声明

本笔记是在观看赵老师关于强化学习视频做的笔记，原视频移步[【一张图讲完强化学习原理】 30分钟了解强化学习的名词脉络*哔哩哔哩*bilibili]

作为入门级视频，赵老师将相关数学讲解的十分透彻，强烈建议想要学习RL的初学者将视频刷完，再次感谢赵老师的无私奉献！

概念

state：状态

state transition：状态改变，可以是确定性的，也可以是不确定性的

Tabular representation: We can use a table to describe the state transition:

	a_1 (upwards)	a_2 (rightwards)	a_3 (downwards)	a_4 (leftwards)	a_5 (unchanged)
s_1	s_1	s_2	s_4	s_1	s_1
s_2	s_2	s_3	s_5	s_1	s_2
s_3	s_3	s_3	s_6	s_2	s_3
s_4	s_1	s_5	s_7	s_4	s_4
s_5	s_2	s_6	s_8	s_4	s_5
s_6	s_3	s_6	s_9	s_5	s_6
s_7	s_4	s_8	s_7	s_7	s_7
s_8	s_5	s_9	s_8	s_7	s_8
s_9	s_6	s_9	s_9	s_8	s_9

$$p\left(s_2 \mid s_1, a_2\right)=1$$
$$p\left(s_i \mid s_1, a_2\right)=0 \quad \forall i \neq 2$$

action：某状态采取的动作，可以用条件概率表示

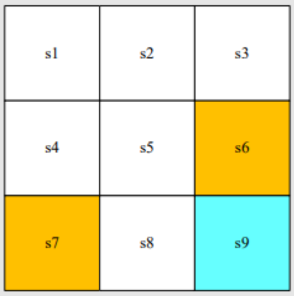
policy：π:策略

确定性概率：

$$\begin{aligned}
\pi\left(a_1 \mid s_1\right)&=0 \\
\pi\left(a_2 \mid s_1\right)&=1 \\
\pi\left(a_3 \mid s_1\right)&=0 \\
\pi\left(a_4 \mid s_1\right)&=0 \\
\pi\left(a_5 \mid s_1\right)&=0
\end{aligned}$$

不确定性：同样是概率

reward：当前状态采取动作对应的奖励/惩罚



Tabular representation of *reward transition*: how to use the table?

	a_1 (upwards)	a_2 (rightwards)	a_3 (downwards)	a_4 (leftwards)	a_5 (unchanged)
s_1	r_{bound}	0	0	r_{bound}	0
s_2	r_{bound}	0	0	0	0
s_3	r_{bound}	r_{bound}	r_{forbid}	0	0
s_4	0	0	r_{forbid}	r_{bound}	0
s_5	0	r_{forbid}	0	0	0
s_6	0	r_{bound}	r_{target}	0	r_{forbid}
s_7	0	0	r_{bound}	r_{bound}	r_{forbid}
s_8	0	r_{target}	r_{bound}	r_{forbid}	0
s_9	r_{forbid}	r_{bound}	r_{bound}	0	r_{target}

return：评价策略好坏，reward总和

discounted return：

1. 防止未来return发散，1+1+1+1+1+1...
2. 平衡现在和未来得到的reward
3. 关于γ：折扣率，∈[0,1)
4. 接近1，远视；接近0，近视

episode：有限步的一次trial，存在terminal state

continuing tasks：没有terminal state，一直交互

统一方法：将episode转化为continuing，

- 无论什么action都会回到当前状态，或者只有留在原地的action，reward=0
- 设置成普通的状态，reward>0/ <0后续可能会跳出来，更一般

MDP

马尔可夫决策过程

马尔可夫性质：和历史无关，状态转移概率和奖励概率都和历史无关

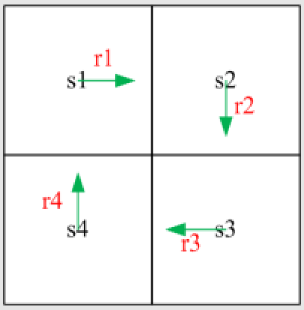
贝尔曼公式

state value

贝尔曼公式

examples

While return is important, how to calculate it?



$$
\begin{aligned}
v_1 &= r_1 + \gamma\left(r_2 + \gamma r_3 + \ldots\right) = r_1 + \gamma v_2 \\
v_2 &= r_2 + \gamma\left(r_3 + \gamma r_4 + \ldots\right) = r_2 + \gamma v_3 \\
v_3 &= r_3 + \gamma\left(r_4 + \gamma r_1 + \ldots\right) = r_3 + \gamma v_4 \\
v_4 &= r_4 + \gamma\left(r_1 + \gamma r_2 + \ldots\right) = r_4 + \gamma v_1
\end{aligned}
$$

v_i表示从某个状态开始计算的return

The returns rely on each other. Bootstrapping!

$$
\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} \gamma v_2 \\ \gamma v_3 \\ \gamma v_4 \\ \gamma v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{\mathbf{r}} + \gamma \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}
$$

$$
\mathbf{v} = \mathbf{r} + \gamma \mathbf{P} \mathbf{v}
$$

This is the Bellman equation (for this specific deterministic problem)!!

- Though simple, it demonstrates the core idea: **the value of one state relies on the values of other states.**
- A matrix-vector form is more clear to see how to solve the **state values**

state value

Consider the following single-step process:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1}$$

- $t, t+1$: discrete time instances
- S_t: state at time t
- A_t: the action taken at state S_t
- R_{t+1}: the reward obtained after taking A_t
- S_{t+1}: the state transited to after taking A_t

Note that S_t, A_t, R_{t+1} are all *random variables*.

This step is governed by the following probability distributions:

- $S_t \to A_t$ is governed by $\pi(A_t = a|S_t = s)$
- $S_t, A_t \to R_{t+1}$ is governed by $p(R_{t+1} = r|S_t = s, A_t = a)$
- $S_t, A_t \to S_{t+1}$ is governed by $p(S_{t+1} = s'|S_t = s, A_t = a)$

At this moment, we assume we know the model (i.e., the probability distributions)!

discounted return is

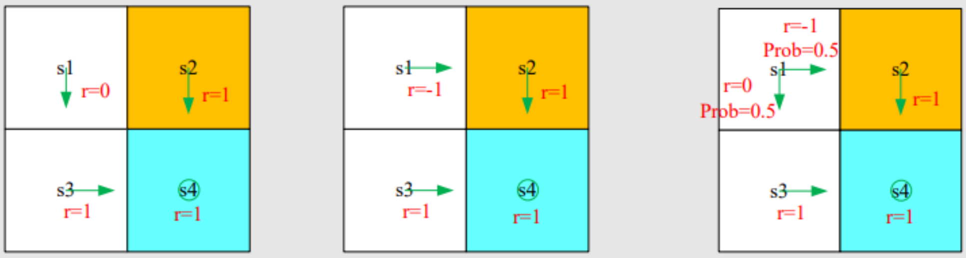
$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$$

The **expectation** (or called **expected value or mean**) of G_t is defined as the state-value function or simply state value:

$$v_\pi(s) = E[G_t|S_t = s]$$

是关于s的函数，**衡量当前状态价值高低**，越大说明当前状态价值越高

Example:



Recall the returns obtained from s_1 for the three examples:

$$v_{\pi_1}(s_1) = 0 + \gamma 1 + \gamma^2 1 + \cdots = \gamma(1 + \gamma + \gamma^2 + \ldots) = \frac{\gamma}{1-\gamma}$$

$$v_{\pi_2}(s_1) = -1 + \gamma 1 + \gamma^2 1 + \cdots = -1 + \gamma(1 + \gamma + \gamma^2 + \ldots) = -1 + \frac{\gamma}{1-\gamma}$$

$$\boxed{v_{\pi_3}(s_1) = 0.5\left(-1 + \frac{\gamma}{1-\gamma}\right) + 0.5\left(\frac{\gamma}{1-\gamma}\right)} = -0.5 + \frac{\gamma}{1-\gamma}$$

期望计算

公式推导

$$v_\pi(s) = \mathbb{E}\left[R_{t+1} \mid S_t = s\right] + \gamma \mathbb{E}\left[G_{t+1} \mid S_t = s\right],$$

$$= \underbrace{\sum_a \pi(a \mid s) \sum_r p(r \mid s, a) r}_{\text{mean of immediate rewards}} + \underbrace{\gamma \sum_a \pi(a \mid s) \sum_{s'} p\left(s' \mid s, a\right) v_\pi\left(s'\right)}_{\text{mean of future rewards}},$$

$$= \sum_a \pi(a \mid s)\left[\sum_r p(r \mid s, a) r + \gamma \sum_{s'} p\left(s' \mid s, a\right) v_\pi\left(s'\right)\right], \quad \forall s \in \mathcal{S}$$

Matrix vector

Sovle the state values

Given a policy, finding out the corresponding state values is called **policy evaluation**!

It is a fundamental problem in RL. It is the foundation to find better policies

- closed-form solution
- iterative solution

不同的策略可以得到相同的state value

通过state value可以评价策略好坏

Action value

选择action value大的值的action更新

state value呢?

计算action value：

- 先求state value，再根据公式计算action value
- 直接计算action value

贝尔曼最优公式

贝尔曼公式的特殊情况

- Core concepts: optimal state value and optimal policy
- A fundamental tool: the Bellman optimality equation (BOE)

EXAMPLE

更新：选择action value最大的action

最优策略：每次都选择action value最大的action

原因：贝尔曼最优公式

What if we select the greatest action value? Then, **a new policy is obtained:**

$$\pi_{\text{new}}\left(a \mid s_1\right)=\begin{cases}1 & a=a^* \\ 0 & a \neq a^*\end{cases}$$

where $a^*=\arg\max_a q_\pi\left(s_1, a\right)=a_3$.

Definition

最优策略：A policy π^* is optimal if $v_{\pi^*}(s) \geq v_\pi(s)$ for all s and for any other policy π .

The definition leads to many questions:

- Does the optimal policy exist? （所有状态state value都大于其他策略，可能过于理想而不存在）
- Is the optimal policy unique? （是否存在多个最优策略）
- Is the optimal policy stochastic or deterministic? （该策略是确定性还是非确定性）
- How to obtain the optimal policy?（怎么得到）

  To answer these questions, we study the Bellman optimality equation.

BOE

$$\begin{aligned} v(s) & =\max_\pi \sum_a \pi(a \mid s)\left(\sum_r p(r \mid s, a) r+\gamma \sum_{s'} p\left(s' \mid s, a\right) v\left(s'\right)\right), \quad \forall s \in \mathcal{S} \\ & =\max_\pi \sum_a \pi(a \mid s) q(s, a) \quad s \in \mathcal{S}\end{aligned}$$

若要max，实际是对应最大的 $q(s, a)$

Inspired by the above example, considering that $\sum_a \pi(a \mid s)=1$, we have

$$\max_\pi \sum_a \pi(a \mid s) q(s, a)=\max_{a \in \mathcal{A}(s)} q(s, a)$$

where the optimality is achieved when

$$\pi(a \mid s)=\begin{cases}1 & a=a^* \\ 0 & a \neq a^*\end{cases}$$

where $a^*=\arg\max_a q(s, a)$.

与example处的结果一致

Solve the optimality equation

固定v，求解 π

实际问题：$v=f(v)$

how to solve the equation?

Contraction mapping theorem

Fixed point（不动点）: $x \in X$ is a fixed point of f : $X \rightarrow X$ if

$$f(x)=x$$

Contraction **mapping** (or contractive **function**): f is a contraction mapping if

$$\|f(x_1)-f(x_2)\| \leq \gamma\|x_1-x_2\|$$

where $\gamma \in (0,1)$.

contraction function在求解$x=f(x)$有三点性质

- Existence: there exists a fixed point $x^* satisfying f(x^*)=x^*$.
- Uniqueness: The fixed point x^* is unique.
- **Algorithm**: Consider a sequence $\{x_k\} where\, x_{k+1}=f(x_k), then\ x_k \rightarrow x^*$ as $k \rightarrow \infty$. Moreover, the convergence rate is exponentially fast.（利用迭代计算出x_k, when k-> ∞)

solve

对于贝尔曼最优问题，其方程为 contractive function

（证明：满足 $\|f(x_1)-f(x_2)\| \leq \gamma\|x_1-x_2\|$ 即可，此处省略证明）

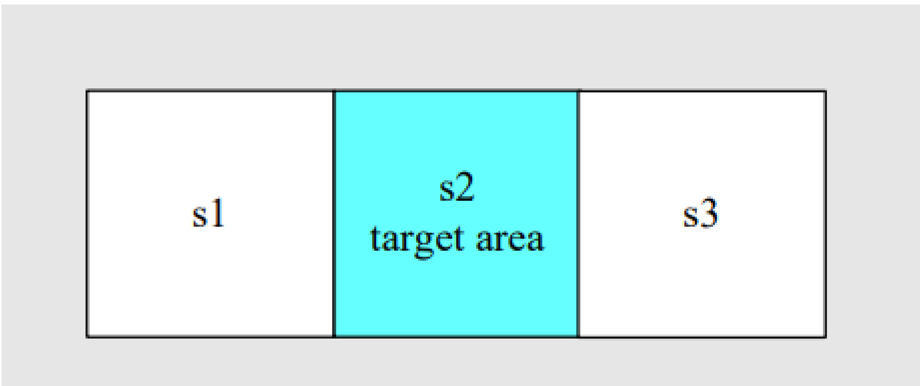
绕路：得到目标奖励越晚！和r等于多少有关，但同时也受到 γ的约束

因此求解：

$$\begin{aligned}
v_{k+1}(s) & =\max_\pi \sum_a \pi(a \mid s)\left(\sum_r p(r \mid s, a) r+\gamma \sum_{s^{\prime}} p\left(s^{\prime} \mid s, a\right) v_k\left(s^{\prime}\right)\right) \\
& =\max_\pi \sum_a \pi(a \mid s) q_k(s, a) \\
& =\max_a q_k(s, a)
\end{aligned}$$

设立初始的v_k，不断迭代得到v_{k+1}即可

Example



The values of $q(s,a)$

q-value table	a_ℓ	a_0	a_r
s_1	$-1+\gamma v(s_1)$	$0+\gamma v(s_1)$	$1+\gamma v(s_2)$
s_2	$0+\gamma v(s_1)$	$1+\gamma v(s_2)$	$0+\gamma v(s_3)$
s_3	$1+\gamma v(s_2)$	$0+\gamma v(s_3)$	$-1+\gamma v(s_3)$

Consider γ**=0.9 以及下面的初始条件**

Our objective is to find $v^*(s_i)$ and $\pi^* k=0$:
v-value: select $v_0(s_1)=v_0(s_2)=v_0(s_3)=0$

q-value (**using the previous table**):

	a_ℓ	a_0	a_r
s_1	-1	0	1
s_2	0	1	0
s_3	1	0	-1

关于policy：采取greedy policy，select the greatest q-value

$$\pi\left(a_r \mid s_1\right)=1, \quad \pi\left(a_0 \mid s_2\right)=1, \quad \pi\left(a_\ell \mid s_3\right)=1$$

v-value: $v_1(s)=\max_a q_0(s, a)$

$$v_1\left(s_1\right)=v_1\left(s_2\right)=v_1\left(s_3\right)=1$$

This this policy good? Yes!

但是注意，此时虽然policy是最好的，但是state value没有到最优！！！因为此时k=1，而对应的state value 要到无穷，实际不用到无穷，只需 $|v_{k+1}-v_k|<\sigma$，因此接下来的iteration，k=1

	a_ℓ	a_0	a_r
s_1	-0.1	0.9	1.9
s_2	0.9	1.9	0.9
s_3	1.9	0.9	-0.1

然后Greedy policy (select the greatest q-value):

$$\pi\left(a_r \mid s_1\right)=1, \quad \pi\left(a_0 \mid s_2\right)=1, \quad \pi\left(a_\ell \mid s_3\right)=1$$

k = 2, 3, . . .

Policy optimality

回答上述的问题

Suppose that v^* is the unique solution to $v=\max_\pi\left(r_\pi+\gamma P_\pi v\right), and v_\pi$ is the state value function satisfying $v_\pi=r_\pi+\gamma P_\pi v_\pi$ for any given policy π , then

$$v^* \geq v_\pi, \quad \forall \pi$$

即最优的policy，**对应的state value大于每个地方的state value**

同时，最优的policy怎么求？贪心规则

For any $s \in \mathcal{S}$, the deterministic **greedy policy**

$$\pi^*(a \mid s)= \begin{cases}1 & a=a^*(s) \\ 0 & a \neq a^*(s)\end{cases}$$

is an optimal policy solving the BOE. Here,

$$a^*(s)=\arg \max_a q^*(a, s),$$

where $q^*(s, a):=\sum_r p(r \mid s, a) r+\gamma \sum_{s^{\prime}} p\left(s^{\prime} \mid s, a\right) v^*\left(s^{\prime}\right).$

Analyzing optimal policies

即一些有趣的情况

比如 γ较大时，会比较远视，当γ较小时，policy 会比较近视

且当reward**线性变化**时，对应的**最优policy不会变化**

Value Iteration& Policy Iteration

model-based

Value iteration

原理

即贝尔曼最优公式的迭代求解法

start from v_0

step1：Policy update（PU）

已知 v_k，求出q-table，然后找到最大的策略 π_{k+1}，然后更新

$$\pi_{k+1} = \arg\max_{\pi} (r_\pi + \gamma P_\pi v_k)$$

step2：value update（VU）

将上面的 π_{k+1}代入求解 v_{k+1}

$$v_{k+1} = r_{\pi_{k+1}} + \gamma P_{\pi_{k+1}} v_k$$

v_k **is not a state value，just a value**

实践算法

Pseudocode: Value iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess v_0.

Aim: Search the optimal state value and an optimal policy solving the Bellman optimality equation.

While v_k has not converged in the sense that $\|v_k - v_{k-1}\|$ is greater than a predefined small threshold, for the kth iteration, do

For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

q-value: $q_k(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a)v_k(s')$

Maximum action value: $a_k^*(s) = \arg\max_a q_k(a,s)$

Policy update: $\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

Value update: $v_{k+1}(s) = \max_a q_k(a,s)$

Policy iteration

原理

start from π_0

step1：policy evaluation（PE）

计算state value，因为state value实际上表征的就是策略的好坏

已知 π_k，求 $v_{\pi k}$

NOTE：**此处有两种计算方法，一种是直接计算，一种是迭代计算**

$$v_{\pi_k} = r_{\pi_k} + \gamma P_{\pi_k} v_{\pi_k}$$

step2：policy improvement（PI）

update policy，用greedy算法得到 π_{k+1}

$$\pi_{k+1} = \arg\max_{\pi} \left(r_\pi + \gamma P_\pi v_{\pi_k}\right)$$

policy iteration 和value iteration的关系

- 证明policy iteration算法收敛时，用到value iteration收敛的结果
- 是 iteration的极端
- ▷ **Q2: In the policy improvement step, why is the new policy π_{k+1} better than π_k?**

  Lemma (Policy Improvement)

  If $\pi_{k+1} = \arg\max_\pi (r_\pi + \gamma P_\pi v_{\pi_k})$, *then* $v_{\pi_{k+1}} \geq v_{\pi_k}$ *for any* k.

实践编程算法

Pseudocode: Policy iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess π_0.

Aim: Search for the optimal state value and an optimal policy.

While the policy has not converged, for the kth iteration, do

Policy evaluation:

Initialization: an arbitrary initial guess $v_{\pi_k}^{(0)}$

While $v_{\pi_k}^{(j)}$ has not converged, for the jth iteration, do

For every state $s \in \mathcal{S}$, do

$$v_{\pi_k}^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}^{(j)}(s')\right]$$

Policy improvement:

For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

$$q_{\pi_k}(s,a) = \sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_{\pi_k}(s')$$

$a_k^*(s) = \arg\max_a q_{\pi_k}(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise

靠近目标的策略会先变好，远离目标的策略会后变好

原因：greedy action，当靠近目标时，target是最greedy的，而greedy则依靠周围的情况，如果周围乱七八糟，得到的策略也不一定是最好的

Truncated policy iteration

上述两个算法的一般化！

	Policy iteration algorithm	Value iteration algorithm	Comments
1) Policy:	π_0	N/A	
2) Value:	$v_{\pi_0} = r_{\pi_0} + \gamma P_{\pi_0} v_{\pi_0}$	$v_0 := v_{\pi_0}$	
3) Policy:	$\pi_1 = \arg\max_\pi(r_\pi + \gamma P_\pi v_{\pi_0})$	$\pi_1 = \arg\max_\pi(r_\pi + \gamma P_\pi v_0)$	The two policies are the same
4) Value:	$v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$	$v_1 = r_{\pi_1} + \gamma P_{\pi_1} v_0$	$v_{\pi_1} \geq v_1$ since $v_{\pi_1} \geq v_{\pi_0}$
5) Policy:	$\pi_2 = \arg\max_\pi(r_\pi + \gamma P_\pi v_{\pi_1})$	$\pi_2' = \arg\max_\pi(r_\pi + \gamma P_\pi v_1)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Consider the step of solving $v_{\pi_1} = r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}$:

$$
\begin{aligned}
& & v_{\pi_1}^{(0)} &= v_0 \\
\text{value iteration} \leftarrow v_1 &\longleftarrow & v_{\pi_1}^{(1)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(0)} \\
& & v_{\pi_1}^{(2)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(1)} \\
& & &\vdots \\
\text{truncated policy iteration} \leftarrow \bar{v}_1 &\longleftarrow & v_{\pi_1}^{(j)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(j-1)} \\
& & &\vdots \\
\text{policy iteration} \leftarrow v_{\pi_1} &\longleftarrow & v_{\pi_1}^{(\infty)} &= r_{\pi_1} + \gamma P_{\pi_1} v_{\pi_1}^{(\infty)}
\end{aligned}
$$

- The **value** iteration algorithm computes **once.**
- The **policy** iteration algorithm computes an **infinite number of iterations.**
- The **truncated** policy iteration algorithm computes a **finite number of iterations** (say j). The rest iterations from j to ∞ are **truncated**.

Pseudocode: Truncated policy iteration algorithm

Initialization: The probability model $p(r|s,a)$ and $p(s'|s,a)$ for all (s,a) are known. Initial guess π_0.

Aim: Search for the optimal state value and an optimal policy.

While the policy has not converged, for the kth iteration, do

Policy evaluation:

Initialization: select the initial guess as $v_k^{(0)} = v_{k-1}$. The maximum iteration is set to be j_{truncate}.

limited times

$\boxed{\text{While } j < j_{\text{truncate}},}$ do

For every state $s \in \mathcal{S}$, do

$v_k^{(j+1)}(s) = \sum_a \pi_k(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k^{(j)}(s') \right]$

Set $v_k = v_k^{(j_{\text{truncate}})}$

Policy improvement:

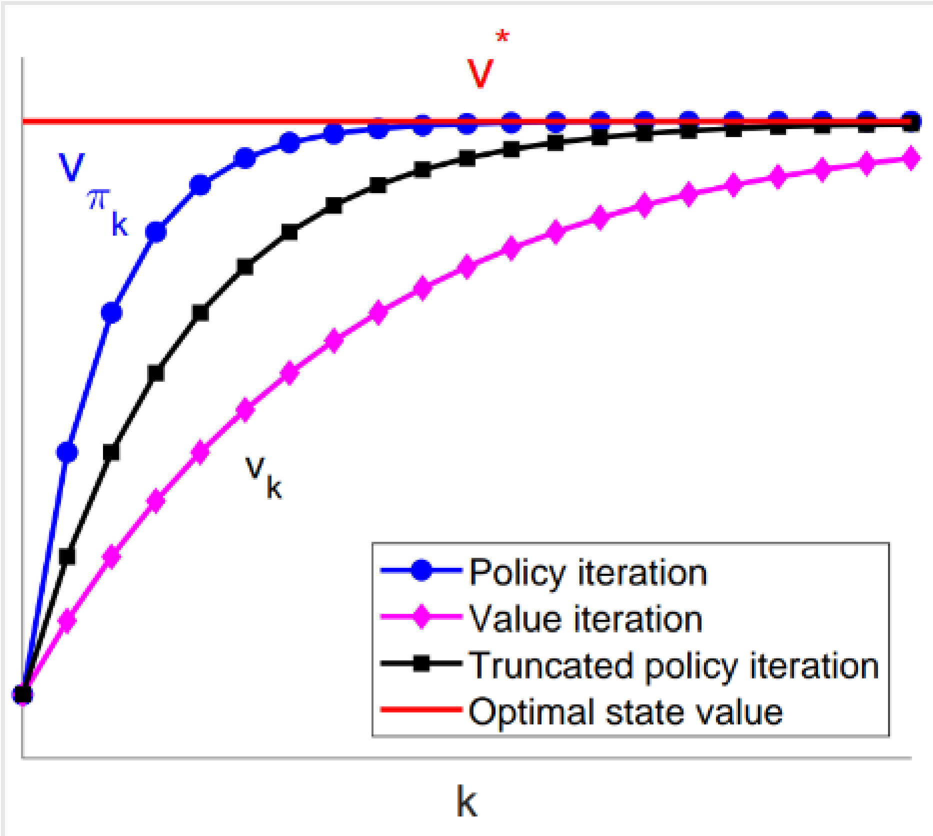
For every state $s \in \mathcal{S}$, do

For every action $a \in \mathcal{A}(s)$, do

$q_k(s,a) = \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_k(s')$

$a_k^*(s) = \arg\max_a q_k(s,a)$

$\pi_{k+1}(a|s) = 1$ if $a = a_k^*$, and $\pi_{k+1}(a|s) = 0$ otherwise



Monte Carlo Learning

Model Free->monte carlo estimation

Core：policy iteration -> model-free

Example

许多次采样！通过平均值来代替期望！数据理论支持：大数定理！

大量实验来近似！为什么蒙特卡罗？因为没有模型，只能实验

Summary:

- Monte Carlo estimation refers to a broad class of techniques that rely on **repeated random sampling** to solve approximation problems.
- Why we care about Monte Carlo estimation? Because **it does not require the model**!
- Why we care about mean estimation? Because **state value and action value** are defined as **expectations of random variables**!

MC Basic

model-free最大区别的点在于PI中的计算action value

Two expressions of action value:

- Expression 1 requires the model:

$$q_{\pi_k}(s, a)=\sum_r p(r \mid s, a) r+\gamma \sum_{s^{\prime}} p\left(s^{\prime} \mid s, a\right) v_{\pi_k}\left(s^{\prime}\right)$$

- **Expression 2 does not require the model:**

$$q_{\pi_k}(s,a)=\mathbb{E}\left[G_t \mid S_t=s, A_t=a\right]$$

Idea to achieve model-free RL: We can use expression 2 to calculate $q_{\pi_k}(s,a)$ based on **data (samples or experiences)**!

计算action value

- Starting from (s,a), following policy π_k, generate an episode.
- The return of this episode is $g(s,a)$
- $g(s,a)$ is a sample of G_t in

$$q_{\pi_k}(s,a)=\mathbb{E}[G_t|S_t=s,A_t=a]$$

- Suppose we have a set of episodes and hence $\{g^{(j)}(s,a)\}$. Then,

$$q_{\pi_k}(s,a)=\mathbb{E}[G_t|S_t=s,A_t=a]\approx\frac{1}{N}\sum_{i=1}^{N}g^{(i)}(s,a).$$

具体Policy iteration

step1: policy evaluation

在求解state value时，用期望代替原本用模型求解的答案

This step is to obtain $q_{\pi_k}(s,a)$ for all (s,a). Specifically, for each action-state pair (s,a), run an infinite number of (or sufficiently many) episodes. The average of their returns is used to approximate $q_{\pi_k}(s,a)$.

step2：policy improvement

NOTE:

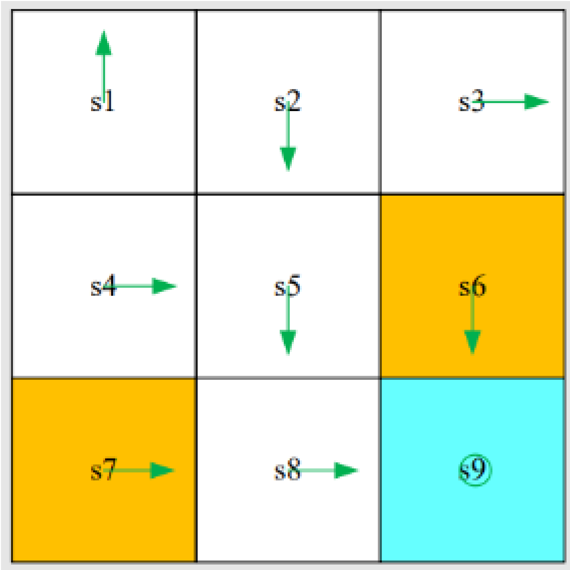
- **useful to reveal the core idea，not practical due to low efficiency**
- **直接估计action value！而不是估计state value**

still is convergent

注意：此处的action value是估计的！

Example1

episode lenth！



Task:

- An initial policy is shown in the figure.
- Use MC Basic to find the optimal policy.
- $r_{\text{boundary}} = -1, r_{\text{forbidden}} = -1, r_{\text{target}} = 1, \gamma = 0.9.$

与model-based区别在哪？不能直接用公式

Step1：policy evaluation

- Since the current policy is **deterministic**, **one episode** would be sufficient to get the action value!
- If the current policy is **stochastic**, **an infinite number of episodes (or at least many) are required**!（统计计算期望！）
- Starting from (s_1, a_1), the episode is $s_1 \xrightarrow{a_1} s_1 \xrightarrow{a_1} s_1 \xrightarrow{a_1} \ldots$. Hence, the action value is

$$q_{\pi_0}(s_1, a_1) = -1 + \gamma(-1) + \gamma^2(-1) + \ldots$$

- Starting from (s_1, a_2), the episode is $s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} s_5 \xrightarrow{a_3} \ldots$. Hence, the action value is

$$q_{\pi_0}(s_1, a_2) = 0 + \gamma 0 + \gamma^2 0 + \gamma^3(1) + \gamma^4(1) + \ldots$$

- Starting from (s_1, a_3), the episode is $s_1 \xrightarrow{a_3} s_4 \xrightarrow{a_2} s_5 \xrightarrow{a_3} \ldots$. Hence, the action value is

$$q_{\pi_0}(s_1, a_3) = 0 + \gamma 0 + \gamma^2 0 + \gamma^3(1) + \gamma^4(1) + \ldots$$

Step2：policy improvement

- By observing the action values, we see that

$$q_{\pi_0}(s_1, a_2) = q_{\pi_0}(s_1, a_3)$$

are the **maximum.**

- As a result, **the policy can be improved as**

$$\pi_1(a_2 \mid s_1) = 1 \text{ or } \pi_1(a_3 \mid s_1) = 1.$$

In either way, the new policy for s_1 becomes optimal.
One iteration is sufficient for this simple example!

Example2

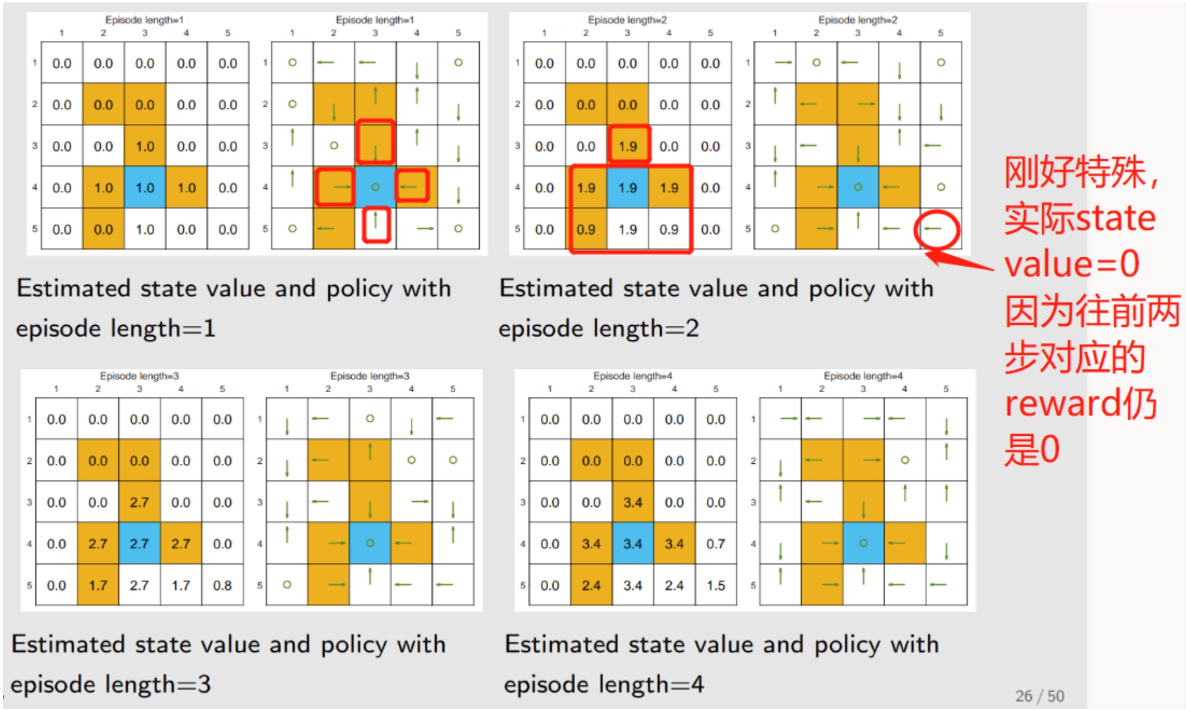
the impact of episode length

所谓episode length，可以理解为探索长度

length=1 -> $q_{\pi_0}(s_1, a_1) = -1$

length=2 -> $q_{\pi_0}(s_1, a_1) = -1 + \gamma(-1)$

且是从target处开始逆向优化！！



注意上面非0的state value，对应为最优的策略

Conclusion：

- The episode length should be sufficiently long.
- The episode length does not have to be infinitely long.

MC Exploring Start

MC Basic的推广

如何更新？引入Visit！

MC Basic：Initial visit

Exploring：在计算一次episode时，其同时访问了其他的**state-action pairs**，因此可以计算其他的action value，**提高效率**

▷ The episode also visits other state-action pairs.

$$
\begin{array}{rl}
s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_4} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} s_5 \xrightarrow{a_1} \ldots & \text{[original episode]} \\
s_2 \xrightarrow{a_4} s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} s_5 \xrightarrow{a_1} \ldots & \text{[episode starting from } (s_2, a_4)] \\
s_1 \xrightarrow{a_2} s_2 \xrightarrow{a_3} s_5 \xrightarrow{a_1} \ldots & \text{[episode starting from } (s_1, a_2)] \\
s_2 \xrightarrow{a_3} s_5 \xrightarrow{a_1} \ldots & \text{[episode starting from } (s_2, a_3)] \\
s_5 \xrightarrow{a_1} \ldots & \text{[episode starting from } (s_5, a_1)]
\end{array}
$$

Can estimate $q_\pi(s_1, a_2)$, $q_\pi(s_2, a_4)$, $q_\pi(s_2, a_3)$, $q_\pi(s_5, a_1)$,...

Data-efficient methods：

- **first-visit method**：只用第一次出现的进行估计！
- **every-visit method**：后面出现的都可以利用来估计！

When to update the policy

- first method：把**所有episode**的return收集后再开始估计，然后改进
- second method：得到**一个episode**的return就开始估计，直接改进，得到一个改进一个（最后仍会收敛）

GPI

GPI：generalized policy iteration

- It refers to the general idea or framework of **switching between policy-evaluation and policy-improvement processes**.
- Many model-based and model-free RL algorithms fall into this framework.
- 不需要十分精确估计！但最后仍能收敛

Exploring的缺点：每一个state action pair都要有一个episode，以防漏掉

如何解决？看下面！

MC ξ-Greedy

为什么要探索？不是按照贪心就可以得到最优策略吗？

为什么用这个策略？**不需要exploring starts**

Soft policy：A policy is called soft if **the probability to take any action is positive**.

此处的soft policy：ξ-Greedy

原因：episode够长，只要用1个或几个就可以覆盖其他所有state action pair

Definition

$$\pi(a \mid s) = \begin{cases} 1 - \frac{\varepsilon}{|\mathcal{A}(s)|}(|\mathcal{A}(s)| - 1), & \text{for the greedy action,} \\ \frac{\varepsilon}{|\mathcal{A}(s)|}, & \text{for the other } |\mathcal{A}(s)| - 1 \text{ actions.} \end{cases}$$

where $\varepsilon \in [0,1]$ and $|\mathcal{A}(s)|$ is the number of actions for s .

- The chance to choose the **greedy action** is always **greater than other actions**, because

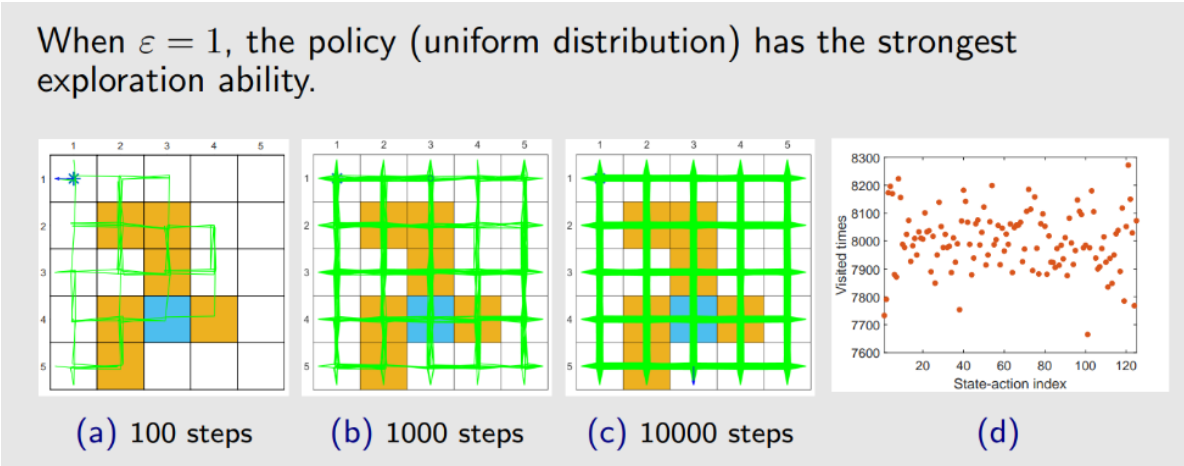
$$1 - \frac{\varepsilon}{|\mathcal{A}(s)|}(|\mathcal{A}(s)| - 1) = 1 - \varepsilon + \frac{\varepsilon}{|\mathcal{A}(s)|} \geq \frac{\varepsilon}{|\mathcal{A}(s)|}$$

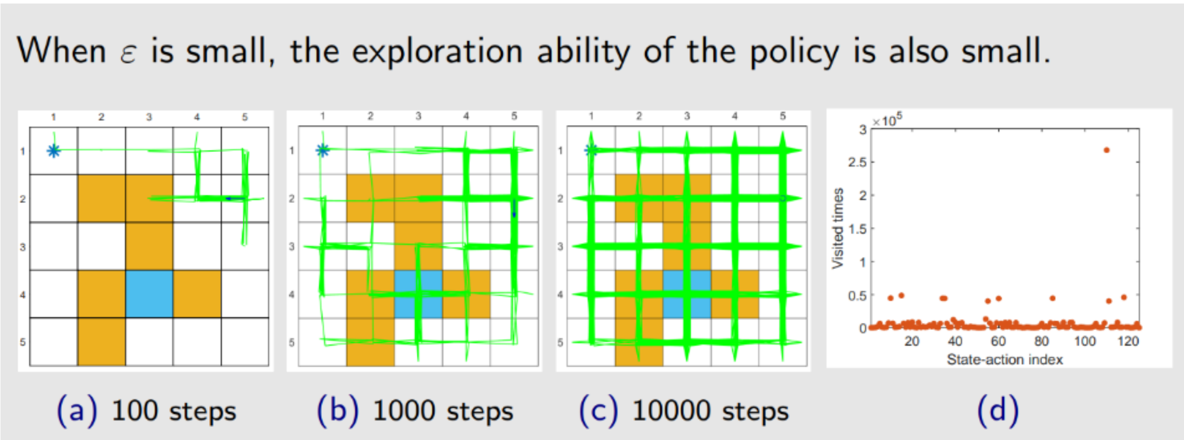
  .

- Why use $\varepsilon - greedy$? **Balance** between **exploitation and exploration！！！（充分利用和探索性）**
- When ε=0 , it becomes **greedy!** Less exploration but more exploitation!
- When ε=1 , it becomes a **uniform distribution（均匀分布）**. More exploration but less exploitation.

在选择数据时，我们利用every visit，因为action pair可能会被访问很多次，如果用first visit，则会导致数据浪费

Example





Conclusion

- The advantage of ε -greedy policies is that they have stronger exploration ability so that the exploring starts condition is not required.

- The disadvantage is that $\varepsilon - greedy$ polices are not optimal in general (we can only show that there always exist greedy policies that are optimal).
- The final policy given by the MC $\varepsilon - Greedy$ algorithm is only optimal in the set Π_ε of all $\varepsilon - greedy$ policies.
- **ε cannot be too large.**
- **当 ε 为0.1或很小时，得到的policy与greedy policy一致，当变大时，得到的最终的policy与greedy有出入**

Stochastic Approximation

Mean estimation Example

how to calculate the mean

- **The first way**, which is trivial, is to **collect all the samples** then calculate the average.
- **The second way** can avoid this drawback because it calculates the average in an **incremental** and **iterative** manner.

We can use

$$w_{k+1} = w_k - \frac{1}{k}(w_k - x_k).$$

to calculate the mean $\bar{x}$ incrementally:(上述公式可推导)

$$\begin{aligned}
w_1 &= x_1 \\
w_2 &= w_1 - \frac{1}{1}(w_1 - x_1) = x_1 \\
w_3 &= w_2 - \frac{1}{2}(w_2 - x_2) = x_1 - \frac{1}{2}(x_1 - x_2) = \frac{1}{2}(x_1 + x_2) \\
w_4 &= w_3 - \frac{1}{3}(w_3 - x_3) = \frac{1}{3}(x_1 + x_2 + x_3) \\
&\vdots \\
w_{k+1} &= \frac{1}{k}\sum_{i=1}^{k} x_i
\end{aligned}$$

将 $\frac{1}{k}$ 替换成 α_k，即为对应的a special **SA algorithm** and also a **special stochastic gradient descent algorithm**

Robbins-Monro algorithm

Description

Stochastic approximation (SA):

- SA is powerful in the sense that it **does not require to know the expression of the objective function nor its derivative**.

Robbins-Monro (RM) algorithm:

- The is a pioneering work in the field of **stochastic approximation.**
- The famous **stochastic gradient descent** algorithm is a special form of the RM algorithm.（SGD）
- It can be used to analyze the **mean estimation algorithms** introduced in the beginning.

用于求解 $g(w)=0$的解

The Robbins-Monro (RM) algorithm can solve this problem:

$$w_{k+1}=w_k-a_k\tilde{g}\left(w_k, \eta_k\right), \quad k=1,2,3, \ldots$$

where

- w_k is the k th estimate of the **root**
- $\tilde{g}\left(w_k, \eta_k\right)=g\left(w_k\right)+\eta_k$ is the kth **noisy observation**
- a_k is a **positive** coefficient.(a_k>0)
  The function $g(w)$ is a **black box!** This algorithm **relies on data**:
- Input sequence: $\left\{w_k\right\}$
- Noisy output sequence: $\left\{\tilde{g}\left(w_k, \eta_k\right)\right\}$
  Philosophy: without model, we need data!
- Here, the model refers to the expression of the function.

Example

Excise: manually solve $g(w)=w-10$ using the RM algorithm.
Set: $w_1=20, a_k \equiv 0.5, \eta_k=0$ (i.e., no observation error)

$$\begin{aligned}
& w_1=20 \Longrightarrow g\left(w_1\right)=10 \\
& w_2=w_1-a_1 g\left(w_1\right)=20-0.5 * 10=15 \Longrightarrow g\left(w_2\right)=5 \\
& w_3=w_2-a_2 g\left(w_2\right)=15-0.5 * 5=12.5 \Longrightarrow g\left(w_3\right)=2.5 \\
& \vdots \\
& w_k \rightarrow 10
\end{aligned}$$

Convergence analysis

A rigorous convergence result is given below

Theorem (Robbins-Monro Theorem)
In the Robbins-Monro algorithm, if

1. $0<c_1 \leq \nabla_w g(w) \leq c_2$ for all w ;
2. $\sum_{k=1}^{\infty} a_k=\infty$and $\sum_{k=1}^{\infty} a_k^2<\infty$;
3. $\mathbb{E}\left[\eta_k \mid \mathcal{H}_k\right]=0$ and $\mathbb{E}\left[\eta_k^2 \mid \mathcal{H}_k\right]<\infty$;
   where $\mathcal{H}_k=\left\{w_k, w_{k-1}, \ldots\right\}$, then w_k **converges with probability 1 (w.p.1)(概率收敛)** to the root w^* satisfying $g\left(w^*\right)=0$

- a_k要收敛到0，但不要收敛太快，

Application to mean estimation

estimation algorithm

$$w_{k+1}=w_k+\alpha_k\left(x_k-w_k\right).$$

We know that

- If $\alpha_k=1 / k$, then $w_{k+1}=1 / k \sum_{i=1}^k x_i$.
- If α_k is not $1 / k$, the convergence was not analyzed.

we show that this algorithm is **a special case of the RM algorithm**. Then, its **convergence naturally** follows

下面将证明上述方程为RM算法

1. Consider a function:

$$g(w) \doteq w-\mathbb{E}[X]$$

Our aim is to solve $g(w)=0$. If we can do that, then we can obtain $\mathbb{E}[X]$.

2. The observation we can get is

$$\tilde{g}(w, x) \doteq w-x$$

because we can only **obtain samples of X** . Note that

$$\begin{aligned} \tilde{g}(w, \eta)=w-x & =w-x+\mathbb{E}[X]-\mathbb{E}[X] \\ & =(w-\mathbb{E}[X])+(\mathbb{E}[X]-x) \doteq g(w)+\eta, \end{aligned}$$

3. The RM algorithm for solving $g(x)=0$ is

$$w_{k+1}=w_k-\alpha_k \tilde{g}\left(w_k, \eta_k\right)=w_k-\alpha_k\left(w_k-x_k\right),$$

which is exactly the **mean estimation algorithm.**
The convergence naturally follows.

SGD

introduction

SGD is a **special RM** algorithm.

The **mean estimation** algorithm is a **special SGD algorithm**

SGD：常用于**解决优化问题**(实际还是求根问题？)

最小化：梯度下降

最大化：梯度上升

- GD（gradient descent)

$$w_{k+1}=w_k-\alpha_k \nabla_w \mathbb{E}\left[f\left(w_k, X\right)\right]=w_k-\alpha_k \mathbb{E}\left[\nabla_w f\left(w_k, X\right)\right]$$

drawback： the **expected value** is difficult to **obtain**.

- BGD：**No model**，**use data to estimate the mean**

$$\begin{aligned} & \mathbb{E}\left[\nabla_w f\left(w_k, X\right)\right] \approx \frac{1}{n} \sum_{i=1}^n \nabla_w f\left(w_k, x_i\right) \\ & w_{k+1}=w_k-\alpha_k \frac{1}{n} \sum_{i=1}^n \nabla_w f\left(w_k, x_i\right) . \end{aligned}$$

Drawback: it requires **many samples** in each iteration for each w_k.

- SGD
- $$w_{k+1}=w_k-\alpha_k \nabla_w f\left(w_k, x_k\right)$$

  compared to the BGD, let $n=1$

example

We next consider an example:

$$\min_w \quad J(w)=\mathbb{E}[f(w,X)]=\mathbb{E}\left[\frac{1}{2}\|w-X\|^2\right],$$

where

$$f(w,X)=\|w-X\|^2/2 \quad \nabla_w f(w,X)=w-X$$

answer

- The SGD algorithm for solving the above problem is

$$w_{k+1}=w_k-\alpha_k \nabla_w f\left(w_k,x_k\right)=w_k-\alpha_k\left(w_k-x_k\right)$$

- Note:
  - It is **the same as the mean estimation** algorithm we presented before.
  - That mean estimation algorithm is a **special SGD** algorithm.

convergence

Core：证明SGD是RM算法，就可以证明其是收敛的

We next show that SGD is a special RM algorithm. Then, the convergence naturally follows.
The aim of SGD is to minimize

$$J(w)=\mathbb{E}[f(w,X)]$$

This problem can be converted to a root-finding problem:

$$\nabla_w J(w)=\mathbb{E}\left[\nabla_w f(w,X)\right]=0$$

Let

$$g(w)=\nabla_w J(w)=\mathbb{E}\left[\nabla_w f(w,X)\right]$$

Then, the aim of SGD is to find the root of $g(w)=0$.

用RM算法解决上述问题

What we can measure is

$$\begin{aligned}
\tilde{g}(w,\eta) & =\nabla_w f(w,x) \\
& =\underbrace{\mathbb{E}\left[\nabla_w f(w,X)\right]}_{g(w)}+\underbrace{\nabla_w f(w,x)-\mathbb{E}\left[\nabla_w f(w,X)\right]}_{\eta}.
\end{aligned}$$

Then, the RM algorithm for solving $g(w)=0$ is

$$w_{k+1}=w_k-a_k \tilde{g}\left(w_k,\eta_k\right)=w_k-a_k \nabla_w f\left(w_k,x_k\right).$$

- It is exactly the SGD algorithm.
- Therefore, SGD is a **special RM algorithm**.

pattern

由于梯度具有随机性，收敛是否存在随机性呢？即 w_k是否会绕一大圈再回到 w^*

不存在

通过**相对误差**来证明

$$\delta_k \doteq \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w f(w_k, X)]|}$$

Since $\mathbb{E}[\nabla_w f(w^*, X)] = 0$, we further have

$$\delta_k = \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w f(w_k, X)] - \mathbb{E}[\nabla_w f(w^*, X)]|} = \frac{|\nabla_w f(w_k, x_k) - \mathbb{E}[\nabla_w f(w_k, X)]|}{|\mathbb{E}[\nabla_w^2 f(\tilde{w}_k, X)(w_k - w^*)]|}.$$

上式用了中值定理

where the last equality is due to the mean value theorem and $\tilde{w}_k \in [w_k, w^*]$

Suppose f is strictly convex such that

$$\nabla_w^2 f \geq c > 0$$

for all w, X, where c is a **positive bound**.
Then, the denominator of δ_k becomes

$$\begin{aligned} \left|\mathbb{E}\left[\nabla_w^2 f(\tilde{w}_k, X)(w_k - w^*)\right]\right| &= \left|\mathbb{E}\left[\nabla_w^2 f(\tilde{w}_k, X)\right](w_k - w^*)\right| \\ &= \left|\mathbb{E}\left[\nabla_w^2 f(\tilde{w}_k, X)\right]\right| |(w_k - w^*)| \geq c|w_k - w^*| \end{aligned}$$

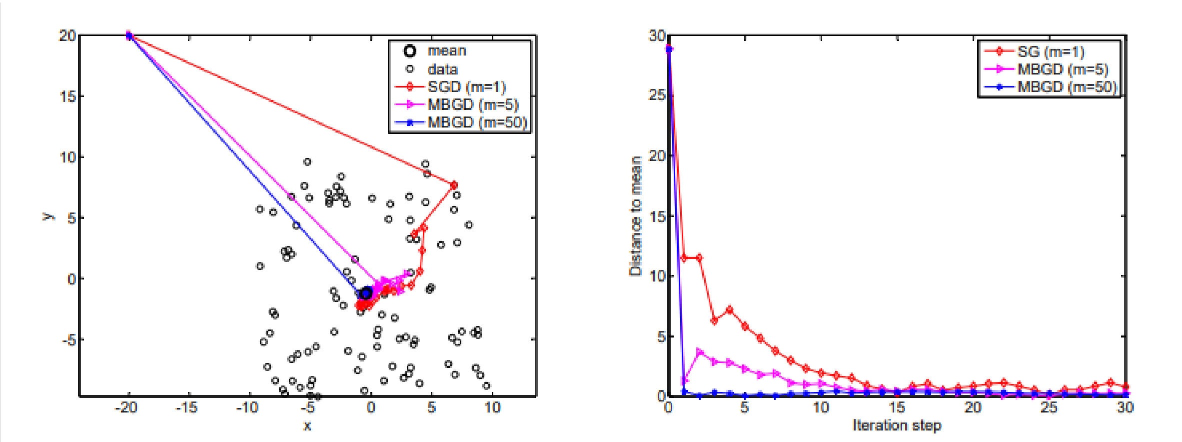
Substituting the above inequality to δ_k gives

$$\delta_k \leq \frac{|\overbrace{\nabla_w f(w_k, x_k)}^{\text{stochastic gradient}} - \overbrace{\mathbb{E}[\nabla_w f(w_k, X)]}^{\text{true gradient}}|}{\underbrace{c|w_k - w^*|}_{\text{distance to the optimal solution}}}.$$

因此，

- 当 w_k与 w^*相距较远时，分母很大，此时从另外一个角度而言，相对误差很小，分子很小，因此随机梯度和真实梯度基本一致，意味着算法的趋势朝着真实值，也就是 w^*前进
- 当 w_k与 w^*相距较近时，分母很小，此时从另外一个角度而言，相对误差较大，分子较大，此时则存在随机性，即其不一定能够准确收敛到 w^*

因此证明了，不会有收敛的随机性！



- Although the initial guess of the mean is **far away from the true value**, the SGD estimate can **approach** the neighborhood of the true value **fast**.
- When the estimate is **close to the true value**, it exhibits certain **randomness** but still approacwhes the true value gradually

Temporal-Difference Learning

Model-free

迭代式算法

Motivating example

三个例子

First, consider the simple mean estimation problem: calculate

$$w=\mathbb{E}[X]$$

based on some iid samples {x} of X .

- By writing $g(w)=w-\mathbb{E}[X]$, we can reformulate the problem to a root-finding problem

$$g(w)=0.$$

- Since we can only obtain samples {x} of X , the noisy observation is

$$\tilde{g}(w, \eta)=w-x=(w-\mathbb{E}[X])+(\mathbb{E}[X]-x) \doteq g(w)+\eta.$$

- Then, according to the last lecture, we know the RM algorithm for solving $g(w)=0$ is

$$w_{k+1}=w_k-\alpha_k \tilde{g}\left(w_k, \eta_k\right)=w_k-\alpha_k\left(w_k-x_k\right)$$

Second

$$w=\mathbb{E}[v(X)]$$

$$\begin{aligned} g(w)&=w-\mathbb{E}[v(X)] \\ \tilde{g}(w, \eta)&=w-v(x)=(w-\mathbb{E}[v(X)])+(\mathbb{E}[v(X)]-v(x)) \doteq g(w)+\eta \end{aligned}$$

$$w_{k+1}=w_k-\alpha_k \tilde{g}\left(w_k, \eta_k\right)=w_k-\alpha_k\left[w_k-v\left(x_k\right)\right]$$

Finally

$$w=\mathbb{E}[R+\gamma v(X)]$$

$$w_{k+1}=w_k-\alpha_k \tilde{g}\left(w_k, \eta_k\right)=w_k-\alpha_k\left[w_k-\left(r_k+\gamma v\left(x_k\right)\right)\right]$$

TD learing of state values

没有模型的情况下，求解贝尔曼公式

求解贝尔曼公式其实就是求解RM算法

Description

The data/experience required by the algorithm:

- $(s_0, r_1, s_1, \ldots, s_t, r_{t+1}, s_{t+1}, \ldots)$ or $\{(s_t, r_{t+1}, s_{t+1})\}_t$ generated following the given policy π.

The TD algorithm can be annotated as

$$\underbrace{v_{t+1}(s_t)}_{\text{new estimate}} = \underbrace{v_t(s_t)}_{\text{current estimate}} - \alpha_t(s_t)\overbrace{\left[v_t(s_t) - \underbrace{[r_{t+1} + \gamma v_t(s_{t+1})]}_{\text{TD target } \bar{v}_t}\right]}^{\text{TD error } \delta_t},$$

Here,

$$\bar{v}_t \doteq r_{t+1} + \gamma v(s_{t+1})$$

is called the TD target.

$$\delta_t \doteq v(s_t) - [r_{t+1} + \gamma v(s_{t+1})] = v(s_t) - \bar{v}_t$$

is called the TD error.
It is clear that the **new estimate** $v_{t+1}(s_t)$ is a **combination** of the **current estimate** $v_t(s_t)$ and the **TD error.**

TD target 实际就是 v_π，即策略的state value，因为没有模型，最开始并不知道完整的state value，需要不断采样，不断更新，到最后的state value（而不是最优策略）

That is because the algorithm drives $v(s_t)$ towards $\bar{v}_t$.

TD error

- It is a **difference** between two consequent time steps.
- It reflects the deficiency between v_t and v_π. To see that, denote

$$\delta_{\pi,t} \doteq v_\pi(s_t) - [r_{t+1} + \gamma v_\pi(s_{t+1})]$$

The idea of the algorithm

Q: What does this TD algorithm do mathematically?

A: It solves the Bellman equation of a given policy π without model.

引入新的贝尔曼公式

First, a new expression of the Bellman equation.
The definition of state value of π is

$$v_\pi(s) = \mathbb{E}[R + \gamma G \mid S = s], \quad s \in \mathcal{S}$$

where G is discounted return. Since

$$\mathbb{E}[G \mid S = s] = \sum_a \pi(a \mid s) \sum_{s'} p(s' \mid s, a) v_\pi(s') = \mathbb{E}[v_\pi(S') \mid S = s],$$

where S' is the next state, we can rewrite (4) as

$$v_\pi(s)=\mathbb{E}\left[R+\gamma v_\pi\left(S^{\prime}\right) \mid S=s\right], \quad s \in \mathcal{S}$$

Equation (5) is another expression of the Bellman equation. It is sometimes called the **Bellman expectation equation**, an important tool to design and analyze TD algorithms.

TD算法是计算贝尔曼公式的一个RM算法

Second, solve the Bellman equation in (5) using the RM algorithm.

In particular, by defining

$$g(v(s))=v(s)-\mathbb{E}\left[R+\gamma v_\pi\left(S^{\prime}\right) \mid s\right],$$

we can rewrite (5) as

$$g(v(s))=0$$

Since we can only obtain the samples r and $s^{\prime}$ of R and $S^{\prime}$, the noisy observation we have is

$$\begin{aligned}
\tilde{g}(v(s)) & =v(s)-\left[r+\gamma v_\pi\left(s^{\prime}\right)\right] \\
& =\underbrace{\left(v(s)-\mathbb{E}\left[R+\gamma v_\pi\left(S^{\prime}\right) \mid s\right]\right)}_{g(v(s))}+\underbrace{\left(\mathbb{E}\left[R+\gamma v_\pi\left(S^{\prime}\right) \mid s\right]-\left[r+\gamma v_\pi\left(s^{\prime}\right)\right]\right)}_\eta .
\end{aligned}$$

therefore

$$\begin{aligned}
v_{k+1}(s) & =v_k(s)-\alpha_k \tilde{g}\left(v_k(s)\right) \\
& =v_k(s)-\alpha_k\left(v_k(s)-\left[r_k+\gamma v_\pi\left(s_k^{\prime}\right)\right]\right), \quad k=1,2,3, \ldots
\end{aligned}$$

To **remove the two assumptions** in the RM algorithm, we can modify it

- One modification is that $\left\{\left(s, r, s^{\prime}\right)\right\}$ is **changed to** $\left\{\left(s_t, r_{t+1}, s_{t+1}\right)\right\}$ so that the algorithm can **utilize the sequential samples** in an episode.
- Another modification is that $v_\pi\left(s^{\prime}\right)$ is **replaced by an estimate of it** because we don't know it in advance.

convergence

Theorem (Convergence of TD Learning)
By the TD algorithm (1), $v_t(s)$ converges with probability 1 to $v_\pi(s)$ for all $s \in \mathcal{S}$ as $t \rightarrow \infty$ if $\sum_t \alpha_t(s)=\infty$ and $\sum_t \alpha_t^2(s)<\infty$ for all $s \in \mathcal{S}$.

Comparison

TD/Sarsa learning	MC learning
Online: TD learning is online. It can update the state/action values immediately after receiving a reward.	Offline: MC learning is offline. It has to wait until an episode has been completely collected.
Continuing tasks: Since TD learning is online, it can handle both episodic and continuing tasks.	Episodic tasks is offline, it can only handle episodic tasks that has terminate states.

TD/Sarsa learning	MC learning
Bootstrapping: TD bootstraps because the update of a value relies on the previous estimate of this value. Hence, it requires initial guesses.	Non-bootstrapping: MC is not bootstrapping, because it can directly estimate state/action values without any initial guess.
Low estimation variance: TD has lower than MC because there are fewer random variables. For instance, Sarsa requires $R_{t+1}, S_{t+1}, A_{t+1}$.	High estimation variance: To estimate $q_\pi\left(s_t, a_t\right)$, we need samples of $R_{t+1}+\gamma R_{t+2}+\gamma^2 R_{t+3}+\ldots$ Suppose the length of each episode is L. There are $\|\mathcal{A}\|^L$ possible episodes.

Sarsa

Description

Core Idea：that is to use **an algorithm to solve the Bellman equation of a given policy.**

The **complication emerges** when we try to **find optimal policies and work efficiently**

Next, we introduce, Sarsa, an algorithm that can directly estimate **action values.**

估计action value，从而更新，改进策略，Policy evaluation+Policy improvement

如何估计呢？not model，need data

也是求解了一个action value相关的贝尔曼公式！

收敛性：$q_t(s,a) -> q_\pi(s,a)$

在policy evaluation（update q-value） 后立马policy improvement（update policy）

First, our aim is to **estimate the action values of a given policy** π **.**
Suppose we have some **experience** $\left\{\left(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}\right)\right\}_t$.（Sarsa）

We can use the following **Sarsa algorithm** to estimate the action values:

$$\begin{aligned} q_{t+1}\left(s_t, a_t\right) & =q_t\left(s_t, a_t\right)-\alpha_t\left(s_t, a_t\right)\left[q_t\left(s_t, a_t\right)-\left[r_{t+1}+\gamma q_t\left(s_{t+1}, a_{t+1}\right)\right]\right], \\ q_{t+1}(s, a) & =q_t(s, a), \quad \forall(s, a) \neq\left(s_t, a_t\right),\end{aligned}$$

where $t=0,1,2, \ldots$

NOTE：第二个条件是当某个state action pair没被访问时，**将保持原状**

- $q_t\left(s_t, a_t\right)$ is an **estimate** of $q_\pi\left(s_t, a_t\right)$;
- $\alpha_t\left(s_t, a_t\right)$ is the **learning rate depending on** s_t, a_t **.**

如何policy improvement?

For each episode, do

- If the current s_t is not the target state, do

- Collect the experience $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$: In particular, take action a_t following $\pi_t(s_t)$, generate r_{t+1}, s_{t+1} , and then take action a_{t+1} following $\pi_t(s_{t+1})$.
- **Update q-value（policy evaluation）：**

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\left[q_t(s_t, a_t) - \left[r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})\right]\right]$$

- **Update policy（policy ）：**

$$\pi_{t+1}(a \mid s_t) = 1 - \frac{\epsilon}{|\mathcal{A}|}(|\mathcal{A}| - 1) \text{ if } a = \arg\max_a q_{t+1}(s_t, a)$$
$$\pi_{t+1}(a \mid s_t) = \frac{\epsilon}{|\mathcal{A}|} \text{ otherwise}$$

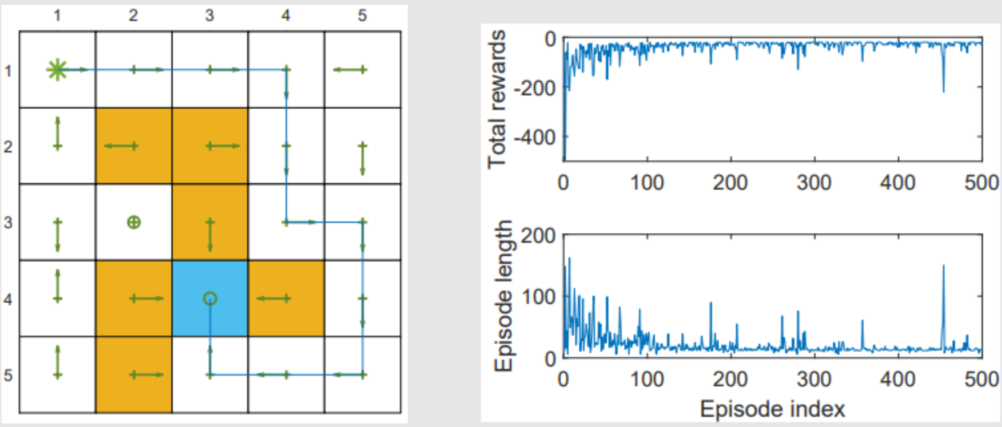
Example

The task is to find a good path **from a specific starting state** to the **target state**

So:

Results:

- The left figures above show the final policy obtained by Sarsa.
  - Not all states have the optimal policy.
- The right figures show the total reward and length of every episode.
  - The metric of total reward per episode will be frequently used.



Expected Sarsa

Description

A variant of Sarsa is the Expected Sarsa algorithm:

$$\begin{aligned} q_{t+1}(s_t, a_t) &= q_t(s_t, a_t) - \alpha_t(s_t, a_t)\left[q_t(s_t, a_t) - (r_{t+1} + \gamma \mathbb{E}\left[q_t(s_{t+1}, A)\right])\right], \\ q_{t+1}(s, a) &= q_t(s, a), \quad \forall (s, a) \neq (s_t, a_t), \end{aligned}$$

where

$$\mathbb{E}\left[q_t(s_{t+1}, A)\right]) = \sum_a \pi_t(a \mid s_{t+1}) q_t(s_{t+1}, a) \doteq v_t(s_{t+1})$$

is the **expected value** of $q_t\left(s_{t+1}, a\right)$ under policy π_t .
Compared to Sarsa:

- The **TD target** is changed from $r_{t+1}+\gamma q_t\left(s_{t+1}, a_{t+1}\right)$ as in Sarsa to $r_{t+1}+\gamma \mathbb{E}\left[q_t\left(s_{t+1}, A\right)\right]$ as in Expected Sarsa.
- Need more **computation**. But it is beneficial in the sense that it **reduces the estimation variances** because it **reduces random variables** in Sarsa from $\left\{s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}\right\}$ to $\left\{s_t, a_t, r_{t+1}, s_{t+1}\right\}$.（因为遍历了所有的action）

n-step Sarsa

n -step Sarsa: can unify Sarsa and Monte Carlo learning The definition of action value is

$$
\begin{aligned}
& \text{Sarsa} \longleftarrow \quad G_t^{(1)}=R_{t+1}+\gamma q_\pi\left(S_{t+1}, A_{t+1}\right), \\
& G_t^{(2)}=R_{t+1}+\gamma R_{t+2}+\gamma^2 q_\pi\left(S_{t+2}, A_{t+2}\right), \\
& \vdots \\
& n\text{-step Sarsa} \longleftarrow \quad G_t^{(n)}=R_{t+1}+\gamma R_{t+2}+\cdots+\gamma^n q_\pi\left(S_{t+n}, A_{t+n}\right), \\
& \vdots \\
& \text{MC} \longleftarrow \quad G_t^{(\infty)}=R_{t+1}+\gamma R_{t+2}+\gamma^2 R_{t+3}+\ldots
\end{aligned}
$$

- Sarsa aims to solve

$$
q_\pi(s, a)=\mathbb{E}\left[G_t^{(1)} \mid s, a\right]=\mathbb{E}\left[R_{t+1}+\gamma q_\pi\left(S_{t+1}, A_{t+1}\right) \mid s, a\right].
$$

- MC learning aims to solve

$$
q_\pi(s, a)=\mathbb{E}\left[G_t^{(\infty)} \mid s, a\right]=\mathbb{E}\left[R_{t+1}+\gamma R_{t+2}+\gamma^2 R_{t+3}+\ldots \mid s, a\right].
$$

- An intermediate algorithm called n -step Sarsa aims to solve
- $$q_\pi(s, a)=\mathbb{E}\left[G_t^{(n)} \mid s, a\right]=\mathbb{E}\left[R_{t+1}+\gamma R_{t+2}+\cdots+\gamma^n q_\pi\left(S_{t+n}, A_{t+n}\right) \mid s, a\right]-$$

  The algorithm of n -step Sarsa is

$$
q_{t+1}\left(s_t, a_t\right)=q_t\left(s_t, a_t\right)-\alpha_t\left(s_t, a_t\right)\left[q_t\left(s_t, a_t\right)-\left[r_{t+1}+\gamma r_{t+2}+\cdots+\gamma^n q_t\left(s_{t+n}, a_{t+n}\right)\right]\right].
$$

n -step Sarsa is **more general** because it becomes the (one-step) Sarsa algorithm when $n=1$ and the MC learning algorithm when $n=\infty$.

Q-learning

Description

Core Idea：求解贝尔曼最优公式

$$
\begin{aligned}
& q_{t+1}\left(s_t, a_t\right)=q_t\left(s_t, a_t\right)-\alpha_t\left(s_t, a_t\right)\left[q_t\left(s_t, a_t\right)-\left[r_{t+1}+\gamma \max _{a \in \mathcal{A}} q_t\left(s_{t+1}, a\right)\right]\right], \\
& q_{t+1}(s, a)=q_t(s, a), \quad \forall(s, a) \neq\left(s_t, a_t\right)
\end{aligned}
$$

引入了 behavior policy，target policy

offline 怎么更新 pi_b(看原书)

off-Policy Vs on-policy

off-policy：比如我behavior policy可以用探索性比较强的，比如action的选择可以均匀分布，以此来得到更多experience

而对应的target policy为了得到最优的策略，直接选择greedy policy，而不是 ε-greedy，因为此时我已经不缺探索性了

on-policy：而behavior policy=target policy，比如 Sarsa，uses ε-greedy policies to **maintain certain exploration ability**, 但由于**一般设置较小，其对应的探索能力有限，因为如果设置较大，最后优化效果并不好**

Value Function

Example

Core Idea：用曲线拟合替代tables 表示state value

最简单：直线拟合

$$\hat{v}(s,w) = as + b = \underbrace{[s,1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{w} = \phi^T(s)w$$

where

- w is the parameter vector
- $\phi(s)$ the feature vector of s
- $\hat{v}(s,w)$ is linear in w

当然，也可以用二阶，三阶，高阶拟合

$$\hat{v}(s,w) = as^2 + bs + c = \underbrace{[s^2,s,1]}_{\phi^T(s)} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{w} = \phi^T(s)w.$$

优点：存储方面，存储的维数大幅减少，

同时，泛化能力很好

When a state s is visited, the parameter w is updated so that the values of some other unvisited states can also be updated.

Algorithm for state value estimation

Objective function

The objective function is

$$J(w) = \mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S,w)\right)^2\right]$$

- Our goal is to find the best w that can **minimize** $J(w)$.

- The **expectation** is with respect to the random variable $S \in \mathcal{S}$. What is the **probability distribution** of S ?
- This is often confusing because we have not discussed the **probability distribution of states** so far in this book.
- There are **several ways to define** the probability distribution of S.

first way：uniform distribution

$$J(w) = \mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S, w)\right)^2\right] = \frac{1}{|\mathcal{S}|}\sum_{s \in \mathcal{S}}\left(v_\pi(s) - \hat{v}(s, w)\right)^2$$

缺点：有些状态离target area较远，并不重要，被访问次数较少，对应的权重应小

second way：stationary distribution.

Let $\{d_\pi(s)\}_{s \in \mathcal{S}}$ denote the stationary distribution of the Markov process under policy π. By definition, $d_\pi(s) \geq 0$ and $\sum_{s \in \mathcal{S}} d_\pi(s) = 1$.

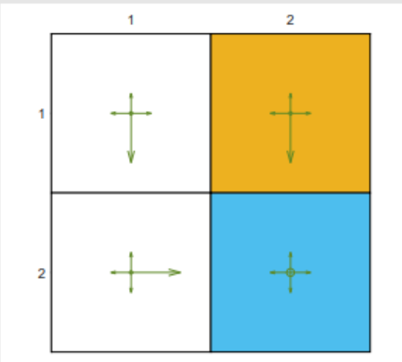
$$J(w) = \mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S, w)\right)^2\right] = \sum_{s \in \mathcal{S}} d_\pi(s)\left(v_\pi(s) - \hat{v}(s, w)\right)^2$$

为什么叫稳态？因为要足够多次的step，等系统稳定后，基本不再改变时

Illustrative example:

- Given a policy shown in the figure.
- Let $n_\pi(s)$ denote the number of times that s has been visited in a very long episode generated by π.
- Then, $d_\pi(s)$ can be approximated by

$$d_\pi(s) \approx \frac{n_\pi(s)}{\sum_{s' \in \mathcal{S}} n_\pi(s')}$$



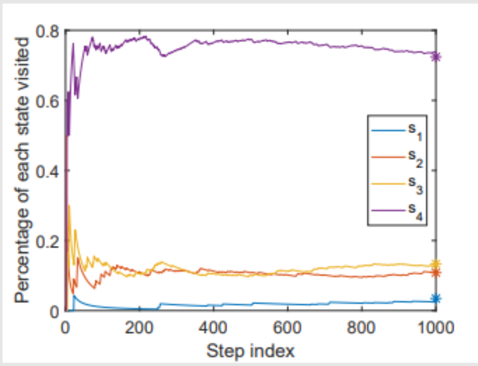


Figure: Long-run behavior of an ϵ-greedy policy with $\epsilon = 0.5$.

可以证明，最后的 $d_\pi(s)$为转移矩阵的特征向量

[Book All-in-one.pdf]

Optimization algorithms

那究竟如何优化呢?

最小化：梯度下降

$$w_{k+1} = w_k - \alpha_k \nabla_w J(w_k)$$

The true gradient is

$$\begin{aligned} \nabla_w J(w) &= \nabla_w \mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S, w)\right)^2\right] \\ &= \mathbb{E}\left[\nabla_w \left(v_\pi(S) - \hat{v}(S, w)\right)^2\right] \\ &= 2\mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S, w)\right)\left(-\nabla_w \hat{v}(S, w)\right)\right] \\ &= -2\mathbb{E}\left[\left(v_\pi(S) - \hat{v}(S, w)\right)\nabla_w \hat{v}(S, w)\right] \end{aligned}$$

use the **stochastic gradient**

$$w_{t+1} = w_t + \alpha_t \left(v_\pi(s_t) - \hat{v}(s_t, w_t)\right)\nabla_w \hat{v}(s_t, w_t)$$

系数2已经合并到常数里了

注意到 v_π未知，因此要进行替代

两种替代方式：Monte Carlo learning和TD Learning（但是此种替代并不严谨，优化的并不是上述true error，而是projected Bellman error）

- First, Monte Carlo learning with function approximation
  Let g_t be the **discounted return starting from s_t in the episode.** Then, g_t can be used to approximate $v_\pi(s_t)$. The algorithm becomes

$$w_{t+1} = w_t + \alpha_t \left(g_t - \hat{v}(s_t, w_t)\right)\nabla_w \hat{v}(s_t, w_t)$$

- Second, TD learning with function approximation
  By the spirit of TD learning, $r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t)$ can be viewed as an approximation of $v_\pi(s_t)$. Then, the algorithm becomes

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \hat{v}(s_{t+1}, w_t) - \hat{v}(s_t, w_t)\right]\nabla_w \hat{v}(s_t, w_t)$$

Selection of function approximators

究竟如何选择相关函数，1阶？2阶?

一阶的好处：简洁，参数少

- The theoretical properties of the TD algorithm in the linear case can be much better understood than in the nonlinear case.
- Linear function approximation is still powerful in the sense that the **tabular representation is merely a special case of linear function approximation.**

坏处：难以拟合非线性情况

Recall that the TD-Linear algorithm is

$$w_{t+1} = w_t + \alpha_t \left[r_{t+1} + \gamma \phi^T(s_{t+1}) w_t - \phi^T(s_t) w_t\right]\phi(s_t),$$

- When $\phi(s_t) = e_s$, the above algorithm becomes

$$w_{t+1} = w_t + \alpha_t (r_{t+1} + \gamma w_t(s_{t+1}) - w_t(s_t)) e_{s_t}.$$

This is a vector equation that merely updates the s_t th entry of w_t .

- Multiplying $e_{s_t}^T$ on both sides of the equation gives

$$w_{t+1}\left(s_t\right)=w_t\left(s_t\right)+\alpha_t\left(r_{t+1}+\gamma w_t\left(s_{t+1}\right)-w_t\left(s_t\right)\right),$$

which is exactly the tabular TD algorithm.

Examples

Summary of the story

theoretical analysis

- The algorithm

$$w_{t+1}=w_t+\alpha_t\left[r_{t+1}+\gamma \hat{v}\left(s_{t+1}, w_t\right)-\hat{v}\left(s_t, w_t\right)\right] \nabla_w \hat{v}\left(s_t, w_t\right)$$

does not minimize the following objective function:

$$J(w)=\mathbb{E}\left[\left(v_\pi(S)-\hat{v}(S, w)\right)^2\right]$$

Different objective functions:

- Objective function 1: **True value error**

$$J_E(w)=\mathbb{E}\left[\left(v_\pi(S)-\hat{v}(S, w)\right)^2\right]=\left\|\hat{v}(w)-v_\pi\right\|_D^2$$

- Objective function 2: **Bellman error**

$$J_{BE}(w)=\left\|\hat{v}(w)-\left(r_\pi+\gamma P_\pi \hat{v}(w)\right)\right\|_D^2 \doteq\left\|\hat{v}(w)-T_\pi(\hat{v}(w))\right\|_D^2$$

where $T_\pi(x) \doteq r_\pi+\gamma P_\pi x$

- Objective function 3: **Projected Bellman error**

$$J_{PBE}(w)=\left\|\hat{v}(w)-M T_\pi(\hat{v}(w))\right\|_D^2$$

where M is a **projection matrix.**（投影变换矩阵，即无论 w怎么选，两者都有距离时，该投影变换矩阵能将二者error变为0）
The TD-Linear algorithm minimizes the projected Bellman error.
Details can be found in the book.

Sarsa with function approximation

Core Idea：利用Sarsa估计action value

So far, we merely considered the problem of **state value estimation**. That is we hope

$$\hat{v} \approx v_\pi$$

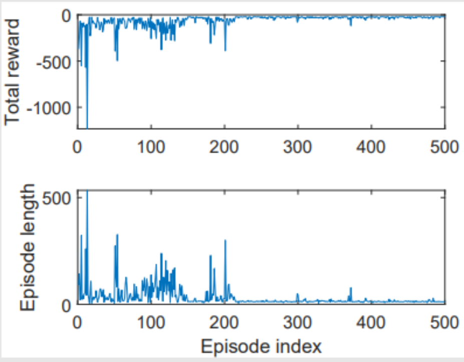
To search for optimal policies, we need to **estimate action values.**
The Sarsa algorithm with value function approximation is

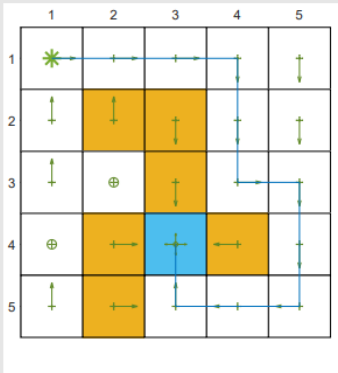
$$w_{t+1}=w_t+\alpha_t\left[r_{t+1}+\gamma\hat{q}\left(s_{t+1},a_{t+1},w_t\right)-\hat{q}\left(s_t,a_t,w_t\right)\right]\nabla_w\hat{q}\left(s_t,a_t,w_t\right).$$

This is **the same as** the algorithm we introduced previously in this lecture **except that** $\hat{v}$ **is replaced by** $\hat{q}$.

Illustrative example:

- Sarsa with *linear function* approximation.
- $\gamma=0.9,\ \epsilon=0.1,\ r_{\text{boundary}}=r_{\text{forbidden}}=-10,\ r_{\text{target}}=1,\ \alpha=0.001.$





Q-learning with function approximation

Core Idea：利用q-learning的方式更新action value

The q-value update rule is

$$w_{t+1}=w_t+\alpha_t\left[r_{t+1}+\gamma\max_{a\in\mathcal{A}(s_{t+1})}\hat{q}\left(s_{t+1},a,w_t\right)-\hat{q}\left(s_t,a_t,w_t\right)\right]\nabla_w\hat{q}\left(s_t,a_t,w_t\right)$$

which is the same as Sarsa except that $\hat{q}\left(s_{t+1},a_{t+1},w_t\right)$ is replaced by $\max_{a\in\mathcal{A}(s_{t+1})}\hat{q}\left(s_{t+1},a,w_t\right)$

Deep Q-learning

DQN：原本算法计算变量梯度，涉及到神经网络底层，因此要进行改进

Deep Q-learning aims to minimize the objective function/loss function:

$$J(w)=\mathbb{E}\left[\left(R+\gamma\max_{a\in\mathcal{A}(S')}\hat{q}\left(S',a,w\right)-\hat{q}(S,A,w)\right)^2\right],$$

where (S,A,R,S') are random variables.

- This is actually the Bellman optimality error. That is because

$$q(s,a)=\mathbb{E}\left[R_{t+1}+\gamma\max_{a\in\mathcal{A}(S_{t+1})}q\left(S_{t+1},a\right)\mid S_t=s,A_t=a\right],\quad\forall s,a$$

The value of $R+\gamma\max_{a\in\mathcal{A}(S')}\hat{q}\left(S',a,w\right)-\hat{q}(S,A,w)$ **should be zero** in the expectation sense

均匀采样的数学原因：

Policy Function

Core Idea：函数表达策略！

value-based to policy based

优化目标函数来求最优策略

θ是参数，可以是神经网络，用以计算 $\pi(a|s)$

Basic idea of Policy gradient

The basic idea of the policy gradient is simple:

- First, metrics (or objective functions) to define **optimal policies**: $J(\theta)$, which can define optimal policies.（**定义目标函数**）
- Second, **gradient-based optimization algorithms** to search for optimal policies:（**优化目标函数**）

$$\theta_{t+1}=\theta_t+\alpha \nabla_\theta J\left(\theta_t\right)$$

Although the idea is simple, the complication emerges when we try to answer the following questions.

- What **appropriate metrics** should be used?（选择什么函数合适）
- How to calculate the gradients of the metrics?（如何优化？）

Metrics to define optimal policies

Two metrics（两种优化函数）

都是关于 π的函数，且 π是 θ的函数

The first metric is the **average state value** or simply called **average value**

Average value

求出每个state的state value然后求mean

$$\bar{v}_\pi=\sum_{s \in \mathcal{S}} d(s) v_\pi(s)$$

- $\bar{v}_\pi$ is a **weighted average** of the state values.
- $d(s) \geq 0$ is the **weight** for state s .
- Since $\sum_{s \in \mathcal{S}} d(s)=1$, we can interpret $d(s)$ as a probability distribution. Then, the metric can be written as

$$\bar{v}_\pi=\mathbb{E}\left[v_\pi(S)\right]$$

where $S \sim d$.

How to select the **distribution d**? There are **two cases.**

- The first case is that d is **independent** of the policy π.（另外一种就是依赖于决策）

不依赖又分两种：均匀（equally important）和非均匀（ only interested in a specific state s_0）

we only care about the long-term return **starting from** s_0

$$d_0\left(s_0\right)=1, \quad d_0\left(s \neq s_0\right)=0$$

- The second case is that d depends on the policy π .
- A common way to select d as $d_\pi(s)$, which is the stationary distribution under π .
- One basic property of d_π is that it satisfies

$$d_\pi^T P_\pi=d_\pi^T$$

where P_π is the **state transition probability matrix.**

- The interpretation of selecting d_π is as follows.
- If one state is frequently visited in the long run, it is **more important and deserves more weight**.
- If a state is **hardly visited**, then we give it less weight.

等价描述：

$$J(\theta)=\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_{t+1}\right]$$

Average reward

求出每个state的immediate reward然后求mean

$$\bar{r}_\pi \doteq \sum_{s \in \mathcal{S}} d_\pi(s) r_\pi(s)=\mathbb{E}\left[r_\pi(S)\right]$$

where $S \sim d_\pi$. Here,

$$r_\pi(s) \doteq \sum_{a \in \mathcal{A}} \pi(a \mid s) r(s, a)$$

is the average of the one-step immediate reward that can be obtained starting from state s , and

$$r(s, a)=\mathbb{E}[R \mid s, a]=\sum_r r p(r \mid s, a)$$

- The weight d_π is the **stationary distribution**.
- As its name suggests, $\bar{r}_\pi$ is simply a weighted average of the one-step immediate rewards.

有一个等价描述

- Suppose an agent follows a given policy and generate a trajectory with the rewards as $\left(R_{t+1}, R_{t+2}, \ldots\right)$.
- The average single-step reward along this trajectory is

$$\begin{aligned}
& \lim _{n \rightarrow \infty} \frac{1}{n} \mathbb{E}\left[R_{t+1}+R_{t+2}+\cdots+R_{t+n} \mid S_t=s_0\right] \\
= & \lim _{n \rightarrow \infty} \frac{1}{n} \mathbb{E}\left[\sum_{k=1}^n R_{t+k} \mid S_t=s_0\right]
\end{aligned}$$

where s_0 is the starting state of the trajectory.

Proof:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{k=1}^{n}R_{t+k}\mid S_t=s_0\right]&=\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\sum_{k=1}^{n}R_{t+k}\right]\\&=\sum_s d_\pi(s)r_\pi(s)\\&=\bar{r}_\pi\end{aligned}$$

- Intuitively, $\bar{r}_\pi$ is more **short-sighted** because it merely considers the immediate rewards, whereas $\bar{v}_\pi$ considers the **total reward overall steps.**
- However, the two metrics are **equivalent** to each other.（两个metric等价，因为当一个达到极值时，另一个必然也到达极值） In the discounted case where $\gamma<1$, it holds that

$$\bar{r}_\pi=(1-\gamma)\bar{v}_\pi$$

Gradients of the metrics

Core Idea：如何求梯度?

Summary of the results about the gradients:

$$\begin{aligned}\nabla_\theta J(\theta)&=\sum_{s\in\mathcal{S}}\eta(s)\sum_{a\in\mathcal{A}}\nabla_\theta\pi(a\mid s,\theta)q_\pi(s,a)\\&=\mathbb{E}\left[\nabla_\theta\ln\pi(A\mid S,\theta)q_\pi(S,A)\right]\end{aligned}$$

where

- $J(\theta)$ can be $\bar{v}_\pi,\bar{r}_\pi$, or $\bar{v}_\pi^0$.
- "=" may denote **strict equality**, **approximation**, or **proportional to**.
- η is a **distribution** or **weight** of the states.

Some remarks: Because we need to calculate $\ln\pi(a\mid s,\theta)$, we must **ensure** that for all s, a, θ

$$\pi(a\mid s,\theta)>0$$

- This can be archived by using **softmax functions** that can normalize the entries in a vector from $(-\infty,+\infty)$ to (0,1) .

Gradient-ascent algorithm

Core Idea：具体怎么优化函数?

对于期望，利用采样近似；对于未知数，比如 $q_\pi(s_t,a_t)$，也要近似替代，两种方法替代

1：Monte-Carlo 对应Reinfoce

2：TD methods 对应 **Actor-Critic**

$$\theta_{t+1} = \theta_t + \alpha \underbrace{\left(\frac{q_t(s_t, a_t)}{\pi(a_t \mid s_t, \theta_t)} \right)}_{\beta_t} \nabla_\theta \pi(a_t \mid s_t, \theta_t)$$

The coefficient β_t can well balance **exploration** and **exploitation.**

- First, β_t is **proportional** to $q_t(s_t, a_t)$.
- If $q_t(s_t, a_t)$ is great, then β_t is great.（**即return大的action，后续选到该action的概率就大！体现剥削性**）
- Therefore, the algorithm intends to **enhance actions with greater values**.
- Second, β_t is **inversely proportional** to $\pi(a_t \mid s_t, \theta_t)$.
- If $\pi(a_t \mid s_t, \theta_t)$ is small, then β_t is large.（即其他action 本身概率小的话，则后续选到他的概率会增大，体现探索性）
- Therefore, the algorithm intends to **explore actions that have low probabilities**.

伪代码！

Pseudocode: Policy Gradient by Monte Carlo (REINFORCE)

Initialization: A parameterized function $\pi(a|s,\theta)$, $\gamma \in (0,1)$, and $\alpha > 0$.
Aim: Search for an optimal policy maximizing $J(\theta)$.

For the kth iteration, do

  Select s_0 and generate an episode following $\pi(\theta_k)$. Suppose the episode is $\{s_0, a_0, r_1, \ldots, s_{T-1}, a_{T-1}, r_T\}$.

  For $t = 0, 1, \ldots, T-1$, do

    Value update: $q_t(s_t, a_t) = \sum_{k=t+1}^{T} \gamma^{k-t-1} r_k$

    Policy update: $\theta_{t+1} = \theta_t + \alpha \nabla_\theta \ln \pi(a_t|s_t, \theta_t) q_t(s_t, a_t)$

  $\theta_k = \theta_T$

Actor-Critic Methods

The Simplest AC（QAC）

实际是Policy gradient，只不过结合了value function

Core Idea：**利用TD估计，称为actor-critic**

何为actor：policy update

何为critic：policy evaluation

对应运用TD算法估计action-value的

The simplest actor-critic algorithm (QAC)

Aim: Search for an optimal policy by maximizing $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\pi(a|s_t, \theta_t)$, observe r_{t+1}, s_{t+1}, and then generate a_{t+1} following $\pi(a|s_{t+1}, \theta_t)$.

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \big[r_{t+1} + \gamma q(s_{t+1}, a_{t+1}, w_t) - q(s_t, a_t, w_t)\big] \nabla_w q(s_t, a_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \nabla_\theta \ln \pi(a_t|s_t, \theta_t) q(s_t, a_t, w_{t+1})$$

A2C

Baseline invariance

Core Idea：introduce a baseline to reduce variance

$$\nabla_\theta J(\theta) = \mathbb{E}_{S\sim\eta, A\sim\pi}\left[\nabla_\theta \ln \pi\left(A \mid S, \theta_t\right) q_\pi(S, A)\right]$$

$$= \mathbb{E}_{S\sim\eta, A\sim\pi}\left[\nabla_\theta \ln \pi\left(A \mid S, \theta_t\right)\left(q_\pi(S, A) - b(S)\right)\right]$$

NOTE：该函数为**S的函数**，且添加后**对期望没有影响，但会影响方差**

relative proof

$$\begin{aligned}
\mathbb{E}_{S\sim\eta, A\sim\pi}\left[\nabla_\theta \ln \pi\left(A \mid S, \theta_t\right) b(S)\right] &= \sum_{s\in\mathcal{S}} \eta(s) \sum_{a\in\mathcal{A}} \pi\left(a \mid s, \theta_t\right) \nabla_\theta \ln \pi\left(a \mid s, \theta_t\right) b(s) \\
&= \sum_{s\in\mathcal{S}} \eta(s) \sum_{a\in\mathcal{A}} \nabla_\theta \pi\left(a \mid s, \theta_t\right) b(s) \\
&= \sum_{s\in\mathcal{S}} \eta(s) b(s) \sum_{a\in\mathcal{A}} \nabla_\theta \pi\left(a \mid s, \theta_t\right) \\
&= \sum_{s\in\mathcal{S}} \eta(s) b(s) \nabla_\theta \sum_{a\in\mathcal{A}} \pi\left(a \mid s, \theta_t\right) \\
&= \sum_{s\in\mathcal{S}} \eta(s) b(s) \nabla_\theta 1 = 0
\end{aligned}$$

- Why? Because $\operatorname{tr}[\operatorname{var}(X)] = \mathbb{E}\left[X^T X\right] - \bar{x}^T \bar{x}$ and

$$\begin{aligned}
\mathbb{E}\left[X^T X\right] &= \mathbb{E}\left[\left(\nabla_\theta \ln \pi\right)^T \left(\nabla_\theta \ln \pi\right)\left(q_\pi(S, A) - b(S)\right)^2\right] \\
&= \mathbb{E}\left[\left\|\nabla_\theta \ln \pi\right\|^2 \left(q_\pi(S, A) - b(S)\right)^2\right]
\end{aligned}$$

Imagine b is huge (e.g., 1 millon)

b存在最优解，但由于过于复杂，我们一般用 $v_\pi(s)$ 替代

algorithm

$$
\begin{aligned}
\theta_{t+1} &= \theta_t + \alpha \mathbb{E}\left[\nabla_\theta \ln \pi\left(A \mid S, \theta_t\right)\left[q_\pi(S, A) - v_\pi(S)\right]\right] \\
&\doteq \theta_t + \alpha \mathbb{E}\left[\nabla_\theta \ln \pi\left(A \mid S, \theta_t\right) \delta_\pi(S, A)\right]
\end{aligned}
$$

where

$$
\delta_\pi(S, A) \doteq q_\pi(S, A) - v_\pi(S)
$$

is called the advantage function (why called advantage?).

当 q_π大于 $v_\pi(S)$，说明该state action-pair优秀！

进行进一步变换

$$
\begin{aligned}
\theta_{t+1} &= \theta_t + \alpha \nabla_\theta \ln \pi\left(a_t \mid s_t, \theta_t\right) \delta_t\left(s_t, a_t\right) \\
&= \theta_t + \alpha \frac{\nabla_\theta \pi\left(a_t \mid s_t, \theta_t\right)}{\pi\left(a_t \mid s_t, \theta_t\right)} \delta_t\left(s_t, a_t\right) \\
&= \theta_t + \alpha \underbrace{\left(\frac{\delta_t\left(s_t, a_t\right)}{\pi\left(a_t \mid s_t, \theta_t\right)}\right)}_{\text{step size}} \nabla_\theta \pi\left(a_t \mid s_t, \theta_t\right)
\end{aligned}
$$

同样能平衡 exploration 和exploitation，而且更好，因为分子是相对值（作差），而QAC是绝对值

进一步，对应的$\delta_\pi(S, A)$可以由TD算法估计得到

伪代码：

Advantage actor-critic (A2C) or TD actor-critic

Aim: Search for an optimal policy by maximizing $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\pi(a|s_t, \theta_t)$ and then observe r_{t+1}, s_{t+1}.

TD error (advantage function):

$$\delta_t = r_{t+1} + \gamma v(s_{t+1}, w_t) - v(s_t, w_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \delta_t \nabla_w v(s_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \delta_t \nabla_\theta \ln \pi(a_t|s_t, \theta_t)$$

由于已经是stochastic，所以不需要ε-greedy

Off-policy AC

有两个概率分布，**用其中一个概率分布计算另外一个概率分布的期望！**

Example

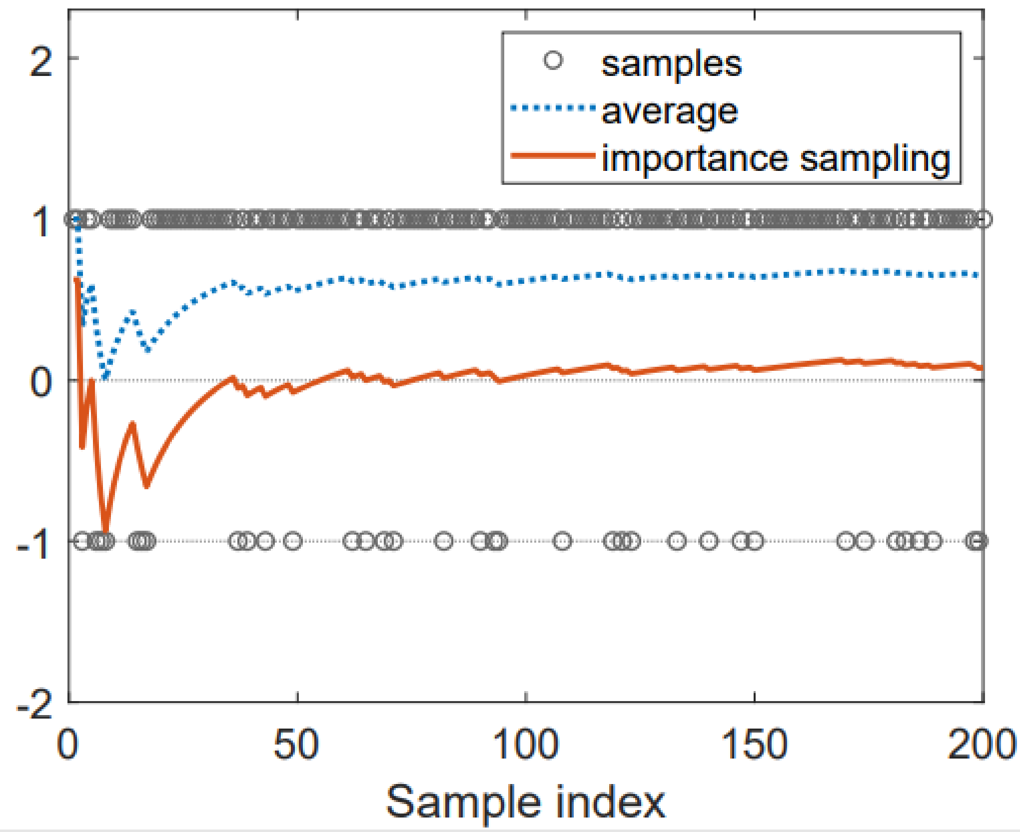
$$p_0(X=+1)=0.5, \quad p_0(X=-1)=0.5$$
$$p_1(X=+1)=0.8, \quad p_1(X=-1)=0.2$$

The expectation is

$$\mathbb{E}_{X\sim p_1}[X]=(+1)\cdot 0.8+(-1)\cdot 0.2=0.6$$

If we use the average of the samples, then without suprising

$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}_{X\sim p_1}[X]=0.6 \neq \mathbb{E}_{X\sim p_0}[X]$$



Importance sampling

Note that

$$\mathbb{E}_{X\sim p_0}[X]=\sum_x p_0(x)x=\sum_x p_1(x)\underbrace{\frac{p_0(x)}{p_1(x)}x}_{f(x)}=\mathbb{E}_{X\sim p_1}[f(X)]$$

- Thus, **we can estimate** $\mathbb{E}_{X\sim p_1}[f(X)]$ **in order to estimate** $\mathbb{E}_{X\sim p_0}[X]$.
- How to estimate $\mathbb{E}_{X\sim p_1}[f(X)]$? Easy. Let （**即对$f(x_i)$采样**）

$$\bar{f} \doteq \frac{1}{n}\sum_{i=1}^{n} f(x_i), \quad \text{where } x_i \sim p_1$$

Then,

$$\mathbb{E}_{X\sim p_1}[\bar{f}] = \mathbb{E}_{X\sim p_1}[f(X)]$$
$$\operatorname{var}_{X\sim p_1}[\bar{f}] = \frac{1}{n}\operatorname{var}_{X\sim p_1}[f(X)]$$

Therefore, $\bar{f}$ is a good approximation for $\mathbb{E}_{X\sim p_1}[f(X)] = \mathbb{E}_{X\sim p_0}[X]$

- $\frac{p_0(x_i)}{p_1(x_i)}$ is called the importance weight.
- If $p_1(x_i) = p_0(x_i)$, the importance weight is **one** and $\bar{f}$ becomes $\bar{x}$.
- If $p_0(x_i) \geq p_1(x_i), x_i$ can be more often sampled by p_0 than p_1 . The importance weight (>1) can emphasize the importance of this sample.
- 举个栗子：当$p_0 > p_1$时，说明原本这个样本概率较大，p_0较大，但是在p_1内较少出现，因此很珍贵，要加大其比重！

off-policy gradient

- Suppose β is the **behavior policy** that generates **experience samples**.
- Our aim is to use these samples to **update** a target policy π that can minimize the metric

$$J(\theta) = \sum_{s\in\mathcal{S}} d_\beta(s)v_\pi(s) = \mathbb{E}_{S\sim d_\beta}[v_\pi(S)]$$

where d_β is the **stationary distribution** under policy β .

So ,in the discounted case where $\gamma \in (0,1)$, the **gradient** of $J(\theta)$ is

$$\nabla_\theta J(\theta) = \mathbb{E}_{S\sim\rho, A\sim\beta}\left[\frac{\pi(A \mid S,\theta)}{\beta(A \mid S)}\nabla_\theta \ln \pi(A \mid S,\theta) q_\pi(S,A)\right]$$

where β is the behavior policy and ρ is a **state distribution**.

The algorithm

Off-policy actor-critic based on importance sampling

Initialization: A given behavior policy $\beta(a|s)$. A target policy $\pi(a|s,\theta_0)$ where θ_0 is the initial parameter vector. A value function $v(s,w_0)$ where w_0 is the initial parameter vector.

Aim: Search for an optimal policy by maximizing $J(\theta)$.

At time step t in each episode, do

Generate a_t following $\beta(s_t)$ and then observe r_{t+1}, s_{t+1}.

TD error (advantage function):

$$\delta_t = r_{t+1} + \gamma v(s_{t+1}, w_t) - v(s_t, w_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \frac{\pi(a_t|s_t,\theta_t)}{\beta(a_t|s_t)}\delta_t \nabla_w v(s_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \frac{\pi(a_t|s_t,\theta_t)}{\beta(a_t|s_t)}\delta_t \nabla_\theta \ln \pi(a_t|s_t,\theta_t)$$

DPG

introduction

Up to now, the policies used in the policy gradient methods are all
stochastic since $\pi(a|s,\theta)$ **> 0 for every (s, a)**.
Can we use deterministic policies in the policy gradient methods?

- Benefit: it can handle **continuous action**.（即action有无数个，此时不能用随机的action，必须确定性的action）
- Now, the deterministic policy is specifically denoted as

$$a = \mu(s,\theta) \doteq \mu(s)$$

- μ is a **mapping** from $\mathcal{S}$ to $\mathcal{A}$.（从state映射到action space，每个状态有确定性的动作）
- μ can be **represented by, for example, a neural network** with the input as s, the output as a, and the parameter as θ.
- We may write $\mu(s,\theta)$ in short as $\mu(s)$.

deterministic policy gradient

$$J(\theta) = \mathbb{E}\left[v_\mu(s)\right] = \sum_{s\in\mathcal{S}} d_0(s) v_\mu(s)$$

where $d_0(s)$ is a probability distribution satisfying $\sum_{s\in\mathcal{S}} d_0(s) = 1$.

- d_0 is selected to be independent of μ. The gradient in this case is easier to calculate.
- There are two special yet important cases of selecting d_0.
-
  - The first special case is that $d_0\left(s_0\right) = 1$ and $d_0\left(s \neq s_0\right) = 0$, where s_0 is a specific starting state of interest.
-
  - The second special case is that d_0 is the stationary distribution of a behavior policy that is different from the μ.

In the discounted case where $\gamma \in (0,1)$, the gradient of $J(\theta)$ is

$$\begin{aligned}
\nabla_\theta J(\theta) &= \sum_{s\in\mathcal{S}} \rho_\mu(s) \nabla_\theta \mu(s) \left(\nabla_a q_\mu(s,a)\right)\Big|_{a=\mu(s)} \\
&= \mathbb{E}_{S\sim\rho_\mu}\left[\nabla_\theta \mu(S) \left(\nabla_a q_\mu(S,a)\right)\big|_{a=\mu(S)}\right]
\end{aligned}$$

Here, ρ_μ is a state distribution.

algorithm

Initialization: A given behavior policy $\beta(a|s)$. A deterministic target policy $\mu(s, \theta_0)$ where θ_0 is the initial parameter vector. A value function $v(s, w_0)$ where w_0 is the initial parameter vector.

Aim: Search for an optimal policy by maximizing $J(\theta)$.

At time step t in each episode, do

Generate a_t following β and then observe r_{t+1}, s_{t+1}.

TD error:

$$\delta_t = r_{t+1} + \gamma q(s_{t+1}, \mu(s_{t+1}, \theta_t), w_t) - q(s_t, a_t, w_t)$$

Critic (value update):

$$w_{t+1} = w_t + \alpha_w \delta_t \nabla_w q(s_t, a_t, w_t)$$

Actor (policy update):

$$\theta_{t+1} = \theta_t + \alpha_\theta \nabla_\theta \mu(s_t, \theta_t) \big(\nabla_a q(s_t, a, w_{t+1})\big)|_{a=\mu(s_t)}$$

Remarks:

- This is an off-policy implementation where the behavior policy β may be different from μ.
- β can also be replaced by μ+noise.
- How to select the function to represent $q(s, a, w)$?
  - Linear function: $q(s, a, w) = \phi^T(s, a)w$ where $\phi(s, a)$ is the feature vector. Details can be found in the DPG paper.
  - Neural networks: deep deterministic policy gradient (DDPG) method.

Over!