CS5002 Discrete Structures	Prof. Rachlin
Spring 2022	February 16, 2022

Homework # 5

Assigned: Wednesday February 16, 2022
Due: Tuesday February 22, 2022 @ 11:59pm ET/Boston
−5%: Wednesday February 23, 2022 @ 11:59pm ET/Boston
−10%: Thursday February 24, 2022 @ 11:59pm ET/Boston

Instructions:

- Homework is due on Tuesday at 11:59pm ET/Boston. Homeworks received up to 24 hours late (11:59pm ET on Wednesday) will be penalized 5 percent. Homeworks received up to 48 hours late (11:59pm ET on Thursday) will be penalized 10 percent. *NO* assignment will be accepted after 48 hours.
- We expect that you will study with friends and fellow students and you are welcome to verbally discuss the problems openly. However, your solution writeup should be the product of your own mind and expressed in your own words. The TAs and I will be available to answer specific questions or address speific points of confusion but we will not verify your answers prior to submission.
- Assignments should be typed using Word or LateX, or hand-written *neatly*. When submitting to gradescope be sure to indicate the page containing your answer to each problem, so that the TAs don't have to search for your solution.
- *To get full credit, explain your solution and show each step of the solution process!* Simply writing down a correct answer will receive little or no credit. We don't need your scratch work or draft solutions, only your final solution explaining your step-by-step reasoning. Recommendation: try to imagine you need to explain your solution to someone not in this class.
- If you think the TA made a clerical error in grading your assignment, you may submit a regrade request on Gradescope within 1 week of the publication of the grades. After 1 week of publication, ALL GRADES ARE FINAL.

Problem 1 [25 (8, 8, 9)]: Ride the MBTA.

i. The worcester line makes 10 inbound trips, 4 of which are express and 6 are local. In how many ways can we order the trips as a sequence of either express or local trips. For example, LLEELLEELL is one possibility.

Solution: $\frac{10!}{6!\cdot 4!}$ This is like the BANANA problem. Now we have 6 indistinguishable L's and 4 indistinguishable E's.

ii. A train engine is to be connected with either 3, 4, or 5 passenger cars chosen from 7 available passenger cars. Assuming the passenger cars are all numbered, and that the same cars linked together in a different sequence constitutes a different train, how many possible trains could we form?

Solution: ${}_7P_3 + {}_7P_4 + {}_7P_5 = \binom{7}{3} \cdot 3! + \binom{7}{4} \cdot 4! + \binom{7}{5} \cdot 5!$

iii. Suppose an inbound train starts at station 1 and ends at station 10. The local train makes all 8 intermediate stops at stations 2 through 9. An express train is any inbound train that makes no more than 3 intermediate stops. (The starting and ending station do not count.) How many different valid express train trips are possible?

Solution: $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 1 + 8 + 28 + 56 = 93$ valid express trips.

Problem 2 [25 pts (6, 6, 6, 7)]**: DNA - The code of life.**

The human genome is a double helix containing some 3 billion nucleotide base pairs. For purposes of discussion we can imagine DNA as a long string of four letters (A,T,G,C) corresponding to the four nucleotides (Adenine, Thymine, Guanine, and Cytosine).

i. How many DNA sequences of length 12 are possible? Note: DNA sequences have a definite direction or orientation, so reversing the sequence produces a different DNA sequence. (Molecular biologists refer to one end as the 3' end and the other as the 5' end.)

Solution: $4^{12} = 16,777,216$ sequences. Each of the 12 positions is one of four possible nucleotides.

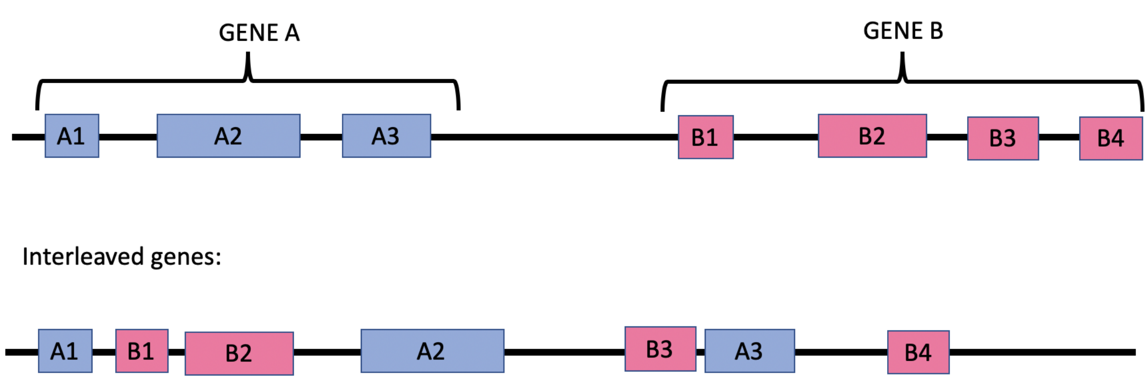
ii. How many DNA sequences of length 12 are possible if no nucleotide ever repeats? By "repeat" we mean that the nucleotide occurs at least twice in a row (AA, CC, GG, TT).

Solution: $4 \cdot 3^{11} = 708,588$ sequences. 4 choices for the first nucleotide but then only 3 choices for each of the remaining 11 nucleotides.

iii. How many DNA sequences have at least one repeating nucleotide somewhere along the sequence?

Solution: From parts 1 and 2 of this problem, there must be $4^{12} - 4 \cdot 3^{11} = 16,777,216 - 708,588 = 16,068,628$ sequences with at least one repeating nucleotide.

iv. Human genes may be fragmented into multiple *exons* as shown in the figure below. In fact the exons of two genes may even be interleaved. Gene A has 3 exons and gene B has 4 exons. In how many ways can the exons of gene A and B be ordered (including interleavings of the two genes) while maintaining the relative sequence of gene A's exons and gene B's exons? That is, exon A1 is always before exon A2 is always before exon A3 and so on.



Solution:

The 7 exons from A and B have 7! permutations but only one of the 3! permutation of A and only one of the 4! permutations of B is valid. This is like treating the exons of A as all identical, and same for the exons of B. So the answer is $\frac{7!}{3!4!} = 35$ possible ways to interleave the two genes. This again is like the BANANA problem, but with only A's and B's.

Problem 3 [25 pts (5, 10, 10)]**: A Long-expected Party**

When Mr. Bilbo Baggins of Bag End announced that he would shortly be celebrating his eleventy-first birthday with a party of special magnificence, there was much talk and excitement in Hobbiton.

i. Suppose Bilbo invited 111 guests. In how many ways might the guests arrive at the party if they arrive one at a time?

Solution: 111! possible ways to sequence the arrivals.

ii. Suppose the guests include 11 Proudfoots (*Proudfeet!*) consisting of 4 women ($W_1 ... W_4$), 4 men ($M_1 ... M_4$), and 3 children ($C_1 ... C_3$). In how many ways might the Proudfeet arrive if all the women enter together (one at a time), all the men enter together, and all the children enter together? The groups (Men, Women, Children) may still enter in any order (Men/Women/Children, Children/Women/Men, etc.) $W_1 W_4 W_2 W_3 C_3 C_2 C_1 M_2 M_4 M_1 M_3$ is one possible arrival sequence.

Solution: There are 4! ways for the men to arrive and 4! ways for the women to arrive and 3! ways for the children to arrive. And the three groups may arrive in 3! ways, so $4! \cdot 4! \cdot 3! \cdot 3! = 20,736$ possible ways to sequence the arrival of the Proudfeet.

iii. What if Frodo Baggins and Lobelia Sackville-Baggins are two of the eleven guests at one of the tables and they must NOT be seated next to each other. How many distinct and valid seating arrangements are there for this table? Rotating guests around the table does not count as a distinct seating arrangement.

Solution: Think of Frodo-Lobelia as one guest and we have a circular permutation on 10 guests, ($\frac{10!}{10} = 9!$) but Frodo could be either to the left or the right of Lobelia, so there are $2 \cdot 9! = 725,760$ *invalid* arrangements. The number of valid arrangements is thus $10! - 2 \cdot 9! = 2,903,0404$.

Problem 4 [25 pts (10, 10, 5)]**: Peanut butter crackers**

i. How many ways are there to pair up four distinct crackers (C_1, C_2, C_3, and C_4) to make two peanut-butter cracker sandwiches? The order in which you form the pairs makes no difference.

It only matters which crackers are paired with which other crackers. One possiblity is C_1 paired with C_4 and C_2 paired with C_3.

Solution:

There are $\binom{4}{2} = 6$ ways to pick a pair of crackers but I might have picked the other two crackers first, resulting in the same cracker pairings, so we are double counting. There are actually just 3 possible ways to make the two cracker sandwiches.

ii. How many ways are there to pair up six distinct crackers ($C_1 ... C_6$) to make three peanut-butter cracker sandwiches? As before, the order in which you form the pairs makes no difference.

Solution: Number the crackers 1 to 6. Consider cracker 1. There are 5 ways to pair it with another cracker. Next, consider the smallest-numbered cracker not already paired up. There are 3 ways to pair it with any of the remaining crackers. The last two crackers have to be paired up together, so we have $5 \cdot 3 \cdot 1 = 15$ possibilities.
A different way to think about this (similar to the class examples) is to choose first 2 crackers to make first sandwich - that's $\binom{6}{2}$ ways, then choose 2 from remaining 4 to make a second sandwich - that's $\binom{4}{2}$ ways, and finally make the last sandwich with last 2 crackers. But since the order we form the pairs (first, second, third) makes no difference, the same sandwiches can be obtained in a different order. Thus we have to divide the count by number of sandwich-permutations.
Total $\binom{6}{2}\binom{4}{2}\binom{2}{2}/3! = 15 \cdot 6 \cdot 1/6 = 15$.

iii. Bonus question: (Only 5 points.) Suppose we had n crackers (n is even). Derive a general expression for the number of sandwich pairings possible when making $\frac{n}{2}$ peanut butter sandwiches. Explain why your expression is correct! (Peanut butter sandwiches will never be the same - sorry!)

Solution: Similar argument as before: Always pair the lowest-numbered cracker available with any remaining cracker. Repeat until you run out of crackers. Cracker 1 has (n-1) pairing possibilities. The next smallest-numbered cracker (either cracker 2 or 3) has (n-3) pairing possibilities. The next has (n-5) possibilities and so on. The last two crackers have to be paired up together.
Total $(n-1) \cdot (n-3) \cdot (n-5) \cdot ... \cdot 3 \cdot 1$ possibilities