Definition 5.7 Correlation

The correlation *of* X *and* Y *is* $r_{X,Y} = \mathrm{E}[XY]$

Theorem 5.16

(a) $\mathrm{Cov}[X, Y] = r_{X,Y} - \mu_X \mu_Y$.

(b) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]$.

(c) If $X = Y$, $\mathrm{Cov}[X, Y] = \mathrm{Var}[X] = \mathrm{Var}[Y]$ and $r_{X,Y} = \mathrm{E}[X^2] = \mathrm{E}[Y^2]$.

Quiz 5.8(A)

Random variables L and T have joint PMF

$P_{L,T}(l,t)$	$t = 40\,\text{sec}$	$t = 60\,\text{sec}$
$l = 1$ page	0.15	0.1
$l = 2$ pages	0.30	0.2
$l = 3$ pages	0.15	0.1.

Find the following quantities.

(a) E$[L]$ and Var$[L]$

(b) E$[T]$ and Var$[T]$

(c) The covariance Cov$[L, T]$

(d) The correlation coefficient $\rho_{L,T}$

Quiz 5.8(A) Solution

It is helpful to first make a table that includes the marginal PMFs.

$P_{L,T}(l,t)$	$t=40$	$t=60$	$P_L(l)$
$l=1$	0.15	0.1	0.25
$l=2$	0.3	0.2	0.5
$l=3$	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	

(a) The expected value of L is

$$\mathrm{E}[L] = 1(0.25) + 2(0.5) + 3(0.25) = 2. \tag{1}$$

Since the second moment of L is

$$\mathrm{E}\left[L^2\right] = 1^2(0.25) + 2^2(0.5) + 3^2(0.25) = 4.5, \tag{2}$$

the variance of L is

$$\mathrm{Var}[L] = \mathrm{E}\left[L^2\right] - (\mathrm{E}[L])^2 = 0.5. \tag{3}$$

(b) The expected value of T is

$$\mathrm{E}[T] = 40(0.6) + 60(0.4) = 48. \tag{4}$$

The second moment of T is

$$\mathrm{E}\left[T^2\right] = 40^2(0.6) + 60^2(0.4) = 2400. \tag{5}$$

[Continued]

Quiz 5.8(A) Solution (Continued 2)

Thus

$$\text{Var}[T] = \text{E}\left[T^2\right] - (\text{E}\left[T\right])^2 = 96. \tag{6}$$

(a) First we need to find

$$\begin{aligned}
\text{E}\left[LT\right] &= \sum_{t=40,60} \sum_{l=1}^{3} lt P_{LT}(lt) \\
&= 1(40)(0.15) + 2(40)(0.3) + 3(40)(0.15) \\
&\quad + 1(60)(0.1) + 2(60)(0.2) + 3(60)(0.1) \\
&= 96.
\end{aligned} \tag{7}$$

The covariance of L and T is

$$\text{Cov}\left[L, T\right] = \text{E}\left[LT\right] - \text{E}\left[L\right] \text{E}\left[T\right] = 96 - 2(48) = 0. \tag{8}$$

(b) Since $\text{Cov}[L, T] = 0$, the correlation coefficient is $\rho_{L,T} = 0$.

Quiz 5.8(B)

The joint probability density function of random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1, 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Find the following quantities.

(a) $\mathsf{E}[X]$ and $\mathsf{Var}[X]$

(b) $\mathsf{E}[Y]$ and $\mathsf{Var}[Y]$

(c) The covariance $\mathsf{Cov}[X,Y]$

(d) The correlation coefficient $\rho_{X,Y}$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy$$

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\ dx$$

$$\mathsf{Cov}[X,Y] = \mathsf{E}[XY] - \mathsf{E}[X]\,\mathsf{E}[Y]$$

$$\mathsf{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\ dx, dy$$

Quiz 5.8(B) Solution

As in the discrete case, the calculations become easier if we first calculate the marginal PDFs $f_X(x)$ and $f_Y(y)$. For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^2 xy\, dy = \frac{1}{2}xy^2\Big|_{y=0}^{y=2} = 2x. \tag{1}$$

Similarly, for $0 \leq y \leq 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^2 xy\, dx = \frac{1}{2}x^2y\Big|_{x=0}^{x=1} = \frac{y}{2}. \tag{2}$$

The complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

From the marginal PDFs, it is straightforward to calculate the various expectations.

(a) The first and second moments of X are

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}. \tag{4}$$

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 2x^3\, dx = \frac{1}{2}. \tag{5}$$

[Continued]

Quiz 5.8(B) Solution (Continued 2)

The variance of X is

$$\text{Var}[X] = \text{E}[X^2] - (\text{E}[X])^2 = \frac{1}{18}.$$

(a) The first and second moments of Y are

$$\text{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^2 \frac{1}{2} y^2\, dy = \frac{4}{3}, \tag{6}$$

$$\text{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^2 \frac{1}{2} y^3\, dy = 2. \tag{7}$$

The variance of Y is

$$\text{Var}[Y] = \text{E}\left[Y^2\right] - (\text{E}[Y])^2 = 2 - \frac{16}{9} = \frac{2}{9}. \tag{8}$$

(b) We start by finding

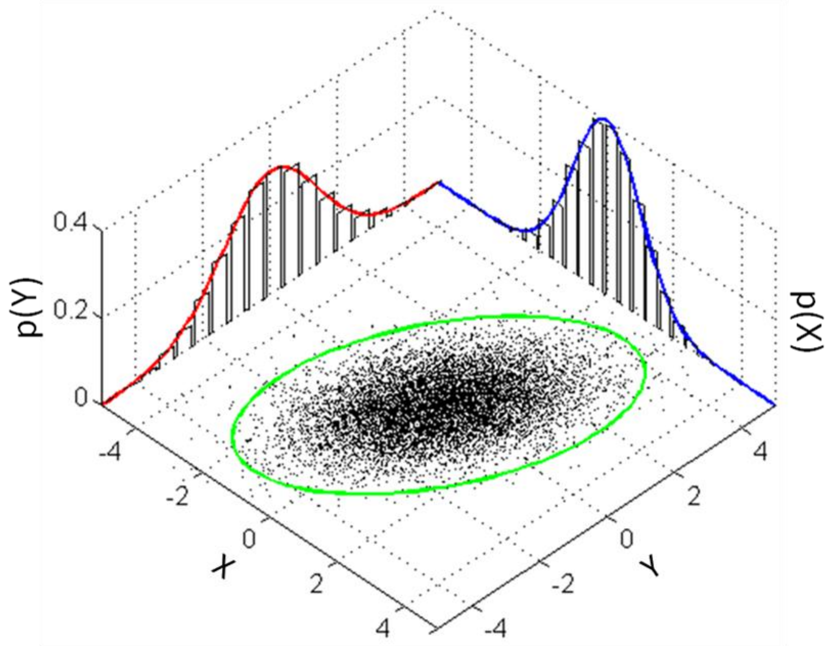
$$\begin{aligned} \text{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\, dx, dy \\ &= \int_0^1\int_0^2 x^2y^2\, dx, dy = \left.\frac{x^3}{3}\right|_0^1 \left.\frac{y^3}{3}\right|_0^2 = \frac{8}{9}. \end{aligned} \tag{9}$$

The covariance of X and Y is then

$$\text{Cov}[X,Y] = \text{E}[XY] - \text{E}[X]\,\text{E}[Y] = \frac{8}{9} - \frac{2}{3}\cdot\frac{4}{3} = 0. \tag{10}$$

(c) Since $\text{Cov}[X,Y] = 0$, the correlation coefficient is $\rho_{X,Y} = 0$.

Bivariate Gaussian Random Variables



Definition 5.10 Bivariate Gaussian Random Variables

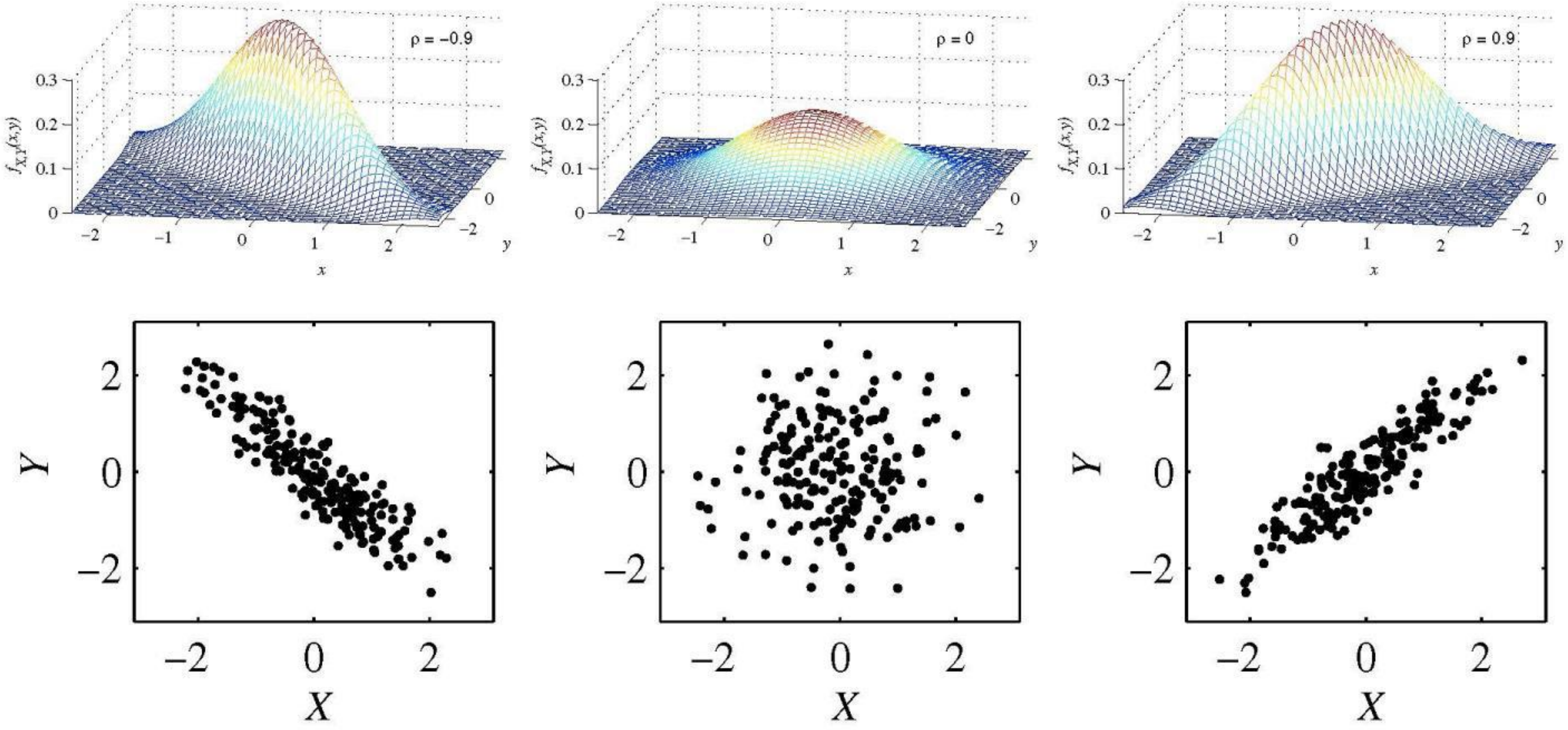
Random variables X *and* Y *have a* bivariate Gaussian PDF *with parameters* μ_X, μ_Y, $\sigma_X > 0$, $\sigma_Y > 0$, *and* $\rho_{X,Y}$ *satisfying* $-1 < \rho_{X,Y} < 1$ *if*

$$f_{X,Y}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{X,Y}^2)}\right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}},$$

$$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}e^{-(x-\mu_X)^2/2\sigma_X^2}, \qquad f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}}e^{-(y-\mu_Y)^2/2\sigma_Y^2}.$$

$$\rho_{X,Y} = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$$

Figure 5.6



The Joint Gaussian PDF $f_{X,Y}(x,y)$ for $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$, and three values of $\rho_{X,Y} = \rho$. Next to each PDF, we plot 200 sample pairs (X,Y) generated with that PDF.

Theorem 5.21

If X and Y are bivariate Gaussian random variables with PDF given by Definition 5.10, and W_1 and W_2 are given by the linearly independent equations

$$W_1 = a_1 X + b_1 Y, \qquad W_2 = a_2 X + b_2 Y,$$

then W_1 and W_2 are bivariate Gaussian random variables such that

$$\begin{aligned}
\mathsf{E}[W_i] &= a_i \mu_X + b_i \mu_Y, && i = 1, 2,\\
\mathsf{Var}[W_i] &= a_i^2 \sigma_X^2 + b_i^2 \sigma_Y^2 + 2 a_i b_i \rho_{X,Y} \sigma_X \sigma_Y, && i = 1, 2,\\
\mathsf{Cov}[W_1, W_2] &= a_1 a_2 \sigma_X^2 + b_1 b_2 \sigma_Y^2 + (a_1 b_2 + a_2 b_1) \rho_{X,Y} \sigma_X \sigma_Y.
\end{aligned}$$

Example 5.19 Problem

For the noisy observation in Example 5.14, find the PDF of $Y = X + Z$. X is Gaussian $(0, \sigma_X)$ and Z is Gaussian $(0, \sigma_Z)$

Example 5.19 Solution

Since X is Gaussian $(0, \sigma_X)$ and Z is Gaussian $(0, \sigma_Z)$ and X and Z are independent, X and Z are jointly Gaussian. It follows from Theorem 5.21 that Y is Gaussian with $\mathrm{E}[Y] = \mathrm{E}[X] + \mathrm{E}[Z] = 0$ and variance $\sigma_Y^2 = \sigma_X^2 + \sigma_Z^2$. The PDF of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi(\sigma_X^2 + \sigma_Z^2)}} e^{-y^2/2(\sigma_X^2+\sigma_Z^2)}. \tag{1}$$

Example 5.20 Problem

Continuing Example 5.19, find the joint PDF of X and Y when $\sigma_X = 4$ and $\sigma_Z = 3$. $Y = X + Z$.

$$f_{X,Y}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{X,Y}^2)}\right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}},$$

$$\mu_X = \mu_Y = 0$$

$$\sigma_Y^2 = \sigma_X^2 + \sigma_Z^2$$

$$\boxed{{\sigma_X}^2 = E[X^2] - (E[X])^2}$$

$$\boxed{\mathrm{E}[XZ] = \mathrm{E}[X]\,\mathrm{E}[Z] = 0.}$$

$$\rho_{X,Y} = \frac{E[XY] - \mu_X\mu_Y}{\sigma_X\sigma_Y} = \frac{E[X(X+Z)] - \mu_X\mu_Y}{\sigma_X\sigma_Y} = \frac{E[X^2] + E[XZ] - \mu_X\mu_Y}{\sigma_X\sigma_Y}$$

Example 5.20 Solution

From Theorem 5.21, we know that X and Y are bivariate Gaussian. We also know that $\mu_X = \mu_Y = 0$ and that Y has variance $\sigma_Y^2 = \sigma_X^2 + \sigma_Z^2 = 25$. Substituting $\sigma_X = 4$ and $\sigma_Z = 3$ in the formula for the correlation coefficient derived in Example 5.18, we have

$$\rho_{X,Y} = \sqrt{\frac{\sigma_X^2/\sigma_Z^2}{1+\sigma_X^2/\sigma_Z^2}} = \frac{4}{5}. \tag{1}$$

Applying these parameters to Definition 5.10, we obtain

$$f_{X,Y}(x,y) = \frac{1}{24\pi} e^{-\left(25x^2/16 - 2xy + y^2\right)/18}. \tag{2}$$

Quiz 5.9

Let X and Y be jointly Gaussian $(0,1)$ random variables with correlation coefficient $1/2$. What is the joint PDF of X and Y?

$$f_{X,Y}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{2\rho_{X,Y}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{X,Y}^2)}\right]}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}},$$

Quiz 5.9 Solution

This problem just requires identifying the various parameters in Definition 5.10. Specifically, from the problem statement, we know $\rho = 1/2$ and

$$\mu_X = 0, \qquad \mu_Y = 0,$$
$$\sigma_X = 1, \qquad \sigma_Y = 1.$$

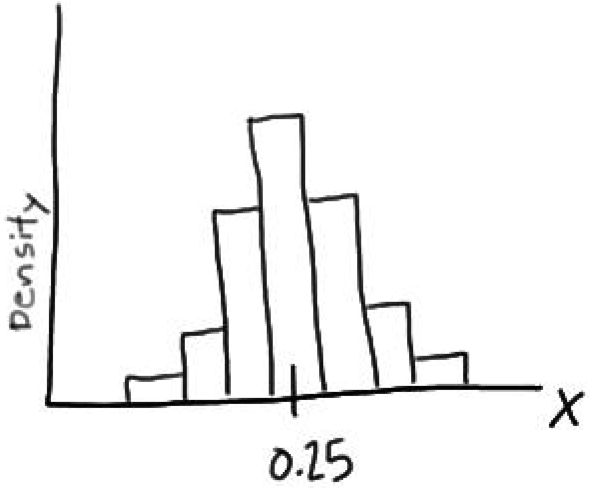
Applying these facts to Definition 5.10, we have

$$f_{X,Y}(x,y) = \frac{e^{-2(x^2 - xy + y^2)/3}}{\sqrt{3\pi^2}}. \tag{1}$$

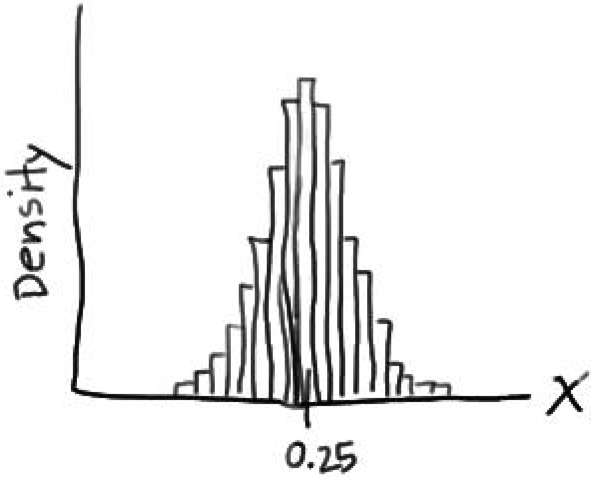
Review of Section 4:
Continuous Random Variable

- Probability Density Function (PDF) & Cumulative Distribution Function (CDF)
- Expected Value, Variance & Standard Deviation)
- Uniform and Gaussian Random Variable

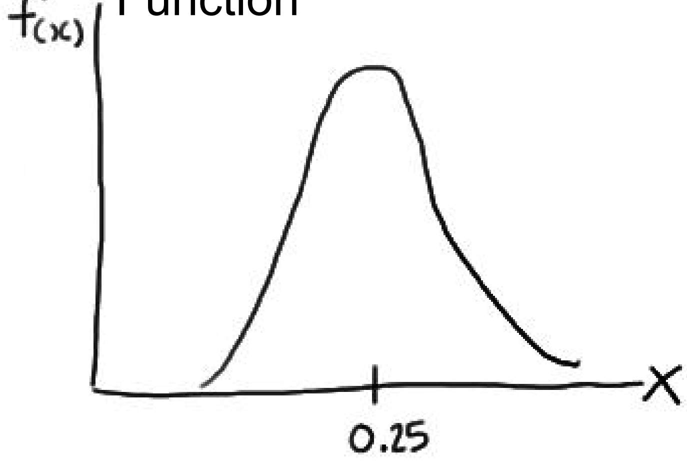
Probability Mass Function



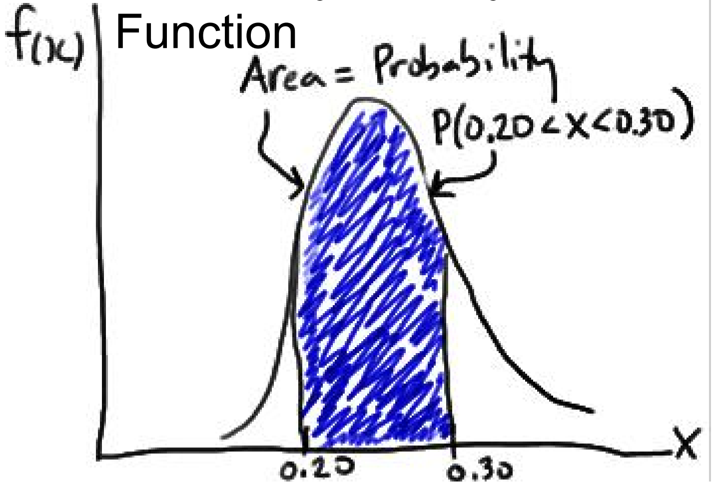
Probability Mass Function



Probability Density Function



Probability Density Function



The cumulative distribution function *(CDF) of random variable X is*

$$F_X(x) = \mathrm{P}\left[X \leq x\right].$$

For any random variable X,

(a) $F_X(-\infty) = 0$

(b) $F_X(\infty) = 1$

(c) $\mathrm{P}[x_1 < X \leq x_2] \quad = \quad F_X(x_2) \; - \; F_X(x_1)$

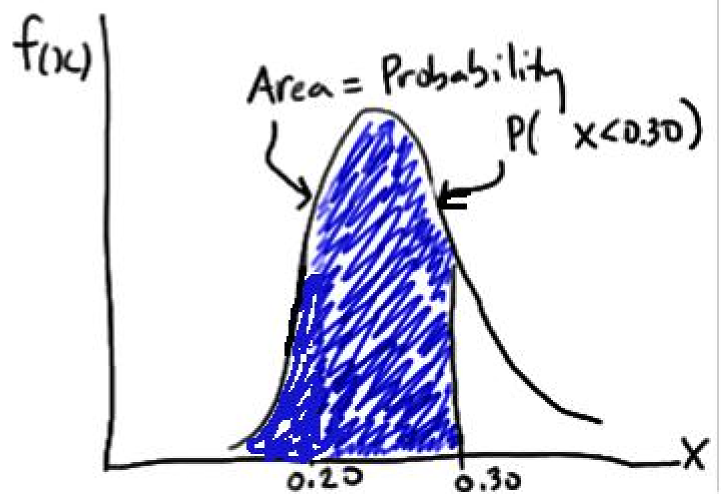
The probability density function *(PDF) of a continuous random variable* X *is*

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

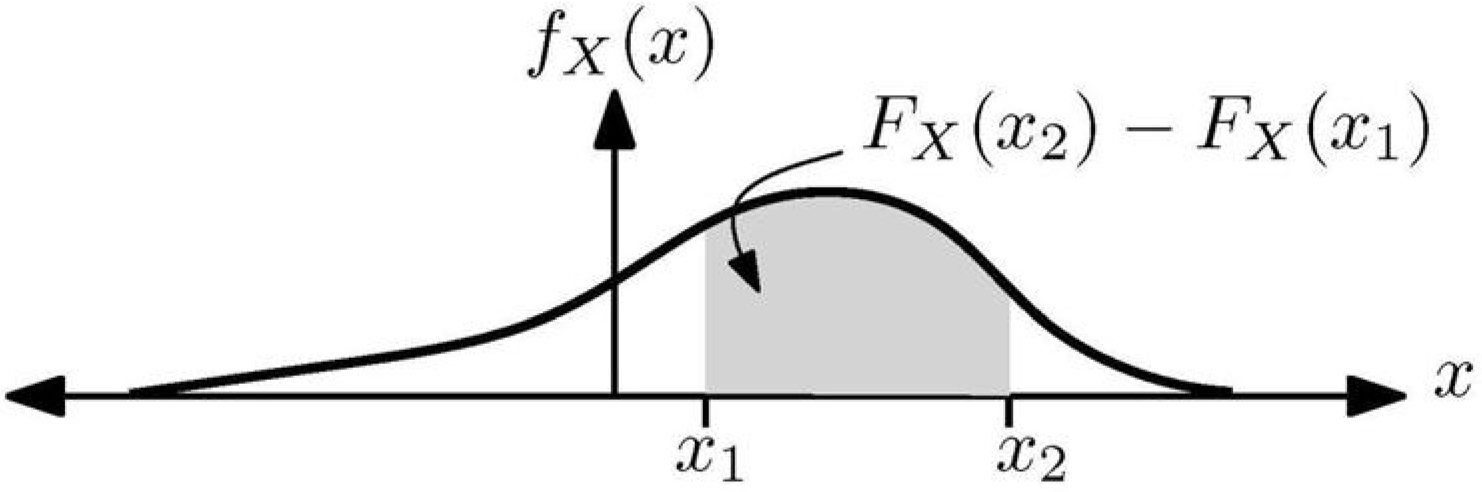
(a) $f_X(x) \geq 0$ for all x,

(b) $F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$,

(c) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.



(d) $$\mathsf{P}\left[x_1 < X \le x_2\right] = \int_{x_1}^{x_2} f_X(x)\, dx.$$



The PDF and CDF of X.

The expected value *of a continuous random variable X is*

$$\mathsf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

The expected value of a function, $g(X)$, of random variable X is

$$\mathsf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

$$\boxed{\mathsf{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx}$$

For any random variable X,

(a) $\mathsf{E}[X - \mu_X] = 0$,

(b) $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$,

(c) $\mathsf{Var}[X] = \mathsf{E}[X^2] - \mu_X^2$,

(d) $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

Uniform Random Variable

X is a uniform (a, b) random variable if the PDF of X is

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x < b, \\ 0 & \textit{otherwise}, \end{cases}$$

where the two parameters are $b > a$.

- The CDF of X is $$F_X(x) = \begin{cases} 0 & x \le a, \\ (x-a)/(b-a) & a < x \le b, \\ 1 & x > b. \end{cases}$$
- The expected value of X is $\mathrm{E}[X] = (b+a)/2.$
- The variance of X is $\mathrm{Var}[X] = (b-a)^2/12.$

Gaussian Random Variable

X is a Gaussian (μ, σ) random variable if the PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2},$$

where the parameter μ can be any real number and the parameter $\sigma > 0$.

If X is a Gaussian (μ, σ) random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

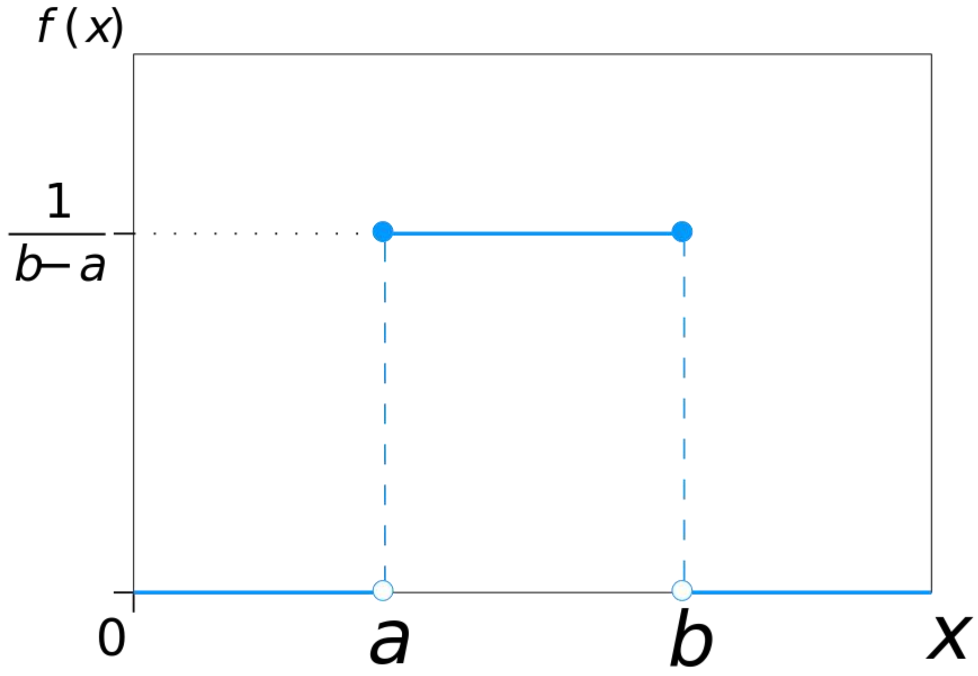
The CDF of the standard normal random variable Z is

Gaussian (0,1)

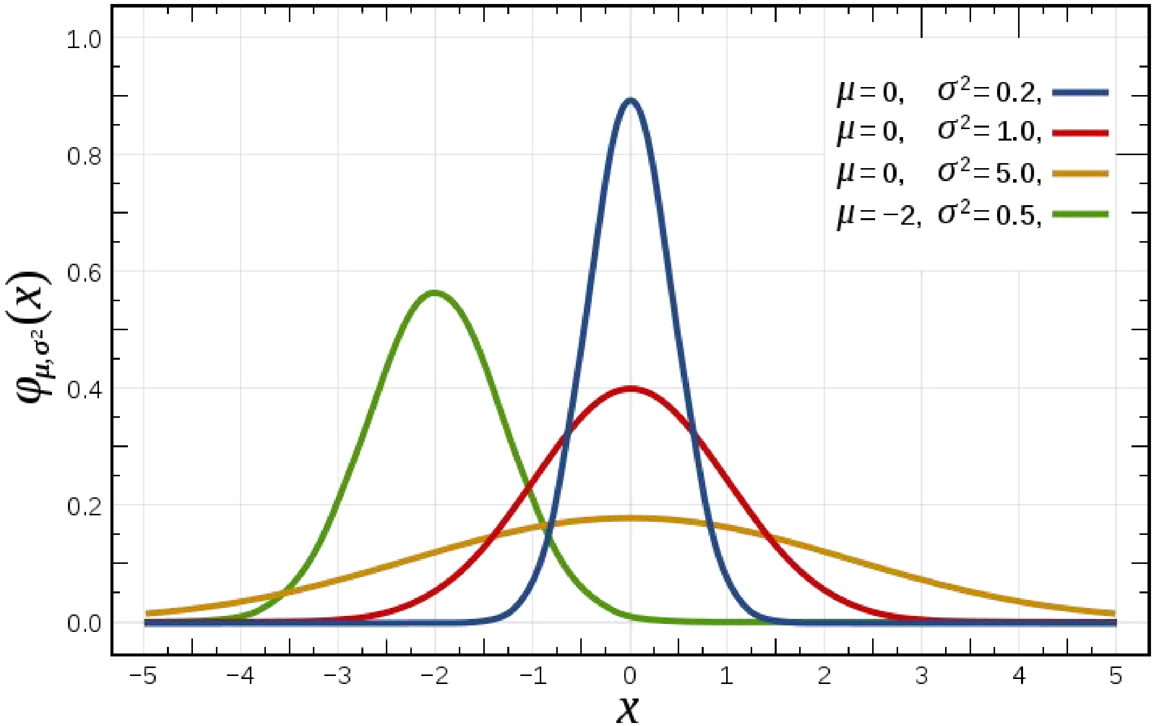
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du.$$

$$\mathrm{E}[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2.$$



f (x)
1
b−a
0
a
b
x



μ = 0, σ² = 0.2,
μ = 0, σ² = 1.0,
μ = 0, σ² = 5.0,
μ = −2, σ² = 0.5,
φ_{μ,σ²}(x)
x

Properties

For any random variable X,

(a) $\mathsf{E}[X - \mu_X] = 0$,

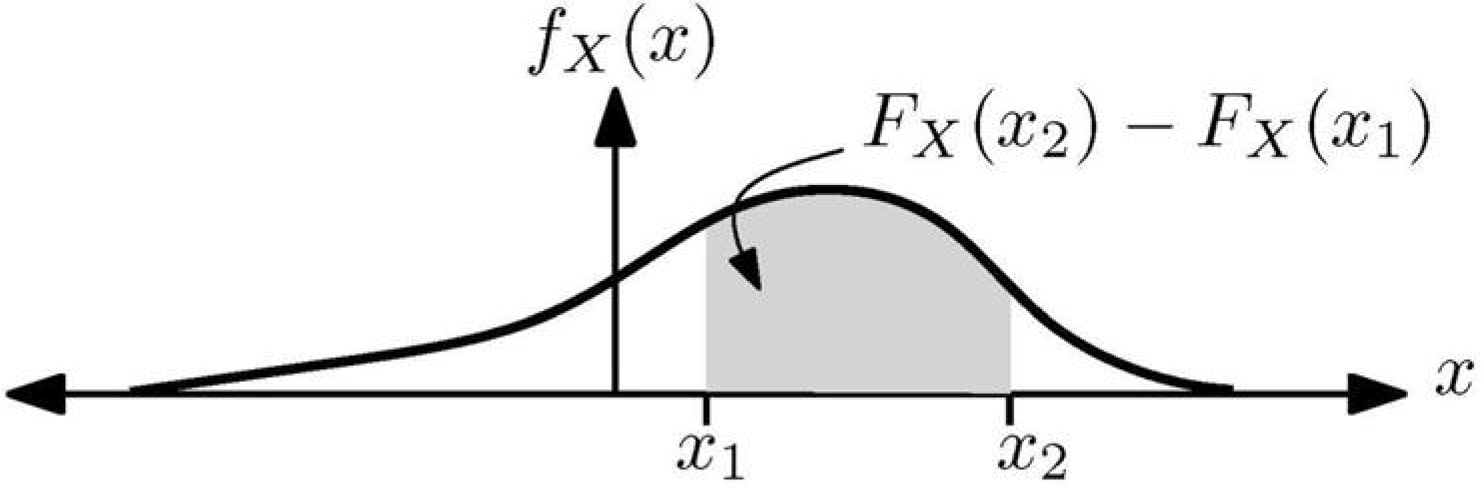
(b) $\mathsf{E}[aX + b] = a\,\mathsf{E}[X] + b$,

(c) $\mathsf{Var}[X] = \mathsf{E}[X^2] - \mu_X^2$,

(d) $\mathsf{Var}[aX + b] = a^2\,\mathsf{Var}[X]$.

If X is Gaussian (μ, σ), $Y = aX + b$ is Gaussian (,   ).

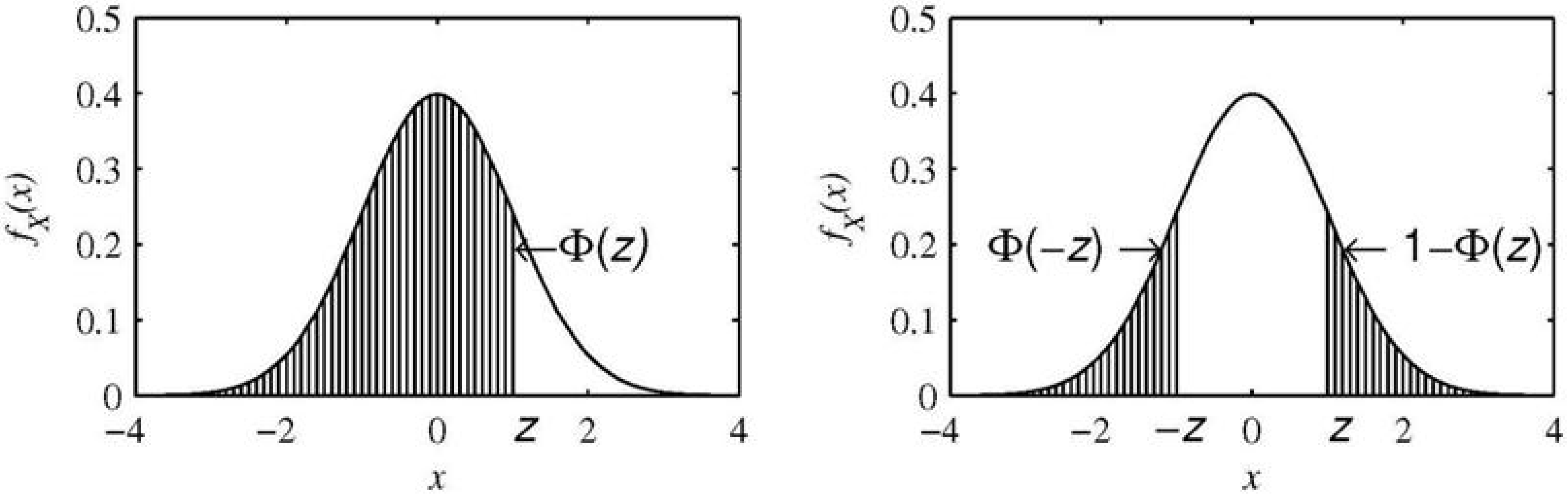
Gaussian's CDF



The PDF and CDF of X.

$$\mathrm{P}\left[a < X \leq b\right] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

$$\boxed{F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right).}$$



$$\Phi(-z) = 1 - \Phi(z).$$

The standard normal complementary CDF *is*

$$Q(z) = \mathrm{P}\left[Z > z\right] = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du = 1 - \Phi(z).$$

Example Problem 1:

The CDF of the continuous random variable X is

$$F_X(x) = \begin{cases} 0 & x < -5, \\ c(x+5)^2 & -5 \le x < 7, \\ 1 & x \ge 7. \end{cases}$$

a) What is c? (Hint: $F_X(7) = 1$)

b) What is $P[X > 2]$?

c) What is $P[-3 < X < 1]$?

d) Determine the PDF $f_X(x)$.

The CDF of the continuous random variable X is

$$F_X(x) = \begin{cases} 0 & x < -5, \\ c(x+5)^2 & -5 \le x < 7, \\ 1 & x \ge 7. \end{cases}$$

a) What is c? (Hint: $F_X(7) = 1$)

b) What is $P[X > 2]$?

c) What is $P[-3 < X < 1]$?

d) Determine the PDF $f_X(x)$.

a) $c \times (7+5)^2 = 1 \Rightarrow c = 1/144$

b) $P[X>2] = F_x(7) - F_x(2)$

$= 1 - \frac{1}{144} \times (2+5)^2$

$= \frac{95}{144}$

c) $P[-3 < X < 1]$

$= F_x(1) - F_x(-3)$

$= \frac{1}{144}(1+5)^2 - \frac{1}{144}(-3+5)^2$

$= \frac{32}{144} = \frac{2}{9}$

d) $f_X(x) = \frac{dF_x(x)}{dx}$

$= \frac{1}{144} \frac{d(x+5)^2}{dx}$

$= \frac{1}{72}(x+5); \quad -5 < x < 7$

Example Problem 2:

X is a standard normal random variable and Y is a Gaussian (1, 2) random variable. determine the following problems using the attached Table.

a) Sketch the PDF Curves of X and Y in one figure

b) P[0<Y<2]

c) P[-1<X<1]

d) P[Y>4.5]

e) Let $Z = \sqrt{2}$X+3. What kind of random variable is Z? Determine E[Z] and Var[Z].

$\phi(\frac{1}{2}) = 0.6915$

$\phi(1) = 0.8413$

$\phi(1.75) = 0.9599$

Example Problem 2:

X is a standard normal random variable and Y is a Gaussian (1, 2) random variable. determine the following problems using the attached Table.

a) Sketch the PDF Curves of X and Y in one figure

b) P[0<Y<2]

c) P[-1<X<1]

d) P[Y>4.5]

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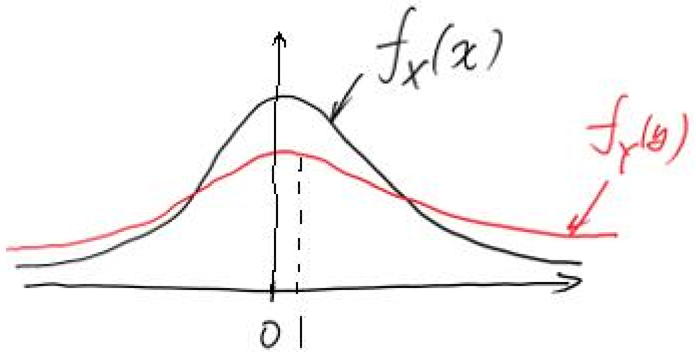

$$
\begin{aligned}
P[0<Y<2] &= \Phi\left(\frac{2-1}{2}\right) - \Phi\left(\frac{0-1}{2}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right) \\
&= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right) \\
&= 2\Phi\left(\frac{1}{2}\right) - 1 \\
&= 2 \times 0.6915 - 1
\end{aligned}
$$

Example Problem 2:

X is a standard normal random variable and Y is a Gaussian (1, 2) random variable. determine the following problems using the attached Table.

a) Sketch the PDF Curves of X and Y in one figure

b) P[0<Y<2]

c) P[-1<X<1]

d) P[Y>4.5]

e) Let $Z = \sqrt{2}$X+3. What kind of random variable is Z? Determine E[Z] and Var[Z].

$$P[-1<X<1] = \phi(1) - \phi(-1)$$
$$= 2\phi(1) - 1$$
$$= 2 \times 0.8413 - 1$$

$$P[Y>4.5] = 1 - P[Y<4.5]$$
$$= 1 - \phi(4.5-1/2)$$
$$= 1 - \phi(1.75) = 1 - 0.9599$$

Example Problem 2:

X is a standard normal random variable and Y is a Gaussian (1, 2) random variable. determine the following problems using the attached Table.

a) Sketch the PDF Curves of X and Y in one figure

b) P[0<Y<2]

c) P[-1<X<1]

d) P[Y>4.5]

e) Let $Z = \sqrt{2}$X+3. What kind of random variable is Z? Determine E[Z] and Var[Z].

Z is also Gaussian random variable.

$E[Z] = E[\sqrt{2}X+3] = 3$

$Var[Z] = Var[\sqrt{2}X+3] = 2$