Prospective testing for the prevalence or transience of a shock effect before it occurs

Abstract

We develop a hypothesis testing procedure to prospectively test whether an anticipated shock is likely to be transient or permanent over a time horizon. We achieve this by borrowing knowledge from other time series that have undergone similar shocks for which post-shock outcomes are observed. These additional time series form a donor pool. For each of the time series in the donor pool we calculate a p-value corresponding to a hypothesis test on the relevance of the inclusion of shock-effect information in predicting the response over the time horizon. These p-values are then combined to form an aggregated p-value which guides one decision in determining whether the shock effect for the time series under study is expected to be prevalent or transient. This p-value can be computed before the shock-effect is observed in the time series under study provided one can form a suitable donor pool. Several simulated data examples, and two real data examples of forecasting Conoco Phillips stock price and .... are provided for verification and illustration.

1 Introduction

We provide forecasting methodology for assessing the lingering effect of an anticipated structural shock to a time series under study. We focus on the setting in which a structural shock has occurred and one desires a prediction for the post-shock response over a set time horizon H. Specific interest is in determining whether the shock is expected to be permanent or transient over H. Standard forecasting methods may not yield any guidance on the post-shock trajectories [Baumeister and Kilian, 2014b]. This is a general problem that has many real life applications. For example, one may acquire terrible or great news about a company and desire to determine whether that news is bound to impact the stock price of that company over a relevant time period. Companies may be interested in forecasting the demand of their products after they were involved in a brand crisis, but they only have recent sales data from pre-crises times. All is not lost in this forecasting setting, one may be able to supplement the present forecast with past data borrowed from other time series which contain post-shock trajectories arising from materially similar structural shocks.

The core idea of our methodology is to sensibly aggregate similar past realized shock effects which arose from other time series, and then incorporate the aggregated shock effect estimator into the present forecast.

Our testing method embraces ideas from forecast aggregation in the post-shock setting [Lin and Eck, 2021], forecast comparison [Diebold and Mariano, 1995, Quaedvlieg, 2021], p-value combination, conditional forecasting [Baumeister and Kilian, 2014b, Kilian and Lütkepohl, 2017], time series pooling using cross-sectional panel data [Ramaswamy et al., 1993, Pesaran et al., 1999, Hoogstrate et al., 2000, Baltagi, 2008, Koop and Korobilis, 2012, Liu et al., 2020], forecasting with judgement and models [Svensson, 2005, Monti, 2008], synthetic control methodology [Abadie et al., 2010, Agarwal et al., 2020], expectation shocks [Croushore and Evans, 2006, Baumeister and Kilian, 2014a, Clements et al., 2019].

2 Setting

We will suppose that a researcher has multivariate time series data $\mathbf{y}_{i,t}$, $t = 1, \ldots, T_i$ and $i = 1, \ldots, n+1$. We let $\mathbf{y}_{i,t} = (y_{i,t},\ \mathbf{x}_{i,t})$ where $y_{i,t}$ is a scalar response and $\mathbf{x}_{i,t}$ is a vector of covariates that are revealed to the analyst prior to the observation of $y_{1,t}$. Suppose that the analyst is interested in forecasting $y_{1,t}$, the first time series in the collection. We will suppose that each time series $\mathbf{y}_{i,t}$ undergoes a shock at time $T_i^* \leq T_i + 1$. To define an interesting setting, we will suppose that $T_1^* = T_1 + 1$, and $1 < T_i^* < T_i + 1$ for $i \geq 2$. We will suppose that $\mathbf{x}_{i,t=T_i^*}$ is observed before the shock takes effect on $y_{i,t=T_i^*}$.

We are interested in point forecasts $y_{i,t}^h$ at multiple horizons, $h = 1, \ldots, H$ with the aim of determining whether the shock has an effect on $y_{i,t}^h$. Quaedvlieg [2021] provided a methodology for comparing forecasts jointly across all horizons of a forecast path, $h = 1, \ldots, H$. In our post-shock setting, we want to compare the forecasts

$$\hat{y}_{i,t}^{1,h} \text{ and } \hat{y}_{i,t}^{2,h}$$

where $y_{i,t}^{1,h}$ is the forecast for $y_{i,t}$ that accounts for the yet-to-be observed structural shock and is based on the information set $\mathcal{F}_{t-h}$, and $\hat{y}_{i,t}^{2,h}$ is defined similarly for the forecast that does not include any shock effect information. We will compare these forecasts in terms of their loss differential

$$\mathbf{d}_{i,t} = \text{Ł}_{i,t,1} - \text{Ł}_{i,t,2},$$

where $\text{Ł}_{i,t,j} \in \mathbb{R}^H$ has elements $L^h(y_{i,t},\ \hat{y}_{i,t}^{j,h})$, $j = 1, 2$, and L is a loss function. Hypothesis tests in Quaedvlieg [2021] are with respect to $E(\mathbf{d}_{i,t}) = \mu_{i,t}$. Conditions for these tests require conditions of Giacomini and White [2006].

We will be interested in $\mu_i = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \mu_{i,t}$.

Note: We need more formality for constructing $\hat{y}_{1,t}^{1,h}$. We could use the forecasts in Lin and Eck [2021] and then consider h-ahead methods after adjusting for the shock. Or we could consider aggregation approaches which average all post-shock responses of the series in the donor pool.

We will consider the average superior predictive ability (aSPA) to assess whether or not a shock is permanent or transitory. The aSPA investigates forecast comparisons based on their weighted average loss difference

$$\mu_i^{(\text{Avg})} = \mathbf{w}_i'\mu_i = \sum_{h=1}^{H} w_{i,h}\mu_i^h,$$

with weights $\mathbf{w}_i$ that sum to one. Note that aSPA requires the user to take a stand on the relative importance of under-performance at one horizon against out-performance at another, and note that it is likely that $\mu_i^h > 0$ for h closer to 1 since the user expects that a structural shock will occur and the structural shock is taken into account by forecast 1.

We consider a simple test for average SPA, based on the weighted-average loss differential. The associated null is

$$H_{i,\text{aSPA}}^0 : \mu_i^{(\text{Avg})} \leq 0. \tag{1}$$

A studentized test statistic corresponding to the null hypothesis (1) is of the form

$$t_{i,\text{aSPA}} = \frac{\sqrt{T}_i \bar{d}_i}{\hat{\zeta}_i}, \tag{2}$$

where $\bar{d}_i = \mathbf{w}_i'\mathbf{d}_i$ and we choose to estimate $\zeta_i = \sqrt{\mathbf{w}_i'\Omega_i\mathbf{w}_i}$ directly based on $\mathbf{w}_i'\mathbf{d}_{i,t}$ using the HAC estimator [Giacomini and White, 2006] where $\Omega_i = \text{avar}\left(\sqrt{T}_i(\bar{d}_i - \mu_i)\right)$.

2.1 Model setup

Note: We need to update the modeling setup. We should present a general modeling class which includes both the decay model and the permanent shift model.

Possible modeling approach

We now describe the assumed autoregressive models with random effects for which post-shock aggregated estimators are provided. The model $\mathcal{M}$ is defined as

$$
\mathcal{M}\colon \begin{array}{l} y_{i,t} = \eta_i + \sum_{j=1}^{q_1} \phi_{i,j} y_{i,t-j} + \sum_{j=0}^{q_2-1} \theta'_{i,j+1}\mathbf{x}_{i,t-j} + \alpha_i D_{i,t} f(t) + \varepsilon_{i,t}, \\ \alpha_i = \mu_\alpha + \delta_i' \mathbf{x}_{i,T_i^*+1} + \tilde{\varepsilon}_i, \end{array} \tag{3}
$$

where $D_{i,t} = I(t \geq T_i^* + 1)$, $I(\cdot)$ is the indicator function, $\mathbf{x}_{i,t} \in \mathbb{R}^p$ are fixed with $p \geq 1$, $f(t)$ is a bounded continuous function which does not cross zero and $\lim_{t\to\infty} g(t) = a \in \mathbb{R}$. Let $\phi_i = (\phi_{i,1}, \ldots, \phi_{i,q_1})'$, $\theta_i = (\theta_{i,1}, \ldots, \theta_{i,q_2})'$, $\delta_i = (\delta_{i,1}, \ldots, \delta_{i,q_3})'$, and suppose that the regression coefficients in (3) have the following random effects structure:

$$
\begin{aligned}
\eta_i &\overset{iid}{\sim} \mathcal{F}_\eta \text{ with } \mathrm{E}_{\mathcal{F}_\eta}(\eta_i) = \mu_\eta, \mathrm{Var}_{\mathcal{F}_\eta}(\eta_i) = \sigma_\eta^2, \\
\phi_i &\overset{iid}{\sim} \mathcal{F}_\phi \text{ where } |\phi_{i,j}| < 1, \\
\theta_i &\overset{iid}{\sim} \mathcal{F}_\theta \text{ with } \mathrm{E}_{\mathcal{F}_\theta}(\theta_i) = \mu_\theta, \mathrm{Var}_{\mathcal{F}_\theta}(\theta_i) = \Sigma_\theta^2, \\
\delta_i &\overset{iid}{\sim} \mathcal{F}_\delta \text{ with } \mathrm{E}_{\mathcal{F}_\delta}(\delta_i) = \mu_\delta, \mathrm{Var}_{\mathcal{F}_\delta}(\delta_i) = \Sigma_\delta, \\
\varepsilon_{i,t} &\overset{iid}{\sim} \mathcal{F}_{\varepsilon_i} \text{ with } \mathrm{E}_{\mathcal{F}_{\varepsilon_i}}(\varepsilon_{i,t}) = 0, \mathrm{Var}_{\mathcal{F}_{\varepsilon_i}}(\varepsilon_{i,t}) = \sigma_i^2, \\
\tilde{\varepsilon}_i &\overset{iid}{\sim} \mathcal{F}_{\tilde{\varepsilon}} \text{ with } \mathrm{E}_{\mathcal{F}_{\tilde{\varepsilon}}}(\tilde{\varepsilon}_i) = 0, \mathrm{Var}_{\mathcal{F}_{\tilde{\varepsilon}}}(\tilde{\varepsilon}_i) = \sigma_\alpha^2, \\
\eta_i &\perp\!\!\!\perp \phi_i \perp\!\!\!\perp \theta_i \perp\!\!\!\perp \delta_i \perp\!\!\!\perp \varepsilon_{i,t} \perp\!\!\!\perp \tilde{\varepsilon}_{i,t}.
\end{aligned}
$$

The model (3) with the above random effects structure is a generalization of both model formulations in Lin and Eck [2021]. **Need to carefully show**. In this formulation $f(t)$ represents either a permanent or transient structural change to the time series that results from the shock.

We further define the parameter sets

$$
\begin{aligned}
\Theta &= \{(\eta_i, \phi_i, \theta_i, \delta_i, \alpha_i, \mathbf{x}_{i,t}, y_{i,t-1})\colon t = 1, \ldots, T_i, i = 2, \ldots, n+1\} \\
\Theta_1 &= \{(\eta_i, \phi_i, \theta_i, \delta_i, \alpha_i, \mathbf{x}_{i,t}, y_{i,t-1})\colon t = 1, \ldots, T_i, i = 1\}
\end{aligned} \tag{4}
$$

Alternative model formulation

We now describe the assumed autoregressive models with random effects for which post-shock aggregated estimators are provided. The model $\mathcal{M}$ is defined as

$$
\mathcal{M}\colon \begin{array}{l} y_{i,t} = \left(\eta_i + \sum_{j=1}^{q_1} \phi_{i,j} y_{i,t-j} + \sum_{j=0}^{q_2-1} \theta'_{i,j+1}\mathbf{x}_{i,t-j}\right)(1 - D_{i,t}) + f(\mathcal{F}_{i,t}, \alpha_i) D_{i,t} + \varepsilon_{i,t}, \\ f(\mathcal{F}_{i,t}, \alpha_i) = \alpha_i + \sum_{j=1}^{q_1} \tilde{\phi}_{i,j} y_{i,t-j} + \sum_{j=0}^{q_2-1} \tilde{\theta}'_{i,j+1}\mathbf{x}_{i,t-j}, \\ \alpha_i = \mu_\alpha + \delta_i' \mathbf{x}_{i,T_i^*+1} + \varepsilon_{\alpha,i}, \end{array} \tag{5}
$$

where $g(t)$ is a known or estimable bounded continuous function which does not cross zero and $\lim_{t\to\infty} g(t) = a \in \mathbb{R}$. Let $\phi_i = (\phi_{i,1}, \ldots, \phi_{i,q_1})'$, $\theta_i = (\theta_{i,1}, \ldots, \theta_{i,q_2})'$, $\tilde{\phi}_i = (\tilde{\phi}_{i,1}, \ldots, \tilde{\phi}_{i,q_1})'$, $\tilde{\theta}_i = (\tilde{\theta}_{i,1}, \ldots, \tilde{\theta}_{i,q_2})'$,

$\delta_i = (\delta_{i,1}, \ldots, \delta_{i,p})'$, and suppose that the regression coefficients in (5) have the following hierarchical random effects structure:

$$
\begin{aligned}
\eta_i &\overset{iid}{\sim} \mathcal{F}_\eta \text{ with } \mathrm{E}_{\mathcal{F}_\eta}(\eta_i) = \mu_\eta, \mathrm{Var}_{\mathcal{F}_\eta}(\eta_i) = \sigma^2_\eta, \\
\phi_{i,j} &\overset{iid}{\sim} \mathcal{F}_{\phi_j} \text{ where } |\phi_{i,j}| < 1, \\
\theta_{i,j} &\overset{iid}{\sim} \mathcal{F}_{\theta_j} \text{ with } \mathrm{E}_{\mathcal{F}_{\theta_j}}(\theta_{i,j}) = \mu_{\theta_j}, \mathrm{Var}_{\mathcal{F}_{\theta_j}}(\theta_{i,j}) = \Sigma^2_{\theta_j}, \\
\delta_i &\overset{iid}{\sim} \mathcal{F}_\delta \text{ with } \mathrm{E}_{\mathcal{F}_\delta}(\delta_i) = \mu_\delta, \mathrm{Var}_{\mathcal{F}_\delta}(\delta_i) = \Sigma_\delta, \\
\varepsilon_{i,t} &\overset{iid}{\sim} \mathcal{F}_{\varepsilon_i} \text{ with } \mathrm{E}_{\mathcal{F}_{\varepsilon_i}}(\varepsilon_{i,t}) = 0, \mathrm{Var}_{\mathcal{F}_{\varepsilon_i}}(\varepsilon_{i,t}) = \sigma^2_i, \\
\varepsilon_{\alpha,i} &\overset{iid}{\sim} \mathcal{F}_{\varepsilon_\alpha} \text{ with } \mathrm{E}_{\mathcal{F}_{\varepsilon_\alpha}}(\varepsilon_{\alpha,i}) = 0, \mathrm{Var}_{\mathcal{F}_{\varepsilon_\alpha}}(\varepsilon_{\alpha,i}) = \sigma^2_\alpha, \\
\tilde{\phi}_{i,j} &\overset{ind}{\sim} \mathcal{F}_{\tilde{\phi}_j}(\mathbf{x}_{i,T_i^*+1}) \text{ where } |\tilde{\phi}_{i,j}| < 1, \\
\tilde{\theta}_{i,j} &\overset{ind}{\sim} \mathcal{F}_{\tilde{\theta}_j}(\mathbf{x}_{i,T_i^*+1}), \\
&\eta_i \perp\!\!\!\perp \phi_{i,j} \perp\!\!\!\perp \theta_{i,j} \perp\!\!\!\perp \tilde{\phi}_{i,j} \perp\!\!\!\perp \lambda_{i,j} \perp\!\!\!\perp \delta_i \perp\!\!\!\perp \varepsilon_{i,t} \perp\!\!\!\perp \tilde{\varepsilon}_i,
\end{aligned}
\tag{6}
$$

where the distributions $\mathcal{F}_{\tilde{\phi}_j}(\mathbf{x}_{i,T_i^*+1})$ and $\mathcal{F}_{\tilde{\theta}_{i'}}(\mathbf{x}_{i,T_i^*+1})$ satisfy $\mathcal{F}_{\tilde{\phi}_j}(\mathbf{x}_{i,T_i^*+1}) \overset{d}{=} \mathcal{F}_{\tilde{\phi}_j}(\mathbf{x}_{i',T_i^*+1})$ and $\mathcal{F}_{\tilde{\theta}_j}(\mathbf{x}_{i,T_i^*+1}) \overset{d}{=} \mathcal{F}_{\tilde{\theta}_j}(\mathbf{x}_{i',T_i^*+1})$ when $\mathbf{x}_{i,T_i^*+1} = \mathbf{x}_{i',T_i^*+1}$. Note that for model (5) to be of use for post-shock forecasting, the variation in $\mathcal{F}_{\tilde{\phi}}$ and $\mathcal{F}_{\tilde{\theta}}$ must be small relative to the signal strength.

We see that model (5) with its accompanying random effects structure is flexible enough to capture changing structural dynamics as well as a mean-shift. These dynamic changes depend heavily on the value of $\mathbf{x}_{i,T_i^*+1}$, the covariates recorded right before the first post-shock response is observed. Under this setup any two series i, j with small $\|\mathbf{x}_{i,T_i^*+1} - \mathbf{x}_{j,T_i^*+1}\|_2$ are expected to experience similar structural changes. This makes distance based weighting an attractive avenue.

2.2 Forecasting and testing for shock persistence

Note: Our forecast needs to be written with respect to our general model. Specifics can be given when we conduct our numerical examples.

In our post-shock setting we consider the following candidate forecasts:

$$
\text{Forecast 1}:
$$
$$
\text{Forecast 2}:
$$

We want to determine which forecast is appropriate over a horizon while the methods in Lin and Eck [2021] were only appropriate in the nowcasting setting in which prediction was only focused on the response immediately following the shock.

Note: We need to explain what the voting method is, possibly in algorithmic format. Also note that the p-values that we obtain are computed using a bootstrap procedure. Perhaps an additional proposition that states the performance of these bootstrap p-values would further guarantee reliability.

Proposition 1. *Suppose $p_2, \ldots, p_{n+1}$ is a sequence of pairwise independent p-values with $\mathbb{P}(p_i \le \alpha) = \kappa$ for $i = 1, \ldots, n+1$, where α is the significance level, κ is a real-valued constant in $[0,1]$, and p_1 is the p-value of the time series of interest. If $\mathbb{P}(p_1 \le \alpha) \neq 0.5$,*

$$
\mathbb{E}\left\{ \left| I\left\{ \frac{1}{n}\sum_{i=2}^{n+1} I(p_i \le \alpha) \ge 0.5 \right\} - I(p_1 \le \alpha) \right| \right\} \to \begin{cases} 1 - \mathbb{P}(p_1 \le \alpha) & \text{if } \mathbb{P}(p_1 \le \alpha) > 0.5 \\ \mathbb{P}(p_1 \le \alpha) & \text{if } \mathbb{P}(p_1 \le \alpha) < 0.5 \end{cases}
$$

Proof. Notice that as $\mathbb{P}(p_i \leq \alpha) = \kappa$ for $i = 1, \ldots, n+1$ and $I(p_i \leq \alpha)$ is a Bernoulli random variable, $I(p_i \leq \alpha)$ is identically distributed for $i = 1, \ldots, n+1$. Since p_i is pairwise independent and p_i are identically distributed for $i = 2, \ldots, n+1$, by Weak Law of Large Numbers,

$$\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \xrightarrow{p} \mathbb{P}(p_i \leq \alpha) = \mathbb{P}(p_1 \leq \alpha),$$

Define

$$f\colon [0,1] \mapsto \{0,1\} \text{ with } f(x) = I(x \geq 0.5).$$

Let $C(f)$ denote the continuity set of f. Suppose that $\mathbb{P}(p_1 \leq \alpha) \neq 0.5$. In this case, notice that

$$\mathbb{P}(\mathbb{P}(p_1 \leq \alpha) \in C(f)) = 1.$$

By Slutsky's Theorem, we have

$$I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} \xrightarrow{p} I\{\mathbb{P}(p_1 \leq \alpha) \geq 0.5\}.$$

It follows that

$$I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} - I(p_1 \leq \alpha) \xrightarrow{p} I\{\mathbb{P}(p_1 \leq \alpha) \geq 0.5\} - I(p_1 \leq \alpha)$$

Since the function $g(x) = |x|$ is continuous in x, by continuous mapping theorem,

$$\left| I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} - I(p_1 \leq \alpha)\right| \xrightarrow{p} |I\{\mathbb{P}(p_1 \leq \alpha) \geq 0.5\} - I(p_1 \leq \alpha)|$$

Moreover, note that

$$\begin{aligned}\mathbb{E}\left\{|I\{\mathbb{P}(p_1 \leq \alpha) \geq 0.5\} - I(p_1 \leq \alpha)|\right\} &= \begin{cases} 1 - \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) > 0.5 \\ \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) < 0.5 \end{cases} \\ &\leq 0.5.\end{aligned}$$

Due to the fact that

$$\left| I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} - I(p_1 \leq \alpha)\right| \leq 1$$

is bounded by 1, by Dominated Convergence Theorem for the version of convergence in measure,

$$\left| I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} - I(p_1 \leq \alpha)\right| \xrightarrow{\mathcal{L}_1} |I\{\mathbb{P}(p_1 \leq \alpha) \geq 0.5\} - I(p_1 \leq \alpha)|.$$

That would imply that

$$\mathbb{E}\left\{\left| I\left\{\frac{1}{n}\sum_{i=2}^{n+1} I(p_i \leq \alpha) \geq 0.5\right\} - I(p_1 \leq \alpha)\right|\right\} \to \begin{cases} 1 - \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) > 0.5 \\ \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) < 0.5 \end{cases}$$

That is, the expected misclassification rate of voting converges to

$$\begin{cases} 1 - \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) > 0.5 \\ \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) < 0.5 \end{cases}$$

□

Lemma 1. *For each $n \in \mathbb{N}$, let $c_{n1}, \ldots, c_{nn}$ be real numbers bounded by some $K > 0$, and let $X_{n1}, \ldots, X_{nn}$ be pairwise independent random variables defined on a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, and let $\mathbb{E}_n$ and $\mathbf{Var}_n$ denote the corresponding expectation and variance. If (b_n) is a sequence of positive numbers such that $b_n \uparrow \infty$ such that*

$$\sum_{i=1}^{n} \mathbb{P}_n(|X_{ni}| > b_n) \to 0 \quad \text{and} \quad \frac{1}{(b_n)^2} \sum_{i=1}^{n} \mathbb{E}_n[X_{ni}^2; |X_{ni}| \leq b_n] \to 0,$$

then

$$\frac{1}{b_n} \sum_{i=1}^{n} c_{ni}(X_{ni} - \mathbb{E}_n[X_{ni}; X_{ni} \leq b_n]) \xrightarrow{p} 0.$$

Proof. For each $n \in \mathbb{N}$ set

$$S_n = \sum_{i=1}^{n} c_{ni} X_{ni}, \quad T_n = \sum_{i=1}^{n} c_{ni} Y_{ni}, \quad Y_{ni} = X_{ni} I(|X_{ni}| \leq b_n), \quad i = 1, \ldots, n.$$

The goal is to show that $(S_n - \mathbb{E}_n[T_n])/b_n \xrightarrow{p} 0$. For this, it suffices to show that (1) $(T_n - S_n)/b_n \xrightarrow{p} 0$ and (2) $(T_n - \mathbb{E}_n[T_n])/b_n \xrightarrow{p} 0$ because

$$(T_n - \mathbb{E}_n[T_n])/b_n - (T_n - S_n)/b_n = (S_n - \mathbb{E}_n[T_n])/b_n.$$

We first prove (1). Notice that for every $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\{|S_n - T_n| > \varepsilon\} \subseteq \bigcup_{i=1}^{n} \{X_{ni} \neq Y_{ni}\} = \bigcup_{i=1}^{n} \{|X_{ni}| > b_n\},$$

and, as a result, by Boole's inequality we have

$$\mathbb{P}_n(|S_n - T_n| > \varepsilon) \leq \sum_{i=1}^{n} \mathbb{P}_n(|X_{ni}| > b_n) \to 0,$$

which proves (1). Next, we prove (2). Since $\mathcal{L}^2$ implies convergence in probability, it suffices to show that $(T_n - \mathbb{E}[T_n])/b_n \xrightarrow{\mathcal{L}^2} 0$, i.e., $\mathbf{Var}[T_n]/(b_n)^2 \to 0$. Since $X_{n1}, \ldots, X_{nn}$ are pairwise independent, so are $Y_{n1}, \ldots, Y_{nn}$, and consequently

$$\mathrm{Var}_n[T_n] = \sum_{i=1}^{n} \mathrm{Var}_n[Y_{ni}] \leq \sum_{i=1}^{n} \mathbb{E}_n[(c_{ni} X_{ni})^2; |X_{ni}| \leq b_n] \leq K^2 \sum_{i=1}^{n} \mathbb{E}_n[(X_{ni})^2; |X_{ni}| \leq b_n].$$

As a result,

$$0 \leq \frac{\mathrm{Var}_n[T_n]}{b_n^2} \leq K^2 \cdot \frac{1}{b_n^2} \sum_{i=1}^{n} \mathbb{E}_n[(X_{ni})^2; |X_{ni}| \leq b_n] \to 0,$$

which proves the result by sandwich theorem. $\square$

Corollary 1. *Let $(w_2, \dots, w_{n+1})$ be weights such that $w_i \in [0,1]$ and $\sum_{i=2}^{n+1} w_i = 1$. Define*

$$\mathcal{I}_n = \{i =\in 2, \dots, n+1 \colon 0 < w_i < 1\}.$$

Suppose that $\mathcal{I}_n$ is non-empty, $|\mathcal{I}_n| \to \infty$ as $n \to \infty$, and $w_i b_n \leq K$ for $i \in \mathcal{I}_n$ and some $K > 0$, where $b_n \geq 1$ and $b_n \to \infty$ as $n \to \infty$. Assume for $i \in \mathcal{I}_n$, p_i are pairwise independent p-values with

$$\sum_{i \in \mathcal{I}_n} \mathbb{P}(p_i \leq \alpha) = \sum_{i \in \mathcal{I}_n} w_i \kappa_i \to \kappa_1,$$

where $\kappa_i = \mathbb{P}(p_i \leq \alpha) \in [0,1]$ for $i \in \mathcal{I}_n$, α is the significance level, $\kappa_1 = \mathbb{P}(p_1 \leq \alpha)$, and p_1 is the p-value of the time series of interest. If $\mathbb{P}(p_1 \leq \alpha) \neq 0.5$,

$$\mathbb{E}\left\{ \left| I\left\{ \sum_{i \in \mathcal{I}_n} w_i I(p_i \leq \alpha) \geq 0.5 \right\} - I(p_1 \leq \alpha) \right| \right\} \to \begin{cases} 1 - \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) > 0.5 \\ \mathbb{P}(p_1 \leq \alpha) & \text{if } \mathbb{P}(p_1 \leq \alpha) < 0.5 \end{cases}$$

Proof. The proof is rather similar to Proposition 1. It suffices to show that

$$\sum_{i \in \mathcal{I}_n} w_i I(p_i \leq \alpha) \xrightarrow{p} \kappa_1 = \mathbb{P}(p_1 \leq \alpha).$$

and the remaining proof is the same as that of Proposition 1 in terms of applying Dominated Convergence Theorem in the version of convergence in probability. As p_i are pairwise independent for $i \in \mathcal{I}_n$, $I(p_i \leq \alpha)$ are pairwise independent for $i \in \mathcal{I}_n$. The idea is to prove the required condition of Lemma 1 holds. As $\mathcal{I}_n$ is non-empty, $|\mathcal{I}_n| \to \infty$, $b_n > 1$, and $b_n \to \infty$ as $n \to \infty$,

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_n} \mathbb{P}\big(I(p_i \leq \alpha) > b_n\big) \to 0 \\
&\frac{1}{b_n^2} \sum_{i \in \mathcal{I}_n} \mathbb{E}\big[I(p_i \leq \alpha) \,|\, I(p_i \leq \alpha) \leq b_n\big] = \frac{1}{b_n^2} \sum_{i \in \mathcal{I}_n} \mathbb{P}\big(p_i \leq \alpha\big) = \frac{1}{b_n^2} \sum_{i \in \mathcal{I}_n} w_i \kappa_i \to 0
\end{aligned}$$

because $\sum_{i \in \mathcal{I}_n} w_i \kappa_i = O(1)$ by the assumption $\sum_{i \in \mathcal{I}_n} w_i \kappa_i \to \kappa_1$. Let $c_{ni} = w_i b_n$ for $i \in \mathcal{I}_n$. Since $w_i b_n \leq K$ for some $K > 0$ and b_n, we have

$$\begin{aligned}
& \frac{1}{b_n} \sum_{i \in \mathcal{I}_n} c_{ni} \big(I(p_i \leq \alpha) - \mathbb{P}(I(p_i \leq \alpha) \,|\, I(p_i \leq \alpha) \leq b_n) \big) \\
= \quad & \frac{1}{b_n} \sum_{i \in \mathcal{I}_n} c_{ni} \big(I(p_i \leq \alpha) - \mathbb{P}(I(p_i \leq \alpha)\big) \xrightarrow{p} 0
\end{aligned}$$

which follows from Lemma 1. The above is equivalent to

$$\sum_{i \in \mathcal{I}_n} w_i I(p_i \leq \alpha) - \sum_{i \in \mathcal{I}_n} w_i \kappa_i \xrightarrow{p} 0$$

As $\sum_{i \in \mathcal{I}_n} w_i \kappa_i \to \kappa_1$ as $n \to \infty$, by Slutsky's Theorem,

$$\sum_{i \in \mathcal{I}_n} w_i I(p_i \leq \alpha) \xrightarrow{p} \kappa_1,$$

which finishes the proof. □

Possible theoretical approach:

Let l be the block length of the moving block bootstrap (MBB) where we assume that $T_i = lK_i$. Let $I_1,\dots,I_{K_i}$ be iid random variables uniformly distributed on $\{1,\dots,T_i-l+1\}$, and define the array

$$\tau_{T_i} = \{I_1+1,\dots,I_1+l,\dots,I_{K_i}+1,\dots,I_{K_i}+l\}.$$

The pseudo time-series is therefore $\mathbf{d}^b_{i,t} = \mathbf{d}^b_{i,\tau_{T_i}}$, with elements $d^{hb}_{i,t}$.

$$\left(\hat{\omega}^{hb}_i\right)^2 = \frac{1}{K_i}\sum_{k=1}^{K_i}\left[\frac{1}{l}\left(\sum_{t=1}^{l} d^{hb}_{i,(k-1)l+t} - \bar{d}^{hb}_i\right)^2\right], \tag{7}$$

where $\bar{d}^{hb}_i = \frac{1}{T_i}\sum_{t=1}^{T_i} d^{hb}_{i,t}$. From the conditions of Theorem 1 and Corollary 1 in Quaedvlieg [2021] we have

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}^b\left[\sqrt{T_i}\frac{\mathbf{w}_i'\bar{\mathbf{d}}^b_i - \mathbf{w}_i'\bar{\mathbf{d}}_i}{\hat{\zeta}^b_i}\right] - \mathbb{P}\left[\sqrt{T_i}\frac{\mathbf{w}_i'\bar{\mathbf{d}}_i - \mathbf{w}_i'\mu_i}{\hat{\zeta}^b_i}\right]\right| \xrightarrow{P} 0, \tag{8}$$

as $T_i \to \infty$ where $l = l_{T_i} = o(\sqrt{T_i})$. We can compute a bootstrap p-value as

$$\hat{p}^B_i = \frac{1}{B}\sum_{b=1}^{B} 1\{t_{i,\text{aSPA}} < t^b_{i,\text{aSPA}}\}.$$

The results (8) implies that

$$|\hat{p}^B_i - \hat{p}_i| \xrightarrow{P} 0, \quad \text{as} \quad T_i, B \to \infty, \tag{9}$$

where $\hat{p}_i = \mathbb{P}(t_{i,\alpha} < t_{i,\text{aSPA}})$ and $t_{i,\alpha}$ is a critical value corresponding to the distribution of $t_{i,\text{aSPA}}$. Therefore bootstrap inference in the donor pool is a good approximation.

This needs some work (ignore for now) Now consider a case where there exists a $2 \le i' < n+1$ where $\mathbf{x}_{1,t} = \mathbf{x}_{i',t}$. Now suppose for simplicity that $T^*_{j'} = T^*_1$ and that $T_{i'} = T_1$ where $T_1 > T^*_1 + q_1 + q_2 + 2$. This assumes that we have data for the first time series beyond the shock point. We will show that when there is a covariate clone of the time series under study, then estimation in this series will approximate estimation in the time series under study provided that the underlying variability is very small relative to the signal. To see this, consider $t > T^*_1$, where

$$y_{1,t} = \eta_1 + \mu_\alpha + \delta_1'\mathbf{x}_{1,T^*_1+1} + \sum_{j=1}^{q_1}(\phi_{1,j} + \tilde{\phi}_{1,j})y_{1,t-j} + \sum_{j=0}^{q_2-1}(\theta_{1,j+1} + \tilde{\theta}_{1,j+1})'\mathbf{x}_{1,t-j} + \tilde{\varepsilon}_1 + \varepsilon_{1,t},$$

$$y_{i',t} = \eta_{i'} + \mu_\alpha + \delta_{i'}'\mathbf{x}_{1,T^*_1+1} + \sum_{j=1}^{q_1}(\phi_{i',j} + \tilde{\phi}_{i',j})y_{i',t-j} + \sum_{j=0}^{q_2-1}(\theta_{i',j+1} + \tilde{\theta}_{i',j+1})'\mathbf{x}_{1,t-j} + \tilde{\varepsilon}_{i'} + \varepsilon_{i',t},$$

$$\hat{y}_{1,t} = \hat{\eta}_1 + \hat{\alpha}_1(\mathbf{x}_{1,T^*_1+1}) + \sum_{j=1}^{q_1}(\hat{\phi}_{1,j} + \hat{\tilde{\phi}}_{1,j})\hat{y}_{1,t-j} + \sum_{j=0}^{q_2-1}(\hat{\theta}_{1,j+1} + \hat{\tilde{\theta}}_{1,j+1})'\hat{\mathbf{x}}_{1,t-j},$$

$$\hat{y}_{i',t} = \hat{\eta}_{i'} + \hat{\alpha}_{i'}(\mathbf{x}_{1,T^*_1+1}) + \sum_{j=1}^{q_1}(\hat{\phi}_{i',j} + \hat{\tilde{\phi}}_{i',j})\hat{y}_{i',t-j} + \sum_{j=0}^{q_2-1}(\hat{\theta}_{i',j+1} + \hat{\tilde{\theta}}_{i',j+1})'\hat{\mathbf{x}}_{1,t-j},$$

where $\alpha_j(\mathbf{x}) = \mu_\alpha + \delta_j'\mathbf{x}$, and we have

$$
\begin{aligned}
|y_{1,t} - \hat{y}_{1,t}| = \Bigg| &(\eta_1 - \hat{\eta}_1) + (\alpha_1 - \hat{\alpha}_1(\mathbf{x}_{1,T_1^*+1})) + \sum_{j=1}^{q_1} \left[(\phi_{1,j} + \tilde{\phi}_{1,j}) y_{1,t-j} - (\hat{\phi}_{1,j} + \hat{\tilde{\phi}}_{1,j}) \hat{y}_{1,t-j} \right] \\
&+ \sum_{j=0}^{q_2-1} \left[(\theta_{1,j+1} + \tilde{\theta}_{1,j+1})' \mathbf{x}_{1,t-j} - (\hat{\theta}_{1,j+1} + \hat{\tilde{\theta}}_{1,j+1})' \hat{\mathbf{x}}_{1,t-j} \right] + \varepsilon_{1,t} \Bigg| \\
|y_{i',t} - \hat{y}_{i',t}| = \Bigg| &(\eta_{i'} - \hat{\eta}_{i'}) + (\alpha_{i'} - \hat{\alpha}_{i'}(\mathbf{x}_{1,T_1^*+1})) + \sum_{j=1}^{q_1} \left[(\phi_{i',j} + \tilde{\phi}_{i',j}) y_{i',t-j} - (\hat{\phi}_{i',j} + \hat{\tilde{\phi}}_{i',j}) \hat{y}_{i',t-j} \right] \\
&+ \sum_{j=0}^{q_2-1} \left[(\theta_{i',j+1} + \tilde{\theta}_{i',j+1})' \mathbf{x}_{1,t-j} - (\hat{\theta}_{i',j+1} + \hat{\tilde{\theta}}_{i',j+1})' \hat{\mathbf{x}}_{1,t-j} \right] + \varepsilon_{i',t} \Bigg|.
\end{aligned}
$$

Needs finishing, but the argument is simple: estimation uncertainty declines as error variance decreases. And differences in the random effects decrease as variability decreases.

We now demonstrate an idealized setting in which the above logic holds. We first present some intermediary technical results and definitions. Let $\lambda_{\max,\theta_k}$, $\lambda_{\max,\theta_\delta}$, $\lambda_{\max,\gamma}$ respectively be the largest eigenvalues of Σ_{θ_k}, Σ_δ, and Σ_γ for $k = 1, \ldots, q_2$. Define

$$
\begin{aligned}
\mu_{\min} &= \min\left(\mu_\eta, \mu_{\phi_j}, \mu_{\tilde{\phi}_j}, \min(\mu_\delta), \min(\mu_{\theta_k}), \min(\mu_{\tilde{\theta}_k}); j = 1, \ldots, q_1, k = 1, \ldots, q_2\right), \\
\mu_{\max} &= \max\left(\mu_\eta, \mu_{\phi_j}, \mu_{\tilde{\phi}_j}, \max(\mu_\delta), \max(\mu_{\theta_k}), \max(\mu_{\tilde{\theta}_k}); j = 1, \ldots, q_1, k = 1, \ldots, q_2\right), \\
\lambda_{\max} &= \max\left(\lambda_{\max,\theta_k}, \lambda_{\max,\theta_\delta}, \lambda_{\max,\gamma}; k = 1, \ldots, q_2\right), \\
\sigma^2_{\max} &= \max\left(\sigma^2_\eta, \sigma^2_{\phi_j}, \sigma^2_i, \sigma^2_\alpha, \sigma^2_{\tilde{\phi}_j}; i = 1, \ldots, n+1, j = 1, \ldots, q_1\right).
\end{aligned}
$$

Proposition 2. *Let $\mathbf{y_{i,t}}$, $i = 1, \ldots, n_i$, $t = 1, \ldots, T$ be a collection of independent time series. Let each time series $\mathbf{y_{i,t}}$ be generated by model (5) with accompanying random effects structure (6), where $y_{i,t}$ is a scalar response $\boldsymbol{x}_{i,t} \in \mathbb{R}^p$ is a fixed vector of covariates. Suppose that there is a common shock time $p + 2 < T^* < T - (p+2)$ where $q_1 = q_2 = 1$, $\|\boldsymbol{x}_{i,t}\| = 1$, each entry of $\boldsymbol{x}_{i,t}$ is non-negative, $\mu_{\min} > 0$, and $y_{i,0} = 0$. Then for every $1 \leq t \leq T$ we have that*

$$
\mathbb{E}(y_{i,t}) \geq C \operatorname{Var}(y_{i,t}), \tag{10}
$$

provided that

$$
\begin{aligned}
\mu_\alpha + 3\mu_{\min} &\geq C\left(3\sigma^2_{\max} + 2\lambda_{\max}\right), \\
2\mu_{\min} &\geq C\left(2\sigma^2_{\max} + \lambda_{\max}\right).
\end{aligned} \tag{11}
$$

Remarks:

1. The conditions (11) govern the signal strength necessary in the expected behavior of the underlying random effects structure relative to variability necessary for the overall signal to noise ratio for the overall time series to be lower bounded by some positive value C.

2. The conditions of Proposition 2 state that there is an increasing trend in $y_{i,t}$ for all i.

Proof. For $t > T^* + 1$ we have

$$\begin{aligned}
\mathbb{E}(y_{i,t}) &= \mathbb{E}\left[\mathbb{E}(y_{i,t}|y_{i,t-1})\right] \\
&= \mathbb{E}\left[\mathbb{E}(\eta_i + \alpha_i + \tilde{\phi}_i y_{i,t-1} + \tilde{\theta}_i' \mathbf{x}_{i,t}|y_{i,t-1})\right] \\
&= \mathbb{E}(\eta_i + \mu_\alpha + \delta_i' \mathbf{x}_{i,T^*+1} + \tilde{\phi}_i y_{i,t-1} + \tilde{\theta}_i' \mathbf{x}_{i,t}) \\
&= \mu_\eta + \mu_\alpha + \mu_\delta' \mathbf{x}_{i,T^*+1} + \mu_{\tilde{\theta}}' \mathbf{x}_{i,t} + \mu_{\tilde{\phi}} \mathbb{E}(y_{i,t-1}) \\
&\geq (\mu_\alpha + 3\mu_{\min}) + \mu_{\min} \mathbb{E}(y_{i,t-1}) \\
&\geq \sum_{j=1}^{t-(T^*+1)} \mu_{\min}^j (\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*} \mathbb{E}(y_{i,T^*}) \\
&= \sum_{j=0}^{t-(T^*+1)} \mu_{\min}^j (\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*} \left(\mathbb{E}\left[\mathbb{E}(y_{i,T^*}|y_{i,T^*-1})\right]\right) \\
&= \sum_{j=0}^{t-(T^*+1)} \mu_{\min}^j (\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*} \left(\mathbb{E}(\eta_i + \theta_i' \mathbf{x}_{i,T^*} + \phi_i y_{i,T^*-1})\right) \\
&\geq \sum_{j=0}^{t-(T^*+1)} \mu_{\min}^j (\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*} \left(2\mu_{\min} + \mu_{\min} \mathbb{E}(y_{i,T^*-1})\right) \\
&\geq \sum_{j=0}^{t-(T^*+1)} \mu_{\min}^j (\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*} \left(\sum_{j=0}^{T^*-1} \mu_{\min}^j (2\mu_{\min})\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(y_{i,t}) &= \mathbb{E}\left[\mathrm{Var}(y_{i,t}|y_{i,t-1})\right] + \mathrm{Var}\left[\mathbb{E}(y_{i,t}|y_{i,t-1})\right] \\
&= \mathbb{E}\left[\mathrm{Var}(\eta_i + \mu_\alpha + \delta_i' \mathbf{x}_{i,T^*+1} + \tilde{\theta}_i' \mathbf{x}_{i,t} + \tilde{\phi}_i y_{i,t-1} + \varepsilon_{i,t} + \varepsilon_{\alpha,i}|y_{i,t-1})\right] \\
&+ \mathrm{Var}\left[\mathbb{E}(\eta_i + \mu_\alpha + \delta_i' \mathbf{x}_{i,T^*+1} + \tilde{\theta}_i' \mathbf{x}_{i,t} + \tilde{\phi}_i y_{i,t-1} + \varepsilon_{i,t} + \varepsilon_{\alpha,i}|y_{i,t-1})\right] \\
&\leq 3\sigma_{\max}^2 + 2\lambda_{\max} + \mathrm{Var}\left[\mathbb{E}(\tilde{\phi}_i y_{i,t-1}|y_{i,t-1})\right] \\
&= 3\sigma_{\max}^2 + 2\lambda_{\max} + \mu_{\tilde{\phi}}^2 \mathrm{Var}(y_{i,t-1}) \\
&\leq \sum_{j=0}^{t-(T^*+1)} (\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*} \mathrm{Var}(y_{i,T^*}) \\
&= \sum_{j=0}^{t-(T^*+1)} (\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*} \left(\mathbb{E}\left[\mathrm{Var}(y_{i,T^*}|y_{i,T^*-1})\right] + \mathrm{Var}\left[\mathbb{E}(y_{i,T^*}|y_{i,T^*-1})\right]\right) \\
&= \sum_{j=0}^{t-(T^*+1)} (\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*} \Big(\mathbb{E}\left[\mathrm{Var}(\eta_i + \theta_i' \mathbf{x}_{i,T^*} + \phi_i y_{i,T^*-1} + \varepsilon_{i,T^*}|y_{i,T^*-1})\right] \\
&\quad + \mathrm{Var}\left[\mathbb{E}(\eta_i + \theta_i' \mathbf{x}_{i,T^*} + \phi_i y_{i,T^*-1} + \varepsilon_{i,T^*}|y_{i,T^*-1})\right]\Big) \\
&\leq \sum_{j=0}^{t-(T^*+1)} (\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*} \left(2\sigma_{\max}^2 + \lambda_{\max} + \mu_\phi^2 \mathrm{Var}(y_{i,T^*-1})\right) \\
&\leq \sum_{j=0}^{t-(T^*+1)} (\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*} \left(\sum_{j=0}^{T^*-1} \left(\mu_\phi^2\right)^j \left(2\sigma_{\max}^2 + \lambda_{\max}\right)\right).
\end{aligned}$$

Combining the above with the condition (11) yields

$$
\begin{aligned}
\frac{\mathbb{E}(y_{i,t})}{\mathrm{Var}(y_{i,t})} &\geq \frac{\sum_{j=0}^{t-(T^*+1)} \mu_{\min}^{j}(\mu_\alpha + 3\mu_{\min}) + \mu_{\min}^{t-T^*}\left(\sum_{j=0}^{T^*-1} \mu_{\min}^{j}(2\mu_{\min})\right)}{\sum_{j=0}^{t-(T^*+1)}(\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*}\left(\sum_{j=0}^{T^*-1}\left(\mu_\phi^2\right)^j \left(2\sigma_{\max}^2 + \lambda_{\max}\right)\right)} \\
&\geq \frac{C\sum_{j=0}^{t-(T^*+1)}(\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + C(\mu_{\tilde{\phi}}^2)^{t-T^*}\left(\sum_{j=0}^{T^*-1}\left(\mu_\phi^2\right)^j \left(2\sigma_{\max}^2 + \lambda_{\max}\right)\right)}{\sum_{j=0}^{t-(T^*+1)}(\mu_{\tilde{\phi}}^2)^j \left(3\sigma_{\max}^2 + 2\lambda_{\max}\right) + (\mu_{\tilde{\phi}}^2)^{t-T^*}\left(\sum_{j=0}^{T^*-1}\left(\mu_\phi^2\right)^j \left(2\sigma_{\max}^2 + \lambda_{\max}\right)\right)} \\
&= C,
\end{aligned}
$$

for all i and t. A similar argument holds when $t \leq T^* + 1$. $\square$

Lemma 2 (Technical Lemma 1)**.** *Let $a_j, b_j \in \mathbb{R}$, $j = 1, 2$ be such that $0 < a_1, a_2 < 1$, $b_1, b_2 > 0$. Let $c_1 = \max(a_1, a_2)$, $c_2 = \min(a_1, a_2)$, $d_1 = \max(b_1, b_2)$, $d_2 = \min(b_1, b_2)$. Then,*

$$|a_1 b_1 - a_2 b_2| \leq |a_1 - a_2| d_1 + |b_1 - b_2| c_2.$$

Proof. Observe that

$$|a_1 b_1 - a_2 b_2| \leq c_1 d_1 - c_2 d_2 = (c_1 - c_2) d_1 + (d_1 - d_2) c_2.$$

The conclusion follows by noting that $c_1 - c_2 = |a_1 - a_2|$ and $d_1 - d_2 = |b_1 - b_2|$. $\square$

Proposition 3. *Let $\mathbf{y_{i,t}}$, $i = 1, \ldots, n_i$, $t = 1, \ldots, T$ be a collection of independent time series. Let each time series $\mathbf{y_{i,t}}$ be generated by model (5) with accompanying random effects structure (6), where $y_{i,t}$ is a scalar response $\boldsymbol{x}_{i,t} \in \mathbb{R}^p$ is a fixed vector of covariates. Suppose that there is a common shock time $p + 2 < T^* < T - (p + 2)$ where $q_1 = q_2 = 1$, $\|\boldsymbol{x}_{i,t}\| = 1$, each entry of $\boldsymbol{x}_{i,t}$ is non-negative, $\mu_{\min} > 0$, and $y_{i,0} = 0$. Let $i' \geq 2$ be an index. Then, for any $\epsilon > 0$ we can choose an $\gamma > 0$ such that*

$$\left|\mathbb{E}(y_{i,t} - y_{i',t})\right| \leq \epsilon,$$

provided that

$$
\begin{aligned}
|\mu_\delta' \boldsymbol{x}_{1,T^*+1} - \mu_\delta' \boldsymbol{x}_{i',T^*+1}| &< \gamma, \\
\left|\mu_{\tilde{\theta}_1}' \boldsymbol{x}_{1,t} - \mu_{\tilde{\theta}_{i'}}' \boldsymbol{x}_{i',t}\right| &< \gamma, \qquad \textit{for all } t, \\
|\mu_{\tilde{\phi}_1}^j - \mu_{\tilde{\phi}_{i'}}^j| &< \gamma, \qquad \textit{for all } j, \\
|\boldsymbol{x}_{1,T^*+1}' \Sigma_\delta \boldsymbol{x}_{1,T^*+1} - \boldsymbol{x}_{i',T^*+1}' \Sigma_\delta \boldsymbol{x}_{i',T^*+1}| &< \gamma, \\
|\boldsymbol{x}_{1,t}' \Sigma_{\tilde{\theta}_1} \boldsymbol{x}_{1,t} - \boldsymbol{x}_{i',t}' \Sigma_{\tilde{\theta}_{i'}} \boldsymbol{x}_{i',t}| &< \gamma, \qquad \textit{for all } t.
\end{aligned}
$$

Proof. A similar recursion to that seen in the proof of Proposition 2 yields

$$
\begin{aligned}
|\mathbb{E}(y_{1,t} - y_{i',t})| = &\left| \sum_{j=0}^{t-(T^*+1)} \left[\left(\mu_{\tilde{\phi}_1}^j (\mu_\delta' \mathbf{x}_{1,T^*+1}) - \mu_{\tilde{\phi}_{i'}}^j (\mu_\delta' \mathbf{x}_{i',T^*+1})\right) + \mu_{\tilde{\phi}_1}^j (\mu_{\tilde{\theta}_1}' \mathbf{x}_{1,t-j}) - \mu_{\tilde{\phi}_{i'}}^j (\mu_{\tilde{\theta}_{i'}}' \mathbf{x}_{i',t-j})\right] \right. \\
&\left. + \sum_{j=0}^{T^*-1} \left[\mu_\eta (\mu_{\tilde{\phi}_1}^{t-T^*} - \mu_{\tilde{\phi}_{i'}}^{t-T^*}) \mu_\phi^j + \left(\mu_{\tilde{\phi}_1}^{t-T^*} (\mu_\theta' \mathbf{x}_{1,t-j}) - \mu_{\tilde{\phi}_{i'}}^{t-T^*} (\mu_\theta' \mathbf{x}_{i',t-j})\right) \mu_\phi^j\right] \right| \\
\leq &\ |\mu_\delta' \mathbf{x}_{1,T^*+1} - \mu_\delta' \mathbf{x}_{i',T^*+1}| + \sum_{j=1}^{t-(T^*+1)} \left|\mu_{\tilde{\phi}_1}^j (\mu_\delta' \mathbf{x}_{1,T^*+1}) - \mu_{\tilde{\phi}_{i'}}^j (\mu_\delta' \mathbf{x}_{i',T^*+1})\right|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |\mu'_{\tilde{\theta}_1}\mathbf{x}_{1,t} - \mu'_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t}| + \sum_{j=1}^{t-(T^*+1)} \left|\mu^j_{\tilde{\phi}_1}(\mu'_{\tilde{\theta}_1}\mathbf{x}_{1,t-j}) - \mu^j_{\tilde{\phi}_{i'}}(\mu'_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-j})\right| \\
&\quad + \mu_\eta \sum_{j=0}^{T^*-1} \left|\mu^{t-T^*}_{\tilde{\phi}_1} - \mu^{t-T^*}_{\tilde{\phi}_{i'}}\right| \mu^j_\phi + \sum_{j=0}^{T^*-1} \left|\mu^{t-T^*}_{\tilde{\phi}_1}(\mu'_\theta \mathbf{x}_{1,T^*-j}) - \mu^{t-T^*}_{\tilde{\phi}_{i'}}(\mu'_\theta \mathbf{x}_{i',T^*-j})\right| \mu^j_\phi \\
&\leq |\mu'_\delta \mathbf{x}_{1,T^*+1} - \mu'_\delta \mathbf{x}_{i',T^*+1}| + |\mu'_{\tilde{\theta}_1}\mathbf{x}_{1,t} - \mu'_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t}| \\
&\quad + \sum_{j=1}^{t-(T^*+1)} \left[|\mu^j_{\tilde{\phi}_1} - \mu^j_{\tilde{\phi}_{i'}}| \max(\mu'_\delta \mathbf{x}_{1,T^*+1}, \mu'_\delta \mathbf{x}_{i',T^*+1}) + \left|\mu'_\delta \mathbf{x}_{1,T^*+1} - \mu'_\delta \mathbf{x}_{i',T^*+1}\right| \mu^j_{\tilde{\phi}_1}\right] \\
&\quad + \sum_{j=1}^{t-(T^*+1)} \left[|\mu^j_{\tilde{\phi}_1} - \mu^j_{\tilde{\phi}_{i'}}| \max(\mu'_\delta \mathbf{x}_{1,t-j}, \mu'_\delta \mathbf{x}_{i',t-j}) + \left|\mu'_{\tilde{\theta}_1}\mathbf{x}_{1,t-j} - \mu'_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-j}\right| \mu^j_{\tilde{\phi}_1}\right] \\
&\quad + \mu_\eta \sum_{j=0}^{T^*-1} \left|\mu^{t-T^*}_{\tilde{\phi}_1} - \mu^{t-T^*}_{\tilde{\phi}_{i'}}\right| \mu^j_\phi + \sum_{j=0}^{T^*-1} \left|\mu^{t-T^*}_{\tilde{\phi}_1}(\mu'_\theta \mathbf{x}_{1,T^*-j}) - \mu^{t-T^*}_{\tilde{\phi}_{i'}}(\mu'_\theta \mathbf{x}_{i',T^*-j})\right| \mu^j_\phi,
\end{aligned}
$$

where the second inequality follows from applications of Lemma 2. Continuing from the second inequality, the conditions of this proposition imply that

$$
\begin{aligned}
&|\mathbb{E}(y_{1,t} - y_{i',t})| \\
&\quad \leq 2\gamma + \sum_{j=1}^{\infty} \left[|\mu^j_{\tilde{\phi}_1} - \mu^j_{\tilde{\phi}_{i'}}| \max(\mu'_\delta \mathbf{x}_{1,T^*+1}, \mu'_\delta \mathbf{x}_{i',T^*+1}) + \gamma \mu^j_{\tilde{\phi}_1}\right] \\
&\qquad + \sum_{j=1}^{\infty} \left[|\mu^j_{\tilde{\phi}_1} - \mu^j_{\tilde{\phi}_{i'}}| \max(\mu'_\delta \mathbf{x}_{1,t-j}, \mu'_\delta \mathbf{x}_{i',t-j}) + \gamma \mu^j_{\tilde{\phi}_1}\right] \\
&\qquad + \mu_\eta \sum_{j=0}^{\infty} \left|\mu^{t-T^*}_{\tilde{\phi}_1} - \mu^{t-T^*}_{\tilde{\phi}_{i'}}\right| \mu^j_\phi + \sum_{j=0}^{\infty} \left|\mu^{t-T^*}_{\tilde{\phi}_1}(\mu'_\theta \mathbf{x}_{1,T^*-j}) - \mu^{t-T^*}_{\tilde{\phi}_{i'}}(\mu'_\theta \mathbf{x}_{i',T^*-j})\right| \mu^j_\phi \\
&\quad \leq 2\gamma + 2\gamma \frac{\max(\mu_\delta)}{(1-\mu_{\tilde{\phi}_1})(1-\mu_{\tilde{\phi}_{i'}})} \\
&\qquad + \mu_\eta \sum_{j=0}^{\infty} \left|\mu^{t-T^*}_{\tilde{\phi}_1} - \mu^{t-T^*}_{\tilde{\phi}_{i'}}\right| \mu^j_\phi + \sum_{j=0}^{\infty} \left|\mu^{t-T^*}_{\tilde{\phi}_1}(\mu'_\theta \mathbf{x}_{1,T^*-j}) - \mu^{t-T^*}_{\tilde{\phi}_{i'}}(\mu'_\theta \mathbf{x}_{i',T^*-j})\right| \mu^j_\phi \\
&\quad \leq 2\gamma + 2\gamma \frac{\max(\mu_\delta)}{(1-\mu_{\tilde{\phi}_1})(1-\mu_{\tilde{\phi}_{i'}})} + \gamma \frac{\mu_\eta}{1-\mu_\phi} + \sum_{j=0}^{\infty} \left|\mu^{t-T^*}_{\tilde{\phi}_1}(\mu'_\theta \mathbf{x}_{1,T^*-j}) - \mu^{t-T^*}_{\tilde{\phi}_{i'}}(\mu'_\theta \mathbf{x}_{i',T^*-j})\right| \mu^j_\phi \\
&\quad \leq 2\gamma + 2\gamma \frac{\max(\mu_\delta)}{(1-\mu_{\tilde{\phi}_1})(1-\mu_{\tilde{\phi}_{i'}})} + \gamma \frac{\mu_\eta}{1-\mu_\phi} + \gamma \frac{\max(\mu_\theta)+1}{1-\mu_\phi},
\end{aligned}
$$

where the last inequality follows from Lemma 2. We can choose γ small enough so that $|\mathbb{E}(y_{1,t} - y_{i',t})| < \varepsilon$. A similar recursion for the variance yields

$$
\begin{aligned}
&\left|\mathrm{Var}(y_{1,t}) - \mathrm{Var}(y_{i',t})\right| = \Big|\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1} + \mathbf{x}'_{1,t}\Sigma_{\tilde{\theta}_i}\mathbf{x}_{1,t} - \mathbf{x}'_{i',t}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t} \\
&\qquad + \mu^2_{\tilde{\phi}_1}\mathrm{Var}(y_{1,t-1}) - \mu^2_{\tilde{\phi}_{i'}}\mathrm{Var}(y_{i',t-1})\Big| \\
&\leq \left|\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1} + \mathbf{x}'_{1,t}\Sigma_{\tilde{\theta}_i}\mathbf{x}_{1,t} - \mathbf{x}'_{i',t}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t}\right| \\
&\qquad + \left|\mu^2_{\tilde{\phi}_1}\mathrm{Var}(y_{1,t-1}) - \mu^2_{\tilde{\phi}_{i'}}\mathrm{Var}(y_{i',t-1})\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1} + \mathbf{x}'_{1,t}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t} - \mathbf{x}'_{i',t}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t}\right| \\
&\quad + \Big|\mu^2_{\tilde{\phi}_1}\left(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2 + \mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} + \mathbf{x}'_{1,t-1}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t-1} + \mu^2_{\tilde{\phi}_1}\operatorname{Var}(y_{1,t-2})\right) \\
&\qquad\quad -\mu^2_{\tilde{\phi}_{i'}}\left(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2 + \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1} + \mathbf{x}'_{i',t-1}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-1} + \mu^2_{\tilde{\phi}_{i'}}\operatorname{Var}(y_{i',t-2})\right)\Big| \\
&\le \left|\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1}\right| + \left|\mathbf{x}'_{1,t}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t} - \mathbf{x}'_{i',t}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t}\right| \\
&\quad + \Big|\mu^2_{\tilde{\phi}_1}\left(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2 + \mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} + \mathbf{x}'_{1,t-1}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t-1}\right) \\
&\qquad\quad -\mu^2_{\tilde{\phi}_{i'}}\left(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2 + \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1} + \mathbf{x}'_{i',t-1}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-1}\right)\Big| \\
&\quad + \left|\mu^4_{\tilde{\phi}_1}\operatorname{Var}(y_{1,t-2}) - \mu^4_{\tilde{\phi}_{i'}}\operatorname{Var}(y_{i',t-2})\right| \\
&\le \sum_{j=0}^{t-(T^*+1)}\Big[\left|\mu^{2j}_{\tilde{\phi}_1}\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mu^{2j}_{\tilde{\phi}_{i'}}\mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1}\right| \\
&\quad + \left|\mu^{2j}_{\tilde{\phi}_1}\mathbf{x}'_{1,t-j}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t-j} - \mu^{2j}_{\tilde{\phi}_{i'}}\mathbf{x}'_{i',t-j}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-j}\right| + \left|\mu^{2j}_{\tilde{\phi}_1} - \mu^{2j}_{\tilde{\phi}_{i'}}\right|(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2)\Big] \\
&\quad + \left|\mu^{2(t-T^*)}_{\tilde{\phi}_1}\operatorname{Var}(y_{1,t-T^*}) - \mu^{2(t-T^*)}_{\tilde{\phi}_{i'}}\operatorname{Var}(y_{i',t-T^*})\right| \\
&\le \sum_{j=0}^{t-(T^*+1)}\Big[\left|\mu^{2j}_{\tilde{\phi}_1} - \mu^{2j}_{\tilde{\phi}_{i'}}\right|\lambda_{\max} + \left|\mathbf{x}'_{1,T^*+1}\Sigma_\delta \mathbf{x}_{1,T^*+1} - \mathbf{x}'_{i',T^*+1}\Sigma_\delta \mathbf{x}_{i',T^*+1}\right|\mu^{2j}_{\tilde{\phi}_1} \\
&\quad + \left|\mu^{2j}_{\tilde{\phi}_1} - \mu^{2j}_{\tilde{\phi}_{i'}}\right|\lambda_{\max} + \left|\mathbf{x}'_{1,t-j}\Sigma_{\tilde{\theta}_1}\mathbf{x}_{1,t-j} - \mathbf{x}'_{i',t-j}\Sigma_{\tilde{\theta}_{i'}}\mathbf{x}_{i',t-j}\right|\mu^{2j}_{\tilde{\phi}_1} + \left|\mu^{2j}_{\tilde{\phi}_1} - \mu^{2j}_{\tilde{\phi}_{i'}}\right|(\sigma^2_\eta + \sigma^2_\alpha + \sigma^2)\Big] \\
&\quad + \Big|\mu^{2(t-T^*)}_{\tilde{\phi}_1}\sum_{j=0}^{T^*-1}\left(\sigma^2_\eta + \sigma^2 + \mathbf{x}'_{1,T^*-j}\Sigma_\theta \mathbf{x}_{1,T^*-j}\right)\mu^{2j}_\phi \\
&\qquad\quad -\mu^{2(t-T^*)}_{\tilde{\phi}_{i'}}\sum_{j=0}^{T^*-1}\left(\sigma^2_\eta + \sigma^2 + \mathbf{x}'_{i',T^*-j}\Sigma_\theta \mathbf{x}_{i',T^*-j}\right)\mu^{2j}_\phi\Big| \\
&\le 2\gamma\frac{\lambda_{\max}}{(1-\mu^2_{\tilde{\phi}_1})(1-\mu^2_{\tilde{\phi}_{i'}})} + 2\gamma\frac{1}{1-\mu^2_{\tilde{\phi}_1}} + \gamma\frac{\sigma^2_\eta + \sigma^2_\alpha + \sigma^2}{(1-\mu^2_{\tilde{\phi}_1})(1-\mu^2_{\tilde{\phi}_{i'}})} \\
&\quad + (\sigma^2_\eta + \sigma^2)\left|\mu^{2(t-T^*)}_{\tilde{\phi}_1} - \mu^{2(t-T^*)}_{\tilde{\phi}_{i'}}\right|\sum_{j=0}^{T^*-1}\mu^{2j}_\phi \\
&\quad + \sum_{j=0}^{2(t-T^*)}\left|\mu^{2(t-T^*)}_{\tilde{\phi}_1}\mathbf{x}'_{1,T^*-j}\Sigma_\theta \mathbf{x}_{1,T^*-j} - \mu^{2(t-T^*)}_{\tilde{\phi}_{i'}}\mathbf{x}'_{i',T^*-j}\Sigma_\theta \mathbf{x}_{i',T^*-j}\right|\mu^{2j}_\phi \\
&\le 2\gamma\frac{\lambda_{\max}}{(1-\mu^2_{\tilde{\phi}_1})(1-\mu^2_{\tilde{\phi}_{i'}})} + 2\gamma\frac{1}{1-\mu^2_{\tilde{\phi}_1}} + \gamma\frac{\sigma^2_\eta + \sigma^2_\alpha + \sigma^2}{(1-\mu^2_{\tilde{\phi}_1})(1-\mu^2_{\tilde{\phi}_{i'}})} \\
&\quad + \gamma\frac{\sigma^2_\eta + \sigma^2}{1-\mu^2_\phi} + \gamma\frac{\lambda_{\max}}{1-\mu^2_\phi} + \gamma\frac{1}{1-\mu^2_\phi}
\end{aligned}
$$

We can choose γ small enough so that $|\operatorname{Var}(y_{1,t}) - \operatorname{Var}(y_{i',t})| < \varepsilon$. $\square$

3 Simulation Setup

Let n denote the donor pool size, p denote the number of covariates used, H denote the number of horizon used, T_i denote the length of time series to be evaluated for time series i, K_i denote the training sample size used for each forecasting time series i, T_i^* denote the time point just before the realization of the shock for time series i for $i = 1, \ldots, n+1$.

In this setting n, p, and H are pre-determined. $T_i, K_i \sim \text{Gamma}(15, 10)$. The total sample size for ith time series is $T_i + K_i + H$. T_i^* is randomly sampled from $\lceil \frac{1}{4} T_i \rceil + 1$ to $\lceil \frac{3}{4} T_i \rceil + K_i + H$. If $T_i, K_i < 90$, we force them to be 90. The adopted model for the data is as below:

$$
\begin{aligned}
y_{i,t} &= \eta_i + \phi_i y_{i,t-1} + \mathbf{x}_{i,t} \boldsymbol{\beta}_i + \alpha_i I(t > T_i^*) + \varepsilon_{i,t}, \\
\alpha_i &= \mu_\alpha + \mathbf{x}_{i,T_i^*+1} \boldsymbol{\gamma}_i + \tilde{\varepsilon}_i,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_i &\sim \text{ indep. } U(0,1) \\
\eta_i &\sim \text{ indep. } \mathcal{N}(0,1) \\
\varepsilon_{i,t} &\sim \text{ indep. } \mathcal{N}(0,\sigma^2) \\
\tilde{\varepsilon}_i &\sim \text{ indep. } \mathcal{N}(0,\sigma_\alpha^2) \\
\boldsymbol{\gamma}_i &\sim \text{ indep. } \mathcal{N}(\mu_\gamma \mathbf{1}_p, \sigma_\gamma^2 \mathbf{I}_p) \\
\boldsymbol{\beta}_i &\sim \text{ indep. } \mathcal{N}(\mathbf{0}_p, \mathbf{I}_p).
\end{aligned}
$$

Moreover, the elements of $\mathbf{x}_{i,t}$ are independently distributed as $\text{Gamma}(1, \delta)$.

Note that K_i is training sample size for time series i. Consider

$$
\begin{aligned}
K_i &\sim \lceil \text{Gamma}(a_K, b_K) \rceil \\
T_i &\sim \lceil \text{Gamma}(a_T, b_T) \rceil \\
T_i^* &\equiv \max\{T_i + 1, \lceil 0.5 \cdot (T_i + K_i + H) \rceil\},
\end{aligned}
$$

$K_i + H + T_i^* > T_i + K_i + H$

Then, we consider the following simulation setup

```
ns <- c(5, 10, 20, 40)
Tscale <- Kscale <- 1 / 2 # b_T, b_K
K.T.shape <- c(200, 400, 800, 1600) # for K_i and T_i
mu.gamma.delta <- 2 # mean for parameter vector of shock
sigma.delta.gamma <- 0.1 # sd for parameter vector of shock
sigma.alpha <- 0.05 # sd for shock noise
sigma <- 0.1 # sd for response noise
mu.alpha <- 50 # intercept for shock (relatively large)
H <- 8
ell <- 4
scale <- 2 # scale for covariates that follow Gamma distribution
```

$$
\begin{aligned}
y_{i,t} &= \eta_i + \phi_i y_{i,t-1} + \mathbf{x}_{i,t} \boldsymbol{\beta}_i + \xi_i \cdot I(t > T_i^*) + \varepsilon_{i,t}, \\
\xi_i &= \alpha_i \cdot e^{-(t - T_i^* - 1)} \\
\alpha_i &= \mu_\alpha + \mathbf{x}_{i,T_i^*+1} \boldsymbol{\gamma}_i + \tilde{\varepsilon}_i,
\end{aligned}
$$

References

Alberto Abadie, Alexis Diamond, and Jens Hainmueller. Synthetic control methods for comparative case studies: Estimating the effect of california's tobacco control program. *Journal of the American Statistical Association*, 105(490):493–505, 2010.

Anish Agarwal, Abdullah Alomar, Arnab Sarker, Devavrat Shah, Dennis Shen, and Cindy Yang. Two burning questions on covid-19: Did shutting down the economy help? can we (partially) reopen the economy without risking the second wave? *arXiv preprint arXiv:2005.00072*, 2020.

Badi H Baltagi. Forecasting with panel data. *Journal of forecasting*, 27(2):153–173, 2008.

Christiane Baumeister and Lutz Kilian. A general approach to recovering market expectations from futures prices with an application to crude oil. 2014a.

Christiane Baumeister and Lutz Kilian. Real-time analysis of oil price risks using forecast scenarios. *IMF Economic Review*, 62(1):119–145, 2014b.

Michael P Clements, Ana Beatriz Galvão, et al. Measuring the effects of expectations shocks. Technical report, Economic Modelling and Forecasting Group, 2019.

Dean Croushore and Charles L Evans. Data revisions and the identification of monetary policy shocks. *Journal of Monetary Economics*, 53(6):1135–1160, 2006.

Francis X Diebold and Robert S Mariano. Comparing predictive accuracy. *Journal of Business & economic statistics*, 13(3):253–263, 1995.

Raffaella Giacomini and Halbert White. Tests of conditional predictive ability. *Econometrica*, 74(6): 1545–1578, 2006.

Andre J Hoogstrate, Franz C Palm, and Gerard A Pfann. Pooling in dynamic panel-data models: An application to forecasting gdp growth rates. *Journal of Business & Economic Statistics*, 18(3):274–283, 2000.

Lutz Kilian and Helmut Lütkepohl. *Structural vector autoregressive analysis*. Cambridge University Press, 2017.

Gary Koop and Dimitris Korobilis. Forecasting inflation using dynamic model averaging. *International Economic Review*, 53(3):867–886, 2012.

Jilei Lin and Daniel J Eck. Minimizing post-shock forecasting error through aggregation of outside information. *International Journal of Forecasting*, 2021.

Laura Liu, Hyungsik Roger Moon, and Frank Schorfheide. Forecasting with dynamic panel data models. *Econometrica*, 88(1):171–201, 2020.

Francesca Monti. Forecast with judgment and models. *National Bank of Belgium Working Paper*, (153), 2008.

M Hashem Pesaran, Yongcheol Shin, and Ron P Smith. Pooled mean group estimation of dynamic heterogeneous panels. *Journal of the American statistical Association*, 94(446):621–634, 1999.

Rogier Quaedvlieg. Multi-horizon forecast comparison. *Journal of Business & Economic Statistics*, 39 (1):40–53, 2021.

Venkatram Ramaswamy, Wayne S DeSarbo, David J Reibstein, and William T Robinson. An empirical pooling approach for estimating marketing mix elasticities with pims data. *Marketing Science*, 12(1): 103–124, 1993.

Lars EO Svensson. Monetary policy with judgment: Forecast targeting. Technical report, National Bureau of Economic Research, 2005.