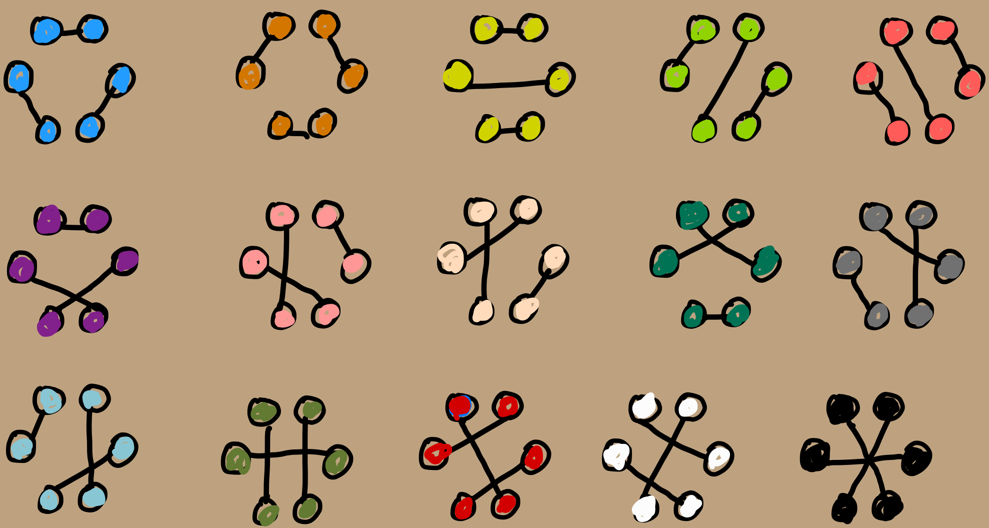
More Counting / Combinatorics



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Lecture Notes

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Permutations $= n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$= n!$

$=$ An ordered arrangement of n objects

k-permutation $= n \cdot (n-1) \cdots (n-k+1)$

$= P(n,k) = C(n,k) \cdot k! = \binom{n}{k} k!$

$= \dfrac{n!}{(n-k)!}$

= ways to choose, then order k of n objects.

combinations $= C(n,k) = \boxed{\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}} = \dfrac{P(n,k)}{k!}$

= n choose k

= # of unordered subsets of size k from n objects.

(The order in which we choose k objects makes no difference)

Fun Facts:

$\binom{n}{n-1} = \binom{n}{1} = n$

$\binom{n}{0} = \binom{n}{n} = 1$

$\binom{n}{k} = \binom{n}{n-k}$

e.g. 01101000

$\binom{8}{3}$ = choose 1 locations

$\binom{8}{5}$ = choose ϕ locations

Examples

① 10 packages: 6 heavy, 4 light.

Choose 3 packages: $\binom{10}{3}$

Choose and Deliver: $\binom{10}{3} 3! = P(10,3) = \frac{10!}{7!}$

② Choose 3 heavy <u>and</u> 3 light.

$$\binom{6}{3}\binom{4}{3} = 80$$

choose <u>and</u> deliver:

$$\binom{6}{3}\binom{4}{3} 6!$$

③ A "byte" has 8 bits.

How many bytes have 3 1-bits, e.g. 01101000

$$\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8\,7\,\cancel{6}\,\cancel{5}\,\cancel{4}\,\cancel{3}\,\cancel{2}\,\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot 1\quad \cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 56$$

④ How many bytes have exactly 5 ones?

$$\binom{8}{5} = \frac{8!}{5!\,3!} = 56$$ [we are just flipping the bits]

⑤ If $|A| = 10$, How many subsets have at least 8 elements?

$$\binom{10}{8} + \binom{10}{9} + \binom{10}{10}$$

$$= \frac{10\cdot 9}{2\cdot 1} + \frac{10}{1} + 1$$

$$= 45 + 10 + 1 = 56$$

(5) Password Spaces

6 - 10 characters : Uppercase, Lowercase, Digits
3 + letters
3 + digits

How many passwords are there?
what are the steps to defining the PW?
Are the decisions AND or OR.

a) Length? 6, 7, 8, 9, or 10 ⇒ Sum Rule.

\# Length 6 + # Length 7 + ... + # Length 10

b) Once I pick length, count legals

\# Legal = # Possible − # illegal

62^9

\# illegals

6 disjoint choices ⇓ sum rule

0	letters	10^9
1	letter	⇒ Pick Letter Pos × Pick Letter × digits^8 $= 9 \cdot 52 \cdot 10^8$
2	letter	⇒ $\binom{9}{2} \cdot 52^2 \cdot 10^7$
0	digit	52^9
1	digit	⇒ Pick digit Position × Pick Digit × letters^8 $= 9 \cdot 10 \cdot 52^8$
2	digit.	⇒ $\binom{9}{2} 10^2 \cdot 52^7$

w/o inclusion/exclusion (2 letters ⇒ 7 digits) so there is no overlap among categories of illegals.

$$\# PW = \sum_{i=6}^{10} \left(62^i - \left(10^i + i \cdot 52 \cdot 10^{i-1} + \binom{i}{2} 52^2 \, 10^{i-2} + 52^i + i \cdot 10 \cdot 52^{i-1} + \binom{i}{2} 10^2 \, 52^{i-2} \right) \right)$$

More passwords:

How many 3-digit PIN^s with at least one even digit.

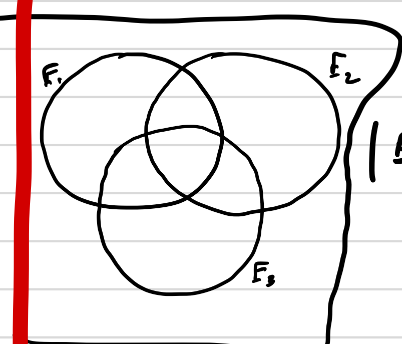
How many total PIN^s? $10 \cdot 10 \cdot 10 = 1000$

a) False Argument:

choose even digit location:	3
# of even digits:	5
Possibilities for remaining digits:	10×10
	1500

1500 > 1000, WRONG!

b) E_1 = even digit in 1st location
E_2 = " " " 2nd "
E_3 = " " " 3rd "



$$|E_1 \cup E_2 \cup E_3| = \overset{500}{|E_1|} + \overset{500}{|E_2|} + \overset{500}{|E_3|}$$

$$- \overset{250}{|E_1 \cap E_2|} - \overset{250}{|E_1 \cap E_3|} - \overset{250}{|E_2 \cap E_3|}$$

$$+ \overset{125}{|E_1 \cap E_2 \cap E_3|} = 875$$

c) # Legal PIN^s = # PIN^s − # Illegal PIN^s

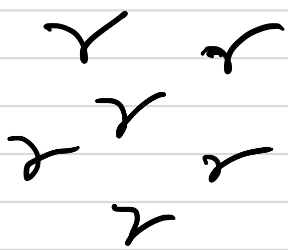
= 1000 − # all odd

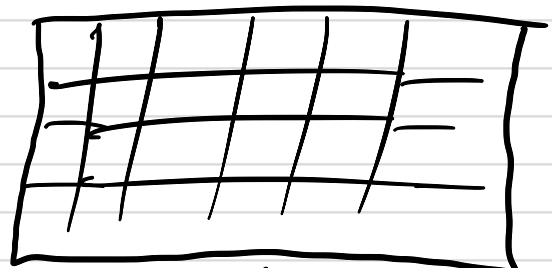
= $1000 - 5 \cdot 5 \cdot 5$

= $1000 - 125 = 875$

Pigeon Hole Principle

A) Imagine you had 101 pigeons and a wall with 100 slots for each of your pigeons to nest in.





What can we say about the occupancy of the holes?

At least one slot has <u>two</u> pigeons!

B) Suppose I flip a coin 5 times. I might get H OR T:

HHTHT
TTTTT
HTTHT

Now my coin flips are the pigeons and I have two slots or "holes" – one for heads and one for tails.

By the Pigeon Hole Principle, I know that I get at least 3+ Heads or 3+ Tails.

PHP - continued.

c) The population of Boston is 600 thousand. A human head has 90-150,000 hairs.

Even ignoring bald people and allowing for up to 500,000 hairs on a head there are 2 people walking around with exactly the same # of hairs!

$$\lceil 600{,}000 / 500{,}000 \rceil = \lceil 1.2 \rceil = 2$$

In general, when we have n objects to assign to k "slots", we know that one of the slots will have

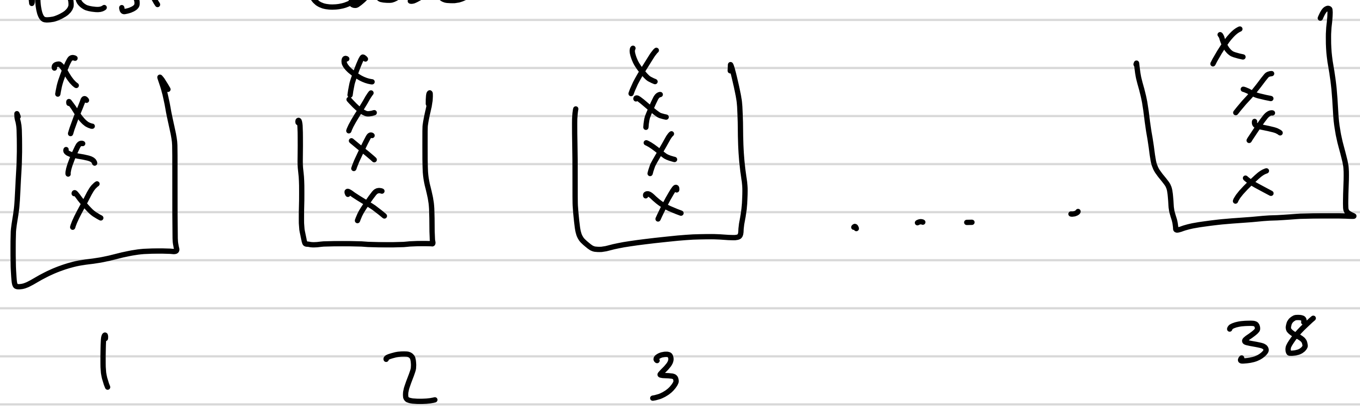
$\lceil n/k \rceil$ objects.

"ceiling" function
round up to the next highest integer.
Objects are discrete.
(No splitting the poor birds across multiple holes!!)

Roulette Again

How many spins before I am guaranteed to see the same # 5 times?

Best Case:



$38 \cdot 4 = 152$ spins and each # appears 4 times.

It only requires 1 more spin

So: $38 \cdot 4 + 1 = 153$ spins.

Note: $\left\lceil \frac{153}{38} \right\rceil = \lceil 4.02 \rceil = \underline{\underline{5}}$

OPTIONAL

Theorem: n objects in k boxes.

Let $x_1, x_2, \ldots x_k$ be

k integers. (non negative)

$\sum_i x_i = n$

Average: $\bar{x} = \frac{n}{k} = \frac{x_1 + x_2 + \cdots + x_k}{k}$

<u>Claim</u>: At least one x_i must be at least as large as $\bar{x}$

$x_i \geq \bar{x}$

(Can't have all values $< \bar{x}$)

<u>Proof by contradiction</u>

Assume that no $x_i \geq \bar{x}$

Then all $x_i < \bar{x}$ for all i

$$
\begin{aligned}
x_1 &< \bar{x} \\
x_2 &< \bar{x} \\
&\vdots \\
x_k &< \bar{x}
\end{aligned}
\quad \Rightarrow \quad x_1 + x_2 + \cdots + x_k < k\bar{x}
$$

$$\Downarrow$$

$$\bar{x} > \frac{x_1 + \cdots + x_k}{k}$$

$$\bar{x} > \bar{x} \quad \text{(contradiction)}$$

OPTIONAL

(7)

Binomial Theorem : $(x+y)^n$

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

		choose x or y	choose x or y	
F	First	x	x	$\to x^2$
O	Outer	x	y	$\to xy$
I	Inner	y	x	$\to yx$
L	Last	y	y	$\to y^2$

$\Rightarrow x^2 + 2xy + y^2$

	x	y
x	x^2	xy
y	yx	y^2

Another way to think about it:

Each Terms can involve 0, 1, or 2 y's
So my choice is: How many y's do we want to expand by?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

For each term I choose x or y for expansion.
The resultant terms I'll have

$$x^3 \qquad x^2y \qquad xy^2 \qquad y^3$$

Q: How many y's do I choose? (Dictates term)
How many ways can you do that
(Dictates coefficient)

OPTIONAL

e.g x^2y term

$\#$ way $= 3$

A:

	# ways		term
0 y's	1	$\binom{3}{0}$	x^3
1 y's	3	$\binom{3}{1}$	x^2y
2 y's	3	$\binom{3}{2}$	xy^2
3 y's	1	$\binom{3}{3}$	y^3

$x^3 + 3x^2y + 3xy^2 + y^3$

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

# y's	# ways	term
0	$\binom{5}{0} = 1$	x^5
1	$\binom{5}{1} = 5$	x^4y
2	$\binom{5}{2} = 10$	x^3y^2
3	$\binom{5}{3} = 10$	x^2y^3
4	$\binom{5}{4} = 5$	xy^4
5	$\binom{5}{5} = 1$	y^5

$\Rightarrow x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

So in general:

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

OPTIONAL

⑧ Application from algebra

$(x+2y^{-2})^6$ Q: what term includes y^{-8} ?

$= (x+2y^{-2})(x+2y^{-2})\cdots\cdots(x+2y^{-2})$

we want to expand by this term 4 times. Out of 6.

So: $\binom{6}{4} x^2 (2y^{-2})^4 = 15 x^2 \, 2^4 y^{-8}$

$= 240 x^2 y^{-8}$

$\dfrac{6\cdot 5 \cdots\cdots}{2\cdot 1 \quad 4\cdot 3\cdot 2\cdot 1}$

$= 15$

⑨ $(x+y)^4 = (x+y)(x+y)^3 = (x+y)(x^3+3x^2y+5xy^2+y^3)$

$= x^4+3x^3y+3x^2y^2+xy^3$

$+ \qquad x^3y+3x^2y^2+3xy^3+y^4$

$x^4+4x^3y+6x^2y^2+4xy^3+y^4$

Coefficients:

$$
\begin{array}{ccccc}
 & & \binom{3}{2} & & \\
1 & 3 & 3 & 1 & \\
 & 1 & 3\ \binom{3}{1} & 3 & 1 \\
\hline
1 & 4 & 6 & 4 & 1 \\
 & & \binom{4}{2} & &
\end{array}
$$

$\binom{4}{2} = \binom{3}{2} + \binom{3}{1}$: I need two ys, so expand by x and choose 2 ys OR expand by y and choose 1 y term.

OPTIONAL

(10) Fun with the binomial theorem.

$$\left.\begin{aligned} 11^0 &= 1 \\ 11^1 &= 11 \\ 11^2 &= 121 \\ 11^3 &= 1331 \\ 11^4 &= 14641 \end{aligned}\right\}\text{ The binomial coefficients!}$$

Why does this work?

$$11^n = (1+10)^n = \sum_{k=0}^{n} \binom{n}{j} 1^{n-k} 10^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} 10^k$$

$$= \binom{n}{n} 10^n + \binom{n}{n-1} 10^{n-1} \cdots + \binom{n}{0} 10$$

e.g.

$$11^3 = (1+10)^3 = \binom{3}{3} 10^3 + \binom{3}{2} 10^2 + \binom{3}{1} 10^1 + \binom{3}{0} 10^0$$

$$1 \cdot 1000 + 3 \cdot 100 + 3 \cdot 10 + 1 \cdot 1$$

$$= 1331$$

The pattern breaks down for $n \geq 5$. Why?

Another Funfact

a) Suppose I need to choose 3 of 8 people
Imagine we line up the 8 people.

$$\binom{8}{3} = \binom{7}{3} + \binom{7}{2}$$

$\binom{7}{3}$: don't choose 1st person

$\binom{7}{2}$: choose 1st person

b) # of length 5 bit strings with exactly 3 ones

$\_\ \_\ \_\ \_\ \_$ $\binom{5}{3}$

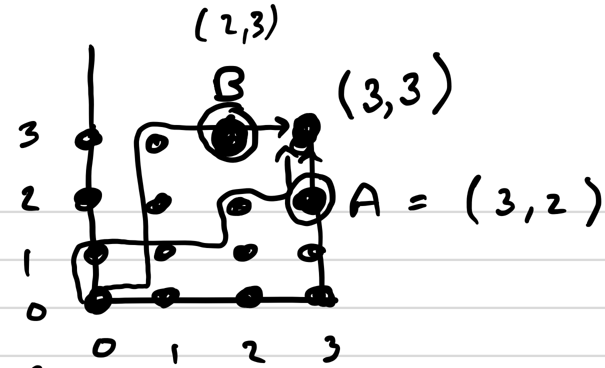
$0\,\_\ \_\ \_\ \_$ $\qquad$ $1\,\_\ \_\ \_\ \_$

$$\binom{4}{3} + \binom{4}{2}$$

$00\,\_\ \_\ \_$ $\qquad$ $01\,\_\ \_\ \_$

$$\binom{3}{3} + \binom{3}{2}$$

c) Lattice Paths



$\binom{6}{3}$ total paths

$=$

$=$ # paths through A + # Paths through B

$= \binom{5}{3} + \binom{5}{2}$

A B

In general:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This recursive formula leads to Pascal's Triangle (next page)

(12) OPTIONAL

PASCAL'S TRIANGLE

$\binom{n}{k}$:

n \ $k=$	0	1	2	3	4			Σ	
0	1						$(x+y)^0$	1	2^0
1	1	1					$(x+y)^1$	2	2^1
2	1	2	1				$(x+y)^2$	4	2^2
3	1	3	3	1			$(x+y)^3$	8	2^3
4	1	4	6	4	1		$(x+y)^4$	16	2^4
5	1	5	10	10	5	1	$(x+y)^5$	32	2^5

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x = 1 \quad y = 1$

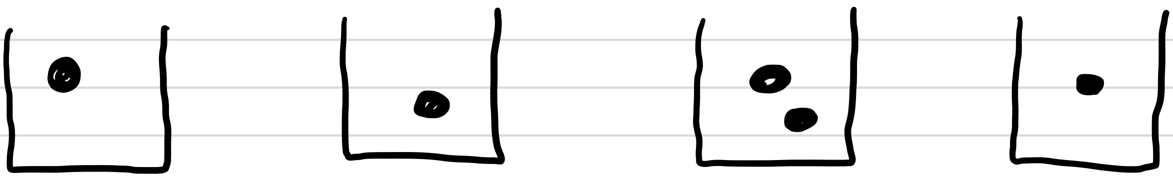
$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{j} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{j} = 2^n$$

Balls and Bins

Suppose you had 4 bins & 5 balls.
The balls are indistinguishable.

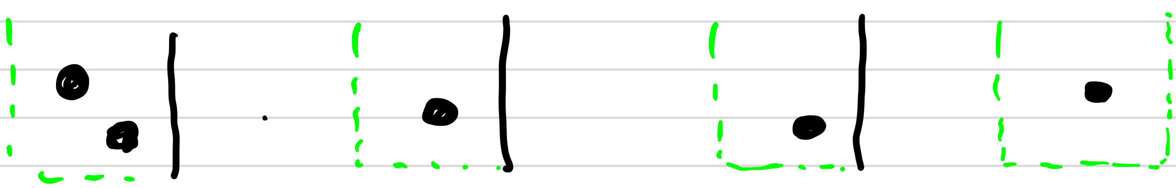
1 2 2 0

1 1 2 1



5 0 0 0

2 1 1 1



$\Downarrow$

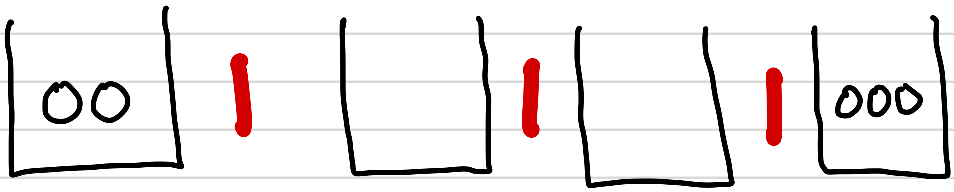
00 | 0 | 0 | 0 $\Rightarrow$ 00101010

$= \binom{8}{3}$ $\leftarrow$ # balls + # dividers $= \binom{\#\text{balls} + \#\text{bins} - 1}{\#\text{bins} - 1}$

$\leftarrow$ # dividers

$= \binom{8}{5}$

$\binom{\#\text{balls} + \#\text{bins} - 1}{\#\text{balls}}$



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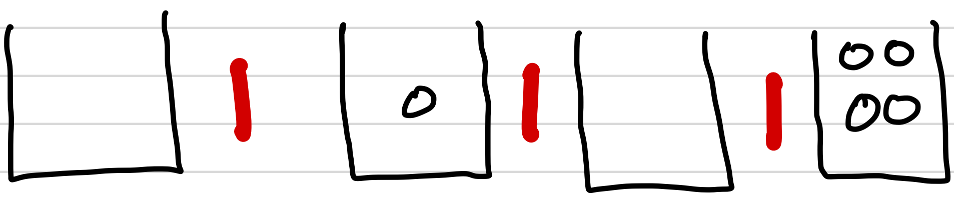

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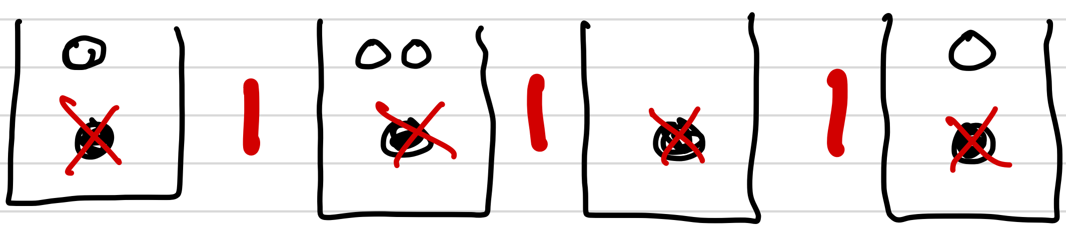
8 balls, At least one in each bin:



○ | ○○ | | ○ (4 balls were "consumed")

$$\cancel{\binom{8+3}{3}} \Rightarrow \binom{4+3}{3} = \binom{7}{3} = 35$$

arrangements of n balls into k bins

$$= \binom{n+k-1}{k-1} = \binom{n+k-1}{n+k-1-(k-1)} = \binom{n+k-1}{n}$$

$\binom{n+k-1}{k-1}$: choosing bin divider locations

$\binom{n+k-1}{n}$: choosing ball locations

Since : $\binom{n}{k} = \binom{n}{n-k}$

5 brands of soda, buy 15 cans.

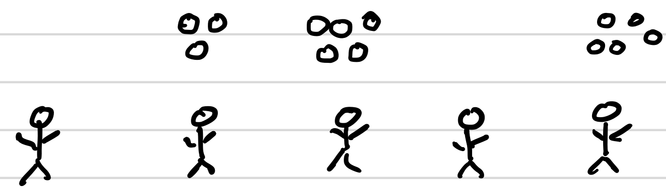
5 brands $\Downarrow$ bins

15 cans $\Downarrow$ balls

$$\binom{15+5-1}{5-1} = \binom{13}{4} = 3876$$

How many ways to distribute 13 cookies to 5 kids

bins = kids
balls = cookies



1 2 3 4 5

$\binom{17}{4}$ |000|00000||00000

a) What if every kid gets 1 cookie? Imagine we just distribute the cookies, leaving 8 left.

$\binom{12}{4}$ e.g ||0000||0000

b) 2 or more cookies each? Pre-distribute the 10 cookies leaving just 3 to distribute to 5 kids

$\binom{7}{4}$ e.g 0||00||

How many integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13$$

This is the same as the cookies & kids problem!

bins = variables
balls = units of one, 13 total

Flower Arrangements:

- Roses, Daisies, Carnation, Irises
- 12 flowes

		R	D	C	I
12	=	12	0	0	0
12	=	0	12	0	0
12	=	6	2	2	2

1. Any Combination

$$\binom{12+3}{3} = \binom{15}{3} = \frac{\overset{5}{\cancel{15}} \cdot \overset{7}{\cancel{14}} \cdot 13}{\cancel{3} \cdot \cancel{2}} = 455$$

2. At least one of each flower

"Consume 4 flowers total

Distribute Another 8 flowers any way you like

$$\binom{8+3}{3} = \binom{11}{3} = \frac{11 \cdot \overset{5}{\cancel{10}} \cdot \overset{3}{\cancel{9}}}{\cancel{3} \cdot \cancel{2}} = 165$$

3. At least 1/2 the flowers are roses:

- Distribute 6 Roses
- Now choose 6 more flowers from R, D, C, I

$$\binom{6+3}{3} = \binom{9}{3} = \frac{\cancel{9}^{3} \cdot \cancel{8}^{4} \cdot 7}{\cancel{3} \cdot \cancel{2}} = 84$$

4. Exactly 1/2 are Roses

- Choose 6 flowers from D, C, I
- $\binom{6+2}{2} = \binom{8}{2} = \frac{8 \cdot 7}{2} = 28$

5. No more than 3 Roses

(a) Method 1: Consider bouquets with 0, 1, 2, or 3 Roses.

$$\underline{0} \qquad \underline{1} \qquad \underline{2} \qquad \underline{3}$$

$$\binom{12+2}{2} + \binom{11+2}{2} + \binom{10+2}{2} + \binom{9+2}{2}$$

$$91 + 78 + 66 + 55 = 290$$

(b) Method 2.

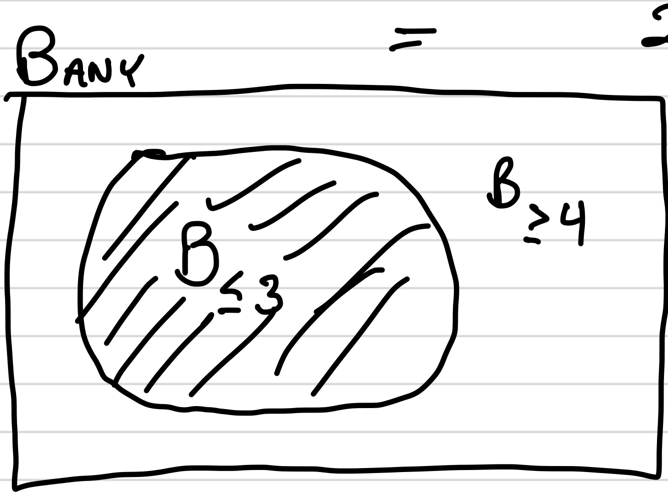
$$\text{Bouquets}_{\leq 3} = \text{Bouquets}_{\text{ANY}} - \text{Bouquets}_{\geq 4}$$

$$= \binom{12+3}{3} - \binom{8+3}{3}$$

$$= \binom{15}{3} - \binom{11}{3}$$

$$= 455 - 165$$

$$= 290$$



6. Roses and Carnations Only.

Max 8 Roses

Max 8 Carnations

$$\left.\begin{array}{cc} R & C \\ \hline 8 & 4 \\ 7 & 5 \\ 6 & 6 \\ 5 & 7 \\ 4 & 8 \end{array}\right\}\ 5 \text{ Bouquets}$$

Set aside 4R, 4C

Distribute Remaining 4 in $\binom{4+1}{1} = \binom{5}{1} = 5$

7. 18 flowers

1 type has	3	R	C	I
1 type has	4	C	R	C
1 type has	5	I	D	D
1 type has	6	D	I	R
	18			

4! possibilities = 24