Haozhe(Jason) Zhang
CS 5002 Discrete Math

Homework #6

Problem 1 [20pts (10,10)]: Pigeonhole Principle

i. What is the minimum number of students that must be assigned to a classroom with 14 tables to guarantee that some table will have at least 3 students?

ii. Suppose a set of 8 numbers are selected from the set {1, 2, 3, … , 13, 14}. Show that two of the selected numbers must sum to 15. (Hint: think about how many subsets of 2 elements you can form such that the sum of the values of the two elements is 15)

Solution:

i. Some table will have at least 3 students means that at least one table has at least 3 students. Let x be the number of students that is needed. (X – 1) / 14 = 2 will give us x = 29. We need at least 29 students to make sure at least one table will have 3 students.

ii. The possible binary sets to form a sum that is equals to 15 are {(1, 14), (2, 13), (3, 12), (4, 11), (5, 10), (6, 9), (7, 8)}. Let n be the number of selections, which is 8 and the m be the number of sum subsets, which are 7 subsets.

Here we get an expression of n>m, which means that if we were to pick 8 numbers and put them in one of the 7 subsets above, then the 8th number that is picked must be put into one of the 7 subsets to form a sum of 15, no exceptions.

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Problem 2 [30 pts (5,5,10,10)]: Weekend trip to Vegas.

For each subproblem, reduce your final answer to a single integer and show your work.

i. A standard 52-card deck has four suits (Hearts, Diamonds, Clubs, and Spades) and each suit has 13 ranks (2,3,4,5,6,7,8,9,10, Jack,Queen,King,Ace). The face cards are Jack, Queen, and King. How many ways are there to be dealt any 2 cards from a 52-card deck? (We are counting as distinct the same two cards received in a different order.)

ii. How many ways are there to be dealt Blackjack? To be dealt Blackjack, either the first card is an Ace and the second card is a face card or a 10, or the first card is a face card or a 10 and the second card is an Ace. (Again, we are counting as distinct the same two cards received in a different order.

iii. How many ways can you be dealt two cards such that the first card is a spade and the second card is a face card?

iv. How many ways can you pick three cards such that the first card is a spade, the second card is a one-eyed Jack, and the third card is a face card? (There are two one-eyed Jacks in a standard deck: the Jack of Hearts and the Jack of Spades. Hint: Break the problem down into 3 disjoint cases for the type of card received 1st, 2nd, and 3rd.

Solution:

i. There is no restriction to pick two cards from the 52-card deck and the order matters, so we are using 2-permutation for 52 and the ways in dealing the 2 cards are 52 * 51 = 2652 ways.

ii. To form a Blackjack, either the first card is an Ace and the second card is a face card or a 10, or the first card is a face card or a 10 and the second card is an Ace.
P1 = the first card is an Ace and the second card is a face card
There are 4 ways to pick an Ace and 12 ways to pick a face card, P1 = 4 * 12
P2 = the first card is an Ace and the second card is a 10, P2 = 4 * 4
P3 = the first card is a face card and the second card is an Ace, P3 = 12 * 4
P4= the first card is a 10 and the second card is an Ace, P4 = 4 * 4
P = P1 + P2 + P3 + P4 = 128

iii. We can break down the problem to find two disjointed sets that first one represents the first card is a spade but not a face card and the second card is a face card, or the first one is a spade and is one of the face card and the second card is one of the remaining 11 face cards.
The total ways are 10 * 12 + 3 * 11 = 120 + 33 = 153

v. spade (not a face card and not a one-eyed jack) = A 2 3 4 5 6 7 8 9 10 ~~J Q K~~ (spade) 10
one-eyed-jack = J(spade) + J(heart) 2
face card (not a one-eyed jack) = JQK (hearts) + JQK (spades)+ JQK (clubs)+ JQK (diamonds) 11

spade (a face card not a jack) = ~~A 2 3 4 5 6 7 8 9 10 J~~ Q K (spade) 2
one-eyed-jack = J(spade) + J(heart) 2
face card = JQK (hearts) + J~~Q~~K (spades)+ JQK (clubs)+ JQK (diamonds) 12 − 1 − 1 = 10

spade (is a one-eyed jack) = ~~A 2 3 4 5 6 7 8 9 10~~ J ~~Q K~~ (spade) 1
one-eyed-jack = ~~J(spade)~~ + J(heart) 1
face card = ~~J~~QK (hearts) + ~~J~~QK (spades)+ JQK (clubs)+ JQK (diamonds) 10

There are 10*2*11 + 2 * 2 * 10 + 1 * 1 * 10 = 220 + 40 + 10 + 270 ways.

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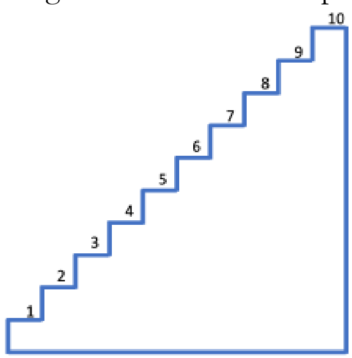
Problem 3 [20 pts (10,10)]: Flights of Fancy.

A flight of stairs has 10 steps numbered 1 to 10 as shown in the figure below.



i. How many ways could you climb up the set of stairs, assuming that you can skip any number of stairs with each step, but you must end on step 10 and you can only go up, never down and never remaining on the same step.

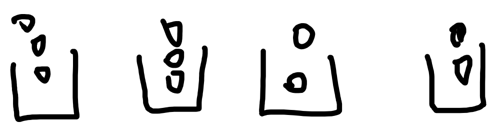
ii. How many ways could you climb up the set of stairs, assuming you take exactly 4 steps. Again, your staircase climb ends on step 10. Although there are different ways in which you could solve this problem, model the problem as a balls and bins problem for full credit.

Solution:

i. There are 2 options for each step, we start at 0 and going up towards to 10, so there are 2^(10 – 1) = 2^9 Ways to climb up the stairs if there is no restrictions on skipping the stairs.

ii. Let each step to be an individual bin and the number of stairs to be the balls, so we have 10 balls to be assigned to 4 bins.
We can have 10 + 4 – 1 choose 4 – 1 ways to assign the balls, which are 286 ways.

One possible answer illustrated as below:



m = 4 bins
n = 10 balls

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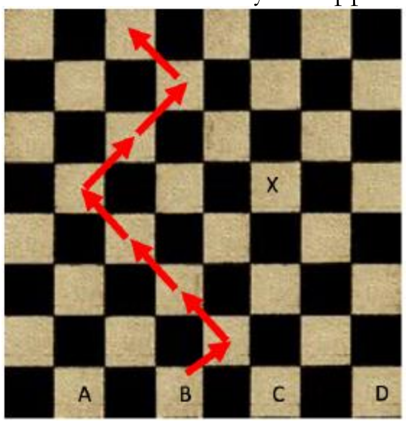
Problem 4 [30 (10,10,10)]: My checkered path

i. In the game of checkers, a game piece is allowed to move diagonally in the upwards direction only. (Let's ignore pieces jumping other pieces.) Starting at square a, b, c, or d, how many paths are there to the opposite end of the board? One such path is shown. Hint: This is basically the application of the sum rule over and over again.



ii. What if all paths MUST go through the square marked X?

iii. What if we exclude paths through the square marked X?

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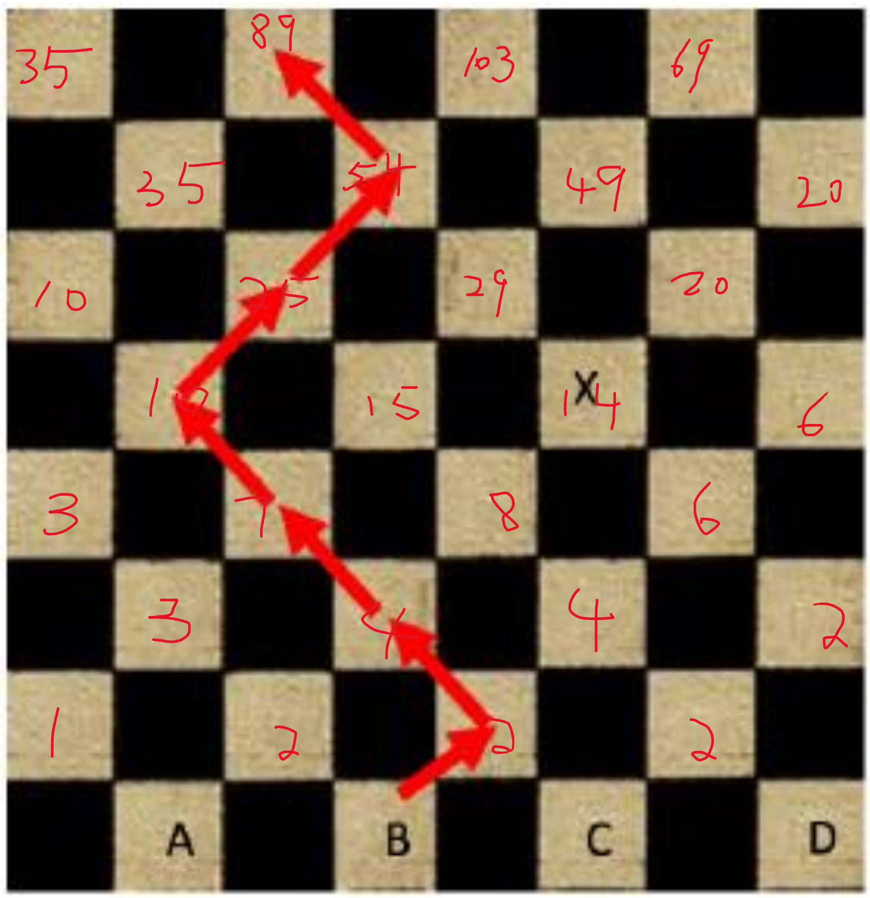
Solution:

i. We can start with any squares from the bottom and count the possible paths to the next square above and adding them up until reach to the top. Illustration shown below:

   At the end we are having 35 + 89 + 103 + 69 = 296 paths.

296



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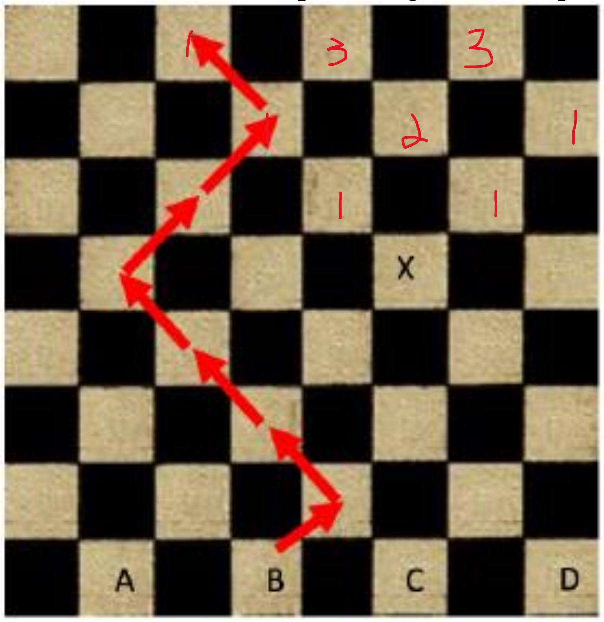
ii. There are 14 ways to get to point x and 7 ways to get to the top.

So the total number of paths to get to the top when must going through x is 14 * 7 = 98 paths



iii. To exclude the paths thought the square marked X, we simply using the answer we got from question 1 and subtracting the answer we got from question 2, which is 296 – 98 = 198 paths.

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spade = A 2 3 4 5 6 7 8 9 10 J Q K
one-eyed-jack = J(h), J(s), J(c), J(d)
face = JQK(hearts) + JQK(spades)+ JQK(clubs)+ JQK(diamonds)