Section 4.5

Families of Continuous Random Variables

Definition 4.5 Uniform Random Variable

X is a uniform (a, b) random variable if the PDF of X is

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x < b, \\ 0 & \textit{otherwise}, \end{cases}$$

where the two parameters are $b > a$.

How did we get this density function of a uniform distribution?

$$\begin{cases} \displaystyle\int_a^b f(x)dx = 1 \\ f(x) = c \end{cases} \Rightarrow \quad c = \frac{1}{b-a}$$

Theorem 4.6

If X is a uniform (a, b) random variable,

- The CDF of X is $$F_X(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/(b-a) & a < x \leq b, \\ 1 & x > b. \end{cases}$$
- The expected value of X is $\mathrm{E}[X] = (b+a)/2$.
- The variance of X is $\mathrm{Var}[X] = (b-a)^2/12$.

Example 4.11 Problem

The phase angle, Θ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. What are the PDF, CDF, expected value, and variance of Θ?

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta < 2\pi, \\ 0 & \text{otherwise}, \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \leq a, \\ (x-a)/(b-a) & a < x \leq b, \\ 1 & x > b. \end{cases}$$

$$\mathrm{E}[X] = (b+a)/2.$$

$$\mathrm{Var}[X] = (b-a)^2/12.$$

Example 4.11 Solution

From the problem statement, we identify the parameters of the uniform (a, b) random variable as $a = 0$ and $b = 2\pi$. Therefore the PDF and CDF of Θ are

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \le \theta < 2\pi, \\ 0 & \text{otherwise}, \end{cases} \qquad F_\Theta(\theta) = \begin{cases} 0 & \theta \le 0, \\ \theta/(2\pi) & 0 < x \le 2\pi, \\ 1 & x > 2\pi. \end{cases} \tag{1}$$

The expected value is $\mathrm{E}[\Theta] = b/2 = \pi$ radians, and the variance is $\mathrm{Var}[\Theta] = (2\pi)^2/12 = \pi^2/3$ rad^2.

Section 4.6

Gaussian Random Variables

Definition 4.8 Gaussian Random Variable

X *is a Gaussian* (μ, σ) *random variable if the PDF of* X *is*

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ *can be any real number and the parameter* $\sigma > 0$.

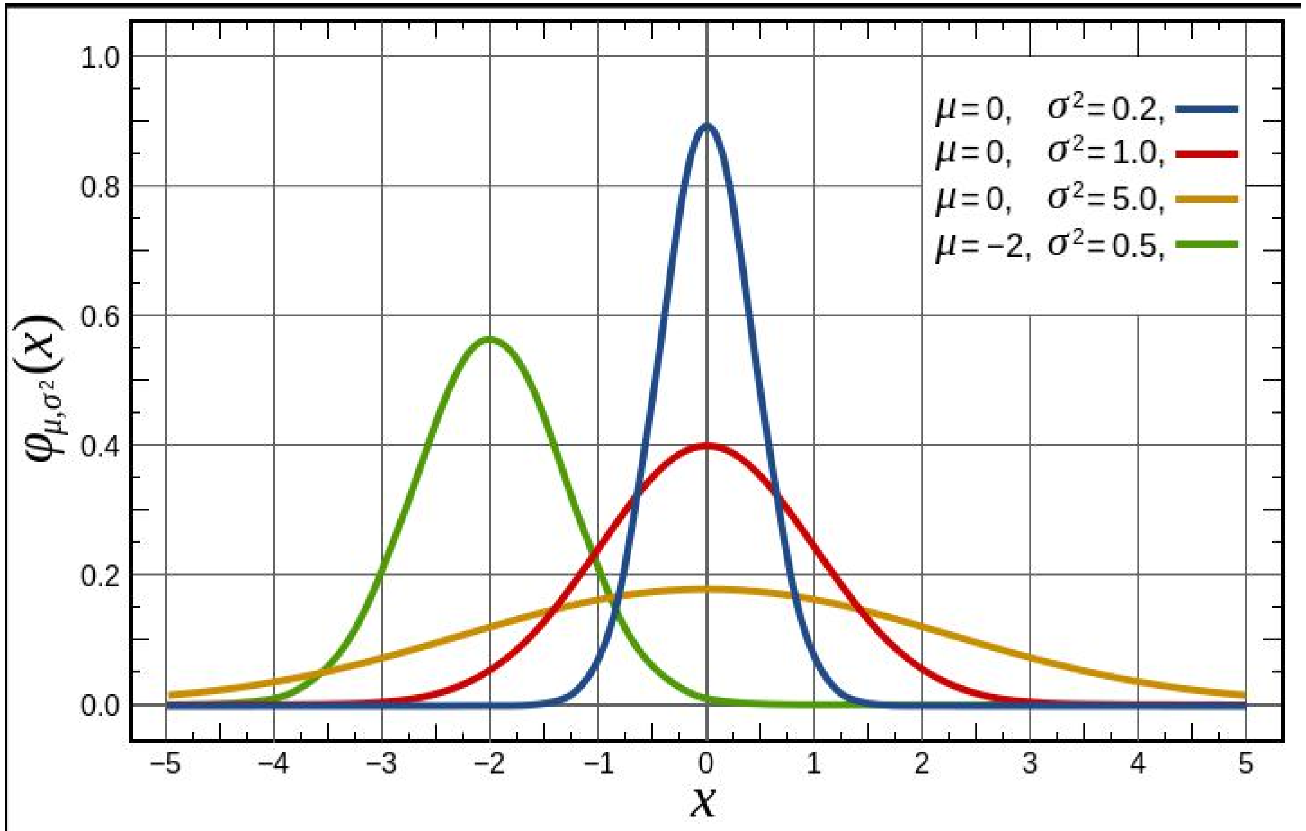
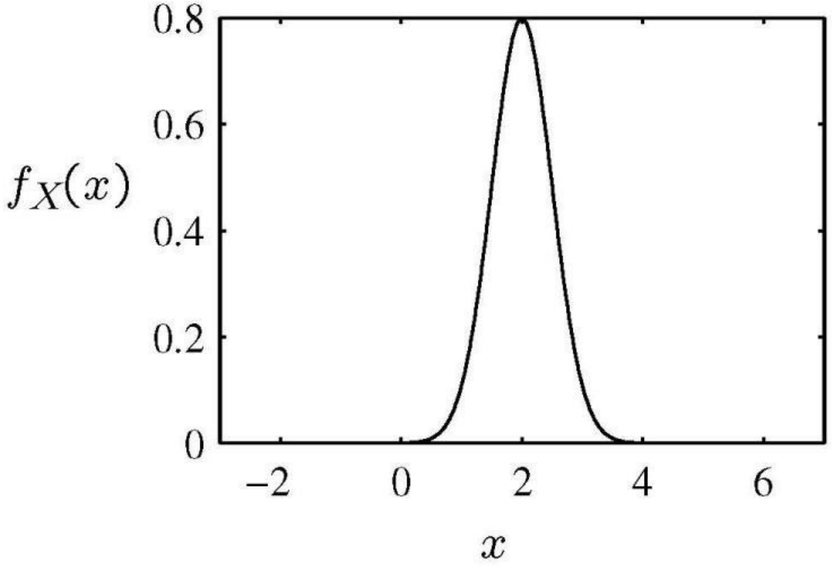
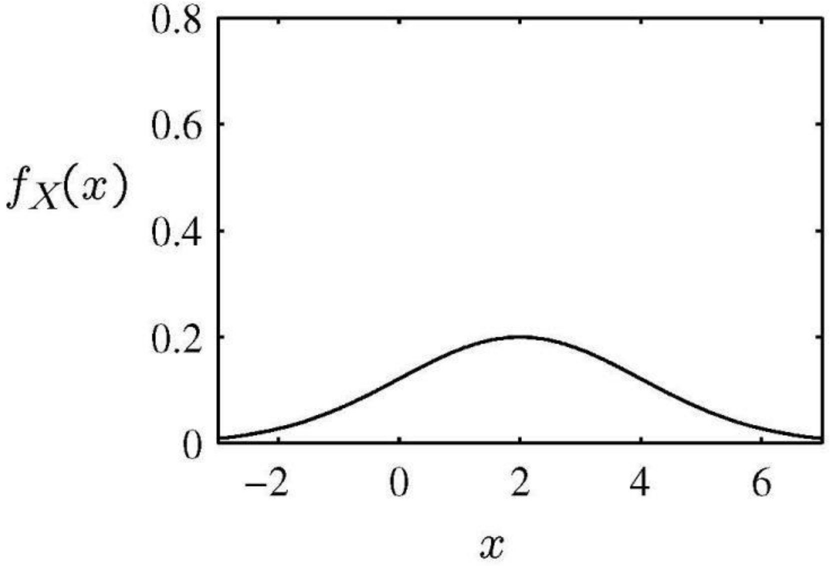


Figure 4.5



(a) $\mu = 2,\ \sigma = 1/2$



(b) $\mu = 2,\ \sigma = 2$

Two examples of a Gaussian random variable X with expected value μ and standard deviation σ.

Theorem 4.12

If X is a Gaussian (μ, σ) random variable,

$$\mathsf{E}[X] = \mu \qquad \mathsf{Var}[X] = \sigma^2.$$

Theorem 4.13

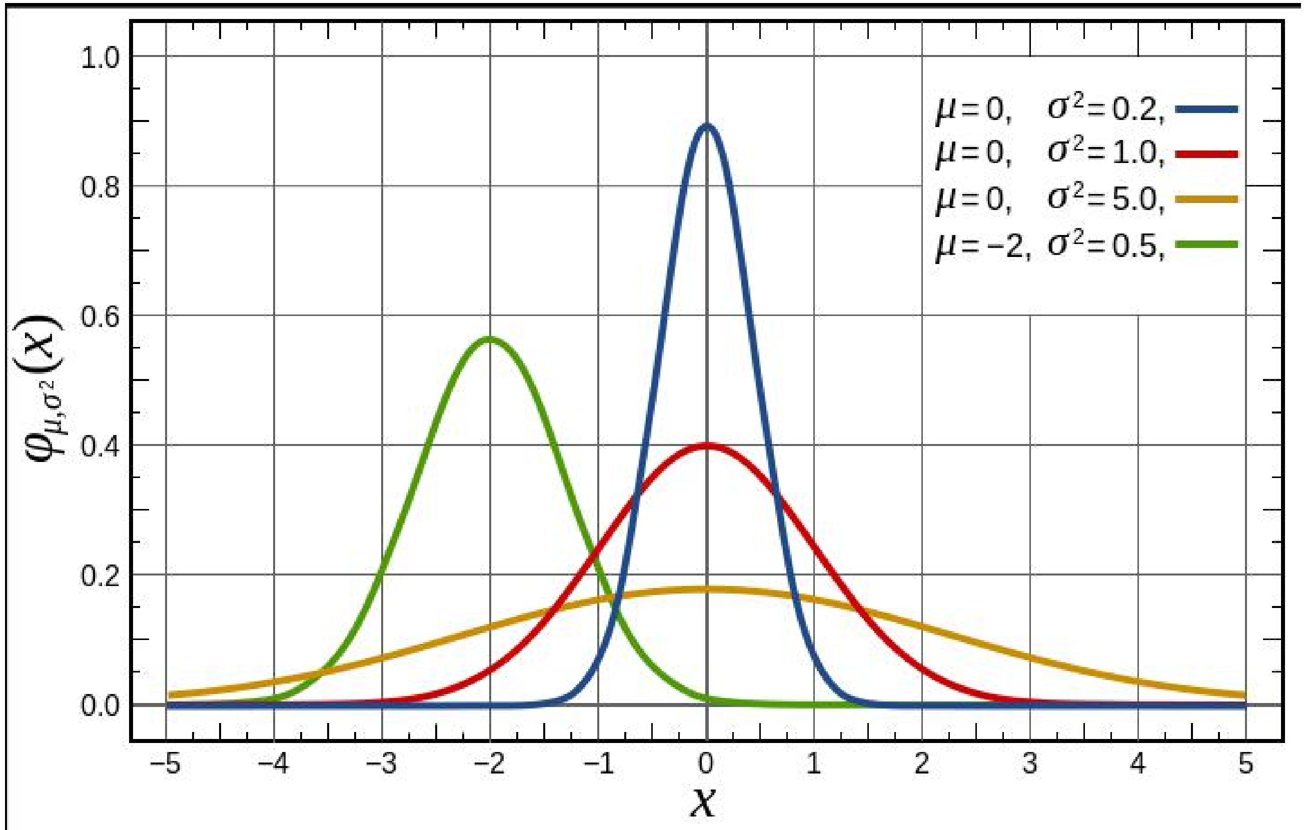
If X is Gaussian (μ, σ), $Y = aX + b$ is Gaussian $(a\mu + b, a\sigma)$.

$$E[Y] = \mathrm{E}[\mathrm{aX} + \mathrm{b}] = \mathrm{a}\mu + b$$

$$Var[Y] = Var[aX + b] = a^2 Var[X] = a^2 \sigma^2$$

Definition 4.9 Standard Normal Random Variable

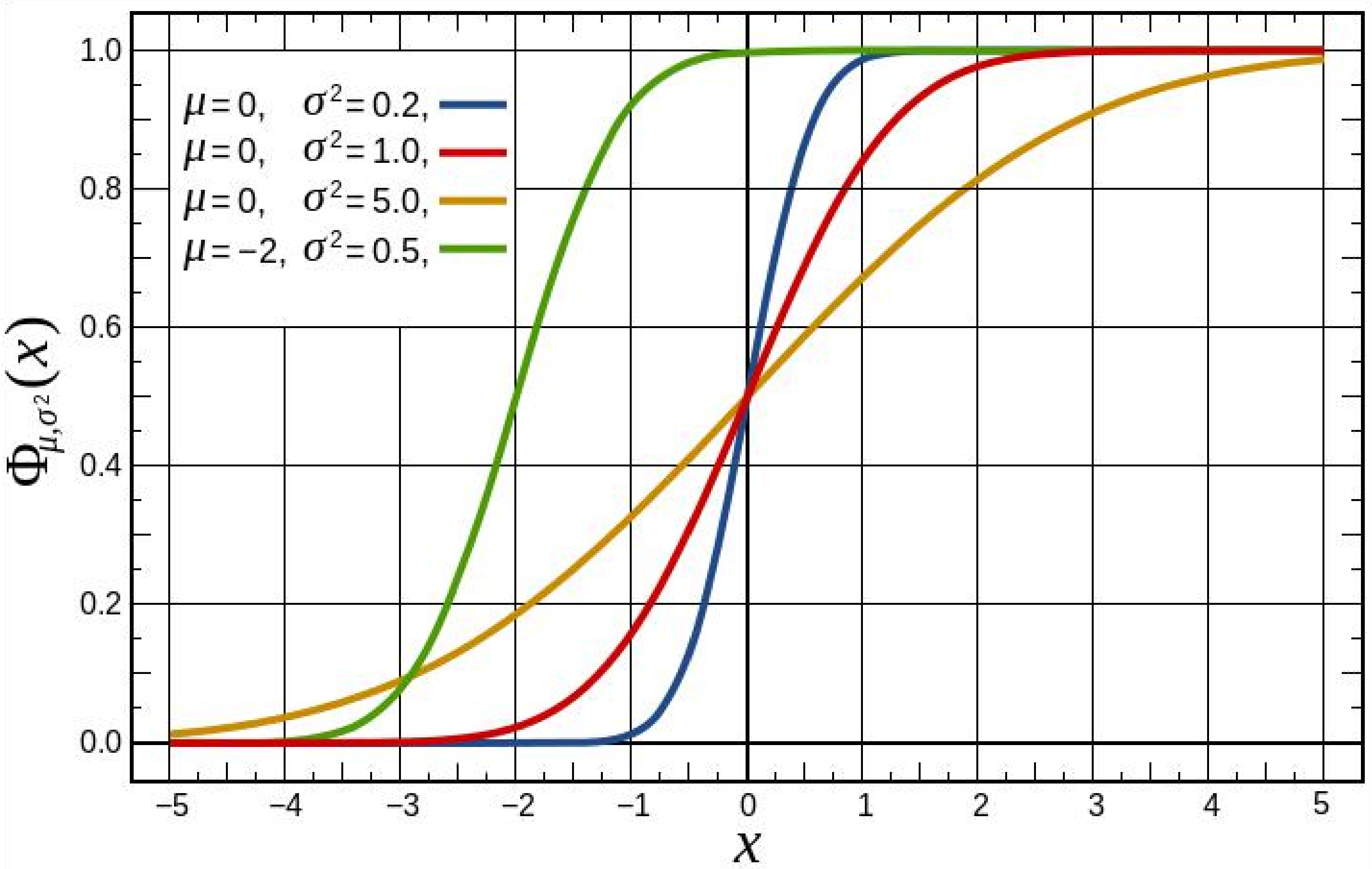
The standard normal random variable Z *is the Gaussian* $(0, 1)$ *random variable.*



Definition 4.10 Standard Normal CDF

The CDF of the standard normal random variable Z is

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$



Theorem 4.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$\mathrm{P}\left[a < X \le b\right] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Example 4.15 Problem

Suppose your score on a test is $x = 46$, a sample value of the Gaussian $(61, 10)$ random variable. Express your test score as a sample value of the standard normal random variable, Z.

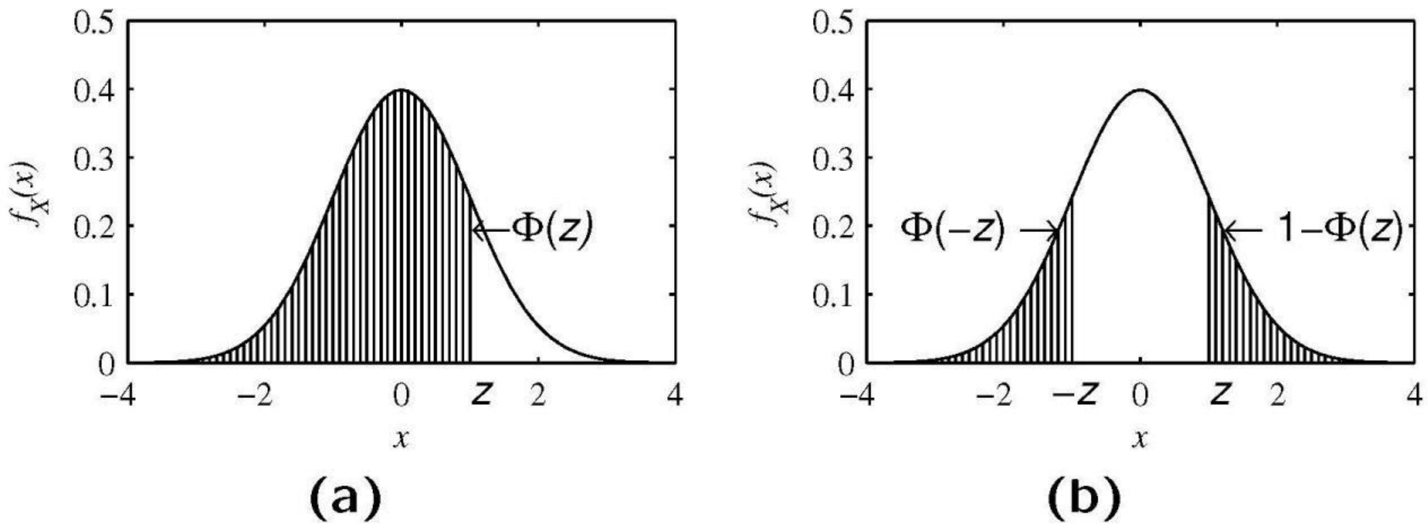
Example 4.15 Solution

Equation (4.50) indicates that $z = (46 - 61)/10 = -1.5$. Therefore your score is 1.5 *standard deviations less than the expected value.*

Theorem 4.15

$$\Phi(-z) = 1 - \Phi(z).$$

Figure 4.6



Symmetry properties of the Gaussian $(0, 1)$ PDF.

Example 4.16 Problem

If X is the Gaussian $(61, 10)$ random variable, what is $\mathrm{P}[X \leq 46]$?

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.2 The standard normal CDF $\Phi(y)$.

z	$\Phi(z)$
1.50	0.9332
1.51	0.9345
1.52	0.9357
1.53	0.9370
1.54	0.9382
1.55	0.9394
1.56	0.9406

Example 4.16 Solution

Applying Theorem 4.14, Theorem 4.15, and the result of Example 4.15, we have

$$\mathrm{P}[X \leq 46] = F_X(46) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067. \quad (1)$$

This suggests that if your test score is 1.5 standard deviations below the expected value, you are in the lowest 6.7% of the population of test takers.

Example 4.17 Problem

If X is a Gaussian ($\mu = 61, \sigma = 10$) random variable, what is P$[51 < X \leq 71]$?

Theorem 4.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$\mathrm{P}\left[a < X \leq b\right] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Example 4.17 **Solution**

Applying Equation (4.50), $Z = (X - 61)/10$ and

$$\{51 < X \leq 71\} = \left\{-1 \leq \frac{X - 61}{10} \leq 1\right\} = \{-1 < Z \leq 1\}. \qquad (1)$$

The probability of this event is

$$\begin{aligned} \mathrm{P}[-1 < Z \leq 1] &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1 = 0.683. \qquad (2) \end{aligned}$$

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.2 The standard normal CDF $\Phi(y)$.

Standard Normal

Definition 4.11 Complementary CDF

The standard normal complementary CDF *is*

$$Q(z) = \mathrm{P}[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du = 1 - \Phi(z).$$

Theorem 4.14

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The probability that X is in the interval $(a, b]$ is

$$\mathrm{P}[a < X \leq b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Example 4.18 Problem

In an optical fiber transmission system, the probability of a bit error is $Q(\sqrt{\gamma/2})$, where γ is the signal-to-noise ratio. What is the minimum value of γ that produces a bit error rate not exceeding 10^{-6}?

Example 4.18 Solution

Referring to Table 4.2, we find that $Q(z) < 10^{-6}$ when $z \geq 4.75$. Therefore, if $\sqrt{\gamma/2} \geq 4.75$, or $\gamma \geq 45$, the probability of error is less than 10^{-6}. Although $10^{(}-6)$ seems a very small number, most practical optical fiber transmission systems have considerably lower binary error rates.

z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$
3.00	$1.35 \cdot 10^{-3}$	3.40	$3.37 \cdot 10^{-4}$	3.80	$7.23 \cdot 10^{-5}$	4.20	$1.33 \cdot 10^{-5}$	4.60	$2.11 \cdot 10^{-6}$
3.01	$1.31 \cdot 10^{-3}$	3.41	$3.25 \cdot 10^{-4}$	3.81	$6.95 \cdot 10^{-5}$	4.21	$1.28 \cdot 10^{-5}$	4.61	$2.01 \cdot 10^{-6}$
3.02	$1.26 \cdot 10^{-3}$	3.42	$3.13 \cdot 10^{-4}$	3.82	$6.67 \cdot 10^{-5}$	4.22	$1.22 \cdot 10^{-5}$	4.62	$1.92 \cdot 10^{-6}$
3.03	$1.22 \cdot 10^{-3}$	3.43	$3.02 \cdot 10^{-4}$	3.83	$6.41 \cdot 10^{-5}$	4.23	$1.17 \cdot 10^{-5}$	4.63	$1.83 \cdot 10^{-6}$
3.04	$1.18 \cdot 10^{-3}$	3.44	$2.91 \cdot 10^{-4}$	3.84	$6.15 \cdot 10^{-5}$	4.24	$1.12 \cdot 10^{-5}$	4.64	$1.74 \cdot 10^{-6}$
3.05	$1.14 \cdot 10^{-3}$	3.45	$2.80 \cdot 10^{-4}$	3.85	$5.91 \cdot 10^{-5}$	4.25	$1.07 \cdot 10^{-5}$	4.65	$1.66 \cdot 10^{-6}$
3.06	$1.11 \cdot 10^{-3}$	3.46	$2.70 \cdot 10^{-4}$	3.86	$5.67 \cdot 10^{-5}$	4.26	$1.02 \cdot 10^{-5}$	4.66	$1.58 \cdot 10^{-6}$
3.07	$1.07 \cdot 10^{-3}$	3.47	$2.60 \cdot 10^{-4}$	3.87	$5.44 \cdot 10^{-5}$	4.27	$9.77 \cdot 10^{-6}$	4.67	$1.51 \cdot 10^{-6}$
3.08	$1.04 \cdot 10^{-3}$	3.48	$2.51 \cdot 10^{-4}$	3.88	$5.22 \cdot 10^{-5}$	4.28	$9.34 \cdot 10^{-6}$	4.68	$1.43 \cdot 10^{-6}$
3.09	$1.00 \cdot 10^{-3}$	3.49	$2.42 \cdot 10^{-4}$	3.89	$5.01 \cdot 10^{-5}$	4.29	$8.93 \cdot 10^{-6}$	4.69	$1.37 \cdot 10^{-6}$
3.10	$9.68 \cdot 10^{-4}$	3.50	$2.33 \cdot 10^{-4}$	3.90	$4.81 \cdot 10^{-5}$	4.30	$8.54 \cdot 10^{-6}$	4.70	$1.30 \cdot 10^{-6}$
3.11	$9.35 \cdot 10^{-4}$	3.51	$2.24 \cdot 10^{-4}$	3.91	$4.61 \cdot 10^{-5}$	4.31	$8.16 \cdot 10^{-6}$	4.71	$1.24 \cdot 10^{-6}$
3.12	$9.04 \cdot 10^{-4}$	3.52	$2.16 \cdot 10^{-4}$	3.92	$4.43 \cdot 10^{-5}$	4.32	$7.80 \cdot 10^{-6}$	4.72	$1.18 \cdot 10^{-6}$
3.13	$8.74 \cdot 10^{-4}$	3.53	$2.08 \cdot 10^{-4}$	3.93	$4.25 \cdot 10^{-5}$	4.33	$7.46 \cdot 10^{-6}$	4.73	$1.12 \cdot 10^{-6}$
3.14	$8.45 \cdot 10^{-4}$	3.54	$2.00 \cdot 10^{-4}$	3.94	$4.07 \cdot 10^{-5}$	4.34	$7.12 \cdot 10^{-6}$	4.74	$1.07 \cdot 10^{-6}$
3.15	$8.16 \cdot 10^{-4}$	3.55	$1.93 \cdot 10^{-4}$	3.95	$3.91 \cdot 10^{-5}$	4.35	$6.81 \cdot 10^{-6}$	4.75	$1.02 \cdot 10^{-6}$
3.16	$7.89 \cdot 10^{-4}$	3.56	$1.85 \cdot 10^{-4}$	3.96	$3.75 \cdot 10^{-5}$	4.36	$6.50 \cdot 10^{-6}$	4.76	$9.68 \cdot 10^{-7}$
3.17	$7.62 \cdot 10^{-4}$	3.57	$1.78 \cdot 10^{-4}$	3.97	$3.59 \cdot 10^{-5}$	4.37	$6.21 \cdot 10^{-6}$	4.77	$9.21 \cdot 10^{-7}$
3.18	$7.36 \cdot 10^{-4}$	3.58	$1.72 \cdot 10^{-4}$	3.98	$3.45 \cdot 10^{-5}$	4.38	$5.93 \cdot 10^{-6}$	4.78	$8.76 \cdot 10^{-7}$
3.19	$7.11 \cdot 10^{-4}$	3.59	$1.65 \cdot 10^{-4}$	3.99	$3.30 \cdot 10^{-5}$	4.39	$5.67 \cdot 10^{-6}$	4.79	$8.34 \cdot 10^{-7}$
3.20	$6.87 \cdot 10^{-4}$	3.60	$1.59 \cdot 10^{-4}$	4.00	$3.17 \cdot 10^{-5}$	4.40	$5.41 \cdot 10^{-6}$	4.80	$7.93 \cdot 10^{-7}$
3.21	$6.64 \cdot 10^{-4}$	3.61	$1.53 \cdot 10^{-4}$	4.01	$3.04 \cdot 10^{-5}$	4.41	$5.17 \cdot 10^{-6}$	4.81	$7.55 \cdot 10^{-7}$
3.22	$6.41 \cdot 10^{-4}$	3.62	$1.47 \cdot 10^{-4}$	4.02	$2.91 \cdot 10^{-5}$	4.42	$4.94 \cdot 10^{-6}$	4.82	$7.18 \cdot 10^{-7}$
3.23	$6.19 \cdot 10^{-4}$	3.63	$1.42 \cdot 10^{-4}$	4.03	$2.79 \cdot 10^{-5}$	4.43	$4.71 \cdot 10^{-6}$	4.83	$6.83 \cdot 10^{-7}$
3.24	$5.98 \cdot 10^{-4}$	3.64	$1.36 \cdot 10^{-4}$	4.04	$2.67 \cdot 10^{-5}$	4.44	$4.50 \cdot 10^{-6}$	4.84	$6.49 \cdot 10^{-7}$
3.25	$5.77 \cdot 10^{-4}$	3.65	$1.31 \cdot 10^{-4}$	4.05	$2.56 \cdot 10^{-5}$	4.45	$4.29 \cdot 10^{-6}$	4.85	$6.17 \cdot 10^{-7}$
3.26	$5.57 \cdot 10^{-4}$	3.66	$1.26 \cdot 10^{-4}$	4.06	$2.45 \cdot 10^{-5}$	4.46	$4.10 \cdot 10^{-6}$	4.86	$5.87 \cdot 10^{-7}$
3.27	$5.38 \cdot 10^{-4}$	3.67	$1.21 \cdot 10^{-4}$	4.07	$2.35 \cdot 10^{-5}$	4.47	$3.91 \cdot 10^{-6}$	4.87	$5.58 \cdot 10^{-7}$
3.28	$5.19 \cdot 10^{-4}$	3.68	$1.17 \cdot 10^{-4}$	4.08	$2.25 \cdot 10^{-5}$	4.48	$3.73 \cdot 10^{-6}$	4.88	$5.30 \cdot 10^{-7}$
3.29	$5.01 \cdot 10^{-4}$	3.69	$1.12 \cdot 10^{-4}$	4.09	$2.16 \cdot 10^{-5}$	4.49	$3.56 \cdot 10^{-6}$	4.89	$5.04 \cdot 10^{-7}$
3.30	$4.83 \cdot 10^{-4}$	3.70	$1.08 \cdot 10^{-4}$	4.10	$2.07 \cdot 10^{-5}$	4.50	$3.40 \cdot 10^{-6}$	4.90	$4.79 \cdot 10^{-7}$
3.31	$4.66 \cdot 10^{-4}$	3.71	$1.04 \cdot 10^{-4}$	4.11	$1.98 \cdot 10^{-5}$	4.51	$3.24 \cdot 10^{-6}$	4.91	$4.55 \cdot 10^{-7}$
3.32	$4.50 \cdot 10^{-4}$	3.72	$9.96 \cdot 10^{-5}$	4.12	$1.89 \cdot 10^{-5}$	4.52	$3.09 \cdot 10^{-6}$	4.92	$4.33 \cdot 10^{-7}$
3.33	$4.34 \cdot 10^{-4}$	3.73	$9.57 \cdot 10^{-5}$	4.13	$1.81 \cdot 10^{-5}$	4.53	$2.95 \cdot 10^{-6}$	4.93	$4.11 \cdot 10^{-7}$
3.34	$4.19 \cdot 10^{-4}$	3.74	$9.20 \cdot 10^{-5}$	4.14	$1.74 \cdot 10^{-5}$	4.54	$2.81 \cdot 10^{-6}$	4.94	$3.91 \cdot 10^{-7}$
3.35	$4.04 \cdot 10^{-4}$	3.75	$8.84 \cdot 10^{-5}$	4.15	$1.66 \cdot 10^{-5}$	4.55	$2.68 \cdot 10^{-6}$	4.95	$3.71 \cdot 10^{-7}$
3.36	$3.90 \cdot 10^{-4}$	3.76	$8.50 \cdot 10^{-5}$	4.16	$1.59 \cdot 10^{-5}$	4.56	$2.56 \cdot 10^{-6}$	4.96	$3.52 \cdot 10^{-7}$
3.37	$3.76 \cdot 10^{-4}$	3.77	$8.16 \cdot 10^{-5}$	4.17	$1.52 \cdot 10^{-5}$	4.57	$2.44 \cdot 10^{-6}$	4.97	$3.35 \cdot 10^{-7}$
3.38	$3.62 \cdot 10^{-4}$	3.78	$7.84 \cdot 10^{-5}$	4.18	$1.46 \cdot 10^{-5}$	4.58	$2.32 \cdot 10^{-6}$	4.98	$3.18 \cdot 10^{-7}$
3.39	$3.49 \cdot 10^{-4}$	3.79	$7.53 \cdot 10^{-5}$	4.19	$1.39 \cdot 10^{-5}$	4.59	$2.22 \cdot 10^{-6}$	4.99	$3.02 \cdot 10^{-7}$

Table 4.3 The standard normal complementary CDF $Q(z)$.

Quiz 4.6

X is the Gaussian $(0,1)$ random variable and Y is the Gaussian $(0,2)$ random variable. Sketch the PDFs $f_X(x)$ and $f_Y(y)$ on the same axes and find:

(a) $P[-1 < X \leq 1]$,

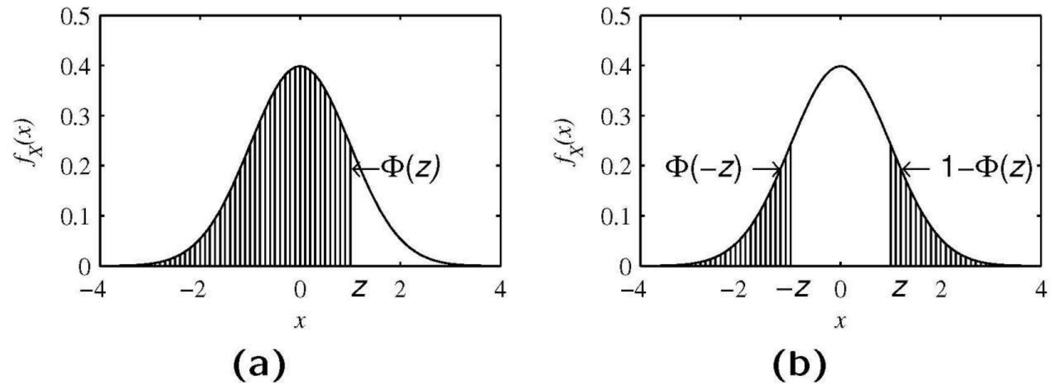
(b) $P[-1 < Y \leq 1]$,

(c) $P[X > 3.5]$,

(d) $P[Y > 3.5]$.

$$P\left[a < X \leq b\right] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

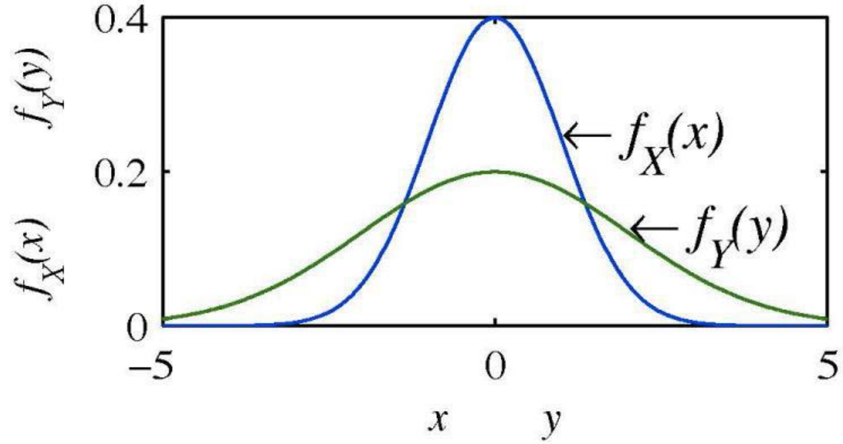
Figure 4.6



Symmetry properties of the Gaussian $(0,1)$ PDF.

Quiz 4.6 Solution

The PDFs of X and Y are:



The fact that Y has twice the standard deviation of X is reflected in the greater spread of $f_Y(y)$. However, it is important to remember that as the standard deviation increases, the peak value of the Gaussian PDF goes down.

Each of the requested probabilities can be calculated using $\Phi(z)$ function and Table 4.2 or $Q(z)$ and Table 4.3.

[Continued]

Quiz 4.6 Solution (Continued 2)

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.2 The standard normal CDF $\Phi(y)$.

(a) Since X is Gaussian $(0,1)$,

$$
\begin{aligned}
\mathrm{P}[-1 < X \le 1] &= F_X(1) - F_X(-1) \\
&= \Phi(1) - \Phi(-1) \\
&= 2\Phi(1) - 1 \\
&= 0.6826. \qquad (1)
\end{aligned}
$$

(b) Since Y is Gaussian $(0,2)$,

$$
\begin{aligned}
\mathrm{P}[-1 < Y \le 1] &= F_Y(1) - F_Y(-1) \\
&= \Phi\left(\frac{1}{\sigma_Y}\right) - \Phi\left(\frac{-1}{\sigma_Y}\right) \\
&= 2\Phi\left(\frac{1}{2}\right) - 1 = 0.383. \qquad (2)
\end{aligned}
$$

(c) Again, since X is Gaussian $(0,1)$, $\mathrm{P}[X > 3.5] = Q(3.5) = 2.33 \times 10^{-4}$.

(d) Since Y is Gaussian $(0,2)$,

$$
\mathrm{P}[Y > 3.5] = Q\left(\frac{3.5}{2}\right) = 1 - \Phi(1.75) = 0.04. \qquad (3)
$$

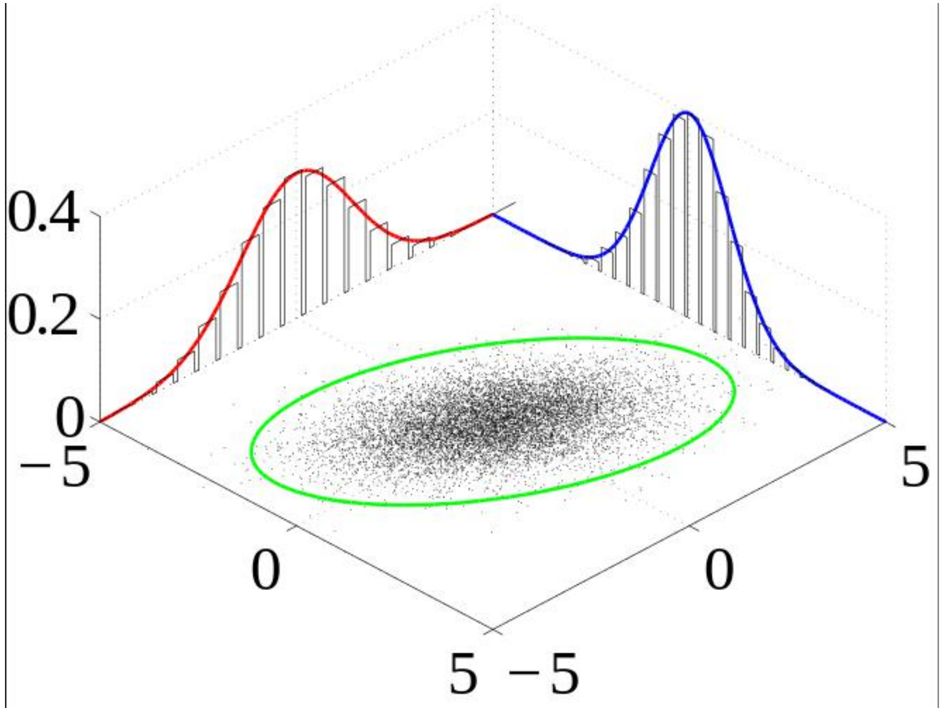
Joint Probability Mass Function

Joint Probability Mass

Definition 5.2 Function (PMF)

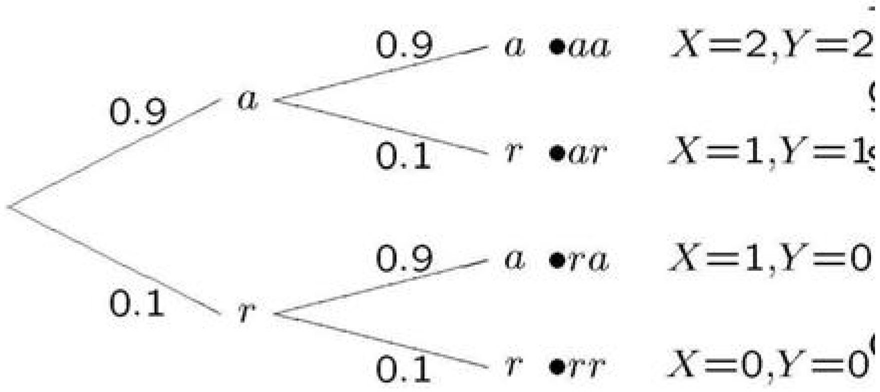
The joint probability mass function *of discrete random variables* X *and* Y *is*

$$P_{X,Y}(x, y) = \mathsf{P}[X = x, Y = y].$$

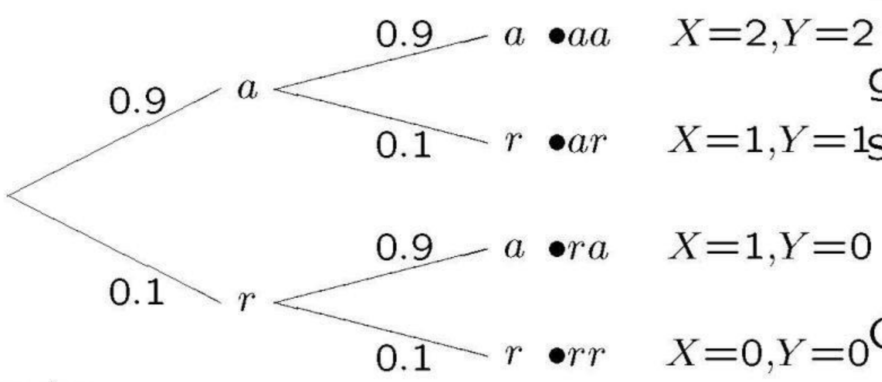


Example 5.3 Problem

Test two integrated circuits one after the other. On each test, the possible outcomes are a (accept) and r (reject). Assume that all circuits are acceptable with probability 0.9 and that the outcomes of successive tests are independent. Count the number of acceptable circuits X and count the number of successful tests Y before you observe the first reject. (If both tests are successful, let $Y = 2$.) Draw a tree diagram for the experiment and find the joint PMF $P_{X,Y}(x, y)$.



Example 5.3 Solution



The experiment has the tree diagram shown to the left. The sample space of the experiment is

$$S = \{aa, ar, ra, rr\}. \tag{1}$$

Observing the tree diagram, we compute

$$\mathsf{P}[aa] = 0.81, \qquad \mathsf{P}[ar] = 0.09, \tag{2}$$

$$\mathsf{P}[ra] = 0.09, \qquad \mathsf{P}[rr] = 0.01. \tag{3}$$

Each outcome specifies a pair of values X and Y. Let $g(s)$ be the function that transforms each outcome s in the sample space S into the pair of random variables (X, Y). Then

$$g(aa) = (2,2), \quad g(ar) = (1,1), \quad g(ra) = (1,0), \quad g(rr) = (0,0). \tag{4}$$

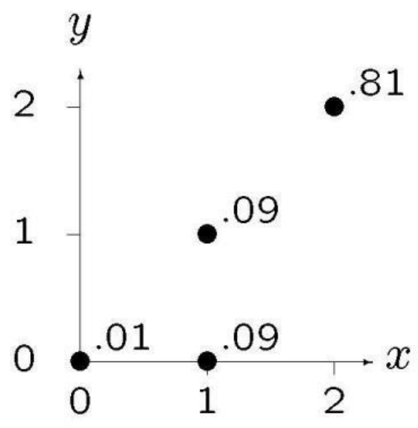
[Continued]

Example 5.3 Solution (Continued 2)

For each pair of values x, y, $P_{X,Y}(x, y)$ is the sum of the probabilities of the outcomes for which $X = x$ and $Y = y$. For example, $P_{X,Y}(1, 1) = \mathrm{P}[ar]$.

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.01	0	0
$x = 1$	0.09	0.09	0
$x = 2$	0	0	0.81

The joint PMF can be represented by the table on left, or, as shown below, as a set of labeled points in the x, y plane where each point is a possible value of the pair (x, y), or as a simple list:

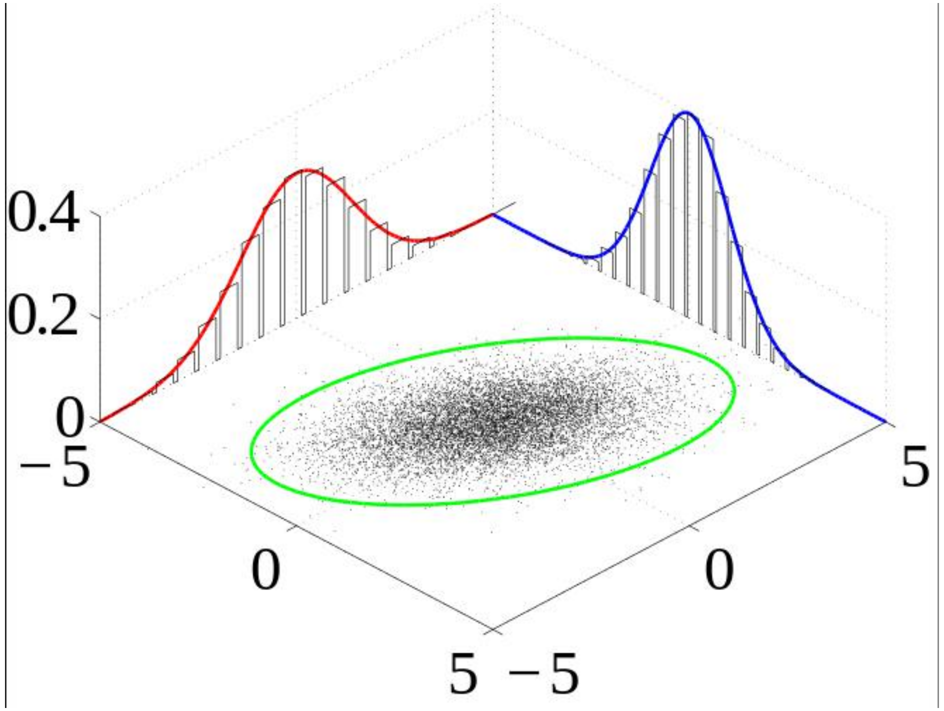


$$P_{X,Y}(x, y) = \begin{cases} 0.81 & x = 2, y = 2, \\ 0.09 & x = 1, y = 1, \\ 0.09 & x = 1, y = 0, \\ 0.01 & x = 0, y = 0. \\ 0 & \text{otherwise} \end{cases}$$

Theorem 5.3

For discrete random variables X and Y and any set B in the X, Y plane, the probability of the event $\{(X, Y) \in B\}$ is

$$\mathsf{P}[B] = \sum_{(x,y) \in B} P_{X,Y}(x, y).$$



Example 5.4 Problem

Continuing Example 5.3, find the probability of the event B that X, the number of acceptable circuits, equals Y, the number of tests before observing the first failure.

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.01	0	0
$x=1$	0.09	0.09	0
$x=2$	0	0	0.81

$$P_{X,Y}(x,y) = \begin{cases} 0.81 & x=2, y=2, \\ 0.09 & x=1, y=1, \\ 0.09 & x=1, y=0, \\ 0.01 & x=0, y=0. \\ 0 & \text{otherwise} \end{cases}$$

Example 5.4 Solution

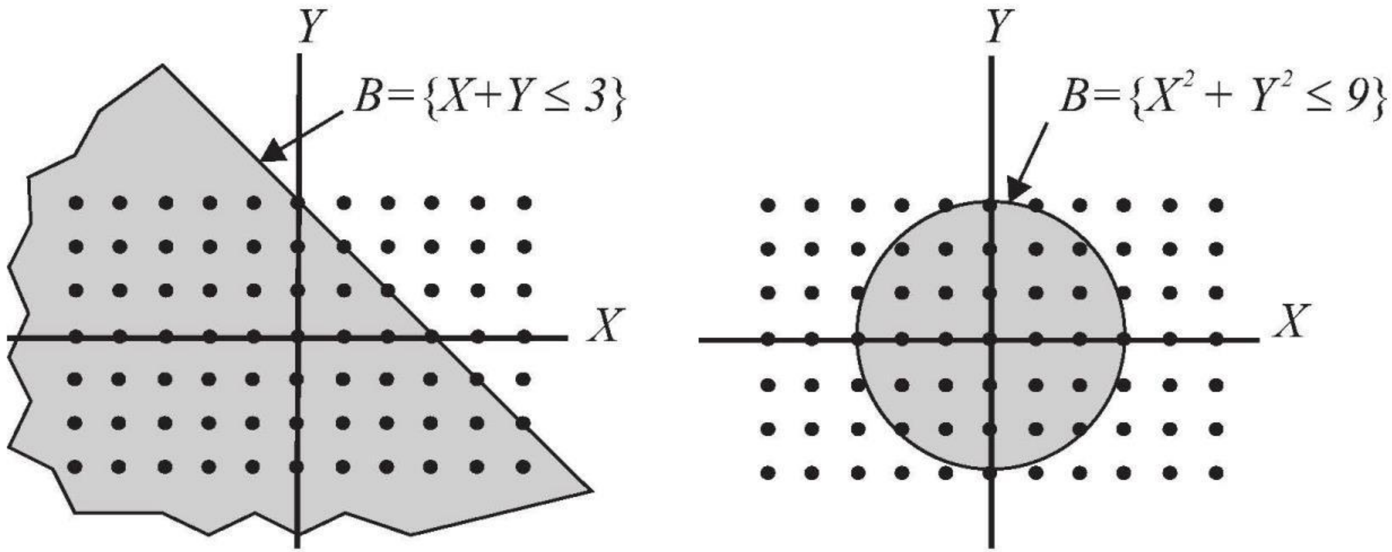
Mathematically, B is the event $\{X = Y\}$. The elements of B with nonzero probability are

$$B \cap S_{X,Y} = \{(0,0), (1,1), (2,2)\}. \tag{1}$$

Therefore,

$$\begin{aligned} \mathrm{P}[B] &= P_{X,Y}(0,0) + P_{X,Y}(1,1) + P_{X,Y}(2,2) \\ &= 0.01 + 0.09 + 0.81 = 0.91. \end{aligned} \tag{2}$$

Figure 5.2



Subsets B of the (X, Y) plane. Points $(X, Y) \in S_{X,Y}$ are marked by bullets.

Quiz 5.2

The joint PMF $P_{Q,G}(q,g)$ for random variables Q and G is given in the following table:

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$
$q=0$	0.06	0.18	0.24	0.12
$q=1$	0.04	0.12	0.16	0.08

Calculate the following probabilities:

(a) $\mathrm{P}[Q=0]$

(b) $\mathrm{P}[Q=G]$

(c) $\mathrm{P}[G>1]$

(d) $\mathrm{P}[G>Q]$

Quiz 5.2 Solution

From the joint PMF of Q and G given in the table, we can calculate the requested probabilities by summing the PMF over those values of Q and G that correspond to the event.

(a) The probability that $Q = 0$ is

$$\begin{aligned} \mathrm{P}[Q = 0] &= P_{Q,G}(0,0) + P_{Q,G}(0,1) + P_{Q,G}(0,2) + P_{Q,G}(0,3) \\ &= 0.06 + 0.18 + 0.24 + 0.12 = 0.6. \end{aligned} \tag{1}$$

(b) The probability that $Q = G$ is

$$\mathrm{P}[Q = G] = P_{Q,G}(0,0) + P_{Q,G}(1,1) = 0.18. \tag{2}$$

(c) The probability that $G > 1$ is

$$\begin{aligned} \mathrm{P}[G > 1] &= \sum_{g=2}^{3} \sum_{q=0}^{1} P_{Q,G}(q,g) \\ &= 0.24 + 0.16 + 0.12 + 0.08 = 0.6. \end{aligned} \tag{3}$$

(d) The probability that $G > Q$ is

$$\begin{aligned} \mathrm{P}[G > Q] &= \sum_{q=0}^{1} \sum_{g=q+1}^{3} P_{Q,G}(q,g) \\ &= 0.18 + 0.24 + 0.12 + 0.16 + 0.08 = 0.78. \end{aligned} \tag{4}$$