Practices for Discrete Random Variable

1. A six-sided die is rolled and observe the roll. Let N_i denote the event that the roll is i. Let O denote the roll of the die is odd-numbered. Let D_j denote the event that the roll is less than j.

a. What is the sample space? Determine the elements of the events O and D_5.
b. What is $P(N_3|D_5)$, the conditional probability that 3 is rolled given that the roll is less than 5?
c. What is the conditional probability that the roll is greater than 3 given that the roll is odd?
d. What is $P(D_4|O)$, the conditional probability that the roll is less than 4 given that the roll is odd?
e. What is $P(O| D_4)$, the conditional probability that the roll is odd given it is less than 4.

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e. What is $P(O| D_4)$, the conditional probability that the roll is odd given it is less than 4.

a. $S = \{N_1, N_2, N_3, N_4, N_5, N_6\}$

$O = \{N_1, N_3, N_5\}$

$D_5 = \{N_1, N_2, N_3, N_4\}$

b. $P[N_3 | D_5] = \frac{P[N_3 \cap D_5]}{P[D_5]} = \frac{1/6}{4/6} = 1/4$

c $P = \frac{P[\bar{N_4} \cap O]}{P[O]} = \frac{1/6}{3/6} = 1/3$

d $P[D_4 | O] = \frac{P[D_4 \cap O]}{P[O]} = \frac{P[\{N_1 \cap N_3\}]}{3/6}$

$= \frac{2/6}{3/6} = 2/3$

e $P[O | D_4] = \frac{P[D_4 \cap O]}{P[D_4]} = \frac{2/6}{3/6} = 2/3$

2. A company has three machines A_1, A_2, and A_3 making 1kΩ resistors. Resistors within 50Ω of the nominal value are considered acceptable. It has been observed that 70% of the resistors produced by A_1, 80% by A_2 and 60% by A_3 are acceptable. Each hour, machine A_1, A_2, and A_3 produce 3000, 2000, 5000 resistors, respectively. All of the resistors are mixed together at random in one bin and packed for shipment.

a. What is the probability that the company ships an UNACCEPTABLE resistor?
b. What is the probability that the unacceptable resistor comes from machine 3?

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a. What is the probability that the company ships an UNACCEPTABLE resistor?
b. What is the probability that the unacceptable resistor comes from machine 3?

Let A & U represent the event that the resistor is acceptable & unacceptable, respectively

$$P[U] = P[U|A_1]P[A_1] + P[U|A_2]P[A_2] + P[U|A_3]P[A_3]$$

$$= 0.3 \times 0.3 + 0.2 \times 0.2 + 0.5 \times 0.4$$

$$= 0.09 + 0.04 + 0.2$$

$$= 0.33$$

$$P[A_3|U] = \frac{P[U|A_3] \times P[A_3]}{P[U]}$$

$$= \frac{0.4 \times 0.5}{0.33} = \frac{20}{33}$$

3. Determine the probability models that can be applied to analyze the following real world scenarios:

   a. The number of packages received by a router in 30 seconds.
   b. The number of dealers you need to visit until you find the dream car.
   c. The number of students who get A in a course with 43 registered students.

Candidate models: Bernoulli, Binomial, Geometric, Pascal, Poisson, Uniform distribution.

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a. Poisson
b. Geometric
c. Binomial

4. The random variable V has the PMF

$$P_V(v) = \begin{cases} cv^2 & v = 1,2,3,4 \\ 0 & otherwise \end{cases}$$

a. Determine c.
b. Find the probability that V is even.
c. Find P[V>2].
d. Find E[V].
e. Given $w = 2v + 1$, find $P_W(w)$ and E[W].

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a. $\because \sum_{v=1}^{4} P_V(v) = 1$

$\therefore c(1+4+9+16) = 1$

$\Rightarrow c = \frac{1}{30}$

(b) $P[V \text{ is even}] = P_V(2) + P_V(4)$

$= \frac{4}{30} + \frac{16}{30}$

$= \frac{2}{3}$

(c) $P[V>2] = P[V=3] + P[V=4]$

$= \frac{9}{30} + \frac{16}{30}$

$= \frac{25}{30} = \frac{5}{6}$

(d) $E[V] = \frac{1}{30} \times 1 + \frac{4}{30} \times 2 + \frac{9}{30} \times 3 + \frac{16}{30} \times 4 = \frac{100}{30}$

(e) $$P_W(w) = \begin{cases} \frac{1}{30} & w = 3 \\ \frac{4}{30} & w = 5 \\ \frac{9}{30} & w = 7 \\ \frac{16}{30} & w = 9 \end{cases}$$

$$E[W] = \frac{1}{30}(3 + 20 + 63 + 144) = \frac{230}{30} = 2E[V] + 1$$

5. In a wireless automatic meter reading system, a base station sends out a wake-up signal to nearby electric meters. On hearing the wake-up signal, a meter transmits a message indicating the electric usage. Each message is repeated eight times.

a. If a single transmission of a message is successful with probability p, what is the PMF of N, the number of successful message transmissions?

b. I is an indicator random variable such that $I = 1$, if at least one message is transmitted successfully; otherwise $I = 0$. Find the PMF of I and $E[I]$.

5. In a wireless automatic meter reading system, a base station sends out a wake-up signal to nearby electric meters. On hearing the wake-up signal, a meter transmits a message indicating the electric usage. Each message is repeated eight times.

a. If a single transmission of a message is successful with probability *p*, what is the PMF of *N*, the number of successful message transmissions?

b. *I* is an indicator random variable such that *I* = 1, if at least one message is transmitted successfully; otherwise *I* = 0. Find the PMF of *I* and *E[I]*.

a. N is a Binomial r.v.

$$P_N(n) = \binom{8}{n} p^n (1-p)^{8-n}, \quad n = 0, 1, \cdots, 8$$

b. The sample space of I is {0, 1}.

Because I = 0 if and only if N = 0, then

$P[I=0] = P[N=0] = (1-p)^8$

Then $P[I=1] = 1 - P[I=0] = 1 - (1-p)^8$

The complete PMF is

$$P_I(i) = \begin{cases} (1-p)^8 & i = 0 \\ 1-(1-p)^8 & i = 1 \end{cases}$$

6. A packet received by your smart phone is error-free with probability 0.8, independent of any other packet.

   a. Out of 10 packets received, let X equal the number of packets received with errors. What is the PMF of X?
   b. What is the probability that there are more than 2 packets received with errors?

6. A packet received by your smart phone is error-free with probability 0.8, independent of any other packet.

a. Out of 10 packets received, let *X* equal the number of packets received with errors. What is the PMF of *X*?

b. What is the probability that there are more than 2 packets received with errors?

a.

$$P_X(x) = \binom{10}{x}(0.2)^x(0.8)^{10-x}$$

b.

$$P[x > 2] = 1 - P[0] - P[1] - P[2]$$

$$= 1 - \binom{10}{0}(0.2)^0(0.8)^{10} - \binom{10}{1}(0.2)^1(0.8)^9 - \binom{10}{2}(0.2)^2(0.8)^8$$

7. The number of database queries processed by a computer in any 10-second interval is 5. What is the probability that there will be no queries processed in a 10-second interval? What is the probability that two queries will be processed in a 2-second interval?

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a) $P[K=0] = 5^0 e^{-5}/0! = e^{-5}$

b) $\alpha' = 5/5 = 1$ query/2-s

$P[K'=2] = 1^2 e^{-1}/2! = \frac{e^{-1}}{2}$

Section 4.4

Expected Values

Definition 4.4 Expected Value

The expected value *of a continuous random variable* X *is*

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

For discrete random variable X

$$E[X] = \sum_{x \in S_x} x P_X(x)$$

Example 4.6 Problem

In Example 4.4, we found that the stopping point X of the spinning wheel experiment was a uniform random variable with PDF

$$f_X(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Find the expected stopping point $\mathrm{E}[X]$ of the pointer.

Example 4.6 Solution

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\, dx = 1/2 \text{ meter.} \tag{1}$$

With no preferred stopping points on the circle, the average stopping point of the pointer is exactly halfway around the circle.

Example 4.7

In Example 4.5, find the expected value of the maximum stopping point Y of the three spins:

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 3y^2 & 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 y(3y^2)\, dy = 3/4 \text{ meter.}$$

Theorem 4.4

The expected value of a function, $g(X)$, of random variable X is

$$\mathsf{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

For discrete random variable

$$E[g(x)] = \sum_{x \in S_X} g(x) P_X(x)$$

Theorem 4.5

For any random variable X,

(a) $\mathsf{E}[X - \mu_X] = 0$,

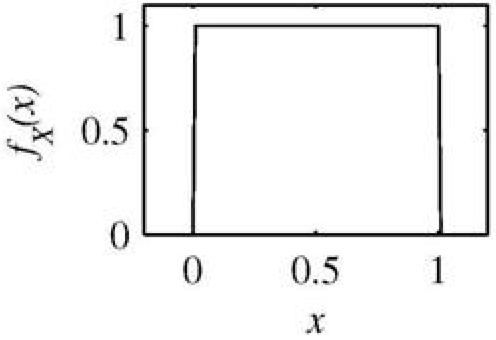
(b) $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$,

(c) $\mathsf{Var}[X] = \mathsf{E}[X^2] - \mu_X^2$,

(d) $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

Example 4.9 Problem

Find the variance and standard deviation of the pointer position in Example 4.1.



$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2,$$

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

$$\mathrm{E}\left[X\right] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Example 4.9 Solution

To compute $\text{Var}[X]$, we use Theorem 4.5(c): $\text{Var}[X] = \text{E}[X^2] - \mu_X^2$. We calculate $\text{E}[X^2]$ directly from Theorem 4.4 with $g(X) = X^2$:

$$\text{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\, dx = 1/3 \text{ m}^2. \tag{1}$$

In Example 4.6, we have $\text{E}[X] = 1/2$. Thus $\text{Var}[X] = 1/3 - (1/2)^2 = 1/12$, and the standard deviation is $\sigma_X = \sqrt{\text{Var}[X]} = 1/\sqrt{12} = 0.289$ meters.

Example 4.10 Problem

Find the variance and standard deviation of Y, the maximum pointer position after three spins, in Example 4.5.

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 3y^2 & 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2,$$

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

$$\mathrm{E}\left[X\right] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Example 4.10 Solution

We proceed as in Example 4.9. We have $f_Y(y)$ from Example 4.5 and $\mathsf{E}[Y] = 3/4$ from Example 4.7:

$$\mathsf{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 y^2 \left(3y^2\right)\, dy = 3/5 \text{ m}^2. \tag{1}$$

Thus the variance is

$$\mathsf{Var}[Y] = 3/5 - (3/4)^2 = 3/80 \text{ m}^2, \tag{2}$$

and the standard deviation is $\sigma_Y = 0.194$ meters.

Quiz 4.4

The probability density function of the random variable Y is

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Sketch the PDF and find the following:

(a) the expected value $\mathsf{E}[Y]$

(b) the second moment $\mathsf{E}[Y^2]$

(c) the variance $\mathsf{Var}[Y]$

(d) the standard deviation σ_Y

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Sketch the PDF and find the following:

(a) the expected value $\mathrm{E}[Y]$ $$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

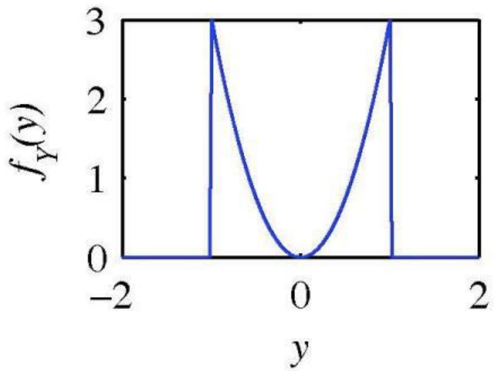
(b) the second moment $\mathrm{E}[Y^2]$ $$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

(c) the variance $\mathrm{Var}[Y]$ $$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2,$$

(d) the standard deviation σ_Y

Quiz 4.4 Solution

The PDF of Y is



$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The expected value of Y is

$$\mathsf{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^3\, dy = (3/8) y^4 \Big|_{-1}^{1} = 0. \tag{2}$$

Note that the above calculation wasn't really necessary because $\mathsf{E}[Y] = 0$ whenever the PDF $f_Y(y)$ is an even function, i.e., $f_Y(y) = f_Y(-y)$.

(b) The second moment of Y is

$$\mathsf{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_{-1}^{1} (3/2) y^4\, dy = (3/10) y^5 \Big|_{-1}^{1} = 3/5. \tag{3}$$

(c) The variance of Y is

$$\text{Var}[Y] = \mathsf{E}\left[Y^2\right] - (\mathsf{E}[Y])^2 = 3/5. \tag{4}$$

(d) The standard deviation of Y is $\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}$.