CS5002 Discrete Structures	Prof. Rachlin
Spring 2022	April 30, 2022

CS52002 Final Exam

Instructions:

1. The exam is open book and open notes. You may use a calculator but it should not be necessary.
2. The exam window opens on Saturday, April 30th at 12:00pm ET/Boston (Noon) and closes 12 hours later at 11:59pm ET/Boston. Once you begin your exam, you will have 4 hours to complete the exam AND submit your exam to gradescope. Therefore, be sure to START your exam before 8pm ET/Boston so that you have the full four hours to complete the exam.
3. Discussing the exam with others is strictly prohibited. Working together on the solutions is strictly prohibited. Giving or receiving help from fellow students earns all parties involved an automatic F for the class. Do not test us on this.
4. *Unless otherwise specified*, you may leave your answer in the form of a fraction or mathematical expression that clearly shows your thought process. For credit you must show your scratch work and clearly explain how you got to your final answer. If your expression isn't clear or obvious you won't earn full credit. A solution is not enough. You must show how you derived your solution.
5. If you have a question, post a PRIVATE question on Piazza. An instructor will respond as soon as possible.
6. When submitting your exam you must specify, for each problem, the page or pages where the solution to that problem is to be found. If we can't see your solution you will not receive credit for the problem.
7. You have been granted extra time solely for the purpose of uploading your exam. Exams emailed to instructors will be penalized 5 percent, so be sure to give yourself enough time to submit your exam! Exams received via email more than a few minutes beyond the 11:59pm will not be graded.

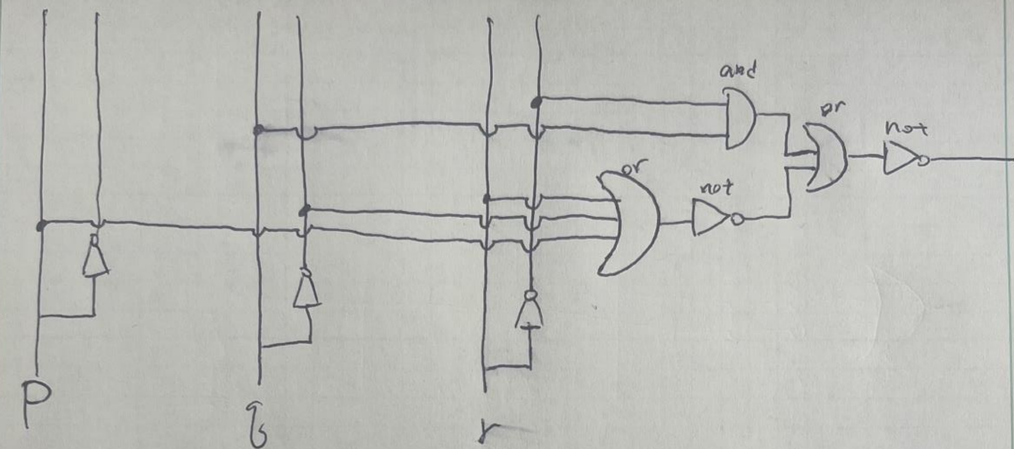
The following may be written out by hand if you are unable to print the coversheet.

PRINT FULL NAME: HAOZHE ZHANG STUDENT ID: 002922066

I have read the instructions above and understand that I may not discuss the exam with other students at any time. SIGN HERE: Haozhe Zhang

Section 1

1: $\neg(\neg(p \vee \neg q \vee r) \vee (q \wedge \neg r))$



2. $\neg(\neg(p \vee \neg q \vee r) \vee (q \wedge \neg r))$ De morgan's law

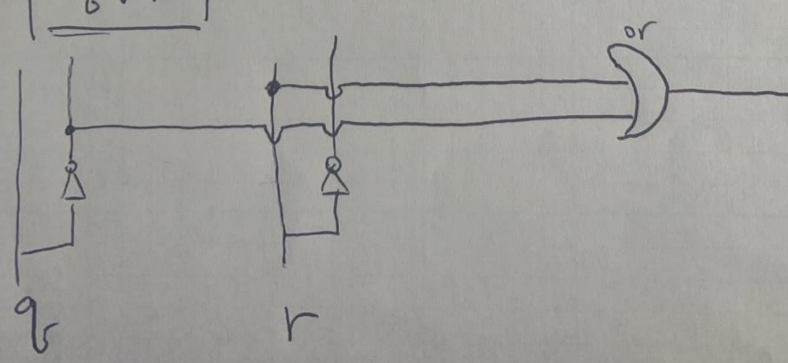
$\neg\neg(p \vee \neg q \vee r) \wedge \neg(q \wedge \neg r)$ Double negation

$(p \vee \neg q \vee r) \wedge \neg(q \wedge \neg r)$ De morgan's law

$(p \vee \neg q \vee r) \wedge (\neg q \vee r)$ Absorption laws ~~Let $(\neg q \vee r)$ be A~~

$\cancel{P \vee}$ $\boxed{\neg q \vee r}$

3.



Section 2: Counting

1. Choose 4 from 9 letters, each selection is independent event,
   The total amount of sequences of length 4 are 9 * 9 * 9 * 9 = 9 ^4.

2. Sequences of length 4 contain exactly THREE different letters can be converted to subtracting sequences of length of 4 with exactly 1/exactly 2/exactly4 letter(s), total possible sequences are 9^4 from question above.

   Exactly 1: 9 (AAAA, BBBB …….IIII)
   Exactly 2: C (9, 2) * 2^4 – 2 (9 choose 2, and each slot can be filled with any of the two, subtracting the common one to avoid overcounting)
   Exactly 4 = 4 different: 9 * 8 * 7 * 6 = P(9, 4), order matters here

   9^4 – 9 – (C (9, 2) * 2^4 – 2) – 9*8*7*6 = 6561 – (36*16 -2) – 3024 = 6561 – 9 – 574 – 3024 = 2954

3. A puzzle can be arranged as 9! Possible ways.
   Only scenarios that “BAD” or “DAB” existed are at diagonal, horizontal and vertical.

   Diagonal: 2 * 2 (two diagonals with forward and backward spelling)
   Horizontal: 2 * 3 (three horizontals with forward and backward spelling)
   Vertical: 2 * 3 (three verticals with forward and backward spelling)

   After grouping “BAD” or “DAB” to one, the rest of 6 letters (CEFGHI) can be chosen randomly, so 6!

   The number of puzzles do not contain the word BAD is
   9! – 14 *6!

Section 3: Probability

NS= not spam, S = spam, F = contains FREE, NF = not contains FREE

1. P(S) = 0.9, P(F|S) = 0.5, P(F|NS) = 0.2

   P(NS) = 1 – P(S) = 0.1
   According to the complement of an event and conditional probability,

   If F and S are both events in the same sample space (Emails), the probability of F and the probability of not F (NF) still sum to 1, even when conditioned on the event S.
   So does apply to F and NS.

   So that
   P(NF|S) = 1 – 0.5 = 0.5
   P(NF|NS) = 1 – 0.2 = 0.8

   P(S|F) = P(F|S) * P(S) / (P(F|S) * P(S) + P(F|NS) * P(NS)) = 0.5 * 0.9 / (0.5 * 0.9 + 0.2 * 0.01) = 45/47

2. 9 intersections can be split into 3 categories, Vertex with 2, 3, 4 intersections.

   Vertex with 2 intersections: 1, 3, 7, 9
   Vertex with 3 intersections: 2, 4, 6, 8
   Vertex with 4 intersections: 5

   |E| = 4/9 * 2 + 4/9 * 3 + 1/9 * 4 = 24 / 9 = 8/3

3. If two distinct intersections are chosen at random from the Vertex above, the probability can be calculated as the |Event| / |Space|
   |Space| = |S| = C (9, 2) = 36
   Event that the chosen cities are not adjacent to each other is any two cities in categories of cities with 2 and 3 intersections.

   For example, 1 and 3 are not adjacent, 2 and 8 are not adjacent

   So, choose 2 from Vertex with 2 intersections, we have C (4, 2);
   Or choose 2 from Vertex with 3 intersections, we have C (4, 2);
   Or choose 1 from Vertex with 2 intersections and choose 1 from Vertex with 4 intersections: C (4,1) * 1

   The probability of choosing two cities that are adjacent are (C (9,2) – C (4, 2) – C (4, 2) – C (4, 1))/ C (9, 2) = (36 – 6 – 6 – 4) / 36 = 5/ 9

4. Corner vertex intersections are 1, 3, 7, 9.

   $|E| = |E1| + |E2| + |E3| + |E4| + |E5| + |E6| + |E7|$

   Each selection is independent to each other, so the $|E1| = |E2| = |E3| = |E4| = |E5| = |E6| = |E7|$

   $|E| = 7|E1|$

   Let corner vertex intersection selection to be 1, and not selected to be 0

   $|E1| = 4/9 * 1 + 5/9 * 0 = 4/9$

   $|E| = 7 * 4/9 = 28/9$

Section 4: Sequence and Recurrences

1. $M_1 \sim M_5 = 2, 7, 15, 26, 40$
2.

$n \geq 1$

2 7 15 26 40

5 8 11 14

3 3 3

Then we can generate a quadratic polynomial function that has a, b, c in a form of $M_n = an^2 + bn + c$

$M_1 = a + b + c = 2$
$M_2 = 4a + 2b + c = 7$
$M_3 = 9a + 3b + c = 15$

$M_2 – M_1 = 3a + b = 5$
$M_3 – M_2 = 5a + b = 8$

$M_3 – M_2 – (M_2 - M_1) = 2a = 3$
After arranging the equations above, we get $a = 3/2$, $b = 1/2$, $c = 0$. Then $M_n = (3/2)*n^2 + (1/2)*n$

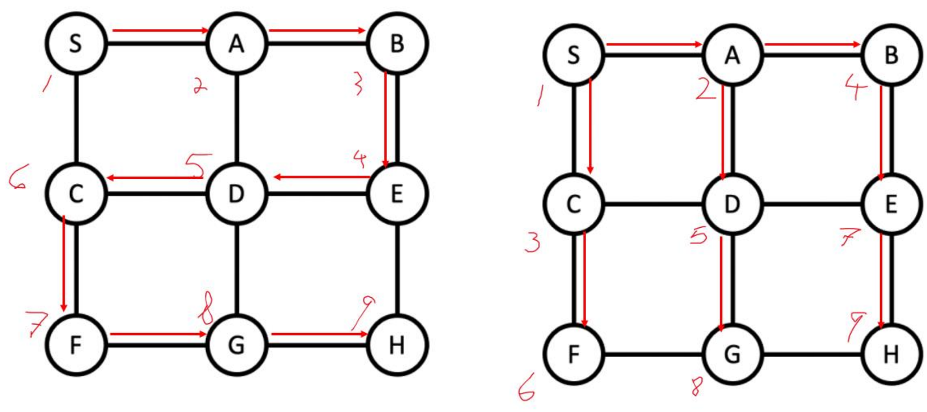
3. It's a quadratic polynomial functions, since the highest degree of the variable in the function is 2.
4. Known the recursive definition of M_n where $M_n = M_{n-1} + 3n – 1$ is true for all $n >= 1$, let $S(n)$ be the statement that $M_n = (3/2)*n^2 + (1/2)*n$
   Base case, $S(1)$ is true because:
   $M_1 = 3/2 + 1/ 2 = 2 = 2$ which is given
   Inductive step:
   We assume $S(k)$ is true for some fixed $k >= 1$. We wish to prove $S(k+1)$
   Consider:

$$\begin{aligned} M(k+1) &= 3/2 * (k + 1)^2 + 1/2 * (k + 1) \\ &= 3/2 * (k^2 + 1 + 2k) + 1/2 * k + 1/2 \\ &= 3/2 *k^2 + 3/2 + 3k + 1/2 * k + 1/ 2 \\ &= 3/2 * k^2 + 1/2 * k + 3k + 2 \qquad \text{known: } 3/2*k^2+ 1/2*k = M_k \\ &= M_k + 3k + 3 – 3 + 2 \\ &= M_k + 3(k + 1) - 1 \end{aligned}$$

   Which $S(k+1)$ corresponds to the recursive definition and is true,
   Thus, by axiom of induction, since $S(1)$ and all $k >=1$, $S(k) \to S(k+1)$, we have that for all $n>=1$, $S(n)$.

Section 5: Graphs

1.



Depth-First Search

Breadth-First Search

2. Adjacency-list representation for graph above.

S →	A	C		
A →	B	D	S	
B →	A	E		
C →	D	F	S	
D →	A	C	E	G
E →	B	D	H	
F →	C	G		
G →	D	F	H	
H →	E	G		

3. Let Mn be the total edges of a n X n array, we can find that for all n >=2.

M(2) = 4,

M(3) = 12,

M(4) = 24,

M(5) = 40…..

The second order of the difference between each M is a constant which equals to 4.

$n \geq 2$:

2x2	3x3	4x4	5x5	6x6
4	12	24	40	60
8	12	16	20	
4	4	4		

Then we can generate a quadratic polynomial function that has a, b, c in a form of Mn = an^2 + bn + c

M2 = 4a + 2b + c = 4
M3 = 9a + 3b + c = 12
M4 = 16a + 4b + c = 24

After arranging the equations above, we get a = 2, b = -2, c = 0. Then Mn = 2n^2 - 2n

If we plug 3 back in to the closed-form formula of Mn, we get M3 = 2 * 3^2 – 2 * 3 = 18 – 6 = 12, which corresponds to the correct amount of the edges we counted on the graph.