# Supplemental Materials for "Minimizing post shock forecasting error using disparate information"

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In the supplementary materials, we provide details for some procedures that are not discussed in the manuscript. Section 1 provides statistical evidence for approximate independence between the shock-effects nested in 2008 September time series in the analysis of Conoco Phillips stock. Section 2 details the algorithms of $\mathcal{B}_f$ and $\mathcal{B}_u$. Section 3 lists the tables for simulations under $\mathcal{M}_1$, whose results are discussed in Section 4 in the manuscript. Section 4 lists an example for the non-uniqueness of $\mathbf{W}^*$ when $2p < n$, where $\mathbf{W}^*$ is very likely to lie in the boundary of parameter space $\mathcal{W}$.

## 1 Supplementary materials for data analysis

The independence of the estimated September, 2008 shock-effects are further tested using likelihood ratio test (LRT) based on their estimated covariance matrix. The estimated covariance matrix is

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 4.012 & 0.362 & -0.062 \\ 0.362 & 3.894 & -0.029 \\ -0.062 & -0.029 & 3.927 \end{pmatrix}.$$

with degrees of freedoms 35. Using hte LRT for independence between blocks of random variables [Marden, 2015, Section 10.2], the LRT test statistic is 0.304 with $p$-value of 0.581. Therefore, we do not reject the null hypothesis that the three estimated shock-effects are independent.

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## 2 Bootstrap algorithms of $\mathcal{B}_f$ and $\mathcal{B}_u$

Algorithm 1 presents the algorithms for the fixed donor pool bootstrapping $\mathcal{B}_f$ and Algorithm 2 outlines the steps for the unfixed donor pool bootstrapping $\mathcal{B}_u$.

**Algorithm 1:** Fixed donor pool bootstrapping $\mathcal{B}_f$ for estimation of shock-effect estimators

**Input:** $B$ – the number of parametric bootstraps
$\{(y_{i,t}, \mathbf{x}_{i,t})\colon i = 2, \dots, n+1, t = 0, \dots, T_i\}$ – the data
$\{T_i^*\colon i = 1, \dots, n+1\}$ – the time point just before the shock
$\{\hat{\varepsilon}_{i,t}\colon t = 1, \dots, T_i\}$ – the collection of residuals for $t = 1, \dots, T_i$
$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i\colon i = 2, \dots, n+1\}$ – the OLS estimates

**Result:** The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

```
for b = 1 : B do
    for i = 2, ..., n+1 do
        Sample with replacement from {ε̂_{i,t} : t = 1, ..., T_i} to obtain {ε̂^{(b)}_{i,t} : t = 1, ..., T_i}
        Define y^{(b)}_{i,0} = y_{i,0}
        for t = 1, ..., T_i do
            Compute y^{(b)}_{i,t} = η̂_i + α̂_i 1(t = T_i^* + 1) + φ̂_i y^{(b)}_{i,t-1} + θ'_i x_{i,t} + β'_i x_{i,t-1} + ε̂^{(b)}_{i,t}
        end
        Compute α̂^{(b)}_i based on OLS estimation with {(y^{(b)}_{i,t}, x_{i,t}) : t = 0, ..., T_i}
    end
    Compute the bth shock-effect estimate α̂^{(b)}_est for est ∈ {adj, wadj, IVW}
end
Compute the sample variance of {α̂^{(b)}_est : b = 1, ..., B} for est ∈ {adj, wadj, IVW}
```

**Algorithm 2:** Unfixed donor pool bootstrapping $\mathcal{B}_u$ for estimation of shock-effect estimators

**Input:** $B$ – the number of parametric bootstraps
$\{(y_{i,t}, \mathbf{x}_{i,t})\colon i = 2, \dots, n+1, t = 0, \dots, T_i\}$ – the data
$\{T_i^*\colon i = 1, \dots, n+1\}$ – the time point just before the shock
$\{\hat{\varepsilon}_{i,t}\colon t = 1, \dots, T_i\}$ – the collection of residuals for $t = 1, \dots, T_i$
$\{\hat{\eta}_i, \hat{\alpha}_i, \hat{\phi}_i, \hat{\theta}_i, \hat{\beta}_i\colon i = 2, \dots, n+1\}$ – the OLS estimates

**Result:** The sample variance of bootstrapped adjustment estimator, inverse-variance weighted estimator, and weighted-adjustment estimator.

```
for b = 1 : B do
    Sample n elements with replacement from I = {2, ..., n+1} to form I^{(b)}, where elements of I^{(b)}
      are not necessarily unique in terms of their indices
    for i ∈ I^{(b)} do
        Sample with replacement from {ε̂_{i,t} : t = 1, ..., T_i} to obtain {ε̂^{(b)}_{i,t} : t = 1, ..., T_i}
        Define y^{(b)}_{i,0} = y_{i,0}
        for t = 1, ..., T_i do
            Compute y^{(b)}_{i,t} = η̂_i + α̂_i 1(t = T_i^* + 1) + φ̂_i y^{(b)}_{i,t-1} + θ'_i x_{i,t} + β'_i x_{i,t-1} + ε̂^{(b)}_{i,t}
        end
        Compute α̂^{(b)}_i based on OLS estimation with {(y^{(b)}_{i,t}, x_{i,t}) : t = 0, ..., T_i}
    end
    Compute the bth shock-effect estimate α̂^{(b)}_est for est ∈ {adj, wadj, IVW}
end
Compute the sample variance of {α̂^{(b)}_est : b = 1, ..., B} for est ∈ {adj, wadj, IVW}
```

# 3 Simulations for $\mathcal{M}_1$

In this section, we present the simulation results for $\mathcal{M}_1$. To make it comparable to $\mathcal{M}_2$, we set $\mu_\alpha = 50$ with other parameter setup the same as that of $\mathcal{M}_2$. The corresponding tables are attached as follows.

**Table 1:** 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_u$ with varying $n$ and $\sigma_\alpha$

| | | Guess | | | LOOCV with $k$ random draws | | | Distance to $y_{1,T_1^*+1}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\sigma_\alpha$ | $\delta_{\hat{\alpha}_{\text{adj}}}$ | $\delta_{\hat{\alpha}_{\text{wadj}}}$ | $\delta_{\hat{\alpha}_{\text{IVW}}}$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$ | Original | $\hat{\alpha}_{\text{adj}}$ | $\hat{\alpha}_{\text{wadj}}$ | $\hat{\alpha}_{\text{IVW}}$ |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.95 (0.02) | 0.95 (0.02) | 0.95 (0.02) | 52.37 (2.81) | 13.52 (2.04) | 14.81 (2.26) | 13.53 (2.06) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.94 (0.02) | 0.91 (0.02) | 0.94 (0.02) | 51.99 (3.05) | 15.32 (2.06) | 16.83 (2.3) | 15.36 (2.09) |
| 5 | 25 | 0.97 (0.03) | 0.97 (0.03) | 0.97 (0.03) | 0.77 (0.03) | 0.78 (0.03) | 0.78 (0.03) | 51.02 (4.81) | 24.15 (3.17) | 25.45 (3.74) | 24.14 (3.25) |
| | 50 | 0.8 (0.07) | 0.77 (0.08) | 0.73 (0.08) | 0.51 (0.05) | 0.56 (0.05) | 0.53 (0.05) | 55.67 (7.6) | 44.98 (5.57) | 48.53 (6.13) | 45.16 (5.6) |
| | 100 | 0.63 (0.09) | 0.63 (0.09) | 0.53 (0.09) | 0.51 (0.04) | 0.52 (0.03) | 0.55 (0.04) | 84.29 (12.32) | 88.51 (11.19) | 97.3 (11.55) | 88.67 (11.22) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.95 (0.02) | 0.93 (0.02) | 0.95 (0.02) | 50.61 (2.75) | 12.03 (1.79) | 12.61 (1.74) | 12.11 (1.82) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.91 (0.02) | 0.9 (0.02) | 0.91 (0.02) | 51.54 (3.05) | 13.84 (1.92) | 13.8 (1.96) | 13.99 (1.95) |
| 10 | 25 | 1 (0) | 1 (0) | 1 (0) | 0.77 (0.03) | 0.77 (0.03) | 0.78 (0.03) | 54.34 (4.69) | 20.97 (3.1) | 20.15 (3.21) | 21.25 (3.12) |
| | 50 | 0.8 (0.07) | 0.8 (0.07) | 0.7 (0.09) | 0.57 (0.03) | 0.59 (0.03) | 0.56 (0.03) | 63.44 (7.02) | 35.39 (5.74) | 34.34 (5.84) | 35.72 (5.76) |
| | 100 | 0.63 (0.09) | 0.6 (0.09) | 0.43 (0.09) | 0.46 (0.05) | 0.45 (0.03) | 0.47 (0.05) | 88.4 (12.01) | 66.89 (11.24) | 65.18 (11.46) | 66.86 (11.37) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.97 (0.01) | 0.95 (0.02) | 0.97 (0.01) | 54.1 (2.7) | 11.47 (2.02) | 12.58 (1.92) | 11.61 (2) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.91 (0.02) | 0.91 (0.02) | 0.91 (0.02) | 54.79 (2.84) | 12.99 (2.39) | 13.48 (2.4) | 13.08 (2.39) |
| 15 | 25 | 1 (0) | 1 (0) | 0.97 (0.03) | 0.73 (0.03) | 0.76 (0.03) | 0.71 (0.03) | 56.91 (4.4) | 21.85 (3.69) | 22.44 (3.58) | 21.85 (3.68) |
| | 50 | 0.93 (0.05) | 0.93 (0.05) | 0.87 (0.06) | 0.56 (0.04) | 0.59 (0.05) | 0.57 (0.04) | 65.97 (6.7) | 39.06 (6.38) | 41.53 (5.74) | 38.85 (6.37) |
| | 100 | 0.7 (0.09) | 0.67 (0.09) | 0.63 (0.09) | 0.44 (0.04) | 0.47 (0.03) | 0.45 (0.04) | 90.79 (12.51) | 74.94 (12.13) | 79.8 (10.84) | 74.17 (12.15) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.97 (0.01) | 0.96 (0.01) | 0.97 (0.01) | 52.01 (2.43) | 10.27 (1.62) | 10.11 (1.66) | 10.41 (1.63) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.96 (0.02) | 0.93 (0.02) | 0.96 (0.02) | 54.22 (2.84) | 12.77 (1.86) | 12.16 (1.91) | 12.8 (1.87) |
| 25 | 25 | 0.97 (0.03) | 0.97 (0.03) | 0.97 (0.03) | 0.84 (0.03) | 0.85 (0.03) | 0.84 (0.03) | 60.85 (4.87) | 22.14 (3.38) | 21.16 (3.37) | 22.21 (3.36) |
| | 50 | 1 (0) | 1 (0) | 0.9 (0.06) | 0.66 (0.05) | 0.6 (0.04) | 0.66 (0.05) | 72.52 (8.76) | 40.45 (6.32) | 39.73 (6.16) | 40.73 (6.25) |
| | 100 | 0.8 (0.07) | 0.73 (0.08) | 0.7 (0.09) | 0.57 (0.05) | 0.53 (0.04) | 0.57 (0.05) | 102.42 (15.73) | 79.07 (12.33) | 78.91 (11.91) | 79.94 (12.12) |

**Table 2:** 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_u$ with varying $\sigma$ and $\sigma_\alpha$

| | | Guess | | | LOOCV with $k$ random draws | | | Distance to $y_{1,T_1^*+1}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma$ | $\sigma_\alpha$ | $\delta_{\hat{\alpha}_{\text{adj}}}$ | $\delta_{\hat{\alpha}_{\text{wadj}}}$ | $\delta_{\hat{\alpha}_{\text{IVW}}}$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$ | Original | $\hat{\alpha}_{\text{adj}}$ | $\hat{\alpha}_{\text{wadj}}$ | $\hat{\alpha}_{\text{IVW}}$ |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 48.77 (1.65) | 7.14 (1.22) | 7.77 (1.25) | 7.26 (1.21) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.97 (0.01) | 0.97 (0.01) | 0.97 (0.01) | 47.1 (2.32) | 10.4 (1.72) | 12.1 (1.73) | 10.51 (1.72) |
| 5 | 25 | 1 (0) | 1 (0) | 1 (0) | 0.81 (0.03) | 0.84 (0.03) | 0.81 (0.03) | 44.03 (4.05) | 22.38 (3.42) | 26.23 (3.54) | 22.5 (3.4) |
| | 50 | 0.97 (0.03) | 0.97 (0.03) | 0.9 (0.06) | 0.64 (0.04) | 0.63 (0.04) | 0.63 (0.04) | 47.72 (6.35) | 42.94 (6.5) | 50.34 (6.81) | 43.03 (6.47) |
| | 100 | 0.67 (0.09) | 0.63 (0.09) | 0.57 (0.09) | 0.5 (0.05) | 0.51 (0.04) | 0.5 (0.04) | 72.88 (11.48) | 84.74 (12.69) | 98.56 (13.51) | 84.85 (12.62) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 49.1 (2.79) | 12.33 (1.96) | 12.84 (1.99) | 12.54 (1.93) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.96 (0.01) | 0.96 (0.01) | 0.96 (0.01) | 47.42 (3.32) | 14.44 (2.43) | 15.77 (2.48) | 14.65 (2.41) |
| 10 | 25 | 1 (0) | 1 (0) | 0.97 (0.03) | 0.8 (0.03) | 0.83 (0.03) | 0.8 (0.03) | 44.87 (4.71) | 24.76 (3.99) | 28.81 (4.05) | 25.09 (3.94) |
| | 50 | 0.97 (0.03) | 0.93 (0.05) | 0.9 (0.06) | 0.65 (0.04) | 0.64 (0.04) | 0.65 (0.04) | 49.9 (6.64) | 44.8 (6.89) | 52.51 (7.13) | 45.06 (6.85) |
| | 100 | 0.63 (0.09) | 0.63 (0.09) | 0.57 (0.09) | 0.52 (0.05) | 0.55 (0.04) | 0.49 (0.05) | 74.62 (11.8) | 85.94 (13.05) | 100.73 (13.67) | 86.14 (12.99) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.82 (0.03) | 0.82 (0.03) | 0.83 (0.03) | 51.82 (5.58) | 28.83 (4.36) | 29.08 (4.47) | 29.2 (4.29) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.8 (0.03) | 0.8 (0.03) | 0.78 (0.04) | 51.37 (5.7) | 29.93 (4.79) | 31.03 (4.85) | 30.44 (4.71) |
| 25 | 25 | 1 (0) | 0.97 (0.03) | 0.93 (0.05) | 0.75 (0.04) | 0.74 (0.04) | 0.75 (0.04) | 51.77 (6.36) | 37.51 (5.98) | 40.78 (6.1) | 37.99 (5.92) |
| | 50 | 0.83 (0.07) | 0.83 (0.07) | 0.8 (0.07) | 0.6 (0.05) | 0.59 (0.04) | 0.59 (0.05) | 57.39 (8.25) | 53.52 (8.71) | 62.34 (8.6) | 54.02 (8.66) |
| | 100 | 0.63 (0.09) | 0.57 (0.09) | 0.57 (0.09) | 0.49 (0.05) | 0.53 (0.04) | 0.5 (0.05) | 80.22 (13.49) | 92.77 (14.41) | 108.3 (14.81) | 93.51 (14.29) |
| | 5 | 0.73 (0.08) | 0.67 (0.09) | 0.67 (0.09) | 0.55 (0.05) | 0.51 (0.04) | 0.55 (0.05) | 67.29 (9.18) | 57.37 (8.43) | 57.45 (8.66) | 58.08 (8.26) |
| | 10 | 0.77 (0.08) | 0.67 (0.09) | 0.67 (0.09) | 0.53 (0.05) | 0.53 (0.04) | 0.53 (0.05) | 67.72 (9.24) | 58.18 (8.8) | 58.99 (8.96) | 58.92 (8.65) |
| 50 | 25 | 0.77 (0.08) | 0.73 (0.08) | 0.7 (0.09) | 0.53 (0.05) | 0.55 (0.05) | 0.53 (0.05) | 69.36 (9.85) | 62.1 (10.11) | 65.26 (10.16) | 63.15 (9.93) |
| | 50 | 0.73 (0.08) | 0.7 (0.09) | 0.63 (0.09) | 0.53 (0.05) | 0.5 (0.04) | 0.53 (0.05) | 75.13 (11.58) | 76.15 (11.97) | 82.62 (12.17) | 77.1 (11.85) |
| | 100 | 0.63 (0.09) | 0.6 (0.09) | 0.53 (0.09) | 0.52 (0.05) | 0.55 (0.04) | 0.5 (0.05) | 98.34 (15.85) | 108.03 (17.49) | 125.76 (17.14) | 108.99 (17.39) |
| | 5 | 0.5 (0.09) | 0.33 (0.09) | 0.4 (0.09) | 0.42 (0.04) | 0.49 (0.04) | 0.41 (0.04) | 114.58 (15.95) | 114.76 (16.6) | 114.79 (17.05) | 115.96 (16.28) |
| | 10 | 0.5 (0.09) | 0.33 (0.09) | 0.37 (0.09) | 0.42 (0.04) | 0.46 (0.04) | 0.42 (0.04) | 115.61 (16) | 115.49 (16.93) | 115.95 (17.35) | 116.81 (16.6) |
| 100 | 25 | 0.5 (0.09) | 0.37 (0.09) | 0.37 (0.09) | 0.39 (0.04) | 0.46 (0.04) | 0.39 (0.05) | 118.76 (16.45) | 117.96 (18.14) | 120.56 (18.37) | 119.47 (17.82) |
| | 50 | 0.57 (0.09) | 0.37 (0.09) | 0.43 (0.09) | 0.46 (0.05) | 0.51 (0.05) | 0.45 (0.05) | 124.94 (17.83) | 125.31 (20.27) | 131.64 (20.36) | 127.15 (19.96) |
| | 100 | 0.53 (0.09) | 0.43 (0.09) | 0.4 (0.09) | 0.48 (0.05) | 0.53 (0.03) | 0.49 (0.05) | 146.49 (20.97) | 153.41 (24.01) | 166.43 (24.35) | 155.32 (23.75) |

**Table 3:** 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_f$ with varying $n$ and $\sigma_\alpha$

| | | Guess | | | LOOCV with $k$ random draws | | | Distance to $y_{1,T_1^*+1}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\sigma_\alpha$ | $\delta_{\hat{\alpha}_{\text{adj}}}$ | $\delta_{\hat{\alpha}_{\text{wadj}}}$ | $\delta_{\hat{\alpha}_{\text{IVW}}}$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$ | Original | $\hat{\alpha}_{\text{adj}}$ | $\hat{\alpha}_{\text{wadj}}$ | $\hat{\alpha}_{\text{IVW}}$ |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.94 (0.02) | 0.94 (0.02) | 0.94 (0.02) | 50.11 (2.43) | 11.11 (1.85) | 13.2 (1.91) | 10.97 (1.85) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.93 (0.02) | 0.93 (0.02) | 0.93 (0.02) | 50.03 (2.81) | 13.68 (1.86) | 15.89 (1.88) | 13.62 (1.86) |
| 5 | 25 | 1 (0) | 1 (0) | 1 (0) | 0.78 (0.03) | 0.77 (0.03) | 0.79 (0.03) | 49.82 (4.54) | 21.75 (2.6) | 24.99 (2.53) | 21.98 (2.61) |
| | 50 | 1 (0) | 1 (0) | 0.93 (0.05) | 0.61 (0.05) | 0.61 (0.04) | 0.61 (0.05) | 57.05 (6.07) | 36.31 (4.58) | 40.79 (4.79) | 36.97 (4.62) |
| | 100 | 0.97 (0.03) | 0.97 (0.03) | 0.67 (0.09) | 0.45 (0.05) | 0.49 (0.05) | 0.45 (0.05) | 88.65 (7.05) | 65.85 (9.15) | 76.22 (9.13) | 67.41 (9.28) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.96 (0.02) | 0.96 (0.02) | 0.96 (0.02) | 50.21 (2.49) | 10.87 (1.57) | 10.89 (1.62) | 10.85 (1.53) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.94 (0.02) | 0.94 (0.02) | 0.94 (0.02) | 49.02 (2.76) | 13.62 (1.6) | 14.02 (1.74) | 13.67 (1.56) |
| 10 | 25 | 1 (0) | 1 (0) | 0.93 (0.05) | 0.75 (0.05) | 0.81 (0.03) | 0.76 (0.05) | 45.91 (4.58) | 24.55 (3.3) | 24.27 (4.09) | 24.43 (3.27) |
| | 50 | 0.97 (0.03) | 0.97 (0.03) | 0.77 (0.08) | 0.61 (0.05) | 0.66 (0.04) | 0.62 (0.05) | 49.9 (7.12) | 44.63 (7.23) | 44.98 (8.51) | 43.98 (7.2) |
| | 100 | 0.87 (0.06) | 0.87 (0.06) | 0.63 (0.09) | 0.53 (0.05) | 0.48 (0.04) | 0.54 (0.05) | 74.74 (13.07) | 89.35 (14.73) | 89.56 (17.32) | 87.87 (14.63) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.94 (0.02) | 0.94 (0.02) | 0.94 (0.02) | 53.03 (2.15) | 9.39 (1.51) | 11.1 (1.66) | 9.4 (1.47) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.91 (0.03) | 0.91 (0.03) | 0.91 (0.03) | 52.46 (2.66) | 11.65 (1.87) | 14 (2.06) | 11.79 (1.82) |
| 15 | 25 | 1 (0) | 1 (0) | 1 (0) | 0.77 (0.04) | 0.76 (0.03) | 0.76 (0.04) | 52.76 (4.47) | 23.88 (3.47) | 26.97 (3.82) | 23.95 (3.47) |
| | 50 | 1 (0) | 1 (0) | 0.9 (0.06) | 0.59 (0.04) | 0.59 (0.04) | 0.59 (0.04) | 59.8 (7.66) | 46.85 (6.7) | 52.97 (6.8) | 47.12 (6.69) |
| | 100 | 0.97 (0.03) | 0.9 (0.06) | 0.8 (0.07) | 0.47 (0.04) | 0.47 (0.03) | 0.47 (0.04) | 95.95 (12.52) | 96.35 (12.82) | 107.42 (12.67) | 97 (12.77) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.92 (0.02) | 0.93 (0.02) | 0.92 (0.02) | 55.1 (2.73) | 11.66 (1.93) | 13.34 (2.08) | 11.77 (1.92) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.89 (0.02) | 0.91 (0.02) | 0.89 (0.02) | 55 (3) | 13.63 (1.93) | 14.94 (2.04) | 13.67 (1.93) |
| 25 | 25 | 1 (0) | 1 (0) | 1 (0) | 0.79 (0.04) | 0.77 (0.04) | 0.79 (0.04) | 55.02 (4.34) | 21.29 (2.6) | 22.14 (2.77) | 21.17 (2.64) |
| | 50 | 1 (0) | 1 (0) | 0.87 (0.06) | 0.59 (0.04) | 0.63 (0.04) | 0.61 (0.05) | 58 (6.77) | 35.88 (4.69) | 38.22 (4.85) | 36.02 (4.69) |
| | 100 | 1 (0) | 0.97 (0.03) | 0.7 (0.09) | 0.49 (0.04) | 0.51 (0.04) | 0.47 (0.04) | 71.18 (12.1) | 68.53 (9.01) | 74.18 (9.38) | 69.17 (8.94) |

**Table 4:** 30 Monte Carlo simulations of $\mathcal{M}_1$ for $\mathcal{B}_f$ with varying $\sigma$ and $\sigma_\alpha$

| | | Guess | | | LOOCV with $k$ random draws | | | Distance to $y_{1,T_1^*+1}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma$ | $\sigma_\alpha$ | $\delta_{\hat{\alpha}_{\text{adj}}}$ | $\delta_{\hat{\alpha}_{\text{wadj}}}$ | $\delta_{\hat{\alpha}_{\text{IVW}}}$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$ | Original | $\hat{\alpha}_{\text{adj}}$ | $\hat{\alpha}_{\text{wadj}}$ | $\hat{\alpha}_{\text{IVW}}$ |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 49.62 (1.34) | 6.69 (0.77) | 6.83 (0.86) | 6.72 (0.74) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.98 (0.01) | 0.97 (0.02) | 0.98 (0.01) | 48.43 (1.96) | 10.21 (1.29) | 10.17 (1.61) | 10.19 (1.28) |
| 5 | 25 | 1 (0) | 1 (0) | 0.97 (0.03) | 0.79 (0.03) | 0.83 (0.02) | 0.79 (0.03) | 45.46 (4.33) | 22.21 (3.62) | 22.48 (4.25) | 21.88 (3.6) |
| | 50 | 0.97 (0.03) | 0.97 (0.03) | 0.73 (0.08) | 0.62 (0.05) | 0.67 (0.04) | 0.63 (0.05) | 48.82 (7.36) | 44.59 (7.37) | 44.76 (8.66) | 43.85 (7.32) |
| | 100 | 0.93 (0.05) | 0.93 (0.05) | 0.63 (0.09) | 0.52 (0.05) | 0.49 (0.05) | 0.53 (0.05) | 74.59 (13.36) | 89.83 (14.92) | 90.85 (17.33) | 88.25 (14.81) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.96 (0.02) | 0.96 (0.02) | 0.96 (0.02) | 50.21 (2.49) | 10.87 (1.57) | 10.89 (1.62) | 10.85 (1.53) |
| | 10 | 1 (0) | 1 (0) | 1 (0) | 0.94 (0.02) | 0.94 (0.02) | 0.94 (0.02) | 49.02 (2.76) | 13.62 (1.6) | 14.02 (1.74) | 13.67 (1.56) |
| 10 | 25 | 1 (0) | 1 (0) | 0.93 (0.05) | 0.75 (0.05) | 0.81 (0.03) | 0.76 (0.05) | 45.91 (4.58) | 24.55 (3.3) | 24.27 (4.09) | 24.43 (3.27) |
| | 50 | 0.97 (0.03) | 0.97 (0.03) | 0.77 (0.08) | 0.61 (0.05) | 0.66 (0.04) | 0.62 (0.05) | 49.9 (7.12) | 44.63 (7.23) | 44.98 (8.51) | 43.98 (7.2) |
| | 100 | 0.87 (0.06) | 0.87 (0.06) | 0.63 (0.09) | 0.53 (0.05) | 0.48 (0.04) | 0.54 (0.05) | 74.74 (13.07) | 89.35 (14.73) | 89.56 (17.32) | 87.87 (14.63) |
| | 5 | 1 (0) | 0.97 (0.03) | 0.97 (0.03) | 0.73 (0.04) | 0.75 (0.03) | 0.74 (0.04) | 53.85 (5.67) | 27.64 (3.65) | 26.5 (3.93) | 27.35 (3.57) |
| | 10 | 1 (0) | 0.97 (0.03) | 0.97 (0.03) | 0.73 (0.04) | 0.75 (0.03) | 0.72 (0.04) | 52.88 (5.55) | 27.72 (3.65) | 27.22 (3.87) | 27.55 (3.58) |
| 25 | 25 | 0.97 (0.03) | 0.97 (0.03) | 0.9 (0.06) | 0.65 (0.05) | 0.69 (0.04) | 0.63 (0.05) | 51.06 (5.81) | 34.43 (3.84) | 35.67 (4.15) | 34.48 (3.76) |
| | 50 | 0.87 (0.06) | 0.77 (0.08) | 0.7 (0.09) | 0.54 (0.05) | 0.59 (0.05) | 0.55 (0.05) | 55.05 (7.13) | 52.38 (6.22) | 52.41 (7.77) | 52.1 (6.2) |
| | 100 | 1 (0) | 0.87 (0.06) | 0.6 (0.09) | 0.51 (0.05) | 0.49 (0.05) | 0.48 (0.05) | 75.13 (12.85) | 91.17 (13.96) | 90.79 (16.82) | 90.39 (13.83) |
| | 5 | 0.9 (0.06) | 0.77 (0.08) | 0.8 (0.07) | 0.51 (0.05) | 0.47 (0.04) | 0.52 (0.05) | 66.08 (10.43) | 55.58 (7.37) | 54.79 (7.57) | 54.65 (7.27) |
| | 10 | 0.87 (0.06) | 0.73 (0.08) | 0.73 (0.08) | 0.49 (0.05) | 0.49 (0.04) | 0.49 (0.04) | 64.11 (10.44) | 55.29 (7.19) | 53.48 (7.69) | 54.85 (7) |
| 50 | 25 | 0.8 (0.07) | 0.7 (0.09) | 0.7 (0.09) | 0.48 (0.05) | 0.47 (0.05) | 0.49 (0.05) | 62.24 (10.1) | 55.85 (7.31) | 56.56 (7.52) | 55.44 (7.24) |
| | 50 | 0.7 (0.09) | 0.67 (0.09) | 0.63 (0.09) | 0.49 (0.05) | 0.54 (0.04) | 0.48 (0.05) | 65.53 (10.01) | 68.53 (7.56) | 71.85 (7.98) | 68.61 (7.44) |
| | 100 | 0.9 (0.06) | 0.8 (0.07) | 0.6 (0.09) | 0.46 (0.05) | 0.53 (0.05) | 0.51 (0.05) | 85.95 (12.65) | 104.55 (12.28) | 105.56 (15.21) | 103.99 (12.25) |
| | 5 | 0.7 (0.09) | 0.43 (0.09) | 0.53 (0.09) | 0.37 (0.04) | 0.42 (0.05) | 0.37 (0.04) | 112.06 (17.96) | 111.81 (14.87) | 111.78 (14.92) | 109.72 (14.68) |
| | 10 | 0.67 (0.09) | 0.47 (0.09) | 0.53 (0.09) | 0.38 (0.04) | 0.43 (0.05) | 0.38 (0.04) | 110.61 (17.81) | 110.68 (14.77) | 110.47 (14.9) | 109 (14.52) |
| 100 | 25 | 0.7 (0.09) | 0.47 (0.09) | 0.5 (0.09) | 0.39 (0.05) | 0.46 (0.05) | 0.39 (0.05) | 106.24 (17.67) | 110.56 (14.13) | 108.12 (15.04) | 109.6 (13.82) |
| | 50 | 0.67 (0.09) | 0.5 (0.09) | 0.53 (0.09) | 0.46 (0.05) | 0.42 (0.05) | 0.46 (0.05) | 102.84 (17.66) | 111.5 (14.54) | 114.27 (14.69) | 110.64 (14.41) |
| | 100 | 0.83 (0.07) | 0.67 (0.09) | 0.5 (0.09) | 0.45 (0.05) | 0.39 (0.05) | 0.46 (0.05) | 118.84 (16.75) | 136.28 (15.17) | 144.32 (15.66) | 136.65 (14.88) |

# 4 Simulation for the Boundary Case ($2p < n$)

In this section, we briefly present the result for the boundary case when $2p < n$ with the example $p = 2$ under $\mathcal{B}_u$ and $\mathcal{M}_2$. Proposition 1 in the main text tells that there are infinitely many solutions of $\mathbf{W}^*$ in this setup. To make it comparable to the main results of Section 4.3 in the main text, we set $\mu_\alpha = 50$. It is because in our simulation, as $p$ decreases, $\mathrm{E}(\alpha_1)$ will decreases as well. The result is attached as below. See discussion in Section 6 in the main text.

**Table 5:** 30 Monte Carlo simulations of $\mathcal{M}_2$ for $\mathcal{B}_u$ with varying $n$ and $\sigma_\alpha$ ($p = 2$, boundary case)

| | | Guess | | | LOOCV with $k$ random draws | | | Distance to $y_{1,T_1^*+1}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $\sigma_\alpha$ | $\delta_{\hat{\alpha}_{\text{adj}}}$ | $\delta_{\hat{\alpha}_{\text{wadj}}}$ | $\delta_{\hat{\alpha}_{\text{IVW}}}$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{adj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{wadj}}})$ | $\bar{\mathcal{C}}^{(k)}(\delta_{\hat{\alpha}_{\text{IVW}}})$ | Original | $\hat{\alpha}_{\text{adj}}$ | $\hat{\alpha}_{\text{wadj}}$ | $\hat{\alpha}_{\text{IVW}}$ |
| | 1 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 59.12 (2.5) | 10.73 (1.34) | 13.54 (1.81) | 10.59 (1.35) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 59.57 (2.63) | 11.24 (1.61) | 15.42 (2.06) | 11.19 (1.61) |
| 5 | 10 | 0.97 (0.03) | 1 (0) | 0.97 (0.03) | 0.91 (0.03) | 0.93 (0.02) | 0.91 (0.02) | 60.13 (3.08) | 13.77 (2.03) | 18.82 (2.68) | 13.85 (2.01) |
| | 25 | 0.83 (0.07) | 0.9 (0.06) | 0.83 (0.07) | 0.72 (0.04) | 0.75 (0.04) | 0.71 (0.04) | 62.49 (5.1) | 26.53 (3.52) | 34.4 (4.71) | 26.62 (3.5) |
| | 50 | 0.67 (0.09) | 0.7 (0.09) | 0.7 (0.09) | 0.49 (0.06) | 0.53 (0.05) | 0.5 (0.06) | 72.46 (7.97) | 50.17 (6.54) | 63.61 (8.47) | 50.25 (6.55) |
| | 1 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 61.13 (2.17) | 9.58 (1.52) | 12.21 (1.89) | 9.58 (1.56) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 61.3 (2.08) | 9.18 (1.44) | 11.89 (1.68) | 9.25 (1.47) |
| 10 | 10 | 1 (0) | 1 (0) | 1 (0) | 0.97 (0.01) | 0.98 (0.01) | 0.97 (0.01) | 61.52 (2.28) | 10.97 (1.39) | 13.32 (1.63) | 11.05 (1.41) |
| | 25 | 0.97 (0.03) | 0.93 (0.05) | 0.97 (0.03) | 0.77 (0.04) | 0.77 (0.04) | 0.78 (0.04) | 62.17 (4.11) | 20.07 (2.64) | 21.84 (3.38) | 20.17 (2.61) |
| | 50 | 0.73 (0.08) | 0.8 (0.07) | 0.77 (0.08) | 0.56 (0.04) | 0.61 (0.03) | 0.55 (0.04) | 63.99 (7.96) | 38.79 (5.64) | 44.19 (6.71) | 38.87 (5.57) |
| | 1 | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 56.41 (1.85) | 9.71 (1.24) | 9.15 (0.99) | 9.77 (1.26) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 1 (0) | 56.41 (2.35) | 11.91 (1.43) | 10.65 (1.32) | 11.96 (1.46) |
| 15 | 10 | 1 (0) | 1 (0) | 1 (0) | 0.98 (0.01) | 0.97 (0.01) | 0.98 (0.01) | 56.42 (3.08) | 14.71 (1.85) | 14.3 (1.74) | 14.77 (1.88) |
| | 25 | 0.87 (0.06) | 0.97 (0.03) | 0.87 (0.06) | 0.82 (0.04) | 0.77 (0.04) | 0.83 (0.04) | 57.92 (4.93) | 24.04 (3.34) | 27.4 (3.46) | 24.2 (3.36) |
| | 50 | 0.7 (0.09) | 0.77 (0.08) | 0.7 (0.09) | 0.63 (0.04) | 0.61 (0.04) | 0.62 (0.04) | 68.27 (6.57) | 42.03 (5.63) | 50.98 (6.57) | 42.24 (5.68) |
| | 1 | 1 (0) | 1 (0) | 1 (0) | 0.99 (0.01) | 0.99 (0.01) | 0.99 (0.01) | 57.87 (2.42) | 10.38 (1.54) | 14.29 (1.36) | 10.35 (1.55) |
| | 5 | 1 (0) | 1 (0) | 1 (0) | 0.98 (0.01) | 0.98 (0.01) | 0.98 (0.01) | 58.2 (2.44) | 10.55 (1.6) | 14.05 (1.62) | 10.51 (1.61) |
| 25 | 10 | 1 (0) | 1 (0) | 1 (0) | 0.95 (0.02) | 0.94 (0.02) | 0.95 (0.02) | 58.6 (2.71) | 11.43 (1.92) | 15.05 (2.08) | 11.41 (1.93) |
| | 25 | 0.97 (0.03) | 0.93 (0.05) | 0.93 (0.05) | 0.77 (0.04) | 0.79 (0.03) | 0.77 (0.04) | 59.81 (4.47) | 19.09 (3.21) | 24.74 (3.61) | 19.05 (3.21) |
| | 50 | 0.8 (0.07) | 0.83 (0.07) | 0.8 (0.07) | 0.61 (0.04) | 0.65 (0.04) | 0.61 (0.04) | 64.73 (7.53) | 35 (6.02) | 44.55 (6.94) | 34.89 (6.02) |

# References


John I Marden. Multivariate statistics: Old school. *University of Illinois*, 2015.