# Logic

"Cogito, ergo sum!"

— Descartes

John Rachlin

discrete structures

Northeastern.

# 1. Logic: The study of formal reasoning

Why do computer scientists care about logic?

a) Programming:

```
if (a > 5 or b = false)
{
        execute code
}
```

b) Mathematical proofs about hard problems.
e.g. prove that comparison-based sorting takes $\geq O(n \log n)$ steps.

- A proof is a series of statements that are "logically connected", i.e, one statement implies the next.
- Statements: True or False

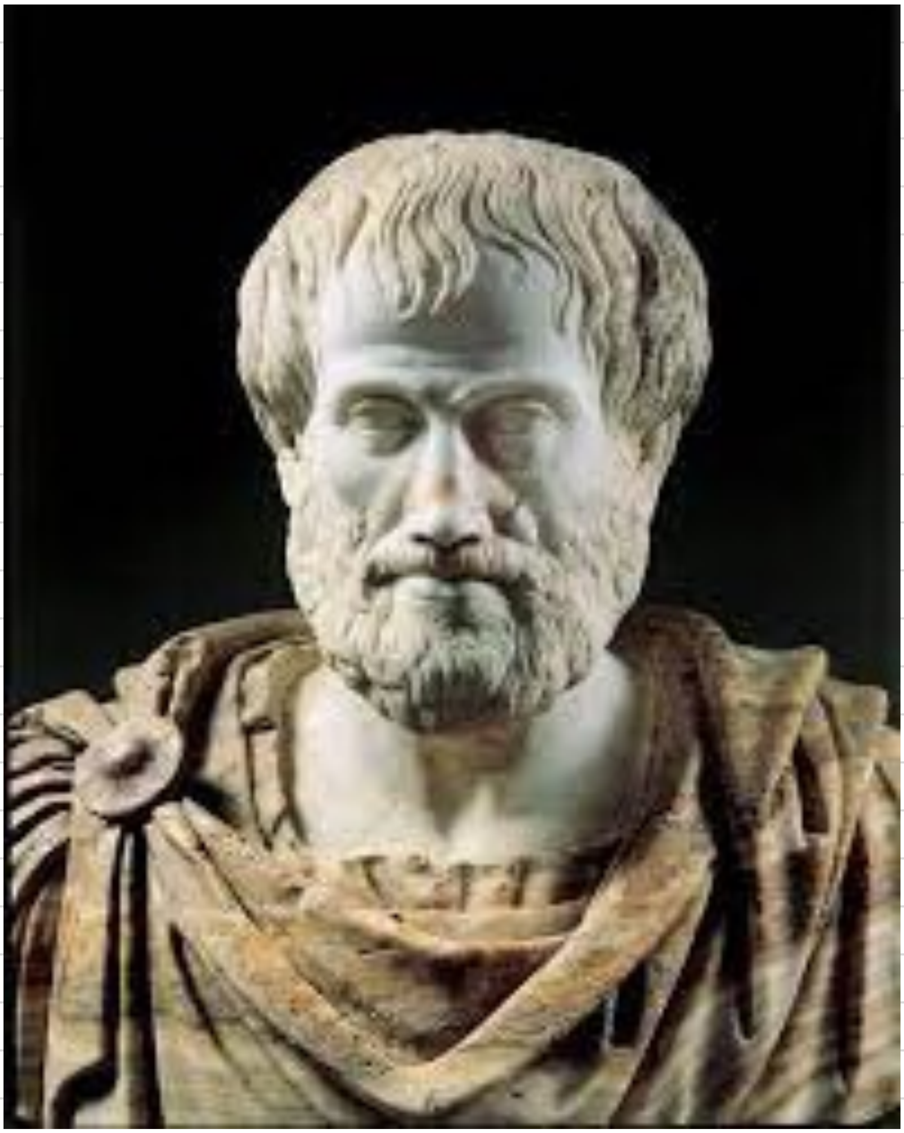
c) Designing digital circuits and even whole CPUs

"Logic circuits"

The inputs are "Boolean" variables that take on one of two values: 0 (false) or 1 (true)

# Aristotle 384 - 322 BC

- Student of Plato
- Father of Western Philosophy.
- Founded his own school Lyceum



"The roots of education are bitter
but the fruit is sweet"
-Aristotle (attributed)

Propositions

- denoted by letters a, b, c, p, q, r ..
- represent a statement
- either true or false
  better: can be assigned a truth value
- opinions or statements with unknown truth value are still propositions

| True Proposition | Not a Proposition | False Proposition |
| --- | --- | --- |
| $2+2=4$ | X is prime. | The moon is purple |
| Asteroid is cute | Will it snow? | Asteroid is cute |
| I will earn alot in CS/DS | Hooray! | |

(Propositional)

## 2. Basic Logic: AND, OR, NOT

### AND :

$p \wedge q$

$p$: it is raining
$q$: I have my umbrella.

$p \wedge q$ is true if and only if both $p$ and $q$ are true.

| | | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|---|
| 0 | 00 | F | F | F |
| 1 | 01 | F | T | F |
| 2 | 10 | T | F | F |
| 3 | 11 | T | T | T |

T = True

F = False

* $p \wedge q \wedge r$
* why FF is on top!

### OR :

$p \vee q$

$p$: It is raining
$q$: The sprinklers are on.

$p \vee q$ : $p$ is true OR $q$ is true OR BOTH.

"inclusive OR"

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

← inclusive

NOT :

$\neg p$ — $\neg p$ is true if and only if $p$ is false.

| $p$ | $\neg p$ |
|---|---|
| F | T |
| T | F |

Logical operators are related to sets.

$A \cup B = \{x : x \in A \vee x \in B\}$ (or)

$A \cap B = \{x : x \in A \wedge x \in B\}$ (and)

$\bar{A} = \{x : \neg(x \in A)\}$ (not)

Bigger Truth tables for more complex expressions

| a | b | c | (a^b) | (a^¬b) | (c^¬a) | (a ^ b) v (a ^ ¬b) v (c ^ ¬a) |
|---|---|---|---|---|---|---|
| F | F | F | F | F | F | F |
| F | F | T | F | F | T | T |
| F | T | F | F | F | F | F |
| F | T | T | F | F | T | T |
| T | F | F | F | T | F | T |
| T | F | T | F | T | F | T |
| T | T | F | T | F | F | T |
| T | T | T | T | F | F | T |

note the sequence. If F=0, T=1 then these are binary #s 0...7!

# Logical Equivalence.

| $a$ | $b$ | $c$ | $(a \wedge b)$ | $\vee\ (a \wedge \neg b)$ | $\vee\ (c \wedge \neg a)$ | $\vee$ |
|---|---|---|---|---|---|---|
| F | F | F | F | F | F | F |
| F | F | T | T | F | T | T |
| F | T | F | F | F | F | F |
| F | T | T | T | F | T | T |
| T | F | F | T | T | F | T |
| T | F | T | T | T | F | T |
| T | T | F | T | F | F | T |
| T | T | T | T | F | F | T |

| $a$ | $b$ | $c$ | $(a \vee c)$ |
|---|---|---|---|
| F | F | F | F |
| F | F | T | T |
| F | T | F | F |
| F | T | T | T |
| T | F | F | T |
| T | F | T | T |
| T | T | F | T |
| T | T | T | T |

If the truth tables are the same, then the two formulas are logically equivalent.

$$(a \wedge b) \vee (a \wedge \neg b) \vee (c \wedge \neg a) \equiv a \vee c$$

# Logical Equivalence - continued

I claim:

$$a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$$

| $a$ | $b$ | $c$ | $a \vee (b \wedge c)$ | $a \vee b$ | $a \vee c$ | $(a \vee b) \wedge (a \vee c)$ |
|---|---|---|---|---|---|---|
| F | F | F | F | F | F | F |
| F | F | T | F | F | T | F |
| F | T | F | F | T | F | F |
| F | T | T | T | T | T | T |
| T | F | F | T | T | T | T |
| T | F | T | T | T | T | T |
| T | T | F | T | T | T | T |
| T | T | T | T | T | T | T |

Similarly, we could show:

$$a \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c)$$

The two logical equivalences are know as "distributive laws".

# Rules of Logical Eq

| | |
|---|---|
| Commutative laws | $p \wedge q \equiv q \wedge p$<br>$p \vee q \equiv q \vee p$ |
| Associative laws | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$<br>$(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| Distributive laws | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$<br>$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| Identity laws | $p \wedge T \equiv p$<br>$p \vee F \equiv p$ |
| Complement laws | $p \wedge \neg p \equiv F$<br>$p \vee \neg p \equiv T$ |
| Annihilator laws | $p \wedge F \equiv F$ |
| | $p \vee T \equiv T$ |
| Idempotence laws | $p \wedge p \equiv p$ |
| | $p \vee p \equiv p$ |
| Absorption laws | $p \wedge (p \vee q) \equiv p$ |
| | $p \vee (p \wedge q) \equiv p$ |
| Double negation law | $\neg(\neg p) \equiv p$ |
| De Morgan's laws | $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ |

Distributive: Prove via truth table
or translate: p: cloudy q: snowing r: raining
p: raining q: hot r: humid

Absorption: The value of the expression clearly depends only on p.
Try p = True, p = False

DeMorgans: Convert to an English example

Its not both hot and humid ≡ Its not hot OR not humid

Today is not monday or tuesday = Today is not Mon & Today is not Tue.

$a \wedge (b \vee c)$
$(a \wedge b) \vee (a \wedge c)$

# Simplifying Logical Formulas.

$(a \wedge b) \vee (a \wedge -b) \vee (c \wedge -a)$

$(a \wedge (b \vee -b)) \vee (c \wedge -a)$ Distrib.

$(a \wedge T) \vee (c \wedge -a)$

$a \vee (c \wedge -a)$ Distrib.

$(a \vee c) \wedge (a \vee -a)$

$a \vee c$

$(a \wedge b \wedge c) \vee (a \wedge b \wedge -c) \vee \neg(a \vee -b)$

$((a \wedge b) \wedge (c \vee -c)) \vee \neg(a \vee -b)$ Distributive

$(a \wedge b) \vee \neg(a \vee -b)$ complement

$(a \wedge b) \vee (-a \wedge --b)$ De Morgan's

$(a \wedge b) \vee (-a \wedge b)$ Double Neg

$b \wedge (a \vee -a)$ Distributive

$b$ identity

$(x_1 \wedge x_2 \wedge x_3 \wedge y) \vee (x_1 \wedge x_2 \wedge x_3 \wedge -y)$

$\Downarrow$

$(x_1 \wedge x_2 \wedge x_3)$

# How many functions are there?

Two inputs, 4 possible combinations

A function is defined by the four values for each combination, so there are $2^4$ possible functions.

| A | 1 | 1 | 0 | 0 | | | |
|---|---|---|---|---|---|---|---|
| B | 1 | 0 | 1 | 0 | | | |
| possible functions | 0 | 0 | 0 | 0 | False | $F$ | |
| | 0 | 0 | 0 | 1 | NOR | $\neg A \wedge \neg B$ | $\neg(A \vee B)$ |
| | 0 | 0 | 1 | 0 | | $\neg(B \to A)$ | |
| | 0 | 0 | 1 | 1 | not A | $\neg A$ | |
| | 0 | 1 | 0 | 0 | | $\neg(A \to B)$ | |
| | 0 | 1 | 0 | 1 | not B | $\neg B$ | |
| | 0 | 1 | 1 | 0 | XOR | $A \oplus B$ | |
| | 0 | 1 | 1 | 1 | NAND | $\neg A \vee \neg B$ | $\neg(A \wedge B)$ |
| | 1 | 0 | 0 | 0 | AND | $A \wedge B$ | |
| | 1 | 0 | 0 | 1 | XNOR | $A \leftrightarrow B$ | |
| | 1 | 0 | 1 | 0 | | $B$ | |
| | 1 | 0 | 1 | 1 | | $A \to B$ | |
| | 1 | 1 | 0 | 0 | | $A$ | |
| | 1 | 1 | 0 | 1 | | $B \to A$ | |
| | 1 | 1 | 1 | 0 | OR | $A \vee B$ | |
| | 1 | 1 | 1 | 1 | True | $T$ | |

2 inputs $\Rightarrow$ 4 output combinations $\Rightarrow 2^4$ functions

$n$ inputs $\Rightarrow 2^n$ output combinations $\Rightarrow 2^{2^n}$ functions

**Tautology** : A statement that is always true.
Simplifies to T

e.g. $p \vee \neg p \equiv T$ Its raining or its not raining.

$a \vee b \vee c \vee (\neg a \wedge \neg b \wedge \neg c) \equiv T$

alice, bob, or colin own a car or none of them do

**Contradiction** : A statement that is always False.
Simplifies to F

e.g. $p \wedge \neg p \equiv F$ Its raining and its not.

$a \wedge b \wedge c \wedge (\neg a \vee \neg b \vee \neg c) \equiv F$

abc all own a car AND not all of them own a car.

Aristotle: Law of non-contradiction: statements are true or false, not both.

Boole: Laws of Thought (1854): Algebraic treatment of logic.

$\Rightarrow$ "Boolean variable": True or False in programming

Implication: $p \to q$ or $p \Rightarrow q$

"If $p$ then $q$." or "$p$ implies $q$"

p. Today is Monday

q. Rachlin has office hours.

| $p$ | $q$ | $p \to q$ | |
|---|---|---|---|
| F | F | T | Friday; not in office $\Rightarrow$ No Lie! |
| F | T | T | Thursday; holding office hours — But I didn't Lie! |
| T | F | F | Rachlin Lied! |
| T | T | T | Its Monday and Rachlin held office hours, so Rachlin is honest. |

$$p \to q \equiv \neg(p \wedge \neg q)$$

$$p \to q \equiv \neg p \vee \neg\neg q \equiv \neg p \vee q$$

Note: Implication $\neq$ Causality.

If $2+2=5$ then elephants are gray.

True!

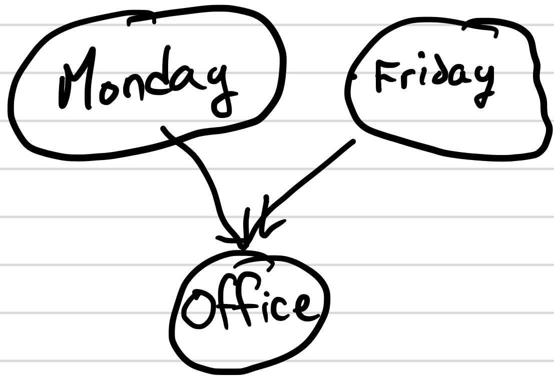
## Converse

$p \to q \not\equiv q \to p$ $\quad$ $q \to p$

: "If Rachlin has office Hours, it's Monday."
NO!! (It might be Friday !)

($p \to q$ is <u>not</u> logically equivalent to $q \to p$.)

| $p$ | $q$ | $p \to q$ | $q \to p$ |
|---|---|---|---|
| F | F | T | T |
| F | T | T | F |
| T | F | F | T |
| T | T | T | T |




## Inverse

If not $p$ then not $q$

$\neg p \to \neg q$

## Contrapositive

$$p \to q \equiv -q \to -p$$

$\underbrace{\text{If its Tuesday,}}_{p}$ $\underbrace{\text{Rachlin has office hours}}_{q}$

If Rachlin doesn't have office hours
Then its not Tuesday. (True!)

$p \to q$
$-p \vee q$ defn
$q \vee -p$ assoc.
$\neg(-q) \vee \neg p$ double neg
$\neg q \to -p$ defn.

$x$ : worked hard on homework
$y$ : turned in home work
$z$ : Earned a good grade.

$x \wedge y \to z$

$-z \to \neg(x \wedge y)$
$-z \to (-x \vee -y)$

If you didnt earn a good grade, you didnt turn it in or you didnt work hard.

$p$ : Tuesday
$q$ : R. has OH

| | |
|---|---|
| Statement<br>$p \to q$ | If Tuesday then Rachlin has office hours<br>OR<br>R. has OH if today is Tuesday. |
| Inverse<br>$\neg p \to \neg q$ | If today is not Tue then R. doesnt have OH<br>$p \to q \not\equiv \neg p \to \neg q$<br>$\neg p \vee q \not\equiv p \vee \neg q$ |
| Converse<br>$q \to p$ | If R. has OH then today is Tuesday.<br>NO!<br>$p \to q \not\equiv q \to p$<br>$\neg p \vee q \not\equiv \neg q \vee p$ |

Contrapositive

$-q \to -p$

If R. doesn't have OH then it's not Tuesday.

Yes!

$p \to q \equiv -q \to -p$

$p \to q$

$-p \vee q$ Defn of Imp

Conditional Ident.

$q \vee -p$ Assoc

$-q \to -p$ Defn of Imp.

Bidirectional Implication

$p \longleftrightarrow q$

$p$ iff $q$

$\equiv p \to q \wedge q \to p$

$\equiv p \to q \wedge -p \to -q$

$\Uparrow$

# Iff (Biconditional)

- "If and only if"
- $p \leftrightarrow q$ or $p \equiv q$
- The truth values are the same on both sides.
- XNOR Gate!
- $p \leftrightarrow q \equiv p \to q \wedge q \to p$

| $p$ | $q$ | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| F | F | T | T | T |
| F | T | T | F | F |
| T | F | F | T | F |
| T | T | T | T | T |

contra-positive

- $p \leftrightarrow q \equiv p \to q \wedge -p \to -q$

$$\equiv q \to p \wedge -q \to -p$$

So Proofs involving "iff", "a iff b" require that we prove both:

a) if a then b  $a \to b$
b) if b then a  $b \to a$

OR:

a) $a \to b$
b) $-a \to -b$ (contrapositive of $b \to a$)