Post-Shock Transition Estimation

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1 Introduction

Suppose that an individual, given time series $\{(\mathbf{x}_{1,t}, y_{1,t}) \colon t = 1, \ldots, T_1\}$, is interested in predicting y_{1,T_1+k} for some $k \in \mathbb{N}$. However, he knows that the time series is about experience a shock at $T_1 + k$, whose signal is not embedded in the data. In this case, statistical methodologies that rely on capturing signal from training data to predict are not expected to perform well.

Comment: Do not use 'he', use something like 'the researcher'.

Based on the synthetic control theory Abadie et al. [2010], Lin and Eck [2020] proposes a post-shock prediction method that borrows information from other similar time series experiencing analogous shocks by constructing a donor pool $\{(\mathbf{x}_{i,t}, y_{i,t}) \colon i = 2, \ldots, n + 1, t = 1, \ldots, T_i\}$, and uses synthetic control method to weight the shock effects $\hat{\alpha}_i$ estimated from the donor pool for $i = 2, \ldots, n + 1$ to provide a scalar adjustment to the naive time series forecast $\hat{y}_{1,T_1+k}$. Lin and Eck [2020] provides conditions when optimality of minimizing mean squared forecast error is attained.

Comment: Our method is not synthetic control, it's a weighted-adjustment estimator that is inspired by synthetic control.

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However, the work of Lin and Eck [2020] is limited to the nowcasting problem. Moreover, it provides no information about how the shock will affect the time series afterwards, let alone whether it will die out. In practice, it is natural to think of applications where we are interested in the dynamics of the shock, whether and how the shock will die out. For example, when a company knows it is going to experience a disastrous shock, managers will be interested in whether the shock will be persistent, how it affects future sales for a number of months, and thus how to adjust cost and production to combat such crisis.

In this paper, we are interested in extending the work of Lin and Eck [2020] to estimate the shock transition after the time series of interest experience a shock, i.e., $\{\hat{y}_{1,T_1+k+j}: j = 1,\ldots,d\}$, extend its optimality propositions of nowcasting to prediction for a period, and develop a hypothesis testing procedure to judge whether the shock is permanent or transitory.

2 Methods

Suppose at $t = T_i^*$, one knows the shock is about to occur at $t = T_i^* + 1$. Consider the following model

$$
\mathcal{M}: \quad
\begin{aligned}
y_{i,t} &= \eta_i + \alpha_{i,t} I(t > T_i^*) + \phi_i y_{i,t-1} + \mathbf{x}_{i,t}\boldsymbol{\theta}_i + \varepsilon_{i,t} \\
\alpha_{i,t} &= \mu_\alpha + \mathbf{x}_{i,t}\boldsymbol{\delta}_i + \tilde{\varepsilon}_i
\end{aligned}
\quad ,
$$

where $D_{i,t} = I(t = T_i^* + t)$, $I(\cdot)$ is an indicator function, and $\mathbf{x}_{i,t}, \boldsymbol{\theta}_i \in \mathbb{R}^p$ with $p \geq 1$. The random effects structure is organized as below.

$$
\tilde{\varepsilon}_i \overset{\text{iid}}{\sim} \mathcal{F}_{\tilde{\varepsilon}} \text{ with } \mathbb{E}_{\mathcal{F}_{\tilde{\varepsilon}}} = 0, \mathbf{Var}_{\mathcal{F}_{\tilde{\varepsilon}}}(\tilde{\varepsilon}_i) = \sigma_\alpha^2 > 0,
$$
$$
\eta_i \perp\!\!\!\perp \alpha_{i,t} \perp\!\!\!\perp \phi_i \perp\!\!\!\perp \boldsymbol{\theta}_i \perp\!\!\!\perp \varepsilon_{i,t} \perp\!\!\!\perp \tilde{\varepsilon}_i.
$$

The candidate forecasts are

$$
\text{Forecast 1}: \hat{y}_{1,T_1^*+k+j}^{(1)} = \hat{\eta}_1 + \hat{\phi}_1 \hat{y}_{1,T_1^*+k+j-1}^{(1)} + \mathbf{x}_{1,T_1^*+k+j-1}\hat{\boldsymbol{\theta}}_1,
$$
$$
\text{Forecast 2}: \hat{y}_{1,T_1^*+k+j}^{(2)} = \hat{\eta}_1 + \hat{\phi}_1 \hat{y}_{1,T_1^*+k+j-1}^{(2)} + \mathbf{x}_{1,T_1^*+k+j-1}\hat{\boldsymbol{\theta}}_1 + \hat{\alpha}^{(j)},
$$

where $\hat{y}_{1,T_1^*}^{(1)} = \hat{y}_{1,T_1^*}^{(2)} = y_{1,T_1^*}$, $\hat{\alpha}^{(j)}$ is an estimate of the shock effect occurring in the model of y_{1,T_1^*+k+j}, and other parameters are estimated in an unbiased fashion, e.g., OLS.

Comment: There is also the h-ahead forecast model, see: https://www.ssc.wisc.

Perhaps their framework is more appropriate. See a more authoritative presentation in Clements and Hendry [1996].

Define

$$
\begin{aligned}
\Theta &= \{(\eta_i, \phi_i, \boldsymbol{\theta}_i, \alpha_{i,t}, \mathbf{x}_{i,t}, y_{i,t-1}, \delta_i) : t = 1, \ldots, T_i, i = 2, \ldots, n+1\}, \\
\Theta_1 &= \{(\eta_i, \phi_i, \boldsymbol{\theta}_i, \alpha_{i,t}, \mathbf{x}_{i,t}, y_{i,t-1}, \delta_i) : t = 1, \ldots, T_i, i = 1\},
\end{aligned}
\tag{1}
$$

For simplicity, assume $k = 0$. Given the post-shock estimated transition, $\{\hat{y}_{1,t}^2 : t = T_1 + 1, \ldots, T_1 + d\}$, to gauge the forecast performance, we consider the forecast risk in the form of MSE as below

$$
R_k = \frac{1}{d} \sum_{j=1}^{d} \mathbb{E}[(\hat{y}_{1,T_1^*+j}^{(k)} - y_{1,T_1+j})^2], \quad k = 1, 2.
$$

Proposition 2.1. *Suppose that $\hat{\alpha}^{(j)}$ estimates α_{1,T_1^*+j} unbiasedly and that $\hat{\alpha}^{(j)}$ is independent of Θ_1. Then, $R_1 > R_2$ when*

Proof. Define

$$
C(\Theta_1, j) = \hat{\eta}_1 + \hat{\phi}_1 \hat{y}_{1,T_1^*+j-1}^{(2)} + \mathbf{x}_{1,T_1^*+j} \hat{\boldsymbol{\theta}}_1 - (\eta_1 + \phi_1 y_{1,T_1^*+j-1} + \mathbf{x}_{1,T_1^*+j} \boldsymbol{\theta}_1 + \varepsilon_{1,T_1^*+j}).
$$

We first consider a simple case when $d = 2$. By Lemma 1 in Lin and Eck [2020],

$$
\begin{aligned}
&\mathbb{E}[(\hat{y}_{1,T_1+1}^{(1)} - y_{1,T_1+1})^2] - \mathbb{E}[(\hat{y}_{1,T_1+1}^{(2)} - y_{1,T_1+1})^2] \\
=\ & \mathbb{E}(\alpha_{1,T_1^*+1}^2) - \mathbb{E}[(\hat{\alpha}^{(1)} - \alpha_{1,T_1^*+1})^2] \\
=\ & -\mathbb{E}[(\hat{\alpha}^{(1)})^2] + 2\mathbb{E}(\hat{\alpha}^{(1)} \alpha_{1,T_1^*+1}) \\
=\ & -\mathbb{E}[(\hat{\alpha}^{(1)})^2] + 2\mathbb{E}[\alpha_{1,T_1^*+1} \cdot \mathbb{E}(\hat{\alpha}^{(1)} \mid \alpha_{1,T_1^*+1})] \\
=\ & 2\mathbb{E}[\alpha_{1,T_1^*+2}^2] - \mathbb{E}[(\hat{\alpha}^{(1)})^2] \\
=\ & \mathbb{E}[\alpha_{1,T_1^*+2}^2] - \mathbf{Var}(\hat{\alpha}^{(1)}).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\mathbb{E}[(\hat{y}_{1,T_1+2}^{(1)} - y_{1,T_1+2})^2] - \mathbb{E}[(\hat{y}_{1,T_1+2}^{(2)} - y_{1,T_1+2})^2] \\
=\ & \mathbb{E}\left\{(C(\Theta_1, 2) - \hat{\phi}_1 \hat{\alpha}^{(2)} - \alpha_{1,T_1^*+2})^2\right\} - \mathbb{E}\left\{(C(\Theta_1, 2) - \alpha_{1,T_1^*+2})^2\right\} \\
=\ & -\mathbb{E}(\alpha_{1,T_1^*+2}^2) + 2\mathbb{E}(C(\Theta_1, 2)\alpha_{1,T_1^*+2}) - 2\mathbb{E}[C(\Theta_1, 2)(\hat{\phi}_1 \hat{\alpha}^{(2)} + \alpha_{1,T_1^*+2})] + \mathbb{E}[(\hat{\phi}_1 \hat{\alpha}^{(2)} + \alpha_{1,T_1^*+2})^2] \\
=\ & \mathbb{E}[(\hat{\phi}_1 \hat{\alpha}^{(2)})^2] + 2\mathbb{E}[\hat{\phi}_1 \hat{\alpha}^{(2)} \alpha_{1,T_1^*+2}] - 2\mathbb{E}[C(\Theta_1, 2)\hat{\phi}_1 \hat{\alpha}^{(2)}]
\end{aligned}
$$

3

By assumption, $\hat{\alpha}^{(2)}$ is independent of Θ_1. Moreover, α_{1,T_1^*+2} is also independent of Θ_1 by construction. Since OLS estimators are unbiased and $\hat{\alpha}^{(2)}$ estimates α_{1,T_1^*+2} unbiasedly,

$$\mathbb{E}[\hat{\phi}_1\hat{\alpha}^{(2)}\alpha_{1,T_1^*+2}] = \phi_1[\mathbb{E}(\hat{\alpha}^{(2)})]^2$$
$$\mathbb{E}[C(\Theta_1,2)\hat{\phi}_1\hat{\alpha}^{(2)}] = \mathbb{E}[C(\Theta_1,2)\hat{\phi}_1] \cdot \mathbb{E}(\hat{\alpha}^{(2)})$$

Thus,

$$\mathbb{E}[(\hat{y}_{1,T_1+2}^{(1)} - y_{1,T_1+2})^2] - \mathbb{E}[(\hat{y}_{1,T_1+2}^{(2)} - y_{1,T_1+2})^2]$$
$$= \mathbb{E}[(\hat{\phi}_1\hat{\alpha}^{(2)})^2] + 2\phi_1 \cdot [\mathbb{E}(\hat{\alpha}^{(2)})]^2 - 2\mathbb{E}[C(\Theta_1,2)\hat{\phi}_1] \cdot \mathbb{E}(\hat{\alpha}^{(2)})$$
$$= \mathbb{E}[(\hat{\phi}_1\hat{\alpha}^{(2)})^2] + 2 \cdot \mathbb{E}(\hat{\alpha}^{(2)}) \cdot \left\{\phi_1\mathbb{E}(\hat{\alpha}^{(2)}) - \mathbb{E}[C(\Theta_1,2)\hat{\phi}_1]\right\}$$

Challenges: How to evaluate $\mathbb{E}[C(\Theta_1,2)\hat{\phi}_1]$. □

2.1 Challenges

Challenge 1. How to define a proper null hypothesis that captures whether the shock transition is transitory or permanent? Define $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}^{(j)})_{j=1}^d$ and $\boldsymbol{\alpha}_1 = (\alpha_{T_1^*+j})_{j=1}^d$. One possible hypothesis is

$$H_0 : \boldsymbol{\alpha}_1 = \mathbf{0} \quad \text{versus} \quad H_1 : \boldsymbol{\alpha}_1 \neq \mathbf{0}$$

In any cases, it is important to find an unbiased estimator of $\boldsymbol{\alpha}_1$. $\hat{\alpha}_{\text{wadj}}$ in [Lin and Eck, 2020] might be possible.

Challenge 2. How to generalize to $\hat{\alpha}_{\text{wadj}}$? In [Lin and Eck, 2020], the weighting is with respect to $\mathbf{x}_{i,T_i^*+1}$ for $i = 1,\ldots,n+1$. However, in this case, we have $\mathbf{x}_{i,T_i^*+j}$ for $i = 1,\ldots,n+1$ and $j = 1,\ldots,d$. Perhaps, we may want to do d times of weighting such that the unbiased property still holds. However, a lot of forecast error for $\mathbf{x}_{T_i^*+j}$ when d is large may be of concern in constructing such estimator.

References

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