Problem 1 [20 pts (10,10)]: Warm-up problems

i. What is the probability, if a die is rolled five times, that only two different values appear?
ii. Which is more likely, rolling an 8 when two dice are rolled, or rolling an 8 when three dice are rolled?

Solution:

i. Only two different values meaning the other three rolls generates same values, there are 6 ways for all other three dice to be same values, which is C(5, 3) * 6 ways to pick three dice with same values. We need to make sure the last two picks are different to qualify the two different values appearing statement, which is 6 * 5 choices. (or 6 out of 36 options to pick two values that are equal, produce the similar formula)
P = C(5, 3) * 6 * 6 * 5 / 6^5 = 0.231

ii. When rolling two dice, the subsets that two values add up to 8 are 2-6, 3-5, 4-4, 5-3, 6, 2.
P(2 rolls) = 5 / 36

When rolling three dice, the subsets that three values add up to 8 are:
Fix first roll at 1, second and third are combined to be summed up to 7, the combinations are 1-6, 2-5, 3-4, 4-3, 5-2, 6-1, 6 ways;

Fix the first roll to be 2, the remaining sum is 6, the combinations are 1-5, 2-4, 3-3, 4-2, 5-1, 5 ways;

Fix the first roll to be 3, the remaining sum is 5, the combinations are, 1-4, 2-3, 3-2, 4-1, 4 ways;

Similar process, fix first roll to be 4 yields 3 ways, first roll to be 5, yields 2 ways, first roll to be 6, yields 1 way.

Total sum of ways are 6 + 5 + 4 + 3 + 2 + 1 = 21;
P(roll 2 dice) = 5/ 36 = 30 / 216 > 21 / 216 = P(roll 3 dice)

Problem 2 [30 pts (6,6,6,12)]: Luigi's House of Random Pizza

At Luigi's House of Random Pizza, you can order only one thing: A medium pizza with three random topping layers chosen by Luigi himself. Luigi has 5 different toppings: Pepperoni, Mushroom, Anchovie, Onion, and Pepper. For example, you might get: Pepperoni-Onion-Mushroom, or maybe Mushroom-Pepperoni-Onion (the order of the layers matters to Luigi) or if you are especially lucky then Anchovie-Anchovie-Anchovie.

1. Today is your lucky day! Luigi hands you an Anchovie-Anchovie-Anchovie pizza! What were the chances?!?
2. It turns out you are allergic to Anchovies. But if you complain then Luigi will respond: No pizza for you! so you decide to take your chances. What are your chances of getting a pizza with 1 or more anchovy toppings?
3. What is the probability that you get a pizza with either two or three distinct toppings? (For example, Mushroom-Onion-Mushroom or Pepper-Onion-Mushroom.) Reduce your answer to a decimal.
4. What is the probability that Luigi gives you a pizza with exactly two distinct toppings (occurring in any order)? Reduce your answer to a decimal.

Solution:

1. P(A-A-A) = 1 / 5^3 = 0.008
2. P(1 or more A) = P(No A-A-A) = 1 – 0.008 = 0.992
3. P(2 or 3 distinct) = 1 – P(not 2 and not 3 distinct) = 1 – P(all the same) = 1 – 5/5^3 = 0.96
4. P(exactly 2 distinct toppings) =
   There are 5 options for first topping picking, second one must be the same as the first topping picking, so the only 1 option, occurring in any order means order don't matter, C(3, 2) ways to place these two toppings(first-second, first-third, second-third), the option for the last topping to be picked is 5-1=4,
   P(exactly 2 distinct toppings) = 5 * C(3, 2) * 4 = 20 * 3 = 60 / 5^3 = 0.48

Problem 3 [20 points (10,10)]: Thinking Independently.

1. A fair coin is flipped 4 times. Let E be the event that the first flip is heads. Let F be the event that the coin landed on heads an even number of times. Are E and F independent? Show your work.

2. A family has three children. Assume that having either a boy or a girl is equally likely. Let the event E be the event that the family has children of both sexes, and F be the event that the family has at most 1 boy. Are the events E and F independent?

Solution

1. Because it's a fair coin, the probability of the first flip is heads is P(E) = 1/2 = 0.5
   Possible E cases are HHHH, HTHH, HHTH, HHHT, HTTH, HHTT, HTTT, HTHT
   The probability of the coin landed on heads an even number of times, total possible cases of coin landed on heads are HHHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, TTTT
   P(F) = 0.5
   E and F = HHHH, HTTH, HHTT, HTHT = 4, P(E and F) = 4 / 16 = 0.25
   P(E) * P(F) = P(E and F) = 0.25
   P(E) and P(F) are independent.

2. Let P(G) =P(B) = 0.5, total cases are GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB.
   Cases that have both sexes are all except GGG and BBB
   P(E) = P(have both sexes) = 6/8 = 3/4 = 0.75
   Cases that family has at most 1 boy are: GGG, GGB, GBG, BGG
   P(F) = 4/8 = 0.5
   Cases that have both sexes and at most 1 boys are: GGB, GBG, BGG
   P(E and F) = 3 /8 =0.375
   P(E) * P(F) = 0.75 * 0.5 = 0.375
   P(E) and P(F) are independent.

Problem 4 [30 points (6,6,6,6,6)]: All your marbles

Susie has a bag of marbles containing 3 Red, 7 Green, and 10 Blue marbles. In this problem, the phrase with replacement means that marbles are drawn one at a time and after the draw it is replaced back into the bag before picking the next marble. The phrase without replacement means that each marble is drawn and held onto until all marbles are drawn.

1. What is the probability of picking 5 marbles and getting at least one red marble? Calculate

the probability (a) with replacement, and (b) without replacement.

2. Pick 8 marbles: 4 green and 4 blue. Calculate the probability (a) with replacement and (b) without replacement.

3. If Susie sells you 7 marbles chosen randomly without replacement, what are your chances of getting at least six marbles of the same color?

4. Susie sells the marbles for 5 cents each. For 5 cents, she'll let you pick out a random marble out of the bag. (There are no taksie-backsies - i.e., no replacements!) How much would you have to pay Susie to be sure of getting at least 6 marbles of the same color?

5. Unfortunately, you only have half the amount you need to guarantee getting six marbles of the same color. (See previous problem to determine what this amount is.) If you buy as many marbles as you can afford, what are your chances of getting six or more marbles of the same color?

Solution:

1. Two scenarios
    a. With replacement(putting it back)
       Picking one out of 20 each time, at least one red is equivalent to subtracting the complement that is no red is selected from 1.
       P(At least 1 Red) = 1 - (17/20) ^ 5 = 0.556
    b. Without replacement(not putting it back)
       Picking one marble without replacement meaning the poll is decremented by 1 marble after each draw, so the remaining marbles after each draw are 20 19 18 17 16, and the remaining marbles that are not red are 17 16 15 14 13.
       P(At least 1 red) = 1 - 17/20 * 16/19 * 15/18 * 14/17 * 13/16 = 0.601

2. Two scenarios
    a. With replacement(putting it back)
       P(4 green and 4 blue) = 7^4 * 10^4 / 20^8 = 0.00094
    b. Without replacement(not putting it back)
       P(4 greens and 4 blues) = C(7, 4) * C(10, 4)/ C(20, 8) = 35 * 210 / 125970 = 0.058

3. Having at least 6 marbles of the same color is equivalent to say that has 6 or 7 marbles of same colors.
   Two colors that could be selected for this scenario, 6/7 Greens or 6/7 blues.
   Without replacement:

   Green:
   For picking 6 green marbles out of 7 picks, we need to choose 6 out of 7 green marbles, and choose 1 marble that is not green from the rest of 13 marbles.
   P(6 same color Green) = C(7, 6) * C(13, 1) / C(20, 7) = 7 * 13 / 77520 = 0.0012
   For picking 7 green marbles out of 7 picks, we need to choose 7 out of 7 green marbles.
   P(7 same color Green) = C(7, 7) / C(20, 7) = 0.00009

   Blue:
   For picking 6 blue marbles out of 7 picks, we need to choose 6 out of 10 blue marbles, and choose 1 marble that is not green from the rest of 10 marbles.
   P(6 same color Blue) = C(10, 6) * C(10, 1) / C(20, 7) =210 * 10 /77520 = 0.027
   For picking 7 blue marbles out of 7 picks, we need to choose 7 out of 10 blue marbles.
   P(7 same color Blue) = C(10, 7) / C(20, 7) = 120 / 77520 = 0.00155

   Sum up the 4 cases above:

P(choose at least 6 marbles of the same color) = 0.0012 + 0.00004+ 0.027 + 0.00155 = 0.0298

4. By applying the pigeonhole principle, if we have picked out all 3 red marbles, 5 green ones and 5 blue ones, the next pick will guarantee we get at least 6 marbles of the same color.
   The amount we must pay Susie is to be (3 + 5 + 5 + 1) * 5 = 70 cents.