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On hyper-Hamiltonicity in graphs

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ABSTRACT

A Hamiltonian graph G is said to be hyper-Hamiltonian when $G-v$ is Hamiltonian for any vertex v of G. In this paper, sufficient conditions for a graph to be hyper-Hamiltonian based on different parameters, such as degree and number of edges, are presented. Furthermore, other sufficient conditions for hyper-Hamiltonicity are obtained from spectral parameters, such as the spectral radii of the adjacency matrix, the signless Laplacian matrix and the distance matrix. Results on hyper-Hamiltonicity of a threshold graph through the eigenvalues of its Laplacian matrix are also presented.

1. Introduction

A cycle passing through all vertices of a graph is called a *Hamiltonian cycle*. A graph containing a Hamiltonian cycle is called a *Hamiltonian graph*. Hamiltonicity is a characteristic of a network required in many applications. The problem of deciding whether a graph is Hamiltonian or not is NP-complete. Several theorems can be found in literature providing sufficient conditions to ensure the existence of a Hamiltonian cycle in a graph. Some of them, as the classics Ore's Theorem [15] and Dirac's Theorem [7] inspired substantial research on Hamiltonicity as can be apprehended by the reading of the recent paper [9], by Gould. In this survey, one may verify that those sufficient conditions for Hamiltonicity of graphs, for the most part, are based on characteristics of the graph structure (minimum degree, sum of degree of non-adjacent vertices, forbidden subgraphs). In the same article, the advances on Hamiltonian problem resulting from the recent approach through spectral properties of matrices associated to graphs are also described.

A Hamiltonian graph G is called *hyper-Hamiltonian* if the subgraph $G-v$ of G, obtained by deleting the vertex v of G is Hamiltonian for each v in G, see [1] by Albert, Aldread, Holton and Sheehan. In the context of architecture of parallel computing machines, the paper [18], by Wu and Duh, studies a kind of network topology named "piramid network", shows that it is Hamiltonian and that remains Hamiltonian with a faulty node.

As far as we are concerned, regarding the hyper-Hamiltonicity of a graph, only few articles are known, as [13] by Mai and Wang, where hyper-Hamiltonian generalized Petersen graphs are investigated. Thus, with the present paper, we hope to provide basis for future research on the theme. Specifically, we prove an "Ore's type" sufficient condition to hyper-Hamiltonicity and give a characterization of graphs with this property through the concept of k-closure. Furthermore, we apply these results to present several conditions on the largest eigenvalue of matrices associated to an arbitrary graph that are sufficient to guarantee that it is hyper-Hamiltonian. We also obtain conditions on Laplacian eigenvalues for hyper-Hamiltonicity (and Hamiltonicity) in the class of threshold graphs.

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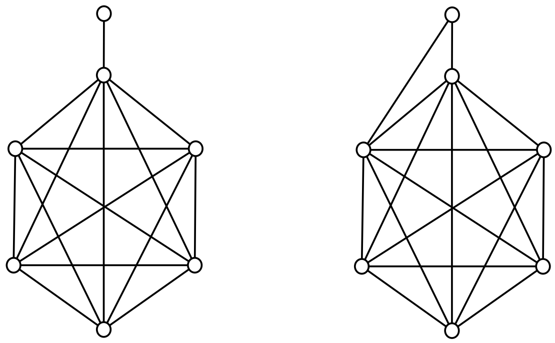

Fig. 1. $\mathbb{P}_{(6)}$ and $\mathbb{P}_{(6)} + e$.

Some of the main results of this article were announced in our paper [6] without proofs. Here, we present the complete proofs as well as several other auxiliary and complementary results.

All graphs considered in this paper are simple and, unless otherwise stated, connected. $V(G)$ and $E(G)$ denote the sets of vertices and edges of the graph G, respectively. We use m_G to indicate the number of edges of the graph G. The degree of vertex v in G is denoted by $d_G(v)$ or $d(v)$, when no ambiguity is possible. By $\delta(G)$ we denote the minimum degree of G and by $\overline{G}$, the complement graph of G. We use $|S|$ to indicate the cardinality of the set S. The *join* of two disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from the union $G_1 + G_2$ by joining each vertex of G_1 to each vertex of G_2. As usually, P_n and K_n denote, respectively, the path and the complete graph of order n, and the star graph $K_1 \vee \overline{K_r}$ is denoted by $K_{1,r}$. We denote by $\mathbb{P}_{(n)}$ the graph (a *pineapple*) obtained from K_n by adding a pendent vertex, and by $\mathbb{P}_{(n)} + e$ the graph obtained from $\mathbb{P}_{(n)}$ by inserting an edge (see Fig. 1).

The paper is organized as follows: the second section is dedicated to establish versions of classic results on Hamiltonicity in the case of hyper-Hamiltonicity. In the third section, we provide several sufficient conditions for hyper-Hamiltonicity depending on the *spectral radius* (the largest eigenvalue) of the adjacency, the signless Laplacian and the distance matrices. Finally, in the fourth section, we establish a relation between hyper-Hamiltonicity and the spectrum of the Laplacian matrix in the particular context of threshold graphs.

2. General conditions for hyper-Hamiltonian graphs

In this section we present results providing conditions for a graph to be hyper-Hamiltonian, which are based on several parameters such as vertex degrees and number of edges. These results, besides worth by themselves, constitute important tools for the theorems in the following sections. First, we may recall the classic Ore's Theorem:

Theorem 2.1 ([15]). *If $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v of the graph G on $n \geq 3$ vertices then G is Hamiltonian.*

Our first theorem is analogous to Ore's Theorem in the case of hyper-Hamiltonian graphs.

Theorem 2.2. *Let G be a graph with $n \geq 3$ vertices such that, for every pair of non-adjacent vertices u and v, $d_G(u)+d_G(v) \geq n + 1$. Then G is hyper-Hamiltonian.*

Proof. It is enough to apply Ore's Theorem to $G' = G - \{w\}$, where $w \in V(G)$ is arbitrary, noting that for any non-adjacent vertices u and v in G' (and consequently, non-adjacent in G) we have $d_{G'}(u)+d_{G'}(v) \geq (d_G(u)-1)+(d_G(v)-1) \geq n-1$. ■

As an immediate consequence we also have an analogue to Dirac's Theorem, see [7].

Corollary 2.1. *If G is a graph on $n \geq 3$ vertices and $\delta(G) \geq \frac{n+1}{2}$ then G is hyper-Hamiltonian.*

The next result is due to Ore, see [15] and also [3], by Bondy and Murty).

Lemma 2.1 ([3, Cor.4.6]). *For a graph G on $n \geq 3$ vertices, if $m_G > \frac{n^2-3n+2}{2}$ then G is Hamiltonian or $G = \mathbb{P}_{(n-1)}$ or $G = K_2 \vee \overline{K_3}$, in case $n = 5$.*

In the sequence, our Theorem 2.3 is the analogue of Lemma 2.1 in hyper-Hamiltonian case. For sake of completeness, we provide its proof, a consequence of the above Theorem 2.2.

Theorem 2.3. *Let G be a graph with $n \geq 3$ vertices. If*

$$m_G \geq \frac{n^2 - 3n + 6}{2}$$

then G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$ or $G = (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$.

Proof. We may note that

$$m_{(\mathbb{P}_{(n-1)}+e)} = \left(\frac{(n-1)(n-2)}{2} + 1\right) + 1 = \frac{n^2 - 3n + 6}{2}.$$

Suppose $G \neq \mathbb{P}_{(n-1)} + e$ and $G \neq (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$. Since, by hypothesis, $m_G > \frac{n^2-3n+2}{2}$, it follows from Lemma 2.1 that G is a Hamiltonian graph or $G = \mathbb{P}_{(n-1)}$ or $G = K_2 \vee \overline{K_3}$, if $n = 5$. Since $m_G > m_{(\mathbb{P}_{(n-1)})}$ then $G \neq \mathbb{P}_{(n-1)}$. Furthermore, in the case $n = 5$, $G \neq K_2 \vee \overline{K_3}$, as this last graph has 7 edges and $m_G \geq 8$. Thus G is Hamiltonian. Now, let $v \in V(G)$ and denote $G' = G - v$. Clearly, $m_{G'}$ is at least equal to $m_G - (n-1)$, with equality in the case $d(v) = n - 1$. Then

$$m_{G'} > \frac{(n-1)^2 - 3(n-1) + 2}{2}.$$

Using again Lemma 2.1, it follows that G' is a Hamiltonian graph or $G' = \mathbb{P}_{(n-2)}$ or $G' = K_2 \vee \overline{K_3}$, if $n - 1 = 5$. If $G' = K_2 \vee \overline{K_3}$ and G has 6 vertices then, by hypothesis, $m_G \geq 12$. As $m_{G'} = 7$, we need to add a universal vertex to G' in order to obtain G, but thus, $G = (K_2 \vee \overline{K_3}) \vee K_1$, a contradiction. If $G' = \mathbb{P}_{(n-2)}$ then, in G, the vertex v is adjacent to the pendent vertex of $\mathbb{P}_{(n-2)}$ and also, to at least one of the vertices of the clique, otherwise G would not be Hamiltonian. We may note that v cannot be a vertex adjacent to all vertices in the maximum clique $K_{(n-2)}$, since $G \neq \mathbb{P}_{(n-1)} + e$. Then

$$m_G < m_{(\mathbb{P}_{(n-1)}+e)} = \frac{n^2 - 3n + 6}{2} = (m_{\mathbb{P}_{(n-2)}} + n - 2) + 1,$$

and it follows that

$$m_G \leq m_{\mathbb{P}_{(n-2)}} + n - 2 = \frac{n^2 - 3n + 4}{2},$$

contradicting the hypothesis. Thus G' is a Hamiltonian graph, and G is hyper-Hamiltonian by definition. ∎

Given a property $\mathcal{P}$ defined for all graphs of order n and a nonnegative integer k, $\mathcal{P}$ is said to be *k-stable* if whenever $G + uv$ has the property $\mathcal{P}$ and $d_G(u) + d_G(v) \geq k$ then G must have property $\mathcal{P}$. This definition was introduced by Bondy and Chvátal in [2], where it is noted that Ore's Theorem implies that among graphs on n vertices, the property of being Hamiltonian is n-stable. By following a similar reasoning and using Theorem 2.2, we can prove that the property of being hyper-Hamiltonian is $(n+1)$-stable.

For an integer $k > 0$, the *k-closure* of G is the (only) graph obtained from G by successively joining pairs of non-adjacent vertices whose sum of degrees is at least k until no such pair remains, see [2] by Bondy and Chvátal for details. The concept of k-closure of a graph allows to establish the following result proved in [2].

Lemma 2.2 ([2]). *If $\mathcal{P}$ is a k-stable property such that the k-closure of G has the property $\mathcal{P}$ then G has the property $\mathcal{P}$.*

Similarly to what happens in investigating the Hamiltonian problem, the following result constitutes an important tool regarding the research on hyper-Hamiltonicity. It is a consequence of Lemma 2.2 and the above mentioned result of Bondy and Chvátal in [2]. The proof is straightforward.

Proposition 2.1. *A graph G on n vertices is hyper-Hamiltonian if, and only if, the $(n+1)$-closure of G is hyper-Hamiltonian.*

3. Spectral conditions for hyper-Hamiltonicity

The *adjacency matrix* $A(G) = [a_{ij}]$ of a graph G with vertices $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix defined by its entries: $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0, otherwise. This is a symmetric matrix and its largest eigenvalue $\lambda(G)$ is called the *spectral radius* of G.

Let $Deg(G)$ be the diagonal matrix whose (i,i)-entry is the degree of the vertex v_i. The *signless Laplacian matrix* of G is $Q(G) = Deg(G) + A(G)$. It is a symmetric matrix and the *signless Laplacian spectral radius* of G is the largest eigenvalue of $Q(G)$, denoted by $q_1(G)$.

By $D(G)$ we denote the *distance matrix* of G, whose (i,j)-entry is $d(v_i, v_j)$, the distance between the vertices v_i and v_j. This matrix is also symmetric and we denote by $\rho(G)$ the *spectral radius of* $D(G)$, that is, its largest eigenvalue.

Accordingly to Perron–Frobenius' Theorem, for connected graphs, the spectral radii of each of the above mentioned matrices are positive and simple (see [12], by Horn and Johnson, for details).

3.1. Adjacency matrix

In [8], Fiedler and Nikiforov presented conditions on the spectral radius of a graph G and of $\overline{G}$, both of them guaranteeing the existence of a Hamiltonian cycle in G. One of these results is here stated in the following lemma.

Lemma 3.1 ([8]). *Let G be a graph with $n \geq 3$ vertices. If $\lambda(G) > n-2$ then G is Hamiltonian unless $G = \mathbb{P}_{(n-1)} + e$.*

By applying the techniques from Fiedler and Nikiforov in [8] and the results of the previous section, we are able to provide the sufficient conditions for hyper-Hamiltonicity of a graph established in the two following theorems.

Theorem 3.1. *Let G be a graph with $n \geq 3$ vertices. If*

$$\lambda(G) > -\frac{1}{2} + \sqrt{\left(n - \frac{3}{2}\right)^2 + 2}$$

then G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$.

Proof. Using Stanley's inequality [16] $\left(\lambda(G) \leq -\frac{1}{2} + \sqrt{2m_G + \frac{1}{4}}\right)$, and the hypothesis, it follows that

$$\sqrt{\left(n - \frac{3}{2}\right)^2 + 2} - \frac{1}{2} < \lambda(G) \leq -\frac{1}{2} + \sqrt{2m_G + \frac{1}{4}},$$

and so $\sqrt{(n - \frac{3}{2})^2 + 2} < \sqrt{2m_G + \frac{1}{4}}$. By algebraic manipulations, we obtain $m_G > \frac{n^2-3n+4}{2}$, and then $m_G \geq \frac{n^2-3n+6}{2}$. Thus, from Theorem 2.3, G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$ or $G = (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$. As $\lambda((K_2 \vee \overline{K_3}) \vee K_1) = 4,16$ and $-\frac{1}{2} + \sqrt{(6 - \frac{3}{2})^2 + 2} = 4,21$, $G \neq (K_2 \vee \overline{K_3}) \vee K_1$, concluding the proof. ∎

Theorem 3.2. *For a graph G on $n > 4$ vertices, if*

$$\lambda(\overline{G}) \leq \sqrt{\binom{n-2}{2} - \left(\frac{n-2}{n}\right)}$$

then G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$.

Proof. Let I be the $(n+1)$-closure of G and assume that G is not hyper-Hamiltonian. By Proposition 2.1, I is not hyper-Hamiltonian. The definition of I tells us that $d_I(u) + d_I(v) \leq (n+1) - 1 = n$ for each pair of non-adjacent vertices u and v of I. Then

$$d_{\overline{I}}(u) + d_{\overline{I}}(v) = (n - 1 - d_I(u)) + (n - 1 - d_I(v)) \geq n - 2,$$

for each edge uv of $\overline{I}$. Summing over all such edges of $\overline{I}$, we obtain

$$\sum_{uv \in E(\overline{I})} (d_{\overline{I}}(u) + d_{\overline{I}}(v)) \geq \sum_{1}^{m_{\overline{I}}} (n-2) = (n-2)m_{\overline{I}}.$$

But each term $d_{\overline{I}}(u)$ occurs exactly $d_{\overline{I}}(u)$ times in the summatory, and therefore $\sum_{u \in V(\overline{I})} d_{\overline{I}}^2(u) \geq (n-2)m_{\overline{I}}$. By applying Hofmeister's inequality ($\lambda \geq \sqrt{\frac{1}{n}\sum_{u \in V} d^2(u)}$, see [11]) for $\overline{I}$, we obtain

$$n\lambda^2(\overline{I}) \geq \sum_{u \in V(\overline{I})} d^2{}_{\overline{I}}(u) \geq (n-2)m_{\overline{I}}.$$

Considering that $\overline{I} \subseteq \overline{G}$ and applying the hypothesis, it follows that

$$n\binom{n-2}{2} - (n-2) \geq n\lambda^2(\overline{G}) \geq n\lambda^2(\overline{I}) \geq (n-2)m_{\overline{I}},$$

and therefore $m_{\overline{I}} \leq \frac{n}{2} - 1$. Hence

$$m_I = \frac{n(n-1)}{2} - m_{\overline{I}} \geq \frac{n(n-1)}{2} - \frac{n}{2} + 1 = \frac{n^2 - 2n + 2}{2} > \frac{n^2 - 3n + 6}{2},$$

as $n > 4$. Since I is not a hyper-Hamiltonian graph, Theorem 2.3 asserts that $I = \mathbb{P}_{(n-1)} + e$ or $I = (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$. Since $m_I > \frac{n^2-3n+6}{2}$ if $n = 6$, I cannot be $(K_2 \vee \overline{K_3}) \vee K_1$. Thus $I = \mathbb{P}_{(n-1)} + e$ and, if $G = I$, the proof is completed. Now, suppose that G is a proper subgraph of $\mathbb{P}_{(n-1)} + e$. Then $\overline{G}$ is the star $K_{1,(n-3)}$ with additional edges and isolated vertices, and this fact tells us that $\overline{G}$ contains a connected supergraph of $K_{1,(n-3)}$. Applying the interlace theorem for the adjacency matrix (see Proposition 3.1.1 of [4], by Brower and Haemers), we obtain

$$\lambda(\overline{G}) \geq \lambda(K_{1,n-3}) = \sqrt{n-3} > \sqrt{\binom{n-2}{2} - \binom{n-2}{n}},$$

where last inequality is true for $n > 4$. Thus we obtain a contradiction and $G = \mathbb{P}_{(n-1)} + e$, completing the proof. ■

The results of paper [8], by Fiedler and Nikiforov, inspired several different spectral conditions for Hamiltonicity as can be found, for instance, in [20], by Zhou, and [19], by Zhong-zhu, Si-si and Guo-qiang, regarding the matrices Q and D, respectively. In the same direction, in the next two sections, we also provide conditions for hyper-Hamiltonicity based on the radii of Q and D.

3.2. Signless Laplacian matrix

In the sequence, we determine conditions on the spectral radius of the signless Laplacian matrix of a graph that are sufficient to guarantee its hyper-Hamiltonicity.

Denote by $Z(G)$ the sum of the squares of the degrees of G, that is, $Z(G) = \sum_{u\in V(G)} d_G(u)^2$. Clearly, $Z(G) = \sum_{uv\in E(G)}(d_G(u) + d_G(v))$. For details about this parameter, see [20], by Zhou, and the references herein. We will need the lower bound of $q_1(G)$ depending on $Z(G)$ stated below and two new lemmas.

Lemma 3.2 ([20]). *If G is a graph with at least one edge then $q_1(G) \geq \frac{Z(G)}{m_G}$.*

Lemma 3.3. *A bipartite k-regular graph on $2k$ vertices is not hyper-Hamiltonian.*

Proof. Indeed, a bipartite k-regular graph G on $2k$ vertices is clearly Hamiltonian, by Ore's Theorem (Theorem 2.1). But for each vertex v, $G - v$ is a graph with an odd number of vertices that is still bipartite, and for this reason it cannot contain a Hamiltonian cycle. Hence, G is not hyper-Hamiltonian. ■

We may recall that a bipartite graph is said to be *semi-regular* if each vertex in the same partition of the graph has equal degree.

Lemma 3.4. *If G is a k-regular non-bipartite graph on $2k$ vertices then G is hyper-Hamiltonian.*

Proof. Let G be a k-regular non-bipartite graph on $n = 2k$ vertices. Since $d(v) = k = \frac{n}{2}$ for all $v \in V(G)$, Theorem 2.1 asserts that G is a Hamiltonian graph. Let $G' = G - u$ for $u \in V(G)$ and suppose that G' is not Hamiltonian. By naming $u = v_{2k}$, we can choose a Hamiltonian path in G' and label its vertices as $v_1v_2\cdots v_{2k-1}$. We may note that G' has k vertices of degree $k-1$ (with v_1 and v_{2k-1} among them) and $k-1$ vertices of degree k. Since G' is not Hamiltonian two cases are possible:

(1) for each i, $2 \leq i \leq 2k-2$, if $v_1v_i \in E(G')$ then $v_{i-1}v_{2k-1} \notin E(G')$, otherwise $v_1v_2\cdots v_{i-1}v_{2k-1}v_{2k-2}\cdots v_i$ is a Hamiltonian cycle.

(2) for each i, $2 \leq i \leq 2k-2$, if $v_1v_i \notin E(G')$ then $v_{i-1}v_{2k-1} \in E(G')$, as $d(v_1) = d(v_{2k-1}) = k-1$. In this case the following situations arise:

  (i) v_1 is adjacent to $v_2, v_3, \ldots, v_k$ and v_{2k-1} is adjacent to $v_k, v_{k+1}, \ldots, v_{2k-2}$: then there is i, $2 \leq i \leq k-1$ such that v_i is adjacent to v_j, for some j, $k+1 \leq j \leq 2k-2$. Indeed, otherwise, since $d(v_i)$ is at least equal to $k-1$, v_i is adjacent to vertices $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$, for all i, $2 \leq i \leq k-1$. Then v_k is adjacent to $v_1, v_2, \ldots, v_{k-1}$ and also is adjacent to v_{k+1} and v_{2k-1}, and so, $d(v_k) = k+1$, which is a contradiction. Therefore $v_1v_2\cdots v_iv_jv_{j+1}\cdots v_{2k-1}v_{j-1}\cdots v_{i+1}$ is a Hamiltonian cycle in G', which is also a contradiction.

  (ii) there is i, $2 \leq i \leq 2k-2$ such that $v_{i+1}v_1 \in E(G')$ but $v_iv_1 \notin E(G')$: in this case, by (2) above, $v_{i-1}v_{2k-1} \in E(G')$. Thus G' contains the $(2k-2)$-cycle C given by $v_{i-1}v_{i-2}\cdots v_1v_{i+1}v_{i+2}\cdots v_{2k-1}$. Relabeling C as $u_1u_2\cdots u_{2k-2}$ and calling u_0 the vertex of G' that is not in C, then u_0 cannot be adjacent to two consecutive vertices of C. Therefore u_0 is adjacent to each one of $u_1, u_3, \ldots, u_{2k-3}$ and $d(u_0) = k-1$. Replacing u_0 by u_{2i}, we obtain another maximum cycle C' in G' and then, u_{2i} must be adjacent to $u_1, u_3, \ldots, u_{2k-3}$. It follows that each vertex with an odd label $(u_1, u_3, \ldots, u_{2k-3})$ is adjacent to the k vertices of even label $(u_0, u_2, \ldots, u_{2k-2})$. Thus $d(u_{2i}) = k-1$ and $d(u_{2i+1}) = k$. Hence, the vertices with an even label cannot be mutually adjacent, neither the vertices with an odd label, as there are exactly k vertices with even label of degree $k-1$ and $k-1$ with odd label of degree k. Thus it follows that G' is a bipartite semi-regular graph and then, the (deleted) vertex u of G must belong to the same partition as $u_1, u_3, \ldots, u_{2k-3}$ and to be adjacent to $u_0, u_2, \ldots, u_{2k-2}$. But this means that G is a bipartite k- regular graph, which is a contradiction.

The assertion is proved. ■

Before proving our next result we need to set some notation. Let $\mathcal{G}_n$ be a family of connected graphs on n vertices such that $G \in \mathcal{G}_n$ if and only if

$G = P_2 \vee (K_a \cup K_{n-a-2})$, with $a \in \mathbb{N}$ and $1 < a < n-2$, or G is a bipartite and $\frac{n}{2}$-regular graph or $G = H \vee F$, where H is a $(\frac{n}{2} - r)$-regular graph and $|V(F)| = r \leq \frac{n}{2}$.

Theorem 3.3. *Let G be a graph with $n \geq 3$ vertices. If $q_1(\overline{G}) \leq n-2$ and $G \notin \mathcal{G}_n$ then G is hyper-Hamiltonian.*

Proof. Let I be the $(n+1)$-closure of G. If $I = K_n$ then the assertion is proved by Proposition 2.1. Suppose that $I \neq K_n$ and, by contradiction, that G is not hyper-Hamiltonian. Then, by Proposition 2.1, I is not hyper-Hamiltonian. From the definition of I we have $d_I(u) + d_I(v) \leq n$, for each of the non-adjacent vertices u and v of I. Then it holds that $d_{\bar{I}}(u) + d_{\bar{I}}(v) \geq n-2$ for all edges uv of $\bar{I}$. Summing over all the edges of $\bar{I}$ and bearing in mind that $Z(\bar{I}) = \sum_{uv \in E(\bar{I})} (d_{\bar{I}}(u) + d_{\bar{I}}(v))$, it follows from Lemma 3.2 that $q_1(\bar{I}) \geq \frac{Z(\bar{I})}{m_{\bar{I}}} \geq n-2$. As $\bar{I} \subset \overline{G}$, by the interlace theorem for the signless Laplacian matrix (see Proposition 3.2.1 of [4], by Brower and Haemers) and the hypothesis, we have

$$n - 2 \geq q_1(\overline{G}) \geq q_1(\bar{I}) \geq \frac{Z(\bar{I})}{m_{\bar{I}}} \geq n - 2,$$

that is, $q_1(\overline{G}) = q_1(\bar{I}) = n-2$, and also $d_{\bar{I}}(u) + d_{\bar{I}}(v) = n-2$, for all edges uv of $\bar{I}$. Since $d_G(u) + d_G(v)$ is constant for all edges $uv \in E(G)$ if and only if G is a regular or a semi-regular (bipartite) graph, then $\bar{I}$ contains a non-trivial component F that is a regular or a semi-regular graph. Let us consider each case.

(1) Let us suppose that F is a regular graph. Then $d(u) = \frac{n-2}{2}$, for all $u \in V(F)$. Suppose firstly that $F = \bar{I}$. Since $q_1(\overline{G}) = q_1(\bar{I}) = n-2$ and $\bar{I}$ is a subgraph of $\overline{G}$, by using again the interlace for signless Laplacian matrix, we obtain $\overline{G} = \bar{I}$, and so, $G = I$, which implies that G is a regular graph with degree $\frac{n}{2}$. Furthermore, if G is bipartite, then $G \in \mathcal{G}_n$, a contradiction with the conditions in the hypothesis; and if G is not bipartite, this contradicts Lemma 3.4. So, $F \neq \bar{I}$. In the sequence, we consider the two possibilities:
   - $\bar{I}$ consists of F and r isolated vertices with $1 \leq r \leq \frac{n}{2}$: since $q_1(\overline{G}) = q_1(\bar{I})$ and $\bar{I}$ is a subgraph of $\overline{G}$, by the interlace theorem for signless Laplacian matrix, $\overline{G} = F + F'$, where F' has r vertices. Then G is the join of $\overline{F}$, which is a regular graph of degree $\frac{n}{2} - r$, and $\overline{F'}$, which is a graph on r vertices, with $1 \leq r \leq \frac{n}{2}$. So $G \in \mathcal{G}_n$, a contradiction.
   - $\bar{I}$ consists of two complete components, each one with $\frac{n}{2}$ vertices: then F is the complete graph with $\frac{n}{2}$ vertices and $\overline{G}$ consists of F and another complete component on $\frac{n}{2}$ vertices. It follows that G is the join of two graphs, which one with $\frac{n}{2}$ isolated vertices, and then G is a bipartite and $\frac{n}{2}$-regular graph. So $G \in \mathcal{G}_n$, a contradiction.

(2) Let F be a (bipartite) semi-regular graph. Then F has $n-2$ vertices, as $d_{\bar{I}}(u) + d_{\bar{I}}(v) = n-2$ for all edges uv of $\bar{I}$. But here $\bar{I}$ is not a connected graph, otherwise we would have a contradiction with the equality $d_{\bar{I}}(u) + d_{\bar{I}}(v) = n-2$, for all uv of $\bar{I}$. Therefore $\bar{I}$ is a disconnected graph consisting of F and two isolated vertices. Since $q_1(\overline{G}) = q_1(\bar{I})$ and $\bar{I}$ is a subgraph of $\overline{G}$, by the interlace for signless Laplacian matrix it follows that $\overline{G} = \bar{I}$. Hence G is the join of the path P_2 and a graph consisting of two complete components. Again, we have $G \in \mathcal{G}_n$, a contradiction.

The assertion is proved. ■

3.3. Distance matrix

We may recall that the *Wiener index* $W(G)$ is defined as the sum of the lengths of shortest paths between all pairs of vertices of the graph G, that is, $W(G) = \sum_{i,j=1}^{n} d(v_i, v_j)$. For more details, see [17], by Wiener. The following lemmas will be applied for proving our next theorem. The first can be found in [19], by Zhong-zhu, Si-si and Guo-qiang, and the second in [21], by Zhou and Trinajstić.

Lemma 3.5 ([19]). *For a graph G with $n \geq 2$ vertices it holds that $\rho(G) \geq \frac{2W(G)}{n}$.*

Lemma 3.6 ([21]). *If G is a graph on $n \geq 2$ vertices and $m_G \geq 1$ then $W(G) \geq n(n-1) - m$.*

Theorem 3.4. *Let G be a connected graph with $n \geq 2$ vertices. If*

$$\rho(G) < \frac{(n-1)(n+2) - 2}{n}$$

then G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$ or $G = (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$.

Proof. Applying Lemmas 3.5 and 3.6 and the hypothesis, we obtain

$$\frac{(n-1)(n+2)-2}{n} > \rho(G) \geq \frac{2W(G)}{n} \geq 2(n-1) - \frac{2m_G}{n},$$

and then $(n-1)(n+2)-2 > 2n(n-1)-2m_G$. It follows that

$$2m_G > 2n(n-1)-(n-1)(n+2)+2 = (n-1)(2n-n-2)+2 = n^2-3n+4.$$

Since n^2-3n+4 is an even integer for all $n \in \mathbb{N}$, we have that $\frac{n^2-3n+4}{2}$ is an integer. Then $m_G > \frac{n^2-3n+4}{2}$ implies $m_G \geq \frac{n^2-3n+6}{2}$. By Theorem 2.3, G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$ or $G = (K_2 \vee \overline{K_3}) \vee K_1$, in case $n = 6$. ■

Remark 3.1. We may note that $\rho((K_2 \vee \overline{K_3}) \vee K_1) = 6,16 < \frac{(6-1)(6+2)-2}{6} = 6,22$, and therefore this graph is not eliminated by the condition on the spectral radius of the distance matrix, established in the hypothesis of Theorem 3.4.

4. Hyper-Hamiltonian threshold graphs

In this section, unlike what was done previously, we will restrict our results to a specific class of graphs, namely, threshold graphs. We will provide conditions for hyper-Hamiltonicity of graphs in this class, based on eigenvalues of their Laplacian matrix.

Recall that the *Laplacian matrix* of G is given by $L(G) = Deg(G) - A(G)$. Since $L(G)$ is a positive semi-definite matrix, the Laplacian eigenvalues of G are nonnegative real numbers. We shall denote the eigenvalues of $L(G)$ in non increasing order as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. The eigenvalue $\mu_{n-1}(G)$ of $L(G)$ is called the *algebraic connectivity* of G and denoted by $a(G)$.

Threshold graphs are graphs free of the path P_4, of the cycle C_4 and of $2K_2$ (see [5], by Chvátal and Hammer for details). Hamiltonicity in threshold graphs is studied in [10], by Harary and Peled, under a non spectral approach. In [14], by Merris, it is shown that Laplacian eigenvalues of a threshold graph can be obtained from its degree sequence. This result, here stated as a lemma, will be applied for proving our next theorem.

Lemma 4.1 ([14]). *Let G be a threshold graph with n vertices, clique number (that is, the size of a maximum clique) ω and degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Then the sequence $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ of Laplacian eigenvalues of G satisfies:*

(i) For $1 \leq i \leq \omega - 1$, then $\mu_i = d_i + 1$;
(ii) For $\omega \leq i \leq n-1$, then $\mu_i = d_{i+1}$ and
(iii) $\mu_n(G) = 0$.

Theorem 4.1. *Let G be a threshold graph with $n \geq 2$ vertices. If $\mu_{n-1}(G) + \mu_{n-2}(G) \geq n+1$ then G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$.*

Proof. Suppose $\mu_{(n-1)} + \mu_{(n-2)} \geq n+1$ and consider the degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. If the clique number is $\omega = n$, then $G = K_n$ and the statement is proved. If $\omega \leq n-2$, by Lemma 4.1, $\mu_{(n-1)} = d_n$ and $\mu_{(n-2)} = d_{n-1}$. This fact implies that for each pair v_1, v_2 of non-adjacent vertices of G, $d(v_1) + d(v_2) \geq d_n + d_{n-1} = \mu_{(n-1)} + \mu_{(n-2)} \geq n+1$. The hyper-Hamiltonicity of G follows then from Theorem 2.2. It remains to consider the case $\omega = n-1$, where G is the graph consisting of a complete graph $K_{(n-1)}$ and one vertex v, connected to r vertices of $K_{(n-1)}$, where $\leq r \leq n-2$.

- If $r = 1$, $G = \mathbb{P}_{(n-1)}$; but, from Lemma 4.1, $\mu_{(n-1)}(\mathbb{P}_{(n-1)}) + \mu_{(n-2)}(\mathbb{P}_{(n-1)}) = n$, contradicting the hypothesis.
- If $r = 2$, then $G = \mathbb{P}_{(n-1)} + e$.
- Now, let $r > 2$ and $G' = G - \{u\}$. If $u = v$, $G' = K_{(n-1)}$, which is Hamiltonian. If $u \neq v$, G' is the graph consisting of $K_{(n-2)}$ with the vertex v adjacent to at least two vertices in $K_{(n-2)}$. Then G' is Hamiltonian and, consequently, G is hyper-Hamiltonian. The assertion is proved. ■

As an immediate consequence we obtain the next corollary.

Corollary 4.1. *For a threshold graph G with $n \geq 2$ vertices, if $a(G) \geq \frac{n+1}{2}$ then G is hyper-Hamiltonian.*

Proof. If $\mu_{n-1}(G) = a(G) \geq \frac{n+1}{2}$ then $\mu_{n-1}(G) + \mu_{n-2}(G) \geq \frac{n+1}{2} + \frac{n+1}{2} = n+1$. As a consequence of Theorem 4.1, G is hyper-Hamiltonian or $G = \mathbb{P}_{(n-1)} + e$. But, from Lemma 4.1, $a(\mathbb{P}_{(n-1)} + e) = \mu_{n-1}(\mathbb{P}_{(n-1)} + e) = 1$, contradicting the hypothesis. Thus the assertion is proved. ■

Remark 4.1. We can also obtain results concerning Hamiltonicity for threshold graphs G with n vertices:

- If $\mu_{n-1}(G) + \mu_{n-2}(G) \geq n$ then G is Hamiltonian.
- If $a(G) \geq \frac{n}{2}$ then G is Hamiltonian or $G = \mathbb{P}_{(n-1)}$.

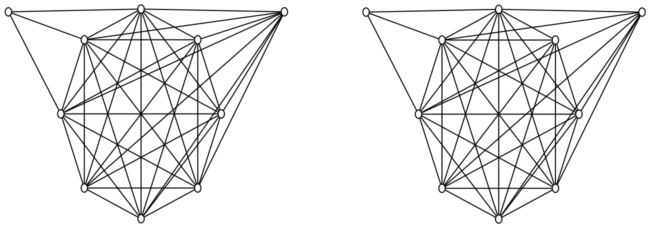

Fig. 2. Two hyper-Hamiltonian graphs of Example 4.1.

We may note that different matrices do not produce equivalent results as can be seen in the following example.

Example 4.1. Both the hyper-Hamiltonian threshold graphs in Fig. 2 have 10 vertices and non connected complements. The graph G_1 on the left has $m = 39$, $\lambda(G_1) = 8,126$, $\lambda(\overline{G_1}) = 2,44$, $q_1(\overline{G_1}) = 7$, $\rho(G_1) = 10,43$, $\mu_{n-1}(G_1) = 3$ and $\mu_{n-2}(G_1) = 9$. G_1 satisfies the conditions of Theorems 2.3, 3.1, 3.3, 3.4 and 4.1, but it neither satisfies conditions of Theorem 3.2 nor Corollary 4.1. The graph G_2 on the right has $m = 38$, $\lambda(G_2) = 7,93$, $\lambda(\overline{G_2}) = 2,68$, $q_1(\overline{G_2}) = 7,13$, $\rho(G_2) = 10,64$, $\mu_{n-1}(G_2) = 3$ and $\mu_{n-2}(G_2) = 7$. Therefore, G_2 satisfies Theorems 2.3 and 3.3 but does not satisfy Theorem 3.1, 3.2, 3.4 or 4.1.

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