Problem 1 [25 points (8,8,9)]: Conditional Probability

We are given 5 cards. 3 of the cards are black and they are numbered 1; 2; 3. The other two cards are red and they are numbered 1; 2.
We pick 2 random cards.

i. What is the probability that both cards are red?
   P(2 reds) = C(2, 2) / C(5, 2) = 1/10
ii. What is the probability that both cards are red, if we know that at least one of them is red?
   Possibilities are = 12, 13, 23, 12, 11, 12, 21, 22, 31, 32

   P(2reds|1red) = P(1red|2reds) * P(2reds) / (P(1red|2reds) * P(2reds) + P(1red|at least one is not red) * P(at least one is not red))

   P(2 reds) = 1/ 10
   P(at least one is not red) = 9/ 10
   P(1red|2 reds) = 1
   P(1red|at least one is not red) = 6 / 9 = 2/3
   P (2 reds| 1 red) = 1 * 1/10/(1 * 1/10 + 2/3 * 9/10) = 1/7

iii. What is the probability that both cards are red, if we know that one of them is red card number 1?
   Similar to problem 2 above.
   Possibilities are = 12, 13, 23, 12, 11, 12, 21, 22, 31, 32

   P(2reds|1red(#1)) = P(1red(#1)) |2reds) * P(2reds) / (P(1red(#1)) |2reds) * P(2reds) + P(1red(#1)) |at least one is not red) * P(at least one is not red))

   P(2 reds) = 1/ 10
   P(at least one is not red) = 9/ 10
   P(1red(#1)) |2 reds) = 1
   P(1red(#1)) |at least one is not red) = 3/ 9 = 1/3
   P (2 reds| 1 red(#1))) = 1 * 1/10/(1 * 1/10 + 1/3 * 9/10) = 1/4

Problem 2 [25 pts (10,15)]: At the carnival!

At a carnival, you are trying to throw 4 balls into 4 colored pots. The pots are colored red, blue, green, and pink. You will throw each of the 4 balls one at a time into these pots. However, you must play this game blindfolded. The game pays out as follows

• $1 for each ball in the red pot

• $2 for each ball in the blue pot

• $3 for each ball in the green pot6

• $4 for each ball in the pink pot

How many points do you expect to score:

i. Assuming every ball lands in some hole with equal probability?
   P(red) = P(blue) = P(green) = P(pink) = 1/ 4
   Let x to be expected points.
   E[x] = 1/4 * 1 + 1/4 * 2 +1/4 * 3 +1/4 * 4 = $2.5
ii. Assuming every ball has a 1 in 3 chance of not landing in any pot (and thus giving you no payout) but is otherwise equally likely to land in any pot?
   E[x]' = $2.5 * (1- 1/ 3) = $1.67

Problem 3 [25 pts (5 pts each)]: Probability

Let W(x) be the number of 1's in the binary representation of x. For example, W(5) = W(00101_2) = 2 because there are 2 1's in the binary representation of 5. This is sometimes called the weight of the binary number. A deck of 32 cards has numbers 0 to 31_{10} written in 5-bit binary (00000_2…11111_2).

1. What is the probability that the weight of a randomly chosen card is exactly 3?
   P(W=3) = C(5, 3)/ 2^5 = 10 / 32 = 5/16

2. What is the probability that the weight of the card is 3 and the number on the card is odd, i.e., P(W = 3 *and* Odd)?
   Number is odd only if the low bit is 1. If known the low bit is 1, the other two choices are chosen from the rest of 4 slots
   C(4, 2) = 6
   P(W =3 and Odd) = 6 / 32 = 3/16

3. Calculate P(Odd|W = 3), the probability that the card represents an odd number given that the weight of the number is 3.
P(Odd|W = 3) = P(W = 3|Odd) * P(Odd) / (P(W = 3|Odd) * P(Odd)+ P(W = 3|Even) * P(Even))
P(Odd) = 1/2
P(Even) = 1/2
Given odd number (16), there are C(4, 2) = 6 ways to have the weight equals 3.
P(W = 3|Odd) = 6 / 16 = 3 /8
Given odd number (16), there are C(4, 3) = 4 ways to have the weight equals 3.
P(W = 3|Even) = 4 /16 = 1 /4

P(Odd|W=3)= 3/8 * 1/2 / (3/8 * 1/2 + 1/4 * 1/2) = 3/5 or P(W=3 and Odd)/P(W=3)= 3/5

4. You are now dealt 3 random cards. What is the expected value for the total weight of your three-card hand?

E[Total weight of three cards] = 1 * C(5, 1)/32 + 2* C(5, 2)/32 + 3* C(5, 3)/32 + 4 * C(5, 4)/32 + 5 * C(5, 5) /32
= 5/32 + 20/32 + 30/32 + 20/32 + 5/32 = 2.5

5. What is the probability that the total weight of the three cards you were dealt is equal to 13? You may leave your answer as a simple expression.
Only two outcome will satisfy this condition {5, 5, 3} and {5, 4, 4}
Since each card is distinct, {5, 5, 3} is eliminated. Only events falls in {5, 4, 4} will produce a weight of 13.
P(W = 5) = 1/32
P(W = 4) = C(5, 4) * 1/ 32 = 5/32
Orders don't matter, but without replacement, the probability of three cards were dealt is equal to 13 is:
P(W = 13(3 cards)) = 1/ 32 * 5/ 31 * 4/30

Problem 4 [25 pts (5,10,10)]: Medical Testing and Bayes
A certain virus is spreading rapidly through the population and doctors have come up with a new but imperfect test to determine if a patient is infected.

- 20 percent of the population is already infected with the virus. P(infected) = 0.2
- 90 percent of infected patients test positive. P(TP|infected) = 0.9
- 50 percent of healthy uninfected patients also test positive. P(TP|uninfected) = 0.5

For this section, express your answer as a simple fraction or number.

P(infected) = 0.2
P(TP|infected) = 0.9
P(TP|uninfected) = 0.5
P(uninfected) = 1 - P(infected) = 0.8

1. What is the probability that a random person tests positive?

P(TP) = P(TP|infected) * P(infected) + P(TP|uninfected) * P(uninfected) = 0.58

2. What is the probability that a random person who tests positive actually has the virus?
   P(infected|TP) = P(TP|infected) * P(infected) / (P(TP|infected) * P(infected) + P(TP|uninfected) * P(uninfected))
   = 0.9 * 0.2 / (0.9 * 0.2 + 0.5 * 0.8) = 0.31

3. Suppose an independent second test is performed on a patient that previously tested positive. This time, the test result is negative. Now what is the probability that the patient is infected with the virus?
   P(TN) = 1 – P(TP) = 0.42
   P(TN|infected) = 1 - P(TP|infected) = 0.1
   P(TN|uninfected) = 0.5
   P(infected|TN) = P(TN|infected) * P(infected) / (P(TN|infected) * P(infected) + P(TN|uninfected) * P(uninfected))
   = 0.1 * 0.2 / (0. 1 * 0. 2 + 0.5 * 0.8) = 0.0498