.33)	9.06 (0.93)
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	55.05 (2.25)	10.47 (1.21)	11.57 (1.54)	10.39 (1.22)
15	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.98 (0.01)	0.95 (0.02)	56.57 (2.83)	13.25 (1.61)	14.78 (1.95)	13.19 (1.6)
	25	0.93 (0.05)	0.93 (0.05)	0.93 (0.05)	0.87 (0.03)	0.89 (0.03)	0.88 (0.03)	61.14 (4.94)	24.01 (2.97)	28.39 (3.26)	23.98 (2.91)
	50	0.83 (0.07)	0.83 (0.07)	0.83 (0.07)	0.65 (0.04)	0.67 (0.03)	0.66 (0.04)	71.22 (8.07)	44.06 (5.29)	52.49 (5.93)	43.98 (5.2)
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	50.84 (2.39)	10.44 (1.51)	11.58 (1.47)	10.43 (1.51)
	5	1 (0)	1 (0)	1 (0)	0.98 (0.01)	0.98 (0.01)	0.98 (0.01)	50.4 (2.22)	9.97 (1.39)	11.77 (1.34)	9.93 (1.39)
25	10	1 (0)	1 (0)	1 (0)	0.96 (0.01)	0.97 (0.01)	0.95 (0.02)	49.84 (2.23)	10.99 (1.21)	12.97 (1.48)	10.98 (1.21)
	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.81 (0.03)	0.81 (0.03)	48.16 (3.49)	16.36 (2.18)	20.67 (2.98)	16.52 (2.15)
	50	0.93 (0.05)	0.93 (0.05)	0.93 (0.05)	0.69 (0.03)	0.69 (0.04)	0.69 (0.03)	48.56 (6.09)	33.14 (3.88)	41.94 (5.18)	33.39 (3.88)

References

John I Marden. Multivariate statistics: Old school. *University of Illinois*, 2015.