And if you listen very hard
The tune will come to
you at last
When all is one
and one is all,
That's what it is
To be a rock and not to roll.

Sequences, Series and Proof by Induction.

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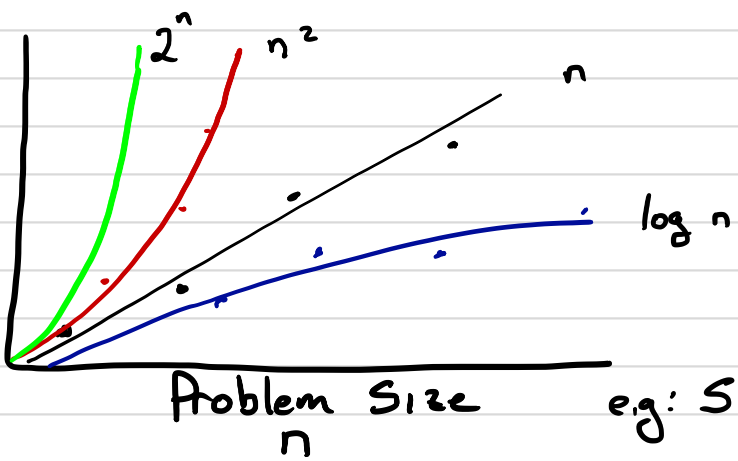
Sequences.

① The study of sequences is motivated by the analysis of algorithms.

- How does memory/run-time increase as we try to tackle bigger and bigger problems.
- How do we prove that one algorithm is more efficient than another?
- What does it mean to be "efficient"?
- what is the best/worst/average case performance?

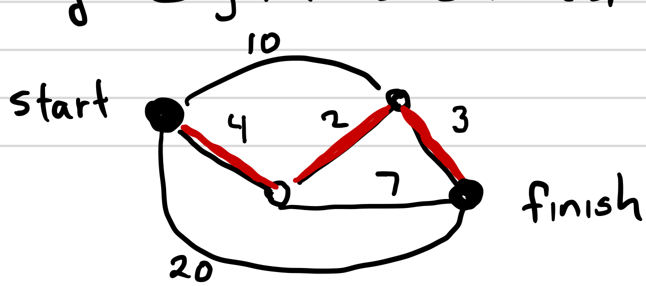
We might try to count the steps of the algorithm for different size inputs.



e.g: Sort n numbers.

The behavior of the algorithm might depend on multiple factors.

e.g Dijkstra's shortest Path : $O(|E| + |V| \log |V|)$



Run time complexity

② What is a sequence?

1) An ordered sequence of numbers:

$$a_1 , a_2 , a_3 , \dots\dots a_n$$

2) We follow the convention that the 1st element is a_1 rather than a_0 as some text book's do.

   - n numbers: $a_1 \dots a_n$

3) This is different than a <u>series</u> of numbers by which we mean the summation over the sequence:

$$a_1 + a_2 + \dots\dots + a_{n-1} + a_n = \sum_n a_i$$

4) Given a sequence, we often want to find a formula for the n_{th} number in the sequence. This can be challenging but there are various tricks to help us.

③ Examples :

a) 2 , 4 , 6 , 8 , 10 , ... $a_n = 2n$ Arithmetic

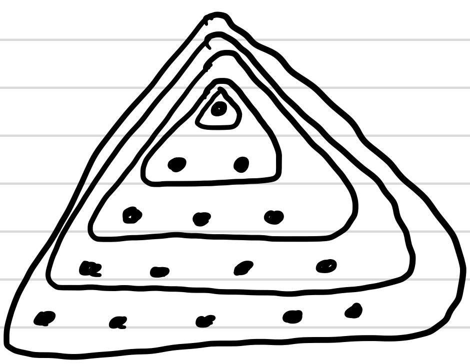
b) 1, 1/2, 1/3, 1/4 ... $a_n = 1/n$ Harmonic

c) 1, 2, 4, 8, 16 ... $a_n = 2^{n-1}$ Geometric

d) 12, 20, 30, 42, 56 .... $X_n = n^2 + 5n + 6$ Quadratic

e) 1, 3, 6, 10, 15 ... $T_n = \frac{n(n+1)}{2}$ Quadratic

aka : Triangular #s.



$T_1 = 1$

$T_2 = 3 = 1 + 2$

$T_3 = 6 = 1 + 2 + 3$

$T_4 = 10 = 1 + 2 + 3 + 4$

$T_5 = 15 = 1 + 2 + 3 + 4 + 5$

etc

f) 1, 1, 2, 3, 5, 8, ... $F_n = F_{n-1} + F_{n-2}$

with $F_1 = 1$, $F_2 = 1$

"Fibonacci Numbers

Arbitrary Sequence (Raw data of algorithmic analysis)

Canonical Form?

- Arithmetic?
  $a_n = a + (n-1)d$
- Geometric?
  $a_n = ar^{(n-1)}$
- Polynomial?
  $a_n = an^2 + n + c$
  $b_n = an^3 + bn^2 + cn + d$

Sequence of Partial Sums

(Series Techniques)

Arithmetic Series

$1 + 2 + 3 + 4 + \cdots + n$

Geometric Series

$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n^2}$

Recurrence Relations.

$F_n = F_{n-1} + F_{n-2}$

Derive Closed Form Formula

$\Downarrow$

Algorithm Run-Time

Complexity: $O(n)$ - linear
$O(n^k)$ - Polynomial
$O(\log n)$ - logarithmic
$O(2^n)$ - exponential

$O()$: Denotes upper bound on growth of function "Big O"

④ **Arithmetic sequence:** $a_k - a_{k-1} = \text{constant}$

The difference between consecutive elements is constant.

n:	a_1	a_2	a_3	a_4	a_5
a_n:	4	7	10	13	16
Δ_1:	3	3	3	3	

4   4+3   4+6   4+9   4+12 . . . .

$$\Downarrow$$

$$a_n = \underset{\substack{\uparrow \\ \text{starting value}}}{a} + (n-1)\underset{\substack{\uparrow \\ \text{difference.}}}{d}$$

In this case:

$a = 4$
$d = 3$

$$\Rightarrow \quad a_n = 4 + (n-1)\cdot 3 = 3n + 1$$

$$a_n = \underset{\substack{\uparrow \\ \text{informative}}}{a + (n-1)d} = \underset{\substack{\uparrow \\ \text{simplified}}}{dn + (a-d)}$$

"linear growth"

a_n : $-7, \quad -1, \quad 5, \quad 11, \quad 17, \quad 23, \ldots$

$$\underbrace{-7, -1}_{6}, \underbrace{-1, 5}_{6}, \underbrace{5, 11}_{6}, \underbrace{11, 17}_{6}, \underbrace{17, 23}_{6}$$

$$a_n = -7 + (n-1)6 = 6n - 13$$

Verify:
$$6 \cdot 1 - 13 = -7$$
$$6 \cdot 2 - 13 = -1$$
$$6 \cdot 3 - 13 = 5$$
$$\vdots$$
etc

(5) Geometric Sequence.

$1, \ 1/2, \ 1/4, \ 1/8, \ 1/16 \ \ldots \quad a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1} = 2 \cdot \left(\frac{1}{2}\right)^n$

$3, \ 6, \ 12, \ 24, \ 48 \ \ldots$

There is a constant multiplier or ratio between consecutive elements.

$$a_{k+1} / a_k = \text{constant} = r$$

$$a_n = a r^{n-1}$$ ← sequences start at a_1 $n=1$

e.g., for second example:

$a = 3$
$r = 2$ $\Rightarrow a_n = 3 \cdot 2^{n-1} = \frac{3}{2} 2^n$

⑥ Quadratic Sequences

T_n : 1 3 6 10 15 21 ...

Δ^1 2 3 4 5 6 (not constant)

Δ^2 1 1 1 1 ... (constant)

Quadratic: The second differences are constant.

This is the discrete analog of derivitives in calculus!

$T_n = an^2 + bn + c$ for some a, b, c.

To solve, get 3 equations with 3 unknowns.

$n=1$: $1 = a\,1^2 + b\cdot 1 + c \Rightarrow a + b + c = 1$

$n=2$: $3 = a\,2^2 + b\cdot 2 + c \Rightarrow 4a + 2b + c = 3$

$n=3$: $6 = a\,3^2 + b\cdot 3 + c \Rightarrow 9a + 3b + c = 6$

Subtract ② - ① : $3a + b = 2$

③ - ② : $5a + b = 3$

$\Rightarrow 2a = 1$

$\boxed{a = 1/2}$

$3/2 + b = 2$

$\boxed{b = 1/2}$

$\boxed{c = 0}$

$$\therefore \boxed{T_n = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}(n^2+n) = \frac{n(n+1)}{2}}$$

(7) Another Quadratic Sequence:

	x_1	x_2	x_3	x_4	x_5	
x_n :	12,	20,	30,	42,	56,	...
Δ^1 :	8	10	12	14		...
Δ^2 :	2	2	2			...

$$x_1 = 12$$

$$x_2 = x_1 + 8 \qquad 6+2 \quad = 6 + 2\cdot 1$$

$$x_3 = x_2 + 10 \qquad 6+4 \quad = 6 + 2\cdot 2$$

$$x_4 = x_3 + 12 \qquad 6+6 \quad = 6 + 2\cdot 3$$

$$\vdots$$

$$\begin{aligned} x_n &= x_{n-1} + C_{n-1} \\ &= x_{n-1} + 6 + 2(n-1) \\ &= x_{n-1} + 2n + 4 \\ &= x_{n-1} + 2(n+2) \end{aligned}$$

e.g $x_5 = 42 + 14 = 56$ ✓

But now lets derive the closed form solution.

$$x_n : 12, 20, 30, 42, 56$$

$$x_n = an^2 + bn + c = d$$

$$x_1 : a + b + c = 12 \qquad (n=1)$$
$$x_2 : 4a + 2b + c = 20 \qquad (n=2)$$
$$x_3 : 9a + 3b + c = 30 \qquad (n=3)$$

$$\left.\begin{array}{ll} x_2 - x_1 : & 3a + b = 8 \\ x_3 - x_2 : & 5a + b = 10 \end{array}\right\} 2a = 2, \ \boxed{a = 1}$$

$$3 + b = 8 \Rightarrow \boxed{b = 5}$$

$$1 + 5 + c = 12 \Rightarrow \boxed{c = 6}$$

$$\therefore x_n = n^2 + 5n + 6$$

OPTIONAL

Note: If we set a_0 as the first value we have:

$$x_0 : \boxed{c = 12}$$

$$\left.\begin{array}{ll} x_1 : & a + b + c = 20 \therefore a + b = 8 \\ x_2 : & 4a + 2b + c = 30 \therefore 4a + 2b = 18 \end{array}\right\rangle 2a = 2 \quad \boxed{a = 1} \quad \boxed{b = 7}$$

$$a'_n = n^2 + 7n + 12$$

$$a_n = (n-1)^2 + 7(n-1) + 12 = n^2 - 2n + 1 + 7n - 7 + 12 = n^2 + 5n + 6 \text{ (as before)}$$

(8) Sums / Series.

Possibly apocryphal story about Carl Friedrich Gauss (German, 1777-1855):
His elementary school teacher asked the students to compute $1 + 2 + 3 + \cdots + 100$ to keep them busy.
Gauss thought for a few seconds and declared: 5050.

How?

$$S = 1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$$
$$S = 100 + 99 + 98 + 97 + \cdots + 3 + 2 + 1$$

$$2S = \underbrace{101 + 101 + 101 + \cdots + 101 + 101 + 101}_{100\times}$$

$$2S = 100 \times 101 = 10100$$

$$S = \frac{10100}{2} = 5050$$

In general: $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

elements, sums

$$= \frac{n(n+1)}{2}$$

adj for double-counting

Which is our closed-form formula for T_n (Triangle #s)

This is a trick for arithmetic sequence.

written forward: $+d$
written backward: $-d$

Method (for arithmetic sequences)

a) write sequence forward and backwards and add together $\Rightarrow 2S$

b) calculate sum $\times$ # of sums (terms)

c) divide by 2

Examples

① $S = 1 + 2 + 3 + 4 + \ldots n$

$S = n + (n-1) + (n-2) \ldots 1$

$$2S = n \cdot (n+1)$$

$$S = \frac{n \cdot (n+1)}{2}$$

} don't just memorize the formula. Understand why its true!

Proof By (Weak) Induction.

Prove $\forall n\, S_n$

1) Base case: Show S_1 is true.

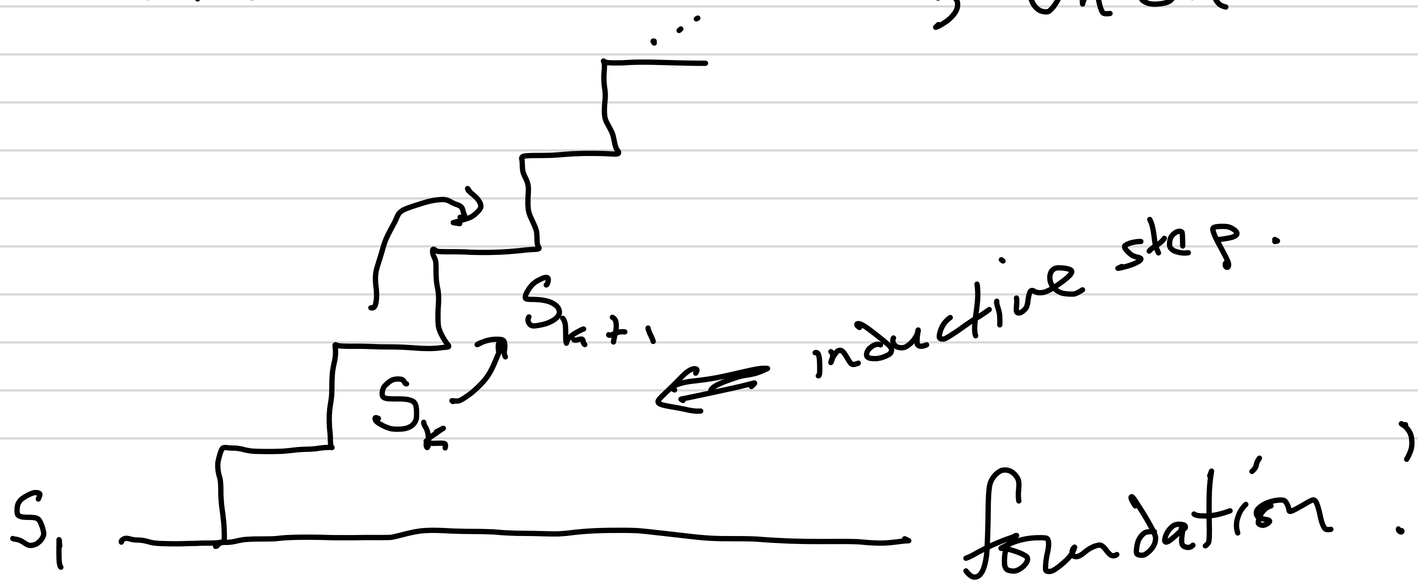
2) Prove: $S_k \rightarrow S_{k+1}$

   2a) Inductive Hypothesis: Assume S_k for some $k > 1$

   2b) Show that S_{k+1} follows.

   (This proves that $S_k \rightarrow S_{k+1}$)

3) State conclusion: "By the principle of mathematical induction, $\forall n\, S_n$"



$$\forall n \sum_{i}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

BASE CASE(S)

$$S_1 = \frac{1(1+1)}{2} = 1 \quad \checkmark$$

$$S_2 = \frac{2(2+1)}{2} = 3 = 1+2 \quad \checkmark$$

$$S_3 = \frac{3(3+1)}{2} = 6 = 1+2+3 \quad -$$

INDUCTIVE HYPOTHESIS

Assume S_k: $\sum_{i=1}^{k} S_i = \frac{k(k+1)}{2}$

We want to show that S_{k+1} follows, i.e,

$$S_{k+1} = \sum_{i}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

For $k+1$:

$$\underbrace{1+2+3+\cdots+k}_{\text{using the I.H.}} + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2+k+2k+2}{2}$$

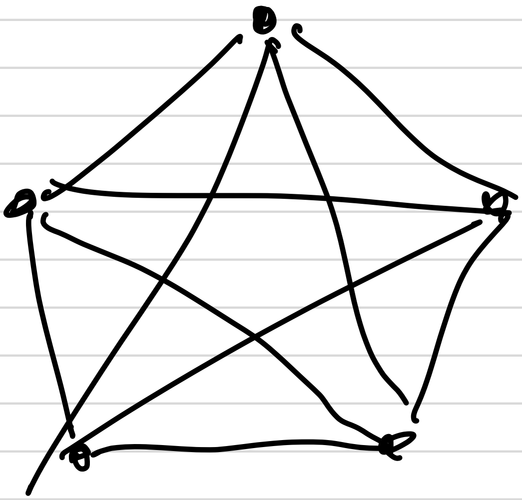
$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2} = S_{k+1}$$

which is what I get when I substitute $k+1$ into the formula, so the formula remains true for S_{k+1}

Consider the graph:

$|V| = 5$ $\quad$ $|E| = 10$



In general $|E| = \frac{|V| \cdot (|V| - 1)}{2}$ for a complete graph.

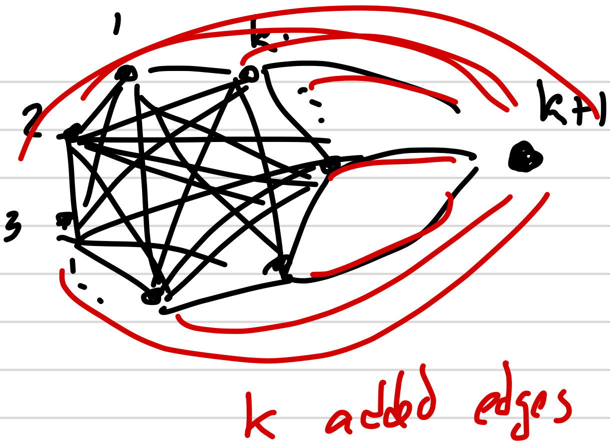
G_n - complete graph w/ n vertices.

$$\forall_n \; |E_n| = \frac{n(n-1)}{2}$$

Assume $|E_k| = \frac{k(k-1)}{2}$ (Inductive Hypothesis)

Show $|E_{k+1}| = \frac{(k+1)(k+1-1)}{2} = \frac{(k+1)k}{2}$

G_{k+1}:



$$|E_{k+1}| = |E_k| + k$$

$$= \frac{k(k-1)}{2} + k \quad \text{(By inductive hypothesis)}$$

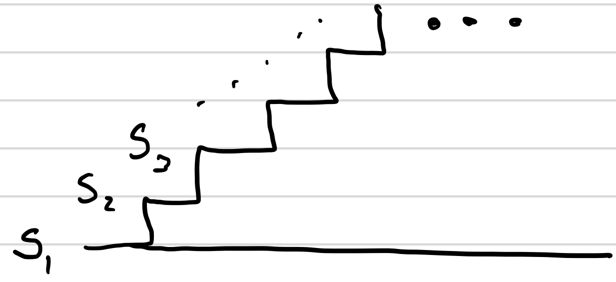
$$= \frac{k^2-k}{2} + \frac{2k}{2}$$

$$= \frac{k^2+k}{2} = \frac{(k+1)k}{2} \quad \checkmark$$

Therefore, By the principle of mathematical induction, $\forall n \; |E_n| = \frac{n(n-1)}{2}$

By the way, don't ignore the importance of the base case.

It is the foundation on which our infinite staircase rests.



e.g. Prove: $2 + 4 + \cdots + 2n = n(n+1) + 2$

Assume $S_k = k(k+1) + 2$

$$
\begin{aligned}
S_{k+1} = S_k + 2(k+1) &= k(k+1) + 2 + 2(k+1) \\
&= k^2 + k + 2 + 2k + 2 \\
&= k^2 + 3k + 2 \quad + 2 \\
&= (k+1)(k+2) \quad + 2 \quad = S_{k+1}
\end{aligned}
$$

But the base case is false!

$$S_1 = 2 \neq 1(1+1) + 2 = 4$$

So always check your base case 1st!

What is the right formula?

$$
\begin{aligned}
S_n &= 2 + 4 + 6 + 8 + \cdots + 2n \\
&= \sum_{i=1}^{n} 2i = 2\sum_{i=1}^{n} i = \frac{2n(n+1)}{2} = n(n+1)
\end{aligned}
$$

We can factor out a constant

We have some additional formulas:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

We can break sums apart:

$$\sum_{k=1}^{n} \left(5k^2 - 2k + 9\right) = 5\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 9$$

$$= \frac{5(n+1)(2n+1)}{6} - \frac{2(n)(n+1)}{2} + 9n$$

$$= \dots \text{(simplify)}$$

Finite Sums

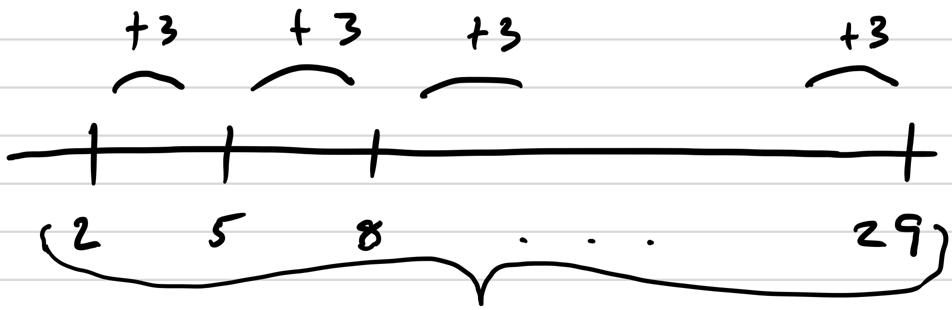
OPTIONAL

$$S = \underbrace{2 + 5}_{3} + \underbrace{5 + 8}_{3} + \underbrace{8 + 11}_{3} + \cdots + 29$$

$$S = 29 + 26 + 23 + \cdots$$

$$2S = 31 + 31 + 31 + \cdots + 31$$

How many terms?



$$\text{span} = 29 - 2 = 27$$

$$\#\text{ gaps} = \frac{\text{span}}{\text{gap size}} = \frac{27}{3} = 9$$

1 gap → 2 terms
2 gaps → 3 terms

$$\#\text{ terms} = \#\text{ gaps} + 1$$

$$\#\text{ terms} = \frac{\text{span}}{\text{gap size}} + 1 = 10$$

$$\therefore \quad S = \frac{10 \cdot 31}{2} = 155$$

We can think of this as

$$\frac{\#\text{ terms} \times \text{sum}}{2}$$

or as $10 \times$ avg value $= 10 \cdot \left(\frac{31}{2}\right)$

$= 155$

(arithmetic series)

$$S = \overbrace{-11 \quad -7}^{4} \overbrace{\quad -3}^{4} + 1 + 3 + 9 + \cdots + 17$$

$$S = 17 + 13 + 9 + \cdots$$

$$2S = 6 + 6 + \cdots$$

$$\#\text{ terms} = \frac{17 - (-11)}{4} + 1 = \frac{28}{4} + 1 = 8 \text{ terms}$$

$$S = \frac{8 \text{ terms} \times 6}{2} = 24$$

---

a_n : 6, 16, 30, 48, 70, ...

Δ^1 : 10, 14, 18, 22

Δ^2 : 4, 4, 4

So we could find a solution for $an^2 + bn + c = a_n$

Alternatively, we can recognize that the sequence terms are all partial sums:

$a_n: \quad 6, \quad 16, \quad 30, \quad 48, \quad 70, \quad \ldots$

$$
\begin{array}{lcll}
a_1 & 6 & = & 6 \\
a_2 & 16 & = & 6 + 10 \\
a_3 & 30 & = & 6 + 10 + 14 \\
a_4 & 48 & = & 6 + 10 + 14 + 18 \\
a_5 & 70 & = & 6 + 10 + 14 + 18 + 22 \\
 & & & \vdots
\end{array}
$$

$$a_n \qquad a_n = \underset{6}{6} + \underset{10}{(6+4)} + \underset{14}{(6+2\cdot 4)} + \underset{18}{(6+3\cdot 4)} + \cdots + \underbrace{6+4(n-1)}_{4n+2}$$

$$S = 6 + 10 + 14 + \cdots + (4n+2)$$

$$S = (4n+2) + (4n-2) + (4n-6) + \cdots + 6$$

$$2S = 4n+8 + 4n+8 + \cdots + 4n+8$$

$$\text{\# terms: } \frac{(4n+2)-6}{4} + 1 = \frac{4n-4}{4} + 1$$

$$= n \quad \text{(of course)}$$

$$a_n: \quad \frac{n(4n+8)}{2} = \frac{4n(n+2)}{2} = 2n(n+2)$$

$$= \underset{\substack{\uparrow \\ a}}{2}n^2 + \underset{\substack{\uparrow \\ b}}{4}n + \underset{\substack{\uparrow \\ c}}{0}$$

Check:

$$
\begin{array}{ll}
n=1 & 2\cdot 3 = 6 \\
2 & 4\cdot 4 = 16 \\
3 & 6\cdot 5 = 30 \\
4 & 8\cdot 6 = 48 \\
5 & 10\cdot 7 = 70 \\
 & \vdots
\end{array}
$$

So: Quadratic Sequence $\Longleftrightarrow$ arithmetic series (Partial Sums)

Geometric Series.

The technique isn't the same but it's similar.

Canonical Form: For some ratio $r \neq 1$

$$\sum_{k=0}^{n} r^k = 1 + r + r^2 + r^3 + \cdots r^n$$

$$
\begin{aligned}
S &= 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + r^n \\
-r \cdot S &= \quad\ \ \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots + r^n + r^{n+1}
\end{aligned}
$$

$$S - rS = 1 - r^{n+1}$$

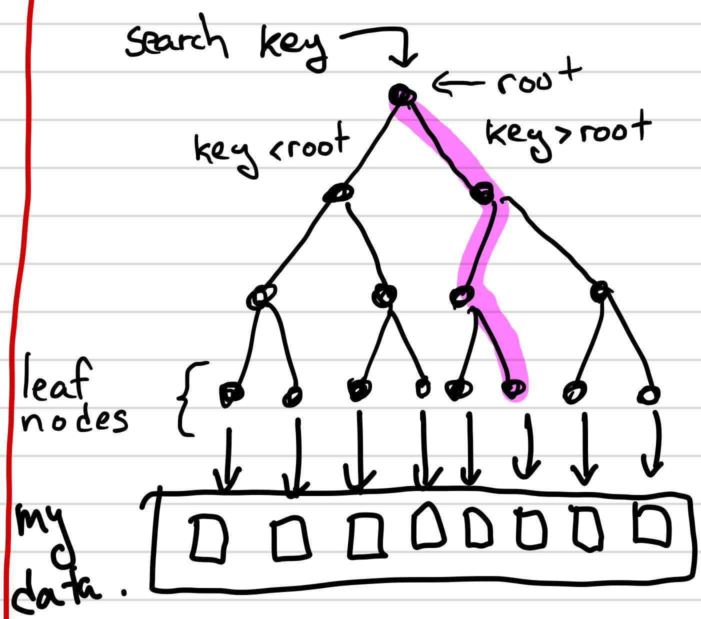
$$S(1-r) = 1 - r^{n+1}$$

$$\therefore \boxed{S = \frac{1 - r^{n+1}}{1 - r}}$$

e.g. $1 + 2 + 4 + 8 + \cdots + 2^n = \dfrac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1$

OPTIONAL

Consider a "binary tree"



Depth	#nodes @ Depth	total
0	1	1 = 1
1	2	3 = 1 + 2
2	4	7 = 1 + 2 + 4
3	8	15 = 1 + 2 + 4 + 8
⋮	⋮	⋮
n	2^n	$2^{n+1} - 1$

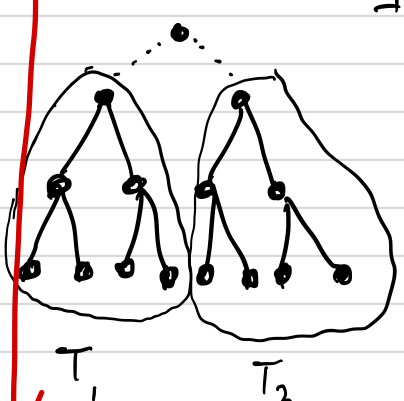
Let L_n = # of leaf nodes in a binary tree of depth n

Prove: $L_n = 2^n$

Base case: $n = 0$ $\quad L_0 = 2^0 = 1$ ✓

Inductive Hypothesis Assume true for L_k

Then for the two subtrees on either side of the root, having depth k each has 2^k leaves.



$$L_{k+1} = \underbrace{L_k}_{T_1} + \underbrace{L_k}_{T_2} = 2L_k = 2 \cdot 2^k = 2^{k+1}$$

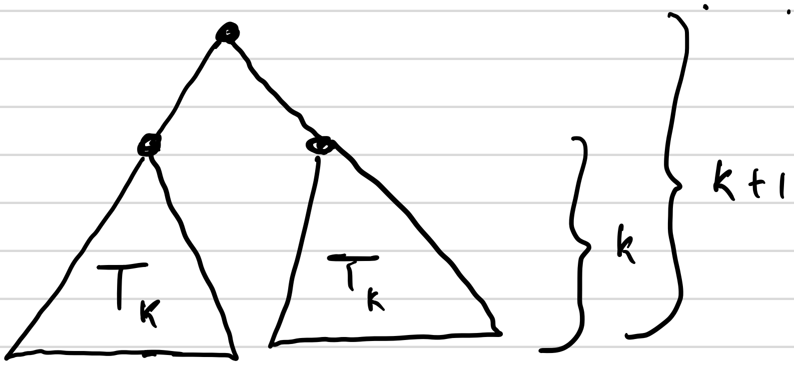
Let S_n = # nodes in a binary tree with depth n

Prove : $S_n = 2^{n+1} - 1$

Proof by induction :

Base Case . $(n = \emptyset)$ $S_0 = 2^{0+1} - 1 = 1$ ✓

Assume true for S_k. That is $S_k = 2^{k+1} - 1$



So $S_{k+1} = \underbrace{\left(2^{k+1} - 1\right)}_{T_k} + \underbrace{\left(2^{k+1} - 1\right)}_{T_k} + \underbrace{1}_{\text{new root}}$

$= 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$

$\therefore$ $S_k \Rightarrow S_{k+1}$ holds

$\therefore$ S_n holds for all $n \geq 0$

If I have 2^n pieces of data

$$\lim_{n\to\infty} \frac{S_n}{L_n} = \lim_{n\to\infty} \frac{2^{n+1}-1}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} - \cancel{\frac{1}{2^n}}_{0}$$

$$= 2$$

An index requires 2x # data elements.

$\times 1/2 \qquad \times 1/2 \qquad \times 1/2$

$$S = 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{32}$$

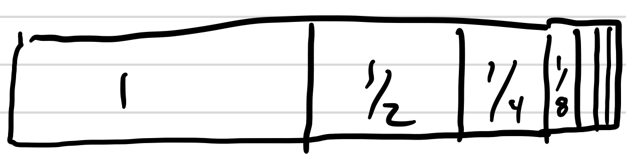
< 6 (How do I know this? - See below)

$$\frac{1}{2}S = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots + \frac{3}{32} + \frac{3}{64}$$

$$S - \frac{1}{2}S = 3 - \frac{3}{64} = \frac{1}{2}S$$

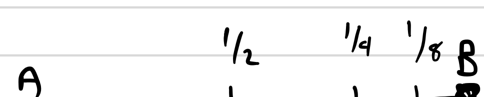
$$\therefore S = 6 - \frac{3}{32} < 6$$

$$S = 3 \cdot \underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots + \frac{1}{32}\right)}_{<2}$$

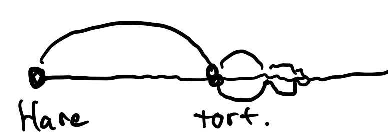


Zeno's Paradox(es):

a) Motion is impossible



b) The hare cannot catch the tortoise



When $|r|<1$, the infinite geometric series always converges.

Infinite Geometric Series

$$k \leq S = 1 + r + r^2 + r^3 + \cdots = \sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$$

Why?

Informally:

$$S = 1 + r + r^2 + r^3 + \cdots$$
$$rS = r + r^2 + r^3 + \cdots$$
$$S - rS = 1$$
$$\therefore S = \frac{1}{1-r} \qquad \text{So } \frac{1}{1-1/2} = 2$$

More precisely:

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k$$

$$= \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r}$$

when $|r|<1$, $\lim_{n\to\infty} r^{n+1} = 0$

In general:

$$\text{Given } S = a + ar + ar^2 + ar^3 \cdots$$
$$- rS = ar + ar^2 + ar^3 \cdots$$

$$\Rightarrow S = \frac{a}{1-r} = \sum_{k=1}^{\infty} ar^k = a\sum_{k=1}^{\infty} r^k \qquad (r<1)$$

Often we can use the formula for an infinite geometric series as a reasonable approximation because each term gets smaller and smaller. $(0<r<1)$

$$\sum_{k=0}^{9}\left(\frac{1}{4}\right)^{k} = 1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^9}$$

$$= \frac{1-(1/4)^{10}}{1-1/4} = 1.333332062$$

$$\approx \frac{a}{1-r} = \frac{1}{(3/4)} = \frac{4}{3} = 1.333\ldots$$

<u>Fun Fact:</u> Rational #s have repeating digits in the fraction. Irrational #s don't!

Consider: $N = 0.2727272\overline{27}\ldots$

$$= 0.27 + \frac{0.27}{100} + \frac{0.27}{10000} + \cdots +$$

$$= \frac{0.27}{10^0} + \frac{0.27}{10^2} + \frac{0.27}{10^4} = .27\left(1+\frac{1}{10^2}+\frac{1}{10^4}\cdots\right)$$

$$= \frac{.27}{1-\frac{1}{100}} = \frac{.27}{.99} = \frac{27}{99}$$

OPTIONAL

Harmonic Series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

$$= \ln(n) + \underbrace{\gamma}_{\gamma \sim 0.577 \text{ Euler's Constant}}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1000} \approx \ln(1000) \approx 6.91$$

$$f(x) = \frac{1}{x}$$

$$\int \frac{1}{x}\, dx = \ln(x)$$

We can sometimes use integrals to find the approximate value of an infinite series.

This comes up in the analysis of quicksort: $O(n \log n)$

OPTIONAL

Theorem: The harmonic series diverges.

Proof (By contradiction):

Suppose the harmonic series converges to H.

$$H = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

Then

$$H \geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \frac{1}{8} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= H + 1/2$$

This is a contradiction.

$\therefore$ Our supposition is false / impossible

$\therefore$ The theorem that the harmonic series diverges is true. $\blacksquare$

Prove $\forall n \in \mathbb{N}$, $6^n - 1$ is multiple of 5

S_n : $6^n - 1$ is a multiple of 5

Base case : S_0 : $6^0 - 1 = 0$ is a multiple of 5 ✓ $(5 \cdot 0 = 0)$

S_1 : $6^1 - 1 = 5$
S_2 : $6^2 - 1 = 35$
S_3 : $6^3 - 1 = 216 - 1 = 215$

etc

Inductive case: Assume S_k true for some arbitrary k.

Then $6^k - 1 = 5j$

$6^k = 5j + 1$

$6^{k+1} = 6(5j+1) = 30j + 6$

$6^{k+1} - 1 = 30j + 5 = 5(6j+1)$ (a multiple of 5)

So S_{k+1} is true

By Principle of mathematical induction, S_n is true for all n.

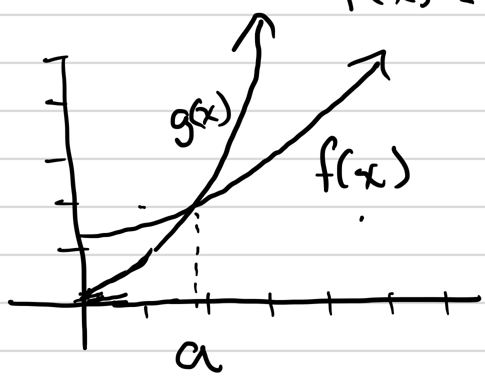
∎

<u>Racetrack Principle</u>: The horse that starts fast and stays fast wins the race

From calculus:

If $f(a) < g(a)$ and $f'(x) < g'(x)$ for all $x > a$ then

$f(x) < g(x)$ for all $x > a$



(once $g(x)$ is ahead it stays ahead).

Prove $n^2 < 2^n$ for all $n \geq 5$

Intuitively,

$$(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = \overbrace{2n+1}^{\text{rate of increase}}$$

$$2^{n+1} - 2^n = 2 \cdot 2^n - 2^n = 2^n > 2n+1 \text{ when } n \text{ is large.}$$

Formally,

Base case S_5: $5^2 < 2^5$ $(25 < 32)$

Inductive case: Assume S_k (arbitrary k)

$$k^2 < 2^k$$

Show $(k+1)^2 < 2^{k+1}$

next page.

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< 2^k + 2k + 1 \\ &< 2^k + 2^k \\ &= 2^{k+1}\end{aligned}$$

$\boxed{\text{(since } 2k+1 < 2^k \text{ for } k \geq 5)}$

$\therefore (k+1)^2 < 2^{k+1}$, so S_{k+1} holds

Prove $\forall n \geq 5$: $2n + 1 < 2^n$

Base $k=5$:

$$2 \cdot 10 + 1 < 2^5$$
$$11 < 32 \checkmark$$

Assume:

$$2 \cdot k + 1 < 2^k \quad (k \geq 5)$$

Show $2(k+1) + 1 < 2^{k+1}$

$$2k + 2 + 1$$
$$(2k+1) + 2 < 2^k + 2 \quad (IH)$$
$$< 2^k + 2^k$$
$$= 2 \cdot 2^k$$
$$= 2^{k+1}$$

OPTIONAL

Power sets: $\mathcal{P}(A)$ = set of all subsets of A

e.g $\mathcal{P}(\{a, b\}) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Prove: S_n: An n-element set has 2^n subsets.

Base: S_1: e.g $\mathcal{P}(\{a\}) = \{\phi, \{a\}\}$

$$|P\{a\}| = 2 = 2^1$$

$$|P\{a,b\}| = 4 = 2^2$$

Assume S_k: k-element set has 2^k subsets.

1) A: set with $k+1$ elements (We show it has 2^{k+1} subsets)

2) $A' = A - \{a\}$ for some element $a \in A$ (A' has k elements)

3) Subsets of A have two types

I: <u>do not contain $\{a\}$.</u> These are the subsets of A' with k elements. There are 2^k such sets.

II: <u>contain $\{a\}$.</u> These sets all have the form $B = B' \cup \{a\}$ where B' is a subset of A'. There are 2^k of these also.

$A = \{a, b, c\}$: $\{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

I: $\{\phi, \{b\}, \{c\}, \{b,c\}\}$ $\mathcal{P}(I) = \mathcal{P}(A - \{a\}) = \mathcal{P}(A')\ 2^k$

II: $\{\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ 2^k

$$\therefore\ |\mathcal{P}(A)| = |\mathcal{P}(I)| + |\mathcal{P}(II)| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

OPTIONAL

De Morgan's Law

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\qquad \overline{A \cap B} = \bar{A} \cup \bar{B}$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\qquad \overline{A \cup B} = \bar{A} \cap \bar{B}$

You may have noticed:

$\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$ $\qquad \overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$

$\neg(p \vee q \vee r) \equiv \neg p \wedge \neg q \wedge \neg r$ $\qquad \overline{A \cup B \cup C} = \bar{A} \cap \bar{B} \cap \bar{C}$

Prove S_n: $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cdots \cap \bar{A}_n$

Base case: S_1 trivially holds. $\bar{A}_1 = \bar{A}_1$

S_2 is DeMorgan's law with $A = A_1$, $B = A_2$

Induction: Assume S_k is true. We seek to show S_{k+1}

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} = \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}}$$

$$= \underbrace{\overline{(A_1 \cup A_2 \cup \cdots A_k)}}_{A} \cap \underbrace{\overline{A_{k+1}}}_{B}$$

$$= \underbrace{(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \bar{A}_k)}_{\text{by inductive hypothesis. This is what we assumed!}} \cap \overline{A_{k+1}}$$

$$= \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k \cap \overline{A_{k+1}}$$

So $S_k \Rightarrow S_{k+1}$. By principle of induction it holds for all n. ∎

OPTIONAL

Outline of Proof by Strong Induction

Proposition: $S_1, S_2, S_3, \ldots$ are all true.

Proof:

Base case: show S_1 is true or first several

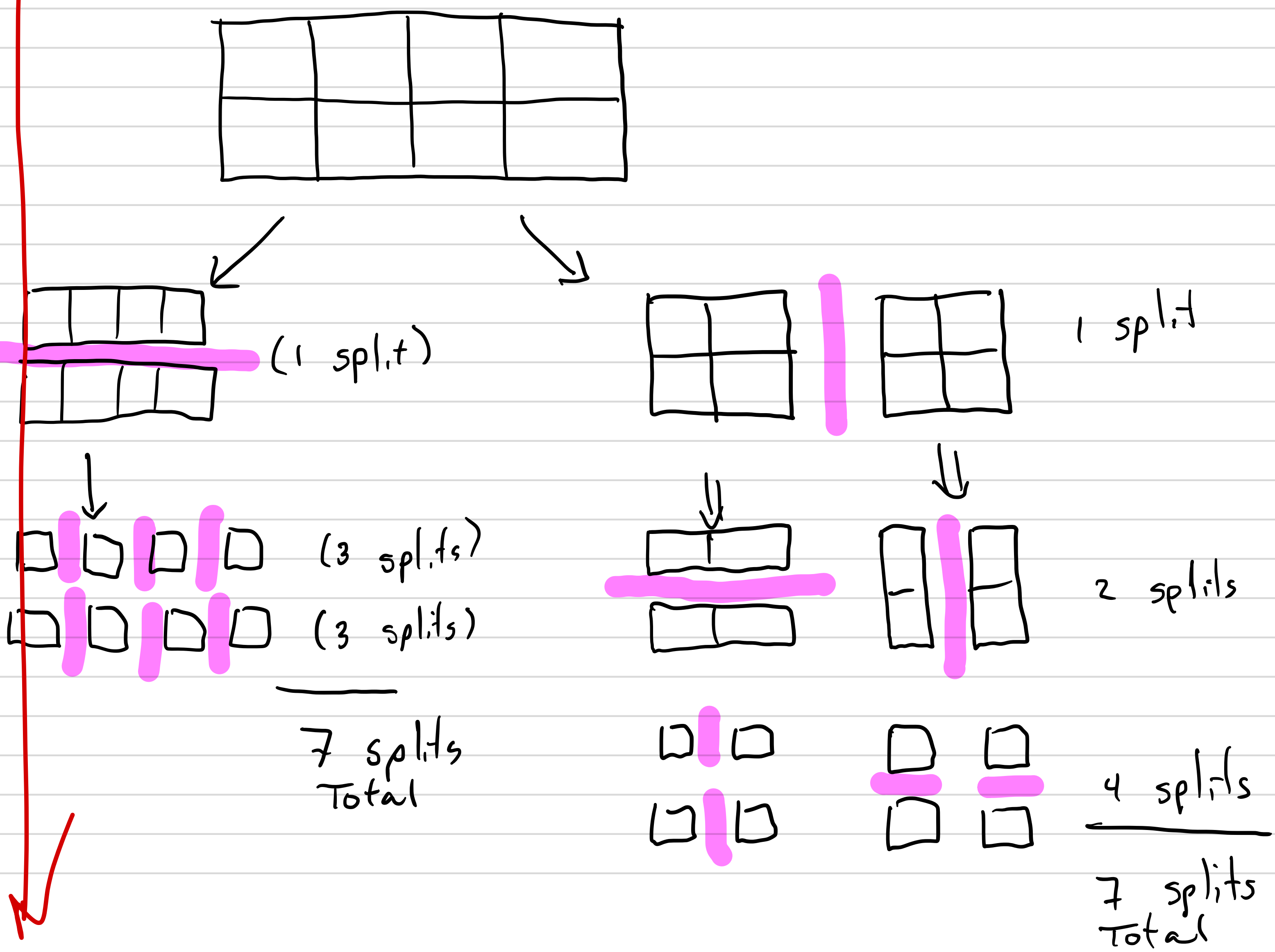
Induction case: Show for $k > 1$

$$S_1 \wedge S_2 \wedge \ldots \wedge S_{k-1} \to S_k$$

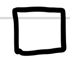
or $$S_1 \wedge S_2 \wedge \ldots \wedge S_k \to S_{k+1}$$

Chocolate Bars!

How many splits are required to reduce the bar to squares?



Prove: S_n: It takes $n-1$ splits to reduce an n-square chocolate bar to squares.

Base: S_1 - no splits required! 

S_2 - 1 split: 

Inductive case: Pick an arbitrary $k > 2$

Assume S_i true for <u>all</u> $i < k$

$S_1, S_2, S_3, \ldots S_{k-1}$ } all true

Show that S_k follows:

- Split k-square chocolate bar along any row or column.

- Two chocolate bars a-square, b-square
  $a < k$, $b < k$

So by the inductive hypothesis: Total # splits:

$$\underbrace{(a-1)}_{\text{a-square chocolate bar}} + \underbrace{(b-1)}_{\text{b-square chocolate bar.}} + \underbrace{1}$$

$= a + b - 1$

$= k - 1$

So S_k is true and by the principle of strong induction, S_n holds for all $n \geq 0$

Prove: Given coin denominations in \$4 and \$5 we can have coins that add up to any integer amount $\geq$ \$12

Base cases:

$S(12)$: $4 + 4 + 4 = 12$

$S(13)$: $4 + 4 + 5 = 13$

$S(14)$: $4 + 5 + 5 = 14$

$S(15)$: $5 + 5 + 5 = 15$

This is sufficient for our base cases (as we will see) but lets continue to observe the pattern:

$S(16)$: $(4 + 4 + 4) + 4 = 12 + 4$

$S(17)$: $(4 + 4 + 5) + 4 = 13 + 4$

$S(18)$: $(4 + 5 + 5) + 4 = 14 + 4$

$S(19) = (5 + 5 + 5) + 4 = 19 + 4$

add \$4 to get the next 4 values.

Inductive Step:

Suppose statement true for $S(12) \ldots S(k-1)$ and $k \geq 16$.

$S(k)$ must be true since we can construct a solution by adding \$4 to $S(k-4)$ and $k-4$ is in the range $12 \ldots k-1$