

# Supplemental Materials for “Minimizing post-shock forecasting error through aggregation of outside information”

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In the supplementary materials, we provide details for some procedures that are not discussed in the manuscript. Section 1 provides statistical evidence for approximate independence between the shock-effects nested in 2008 September time series in the analysis of Conoco Phillips stock. Section 2 details the algorithms of  $\mathcal{B}_f$  and  $\mathcal{B}_u$ . Section 3 lists the tables for simulations under  $\mathcal{M}_1$ , whose results are discussed in Section 4 in the manuscript. Section 4 lists an example for the non-uniqueness of  $\mathbf{W}^*$  when  $p < n$ , where  $\mathbf{W}^*$  is very likely to lie in the boundary of parameter space  $\mathcal{W}$ .

## 1 Supplementary materials for data analysis

The independence of the estimated September, 2008 shock-effects are further tested using likelihood ratio test (LRT) based on their estimated covariance matrix. The estimated covariance matrix is

$$\hat{\Sigma} = \begin{pmatrix} 4.836 & 0.418 & -0.552 \\ 0.418 & 4.269 & 0.161 \\ -0.552 & 0.161 & 4.170 \end{pmatrix}.$$

with degrees of freedoms 14. Using the LRT for independence between blocks of random variables [?, Section 10.2], the LRT test statistic is 0.367 with  $p$ -value of 0.545. Therefore, we do not reject the null hypothesis that the three estimated shock-effects are independent under a significance level of 5%.

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## 2 Bootstrap algorithms of $\mathcal{B}_f$ and $\mathcal{B}_u$

Algorithm 1 presents the algorithms for the fixed donor pool bootstrapping  $\mathcal{B}_f$  and Algorithm 2 outlines the steps for the unfixed donor pool bootstrapping  $\mathcal{B}_u$ .

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**Algorithm 1:** Fixed donor pool bootstrapping  $\mathcal{B}_f$  for estimation of shock-effect estimators

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**Input:**  $B$  – the number of parametric bootstraps

$\{(y_{i,t}, \mathbf{x}_{i,t}) : i = 2, \dots, n+1, t = 0, \dots, T_i\}$  – the data

$\{T_i^* : i = 1, \dots, n+1\}$  – the time point just before the shock

$\{\hat{\varepsilon}_{i,t} : t = 1, \dots, T_i\}$  – the collection of residuals for  $t = 1, \dots, T_i$

$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i : i = 2, \dots, n+1\}$  – the OLS estimates

**Result:** The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

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1 for  $b = 1 : B$  do
2   for  $i = 2, \dots, n+1$  do
3     Sample with replacement from  $\{\hat{\varepsilon}_{i,t} : t = 1, \dots, T_i\}$  to obtain  $\{\hat{\varepsilon}_{i,t}^{(b)} : t = 1, \dots, T_i\}$ 
4     Define  $y_{i,0}^{(b)} = y_{i,0}$ 
5     for  $t = 1, \dots, T_i$  do
6       Compute  $y_{i,t}^{(b)} = \hat{\eta}_i + \hat{\alpha}_i 1(t = T_i^* + 1) + \hat{\phi}_i y_{i,t-1}^{(b)} + \theta_i' \mathbf{x}_{i,t} + \beta_i' \mathbf{x}_{i,t-1} + \hat{\varepsilon}_{i,t}^{(b)}$ 
7     end
8     Compute  $\hat{\alpha}_i^{(b)}$  based on OLS estimation with  $\{(y_{i,t}^{(b)}, \mathbf{x}_{i,t}) : t = 0, \dots, T_i\}$ 
9   end
10  Compute the  $b$ th shock-effect estimate  $\hat{\alpha}_{\text{est}}^{(b)}$  for  $\text{est} \in \{\text{adj}, \text{wadj}, \text{IVW}\}$ 
11 end
12 Compute the sample variance of  $\{\hat{\alpha}_{\text{est}}^{(b)} : b = 1, \dots, B\}$  for  $\text{est} \in \{\text{adj}, \text{wadj}, \text{IVW}\}$ 

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**Algorithm 2:** Unfixed donor pool bootstrapping  $\mathcal{B}_u$  for estimation of shock-effect estimators

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**Input:**  $B$  – the number of parametric bootstraps

$\{(y_{i,t}, \mathbf{x}_{i,t}) : i = 2, \dots, n+1, t = 0, \dots, T_i\}$  – the data

$\{T_i^* : i = 1, \dots, n+1\}$  – the time point just before the shock

$\{\hat{\varepsilon}_{i,t} : t = 1, \dots, T_i\}$  – the collection of residuals for  $t = 1, \dots, T_i$

$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i : i = 2, \dots, n+1\}$  – the OLS estimates

**Result:** The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

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1 for  $b = 1 : B$  do
2   Sample  $n$  elements with replacement from  $I = \{2, \dots, n+1\}$  to form  $I^{(b)}$ , where elements of  $I^{(b)}$ 
   are not necessarily unique in terms of their indices
3   for  $i \in I^{(b)}$  do
4     Sample with replacement from  $\{\hat{\varepsilon}_{i,t} : t = 1, \dots, T_i\}$  to obtain  $\{\hat{\varepsilon}_{i,t}^{(b)} : t = 1, \dots, T_i\}$ 
5     Define  $y_{i,0}^{(b)} = y_{i,0}$ 
6     for  $t = 1, \dots, T_i$  do
7       Compute  $y_{i,t}^{(b)} = \hat{\eta}_i + \hat{\alpha}_i 1(t = T_i^* + 1) + \hat{\phi}_i y_{i,t-1}^{(b)} + \theta_i' \mathbf{x}_{i,t} + \beta_i' \mathbf{x}_{i,t-1} + \hat{\varepsilon}_{i,t}^{(b)}$ 
8     end
9     Compute  $\hat{\alpha}_i^{(b)}$  based on OLS estimation with  $\{(y_{i,t}^{(b)}, \mathbf{x}_{i,t}) : t = 0, \dots, T_i\}$ 
10  end
11  Compute the  $b$ th shock-effect estimate  $\hat{\alpha}_{\text{est}}^{(b)}$  for  $\text{est} \in \{\text{adj}, \text{wadj}, \text{IVW}\}$ 
12 end
13 Compute the sample variance of  $\{\hat{\alpha}_{\text{est}}^{(b)} : b = 1, \dots, B\}$  for  $\text{est} \in \{\text{adj}, \text{wadj}, \text{IVW}\}$ 

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### 3 Simulations for $\mathcal{M}_1$

In this section, we present the simulation results for  $\mathcal{M}_1$ . To make it comparable to  $\mathcal{M}_2$ , we set  $\mu_\alpha = 50$  with other parameter setup the same as that of  $\mathcal{M}_2$ . The corresponding tables are attached as follows.

**Table 1:** 30 Monte Carlo simulations of  $\mathcal{M}_1$  for  $\mathcal{B}_u$  with varying  $n$  and  $\sigma_\alpha$

$n$	$\sigma_\alpha$	Guess			LOOCV with $k$ random draws			Distance to $y_{1,T_1^*+1}$			
		$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
4	5	1 (0)	1 (0)	1 (0)	0.97 (0.01)	0.97 (0.01)	0.97 (0.01)	50.44 (3.04)	14.42 (2.64)	14.58 (2.69)	14.37 (2.64)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.93 (0.02)	0.93 (0.02)	51.17 (3.43)	16.97 (2.88)	16.82 (2.91)	17.04 (2.87)
	25	1 (0)	1 (0)	1 (0)	0.85 (0.02)	0.83 (0.03)	0.85 (0.02)	53.63 (5.49)	27.12 (4.56)	28.8 (4.26)	27.31 (4.54)
	50	0.97 (0.03)	0.97 (0.03)	0.9 (0.06)	0.61 (0.05)	0.63 (0.05)	0.63 (0.05)	63.4 (8.74)	47.94 (8.09)	52.66 (7.56)	48.28 (8.06)
	100	0.7 (0.09)	0.63 (0.09)	0.63 (0.09)	0.55 (0.05)	0.6 (0.04)	0.55 (0.05)	92.54 (15.54)	91.55 (15.74)	102.28 (14.95)	92.35 (15.7)
	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.93 (0.02)	51.41 (2.47)	11.95 (1.75)	12.26 (1.88)	12.02 (1.78)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.9 (0.02)	0.92 (0.02)	50.22 (3.12)	14.17 (2.41)	14.54 (2.46)	14.17 (2.47)
	25	1 (0)	0.97 (0.03)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	47 (5.94)	28.21 (4.39)	26.37 (4.8)	28.37 (4.46)
	50	0.87 (0.06)	0.9 (0.06)	0.73 (0.08)	0.65 (0.04)	0.64 (0.04)	0.63 (0.04)	52.83 (9.71)	54.55 (8.17)	51.15 (8.89)	54.71 (8.26)
	100	0.77 (0.08)	0.73 (0.08)	0.57 (0.09)	0.47 (0.04)	0.53 (0.05)	0.47 (0.04)	85.79 (17.29)	108.57 (15.99)	106.48 (16.44)	108.85 (16.09)
	5	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	47.84 (2.91)	13.2 (1.81)	12.78 (1.63)	13.04 (1.82)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.93 (0.02)	0.95 (0.02)	48.16 (3.24)	14.63 (2)	14.14 (1.91)	14.58 (1.98)
	25	1 (0)	1 (0)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	49.39 (4.94)	21.64 (3.28)	23.4 (3.27)	21.59 (3.29)
	50	0.9 (0.06)	0.9 (0.06)	0.83 (0.07)	0.57 (0.05)	0.62 (0.04)	0.55 (0.04)	56.69 (7.53)	38.83 (5.67)	45.4 (5.59)	39.2 (5.56)
	100	0.67 (0.09)	0.63 (0.09)	0.47 (0.09)	0.47 (0.04)	0.44 (0.04)	0.46 (0.04)	85 (12.02)	77.64 (10.39)	91.43 (10.65)	77.98 (10.23)
	5	1 (0)	1 (0)	1 (0)	0.98 (0.01)	0.97 (0.01)	0.98 (0.01)	47.68 (3.18)	12.86 (1.96)	12.6 (2.09)	12.75 (1.98)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	46.14 (3.34)	14.31 (1.86)	13.74 (2.1)	14.33 (1.85)
	25	1 (0)	1 (0)	0.97 (0.03)	0.79 (0.03)	0.8 (0.03)	0.79 (0.03)	42.26 (4.45)	21.32 (2.55)	22.65 (2.87)	21.46 (2.52)
	50	0.93 (0.05)	0.9 (0.06)	0.87 (0.06)	0.62 (0.04)	0.61 (0.04)	0.63 (0.04)	44.43 (6.07)	38.19 (4.48)	43.57 (4.93)	38.6 (4.44)
	100	0.9 (0.06)	0.8 (0.07)	0.73 (0.08)	0.53 (0.04)	0.54 (0.04)	0.51 (0.04)	71.69 (9.1)	72.64 (9.64)	87.82 (9.85)	73.73 (9.57)

**Table 2:** 30 Monte Carlo simulations of  $\mathcal{M}_1$  for  $\mathcal{B}_u$  with varying  $\sigma$  and  $\sigma_\alpha$

$\sigma$	$\sigma_\alpha$	Guess			LOOCV with $k$ random draws			Distance to $y_{1,T_1^*+1}$			
		$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
$\sigma$	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	49.68 (1.63)	7.62 (1.21)	7.77 (1.28)	7.62 (1.24)
	10	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	48.5 (2.53)	11.95 (1.85)	11.49 (2)	12.03 (1.88)
	25	1 (0)	1 (0)	0.93 (0.05)	0.85 (0.03)	0.85 (0.03)	0.85 (0.03)	44.99 (5.78)	27.75 (4.03)	26.06 (4.42)	27.85 (4.07)
	50	0.9 (0.06)	0.93 (0.05)	0.8 (0.07)	0.65 (0.03)	0.66 (0.03)	0.63 (0.04)	52.5 (9.41)	54.76 (7.92)	53.73 (8.17)	54.92 (7.97)
	100	0.73 (0.08)	0.73 (0.08)	0.57 (0.09)	0.48 (0.04)	0.51 (0.04)	0.47 (0.04)	86.04 (17.13)	110.35 (15.5)	109.21 (15.79)	110.42 (15.62)
	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.93 (0.02)	51.41 (2.47)	11.95 (1.75)	12.26 (1.88)	12.02 (1.78)
	10	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.9 (0.02)	0.92 (0.02)	50.22 (3.12)	14.17 (2.41)	14.54 (2.46)	14.17 (2.47)
	25	1 (0)	0.97 (0.03)	0.97 (0.03)	0.79 (0.03)	0.79 (0.03)	0.79 (0.03)	47 (5.94)	28.21 (4.39)	26.37 (4.8)	28.37 (4.46)
	50	0.87 (0.06)	0.9 (0.06)	0.73 (0.08)	0.65 (0.04)	0.64 (0.04)	0.63 (0.04)	52.83 (9.71)	54.55 (8.17)	51.15 (8.89)	54.71 (8.26)
	100	0.77 (0.08)	0.73 (0.08)	0.57 (0.09)	0.47 (0.04)	0.53 (0.05)	0.47 (0.04)	85.79 (17.29)	108.57 (15.99)	106.48 (16.44)	108.85 (16.09)
	5	0.97 (0.03)	0.93 (0.05)	0.9 (0.06)	0.77 (0.03)	0.79 (0.03)	0.77 (0.03)	56.81 (5.73)	28.42 (3.9)	28.84 (4.24)	28.31 (4)
	10	0.97 (0.03)	0.93 (0.05)	0.9 (0.06)	0.77 (0.03)	0.79 (0.03)	0.77 (0.03)	55.69 (5.99)	29.55 (4.17)	29.47 (4.56)	29.6 (4.26)
	25	0.93 (0.05)	0.9 (0.06)	0.87 (0.06)	0.71 (0.04)	0.69 (0.04)	0.69 (0.04)	54.03 (7.4)	35.79 (6.05)	35.34 (6.29)	35.8 (6.21)
	50	0.8 (0.07)	0.73 (0.08)	0.67 (0.09)	0.59 (0.05)	0.59 (0.05)	0.59 (0.05)	58.47 (10.63)	57.13 (9.47)	53.81 (10.1)	57.46 (9.61)
	100	0.7 (0.09)	0.73 (0.08)	0.6 (0.09)	0.48 (0.05)	0.53 (0.04)	0.47 (0.04)	89.65 (17.56)	108.87 (16.85)	100.54 (18.43)	109.32 (17.01)
	5	0.77 (0.08)	0.73 (0.08)	0.7 (0.09)	0.48 (0.04)	0.49 (0.04)	0.48 (0.04)	74.3 (9.4)	56.62 (7.74)	57.8 (8.28)	56.64 (7.83)
	10	0.77 (0.08)	0.73 (0.08)	0.7 (0.09)	0.47 (0.05)	0.49 (0.04)	0.47 (0.04)	73.97 (9.36)	57.3 (7.84)	57.88 (8.45)	57.29 (7.97)
	25	0.73 (0.08)	0.63 (0.09)	0.67 (0.09)	0.49 (0.04)	0.5 (0.04)	0.5 (0.04)	74.15 (9.79)	61.5 (8.75)	60.05 (9.58)	61.64 (8.95)
	50	0.67 (0.09)	0.6 (0.09)	0.57 (0.09)	0.5 (0.04)	0.55 (0.04)	0.49 (0.04)	77.11 (12.25)	72.31 (12.11)	71.3 (12.48)	72.47 (12.4)
	100	0.67 (0.09)	0.6 (0.09)	0.5 (0.09)	0.47 (0.05)	0.51 (0.04)	0.47 (0.04)	100.81 (18.65)	114.75 (18.98)	107.41 (20.26)	115.39 (19.27)
100	5	0.57 (0.09)	0.5 (0.09)	0.53 (0.09)	0.51 (0.04)	0.49 (0.04)	0.53 (0.05)	120.39 (16.06)	113.53 (15.4)	116.64 (16.14)	113.57 (15.53)
	10	0.57 (0.09)	0.5 (0.09)	0.53 (0.09)	0.52 (0.05)	0.49 (0.04)	0.53 (0.05)	119.9 (16.04)	113.73 (15.47)	115.28 (16.51)	113.78 (15.63)
	25	0.6 (0.09)	0.5 (0.09)	0.53 (0.09)	0.51 (0.04)	0.55 (0.04)	0.53 (0.04)	120.59 (15.93)	115.82 (15.92)	115.95 (17.06)	115.74 (16.23)
	50	0.6 (0.09)	0.47 (0.09)	0.53 (0.09)	0.53 (0.05)	0.58 (0.05)	0.53 (0.04)	122.37 (17.24)	123.37 (17.55)	120.3 (18.99)	123.69 (17.93)
	100	0.63 (0.09)	0.47 (0.09)	0.5 (0.09)	0.47 (0.05)	0.53 (0.05)	0.48 (0.04)	137.56 (21.94)	145.22 (24.23)	143.1 (24.73)	145.67 (24.77)

**Table 3:** 30 Monte Carlo simulations of  $\mathcal{M}_1$  for  $\mathcal{B}_f$  with varying  $n$  and  $\sigma_\alpha$

$n$	$\sigma_\alpha$	Guess			LOOCV with $k$ random draws			Distance to $y_{1,T_1^*+1}$			
		$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
5	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.93 (0.02)	0.93 (0.02)	49.36 (2.51)	12.52 (2.39)	12.82 (2.25)	12.27 (2.39)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.89 (0.02)	49.62 (2.89)	15.15 (2.61)	14.75 (2.55)	14.93 (2.59)
	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.79 (0.03)	0.81 (0.03)	50.39 (5.33)	28.01 (3.76)	28 (3.52)	27.66 (3.72)
	50	0.97 (0.03)	0.97 (0.03)	0.97 (0.03)	0.66 (0.03)	0.65 (0.04)	0.67 (0.03)	61.3 (8.41)	51.79 (6.68)	51.84 (6.45)	51.36 (6.62)
	100	0.93 (0.05)	0.9 (0.06)	0.9 (0.06)	0.53 (0.05)	0.45 (0.05)	0.55 (0.05)	102.54 (13.46)	100.42 (13.26)	99.48 (13.38)	99.81 (13.15)
10	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.85 (2.66)	11.93 (1.97)	13.19 (2.07)	11.85 (2.01)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.9 (0.02)	53.22 (3)	13.55 (2.1)	14.51 (2.19)	13.43 (2.15)
	25	1 (0)	1 (0)	1 (0)	0.75 (0.04)	0.79 (0.04)	0.76 (0.04)	54.31 (4.77)	23.71 (2.71)	21.37 (3.47)	23.58 (2.72)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.59 (0.04)	0.64 (0.04)	0.59 (0.04)	58.73 (7.88)	41.53 (5.24)	37.18 (6.44)	41.37 (5.15)
	100	1 (0)	0.97 (0.03)	0.73 (0.08)	0.48 (0.05)	0.48 (0.05)	0.48 (0.04)	82.13 (12.74)	77.4 (11.24)	72.81 (12.65)	77.07 (11.02)
15	5	1 (0)	1 (0)	1 (0)	0.94 (0.02)	0.93 (0.02)	0.94 (0.02)	46.76 (2.5)	11.39 (1.4)	13.23 (1.68)	11.38 (1.39)
	10	1 (0)	1 (0)	1 (0)	0.92 (0.02)	0.91 (0.02)	0.92 (0.02)	46.37 (2.59)	11.65 (1.56)	13.88 (1.96)	11.66 (1.55)
	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.81 (0.03)	0.81 (0.03)	45.21 (3.62)	17.31 (2.23)	21.19 (2.76)	17.16 (2.33)
	50	1 (0)	1 (0)	0.87 (0.06)	0.64 (0.05)	0.67 (0.04)	0.65 (0.04)	44.29 (6.1)	30.7 (4.15)	36.42 (4.91)	31.17 (4.18)
	100	0.9 (0.06)	0.87 (0.06)	0.7 (0.09)	0.55 (0.04)	0.52 (0.05)	0.56 (0.04)	57.73 (9.79)	61.28 (8.18)	71.42 (9.17)	62.33 (8.22)
25	5	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.95 (0.02)	0.95 (0.02)	47.87 (3.13)	12.4 (2.09)	13.37 (1.95)	12.4 (2.09)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.93 (0.02)	0.95 (0.02)	46.81 (3.5)	14.31 (2.26)	15.8 (2.17)	14.23 (2.26)
	25	1 (0)	1 (0)	0.93 (0.05)	0.74 (0.03)	0.77 (0.03)	0.74 (0.03)	45.32 (4.84)	23.24 (3.26)	25.45 (3.64)	23.12 (3.23)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.61 (0.03)	0.59 (0.03)	0.6 (0.03)	52.43 (6.48)	43.31 (5.03)	46.98 (6.12)	43.24 (4.93)
	100	1 (0)	0.97 (0.03)	0.67 (0.09)	0.53 (0.04)	0.53 (0.03)	0.53 (0.04)	85.99 (9.26)	83.52 (9.56)	90.95 (11.69)	83.53 (9.39)

**Table 4:** 30 Monte Carlo simulations of  $\mathcal{M}_1$  for  $\mathcal{B}_f$  with varying  $\sigma$  and  $\sigma_\alpha$

$\sigma$	$\sigma_\alpha$	Guess			LOOCV with $k$ random draws			Distance to $y_{1,T_1^*+1}$			
		$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
5	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.6 (1.52)	6.91 (1.05)	7.35 (1.09)	6.85 (1.07)
	10	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.96 (2.07)	10.22 (1.19)	9.5 (1.47)	10.18 (1.2)
	25	1 (0)	1 (0)	1 (0)	0.79 (0.03)	0.82 (0.03)	0.79 (0.03)	53.05 (4.27)	20.92 (2.62)	18.7 (3.21)	20.87 (2.57)
	50	1 (0)	1 (0)	0.87 (0.06)	0.59 (0.04)	0.66 (0.04)	0.58 (0.04)	57.88 (7.51)	38.89 (5.61)	36.51 (6.32)	38.74 (5.51)
	100	1 (0)	1 (0)	0.73 (0.08)	0.45 (0.04)	0.46 (0.05)	0.46 (0.04)	80.3 (12.8)	76.09 (11.54)	72.77 (12.71)	75.51 (11.37)
	5	1 (0)	1 (0)	1 (0)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.85 (2.66)	11.93 (1.97)	13.19 (2.07)	11.85 (2.01)
	10	1 (0)	1 (0)	1 (0)	0.89 (0.02)	0.9 (0.02)	0.9 (0.02)	53.22 (3)	13.55 (2.1)	14.51 (2.19)	13.43 (2.15)
	25	1 (0)	1 (0)	1 (0)	0.75 (0.04)	0.79 (0.04)	0.76 (0.04)	54.31 (4.77)	23.71 (2.71)	21.37 (3.47)	23.58 (2.72)
	50	1 (0)	0.97 (0.03)	0.77 (0.08)	0.59 (0.04)	0.64 (0.04)	0.59 (0.04)	58.73 (7.88)	41.53 (5.24)	37.18 (6.44)	41.37 (5.15)
	100	1 (0)	0.97 (0.03)	0.73 (0.08)	0.48 (0.05)	0.48 (0.05)	0.48 (0.04)	82.13 (12.74)	77.4 (11.24)	72.81 (12.65)	77.07 (11.02)
10	5	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.77 (0.03)	0.75 (0.03)	0.76 (0.03)	58.95 (5.73)	28.82 (4.87)	30.78 (5.44)	29.02 (4.91)
	10	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.78 (0.04)	0.76 (0.04)	0.77 (0.04)	59.25 (5.89)	28.98 (4.93)	31.55 (5.31)	29.1 (4.99)
	25	0.97 (0.03)	0.97 (0.03)	0.9 (0.06)	0.67 (0.04)	0.67 (0.04)	0.67 (0.04)	61.16 (6.63)	33.6 (5.22)	35.64 (5.47)	33.34 (5.33)
	50	0.9 (0.06)	0.83 (0.07)	0.77 (0.08)	0.55 (0.04)	0.59 (0.04)	0.54 (0.05)	66.95 (8.6)	49.6 (6.03)	46.13 (7.36)	49.25 (6.08)
	100	0.9 (0.06)	0.87 (0.06)	0.7 (0.09)	0.49 (0.05)	0.46 (0.05)	0.47 (0.05)	89.11 (12.91)	85.23 (10.31)	75.55 (13.03)	84.46 (10.25)
	5	0.77 (0.08)	0.6 (0.09)	0.67 (0.09)	0.62 (0.04)	0.57 (0.05)	0.63 (0.04)	77.73 (9.69)	58.31 (9.66)	61.13 (11.05)	59.11 (9.66)
	10	0.77 (0.08)	0.67 (0.09)	0.73 (0.08)	0.64 (0.04)	0.58 (0.05)	0.62 (0.04)	78.36 (9.65)	57.79 (9.73)	61.58 (10.86)	58.54 (9.74)
	25	0.8 (0.07)	0.67 (0.09)	0.73 (0.08)	0.63 (0.04)	0.57 (0.04)	0.61 (0.05)	81.3 (9.66)	59.28 (9.82)	64.17 (10.5)	59.24 (9.98)
	50	0.87 (0.06)	0.7 (0.09)	0.67 (0.09)	0.51 (0.05)	0.51 (0.05)	0.53 (0.05)	86.76 (10.76)	67.58 (10.33)	70.73 (10.98)	67.06 (10.54)
	100	0.87 (0.06)	0.7 (0.09)	0.63 (0.09)	0.48 (0.05)	0.53 (0.04)	0.47 (0.05)	102.61 (14.87)	98.97 (12.06)	91.38 (14.79)	98.17 (12.16)
50	5	0.63 (0.09)	0.53 (0.09)	0.5 (0.09)	0.55 (0.04)	0.54 (0.03)	0.53 (0.04)	123.93 (17.91)	117.54 (19.27)	122.82 (22.17)	119.19 (19.23)
	10	0.6 (0.09)	0.5 (0.09)	0.5 (0.09)	0.54 (0.04)	0.54 (0.03)	0.53 (0.04)	124.06 (17.94)	117.01 (19.28)	122.57 (22.05)	118.62 (19.25)
	25	0.63 (0.09)	0.47 (0.09)	0.5 (0.09)	0.51 (0.04)	0.53 (0.04)	0.52 (0.04)	125.25 (18.06)	115.88 (19.47)	123.94 (21.52)	116.93 (19.56)
	50	0.67 (0.09)	0.43 (0.09)	0.53 (0.09)	0.51 (0.05)	0.51 (0.04)	0.52 (0.05)	130.4 (18.23)	119.1 (19.58)	128.78 (20.9)	119.01 (19.88)
	100	0.57 (0.09)	0.53 (0.09)	0.43 (0.09)	0.49 (0.04)	0.49 (0.04)	0.45 (0.04)	143.5 (20.29)	135.98 (20.53)	141.58 (21.93)	134.9 (20.94)
100	5	0.63 (0.09)	0.53 (0.09)	0.5 (0.09)	0.55 (0.04)	0.54 (0.03)	0.53 (0.04)	123.93 (17.91)	117.54 (19.27)	122.82 (22.17)	119.19 (19.23)
	10	0.6 (0.09)	0.5 (0.09)	0.5 (0.09)	0.54 (0.04)	0.54 (0.03)	0.53 (0.04)	124.06 (17.94)	117.01 (19.28)	122.57 (22.05)	118.62 (19.25)
	25	0.63 (0.09)	0.47 (0.09)	0.5 (0.09)	0.51 (0.04)	0.53 (0.04)	0.52 (0.04)	125.25 (18.06)	115.88 (19.47)	123.94 (21.52)	116.93 (19.56)
	50	0.67 (0.09)	0.43 (0.09)	0.53 (0.09)	0.51 (0.05)	0.51 (0.04)	0.52 (0.05)	130.4 (18.23)	119.1 (19.58)	128.78 (20.9)	119.01 (19.88)
	100	0.57 (0.09)	0.53 (0.09)	0.43 (0.09)	0.49 (0.04)	0.49 (0.04)	0.45 (0.04)	143.5 (20.29)	135.98 (20.53)	141.58 (21.93)	134.9 (20.94)

## 4 Simulation for the Boundary Case ( $p < n$ )

In this section, we briefly present the result for the non-uniqueness and the boundary case when  $p < n$  with the example  $p = 2$  under  $\mathcal{B}_u$  and  $\mathcal{M}_2$ . Proposition 1 in the main text tells that there are infinitely many solutions of  $\mathbf{W}^*$  in this setup. To make it comparable to the main results of Section 4.3 in the main text, we set  $\mu_\alpha = 50$ . It is because in our simulation, as  $p$  decreases,  $E(\alpha_1)$  will decrease as well. The result is attached as below. See discussion in Section 3.2 and Section 6 in the main text. The following table verifies the claim at the end of Section 3.2 that non-uniqueness is not a serious problem for inferential purposes and the point in the discussion that boundary problems for  $\mathcal{B}_u$  do not compromise inference.



**Table 5:** 30 Monte Carlo simulations of  $\mathcal{M}_2$  for  $\mathcal{B}_u$  with varying  $n$  and  $\sigma_\alpha$  ( $p = 2$ , boundary case)

$n$	$\sigma_\alpha$	Guess			LOOCV with $k$ random draws			Distance to $y_{1,T_1^*+1}$			
		$\delta_{\hat{\alpha}_{\text{adj}}}$	$\delta_{\hat{\alpha}_{\text{wadj}}}$	$\delta_{\hat{\alpha}_{\text{IVW}}}$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$	$\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$	Original	$\hat{\alpha}_{\text{adj}}$	$\hat{\alpha}_{\text{wadj}}$	$\hat{\alpha}_{\text{IVW}}$
5	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	51.71 (1.69)	8.84 (1.23)	9.55 (1.31)	8.81 (1.26)
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	52.63 (1.87)	10.14 (1.4)	10.33 (1.39)	10.17 (1.42)
	10	1 (0)	1 (0)	1 (0)	0.97 (0.01)	0.97 (0.01)	0.96 (0.02)	53.78 (2.42)	12.44 (1.84)	13.16 (1.83)	12.44 (1.86)
	25	0.9 (0.06)	1 (0)	0.9 (0.06)	0.75 (0.04)	0.79 (0.04)	0.75 (0.04)	57.62 (4.64)	22.34 (3.43)	25.19 (4.13)	22.33 (3.44)
	50	0.7 (0.09)	0.7 (0.09)	0.7 (0.09)	0.59 (0.05)	0.57 (0.04)	0.57 (0.05)	68.41 (7.84)	42.53 (5.97)	49.34 (8.12)	42.44 (6.01)
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	52.19 (2.18)	11.1 (1.32)	12.18 (1.81)	11.19 (1.32)
	5	1 (0)	1 (0)	1 (0)	0.97 (0.02)	0.96 (0.01)	0.97 (0.02)	52.33 (2.5)	12.51 (1.58)	14.19 (1.94)	12.61 (1.58)
	10	0.97 (0.03)	0.97 (0.03)	0.93 (0.05)	0.93 (0.02)	0.92 (0.02)	0.93 (0.02)	52.5 (3.09)	14.69 (2.12)	17.46 (2.25)	14.76 (2.13)
	25	0.9 (0.06)	0.9 (0.06)	0.9 (0.06)	0.74 (0.03)	0.76 (0.03)	0.74 (0.03)	54.62 (4.81)	24.86 (3.7)	28.35 (3.96)	24.94 (3.73)
	50	0.7 (0.09)	0.73 (0.08)	0.73 (0.08)	0.59 (0.04)	0.55 (0.04)	0.57 (0.04)	62.89 (7.56)	44.32 (6.5)	49.12 (7.13)	44.44 (6.54)
15	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	1 (0)	0.99 (0.01)	53.83 (1.89)	9.12 (0.92)	9.95 (1.33)	9.06 (0.93)
	5	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.99 (0.01)	0.99 (0.01)	55.05 (2.25)	10.47 (1.21)	11.57 (1.54)	10.39 (1.22)
	10	1 (0)	1 (0)	1 (0)	0.95 (0.02)	0.98 (0.01)	0.95 (0.02)	56.57 (2.83)	13.25 (1.61)	14.78 (1.95)	13.19 (1.6)
	25	0.93 (0.05)	0.93 (0.05)	0.93 (0.05)	0.87 (0.03)	0.89 (0.03)	0.88 (0.03)	61.14 (4.94)	24.01 (2.97)	28.39 (3.26)	23.98 (2.91)
	50	0.83 (0.07)	0.83 (0.07)	0.83 (0.07)	0.65 (0.04)	0.67 (0.03)	0.66 (0.04)	71.22 (8.07)	44.06 (5.29)	52.49 (5.93)	43.98 (5.2)
	1	1 (0)	1 (0)	1 (0)	0.99 (0.01)	0.98 (0.01)	0.99 (0.01)	50.84 (2.39)	10.44 (1.51)	11.58 (1.47)	10.43 (1.51)
25	5	1 (0)	1 (0)	1 (0)	0.98 (0.01)	0.98 (0.01)	0.98 (0.01)	50.4 (2.22)	9.97 (1.39)	11.77 (1.34)	9.93 (1.39)
	10	1 (0)	1 (0)	1 (0)	0.96 (0.01)	0.97 (0.01)	0.95 (0.02)	49.84 (2.23)	10.99 (1.21)	12.97 (1.48)	10.98 (1.21)
	25	1 (0)	1 (0)	1 (0)	0.81 (0.03)	0.81 (0.03)	0.81 (0.03)	48.16 (3.49)	16.36 (2.18)	20.67 (2.98)	16.52 (2.15)
	50	0.93 (0.05)	0.93 (0.05)	0.93 (0.05)	0.69 (0.03)	0.69 (0.04)	0.69 (0.03)	48.56 (6.09)	33.14 (3.88)	41.94 (5.18)	33.39 (3.88)