CS5002 Discrete Structures	Prof. Rachlin
Spring 2022	February 23, 2022

Homework # 6

Assigned: Wednesday February 23, 2022
Due: Tuesday March 1, 2022 @ 11:59pm ET/Boston
−5%: Wednesday March 2, 2022 @ 11:59pm ET/Boston
−10%: Thursday March 3, 2022 @ 11:59pm ET/Boston

Instructions:

- Homework is due on Tuesday at 11:59pm ET/Boston. Homeworks received up to 24 hours late (11:59pm ET on Wednesday) will be penalized 5 percent. Homeworks received up to 48 hours late (11:59pm ET on Thursday) will be penalized 10 percent. *NO* assignment will be accepted after 48 hours.
- We expect that you will study with friends and fellow students and you are welcome to verbally discuss the problems openly. However, your solution writeup should be the product of your own mind and expressed in your own words. The TAs and I will be available to answer specific questions or address speific points of confusion but we will not verify your answers prior to submission.
- Assignments should be typed using Word or LateX, or hand-written *neatly*. When submitting to gradescope be sure to indicate the page containing your answer to each problem, so that the TAs don't have to search for your solution.
- *To get full credit, explain your solution and show each step of the solution process!* Simply writing down a correct answer will receive little or no credit. We don't need your scratch work or draft solutions, only your final solution explaining your step-by-step reasoning. Recommendation: try to imagine you need to explain your solution to someone not in this class.
- If you think the TA made a clerical error in grading your assignment, you may submit a regrade request on Gradescope within 1 week of the publication of the grades. After 1 week of publication, ALL GRADES ARE FINAL.

Problem 1 [20pts (10,10)]: Pigeonhole Principle

i. What is the minimum number of students that must be assigned to a classroom with 14 tables to guarantee that some table will have at least 3 students?

Solution: $(14 * 2) + 1 = 29$

ii. Suppose a set of 8 numbers are selected from the set $\{1, 2, 3, ..., 13, 14\}$. Show that two of the selected numbers must sum to 15. (Hint: think about how many subsets of 2 elements you can form such that the sum of the values of the two elements is 15)

Solution: Partition the integers in the range 1 through 14 into seven sets: $\{1, 14\}, \{2, 13\}, \{3, 12\}, \{4, 11\}, \{5, 10\}, \{6, 9\}, \{7, 8\}$. Map each selected number to the set that contains that number. Since there are 8 numbers selected and 7 sets, there must be at least two selected numbers that map to the same set. The two numbers in each set sum to 15. Therefore, two of the numbers selected must sum to 15.

Problem 2 [30 pts (5,5,10,10)]**: Weekend trip to Vegas.**

For each subproblem, reduce your final answer to a single integer and show your work.

i. A standard 52-card deck has four suits (Hearts, Diamonds, Clubs, and Spades) and each suit has 13 ranks (2,3,4,5,6,7,8,9,10,Jack,Queen,King,Ace). The face cards are Jack, Queen, and King. How many ways are there to be dealt any 2 cards from a 52-card deck? (We are counting as distinct the same two cards received in a different order.)

Solution: 52 choices for the first card x 51 choices for the second card $= 52 \cdot 51 = 2652$.

ii. How many ways are there to be dealt Blackjack? To be dealt Blackjack, either the first card is an Ace and the second card is a face card or a 10, or the first card is a face card or a 10 and the second card is an Ace. (Again we are counting as distinct the same two cards received in a different order.)

Solution: There is no overlap between 4 Aces and 16 face cards or tens, and the two cards could be received in either order so $(4 \cdot 16) + (16 \cdot 4) = 128$ ways.

iii. How many ways can you be dealt two cards such that the first card is a spade and the second card is a face card?

Solution:

1. First card is a face card spade: $3 \cdot 11 = 33$
2. First card is a non-face card spade: $10 \cdot 12 = 120$

So 153 ways altogether.

iv. How many ways can you pick three cards such that the first card is a spade, the second card is a one-eyed Jack, and the third card is a face card? (There are two one-eyed Jacks in a standard deck: the Jack of Hearts and the Jack of Spades. *Hint: Break the problem down into 3 disjoint cases for the type of card received 1st, 2nd, and 3rd.*

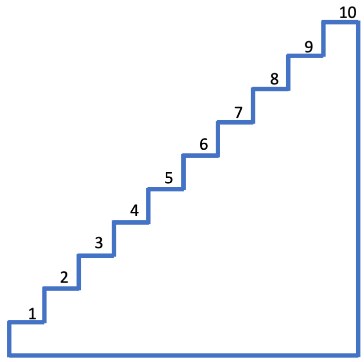
Solution: Break the problem down into disjoint cases for the type of card received 1st, 2nd, and 3rd.

1. (Non-Face Spade, Any one-eyed Jack, Face): $10 \cdot 2 \cdot 11 = 220$
2. (Face Spade non-one-eyed-Jack, Any one-eyed Jack, Face): $2 \cdot 2 \cdot 10 = 40$
3. (Face Spade one-eyed Jack, other one-eyed Jack, Face): $1 \cdot 1 \cdot 10 = 10$

So 270 ways altogether.

Problem 3 [20 pts (10,10)]**: Flights of Fancy.**

A flight of stairs has 10 steps numbered 1 to 10 as shown in the figure below.



i. How many ways could you climb up the set of stairs, assuming that you can skip any number of stairs with each step, but you must end on step 10 and you can only go up, never down and never remaining on the same step.

Solution: Model the 10 steps as a 10-bit binary number where a 1 denotes touching a particular step and 0 denotes skipping the particular step. There are $2^9 = 512$ 10-bit binary numbers with a 1 in the lowest-order bit, so 512 ways to climb the stairs.

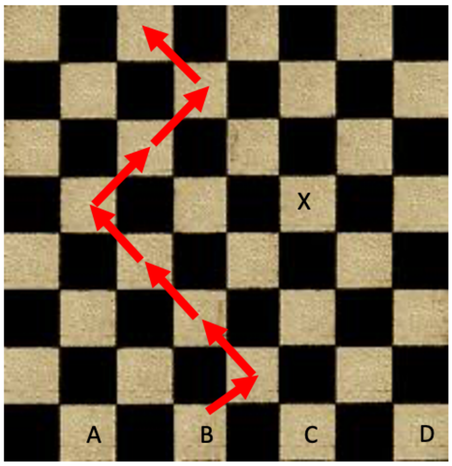
ii. How many ways could you climb up the set of stairs, assuming you take exactly 4 steps. Again, your staircase climb ends on step 10. Although there are different ways in which you could solve this problem, model the problem as a balls and bins problem for full credit.

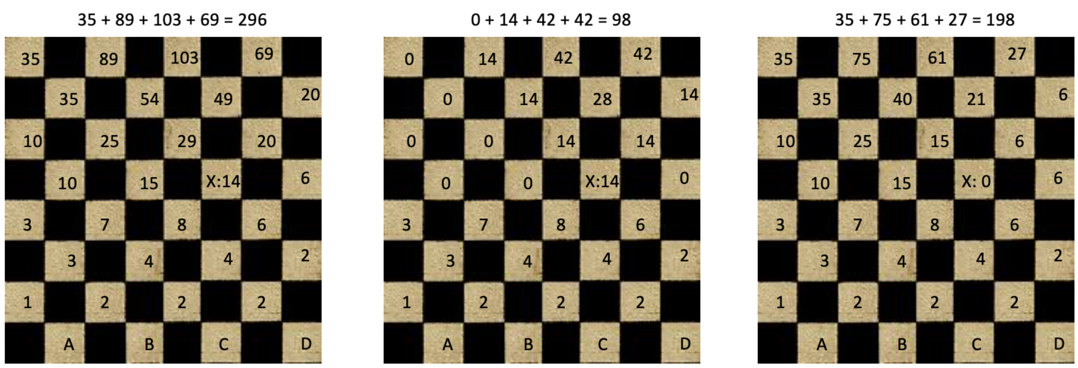
Solution:

The easiest way to model this is as before: a 10 bit number with 1 denoting we touched that step, and 0 meaning we skipped the step. Again, the last (lowest-order) bit is a 1. We have to decide which of the other 9 steps we touched, so $\binom{9}{3} = 84$. But to model this as a balls and bins problem, we instead imagine taking 4 strides (each extending across 1 or more steps). The total number of steps covered is 10. So we have $x_1 + x_2 + x_3 + x_4 = 10$ and $x_i \geq 1$. This is classic balls and bins: The x_i's are the bins and the balls are the ten total steps crossed. Each variable (stride) consumes 1 step leaving 6 more steps to assign to the 4 variables. So we have $\binom{6+4-1}{4-1} = \binom{9}{3} = 84$ as before.

Problem 4 [30 (10,10,10)]: My checkered path

i. In the game of checkers, a game piece is allowed to move diagonally in the upwards direction only. (Let's ignore pieces jumping other pieces.) Starting at square a, b, c, or d, how many paths are there to the opposite end of the board? One such path is shown. Hint: This is basically the application of the sum rule over and over again.





Any path allowed

All paths go through X

Any path avoid X
= All paths – All paths through X
= 296 – 98
= 198

<u>*Solution:*</u> Working row by row, count the number of paths through each square as the sum of the paths from the one or two paths leading into the square.

ii. What if all paths MUST go through the square marked X?

<u>*Solution:*</u> The number of paths going through the square X is the number of paths *to* X plus the number of paths to the back row *from* X. No paths through other squares on the same row as X are allowed, so they are set to 0. We proceed as before for the 3 remaining rows.

iii. What if we exclude paths through the square marked X?

Solution: Follow the same procedure but simply set the path count for square X to zero. Or alternatively, recognize that the number of paths that avoid X are the number of total paths - the paths through X.