Problem 1 [20 points] Quilting Mathematics
Find a closed form formula for the sequence depicted in the diagram below. For this problem, an denotes the growing number of squares in each figure, so a1 = 1; a2 = 5; a3 = 13, etc.)
According to the figure given,
a1 = 0^2 + 1^2 = 1
a2 = 1^2 + 2^2 = 5
a3 = 2^2 + 3^2 = 13
a4 = 3^2 + 4^2 = 25
…
The difference between each term is 4, 8, 12……
Each time we are adding a 4(n-1) to the previous term.
So, the closed form formula for the sequence is

$$a_n = \sum_{k=1}^{n} (k-1)^2 + k^2$$

We get a(n) = ((2n^2 – 1) + 1)/ 2

Problem 2 [25 points]: The Pirates of Penzance
The pirates of Penzance have built a tetrahedral pyramid of cannon balls (i.e, a pyramid with a triangular base). Let C(n) be the total number of cannon balls where n is the height (number of levels) in the pyramid. (C(1)=1, C(2)=4, C(3)=10, etc.). Give a closed-form formula for C(n). What is C(15)? (Hint, to solve this problem you must be very well acquainted, too, with matters mathematical, and understand equations, both the simple and quadratical ... or rather cubical!)

Similar to the problem above,
C1 = 1 =1
C2 = 1 + 3 = 4
C3 = 1 + 3 + 6 = 11
C4 = 1 + 3 + 6 + 10 = 20
…
The difference between each term is 3, 6, 10……
Each time we are adding a (n + 1)* n/2 term to the previous term
So, the closed form formula for the sequence is

$$a_n = \sum_{k=1}^{n} \frac{n(n+1)}{2}$$

C(n) = 1 + 3 + 6 + 10 + 15 + …. +n*(n+1)/2
We get C(n) = n(n +1)(n+2)/6
C15 = 1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91 + 105 + 120 = 680.
Or C15 = 15 * 16 * 17 / 6 = 680

Problem 3 [25 points] Wait, is this Physics 5002?
A rubber ball is dropped from a height of 1 meter. After each bounce it rises to 75% of it's previous height as the kinetic and potential energy of the ball dissipates in the form of heat with each bounce. Find the total vertical distance (moving both up and down) traveled by the ball before it comes to a rest.
Let r be total vertical distance of after each bounce (including up and down)
Split the distance traveled from each bounce to the sum of up and the sum of down

# of bounces /Direction	Up(m)	Down(m)
0 (starting)	0	1
1	0.75	0.75
2	0.75^2	0.75^2
3	0.75^3	0.75^3
…	…	…
k	0.75^k	0.75^k

Assume there are k up and k down travels, the total number of travels are 1 + k(up) + k(down) = n
Therefore:
Bounce efficiency = 0.75, k = (n – 1) /2

$$Total\ traveled\ distance = 2 * \lim_{(n-1)/2\to\infty} \left(\sum_{k=0}^{(n-1)/2} 0.75^k\right) - 1$$

According to the definition of the infinite geometric series,

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

When |r| < 1, lim(n->infinite) r^n+1 = 0.

So the upper most equation can be simplified to 2 * (1/(1-r)) - 1where r = 0.75.
The total traveled distance is to be 2 * 1/0.25 - 1= 7 meters

Problem 4 [30 points] Stairways to Heaven

1. Prove by mathematical induction that 1 + 3 + 5 + …. + (2n - 1) = n^2 for all n >= 1:

1 + 3 + 5 + … + (2n-1) = $\sum_{i=1}^{n}(2n-1)$ = n^2
Theorem, for n >= 1,

$$\sum_{i=1}^{n}(2n-1) = n^2$$

Proof.
By induction on n,

Base case: n = 1.
When n = 1, the left side of sequence equals to 1,
When n = 1, the right side of sequence equals to 1.

Therefore, $\sum_{i=1}^{1}(2 * 1 - 1) = 1^2 = 1$

Inductive step:

Suppose that for positive integer k where k >=1, $\sum_{i=1}^{k+1}(2 * \mathrm{k} - 1) = k^2$

Then we will show that, $\sum_{i=1}^{k+1}(2\mathrm{k} - 1) = (k+1)^2$

Starting with the left side of the equation to be proven:

$\sum_{i=1}^{k}(2 * \mathrm{k} - 1) + 2(k+1) - 1$	by separating out the last term
$= k^2 + 2(k + 1) - 1$	by the inductive hypothesis
$= k^2 + 2k - 1$	
$= (k+1)^2$	by algebra

Therefor, $\sum_{i=1}^{k+1}(2\mathrm{k} - 1) = (k+1)^2$

2. Prove using mathematical induction that 7^n - 1 is a multiple of 6 for all n belongs to N(natural):

Similar to previous problem, we can simplify the induction proving process

Theorem: for all n belongs to natural numbers where natural numbers are 1, 2, 3, …, n.
7^n -1 is a multiple of 6.

Proof.

By induction on n

Base case: n = 1

7^1 – 1 = 6 where is a multiple of 6

Inductive step:

Suppose that for all n belongs to natural numbers N, 7^n -1 is a multiple of 6, let say, 7^n – 1 = 6m we will show that 7^(n + 1) is a multiple of 6

Starting with 7^(n+1) - 1,

7^(n+1) - 1= 7 * 7^n – 1	by separating the last term
7*7^n – 1 + 6 – 6 = 7(7^n – 1) + 6, where 7 ^n -1 = 6m	by inductive hypothesis

7 * (6m) + 6 =6 *(7m + 1), let 7m + 1 = n

7^(n+1) – 1 = 6n	by substitution

Therefore, 7^(n + 1) – 1 is a multiple of 6.

3. Prove by mathematical induction that $2^n < n!$ for all n >= 4:

Theorem: for all n >=4, 2^n < n!

Proof.

By induction on n

Base case: n = 4

2^4 = 16, 4! = 4 * 3 * 2 * 1 = 24

Therefore, 2^4 < 4!

Inductive step:

Suppose that for all n >= 4, 2^n < n!, then we will show that 2^(n + 1) < (n + 1)!

Starting with right side of the equation to be proven,

(n + 1)! = (n + 1) * n! by separating the last term

= (n + 1 – 2 + 2) * n!

= (n – 1) * n! + 2 * n!

> 2 * n!

> 2 * 2^n by inductive hypothesis

> 2^(n + 1) by algebra

Therefore, 2^(n + 1) < (n + 1)!, which also proves that for all n >=4, 2^n < n!