CS5002 Discrete Structures	Profs. Amor-Tijani and Rachlin
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CS52002 Practice Problems

Instructions (for final exam):

1. The exam is open book and open notes. You may use a calculator but it should not be necessary.
2. Discussing the exam with others is strictly prohibited. Giving or receiving help from fellow students earns all parties involved an automatic F for the class. Do not test us on this.
3. *Unless otherwise specified*, you may leave your answer in the form of a fraction or mathematical expression that clearly shows your thought process. For credit you must show your scratch work and clearly explain how you got to your final answer. If this isn't clear and obvious you won't get full credit for your response.
4. The exam is worth 100 total points. The points for each problemn and sub-problem are provided.
5. When submitting your exam you must specify, for each problem, the page or pages where the solution to that problem is to be found. If we can't see your solution you will not receive credit for the problem.
6. You have been granted an extra 15 minutes solely for the purpose of uploading your exam. Exams emailed to instructors will receive a 5 percent penalty, so be sure to give yourself enough time to submit your exam! Exams received via email more than a few minutes beyond the deadline will not be graded.
7. The exam window starts at 1:35pm ET/Boston and all exams must be received by 9:15pm ET/Boston. Once started you will have 3 hours and 15 minutes to complete the exam AND submit your exam to gradescope.

PRINT FULL NAME: ________________________ STUDENT ID: __________

I have read the instructions above and understand that I may not discuss the exam with other students at any time. SIGN HERE: ________________________

Problem 1 : Number Representations

1. Convert $F3A_{16}$ to binary (base 2).

   Solution: 111100111010_2

2. Convert 101110110_4 (base 4) to hexadecimal (base 16).

   Solution: Chunk 2 digits at a time: $01|01|11|01|10_4 = 11514_{16}$

Problem 2 : Logic

Use rules of logical equivalence to show that $(p \wedge q \wedge r) \vee \neg(\neg p \vee \neg q \vee r) \equiv (p \wedge q)$. Indicate when using the Distributive, Double Negative, or De Morgan's Law.

Solution:

$(p \wedge q \wedge r) \vee \neg(\neg p \vee \neg q \vee r)$ (Start)
$(p \wedge q \wedge r) \vee (\neg\neg p \wedge \neg\neg q \wedge \neg r)$ (De Morgan's)
$(p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r)$ (Double Neg)
$(p \wedge q) \wedge (r \vee \neg r)$ (Distributive)
$(p \wedge q)$ (Identity)

Problem 3: Sets

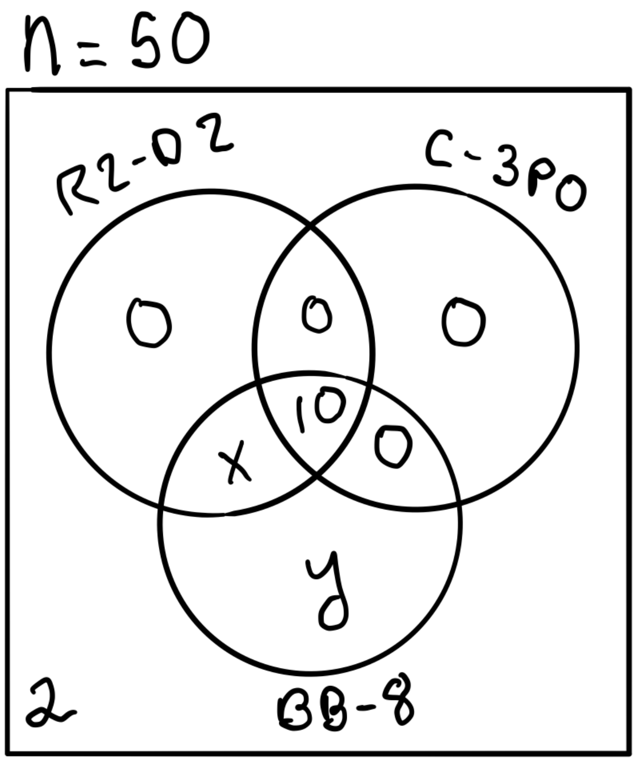
Fifty (50) fans of the *Star Wars* movie franchise were asked which droids they liked. Specifically, they were asked if they liked C-3PO, R2-D2, and/or BB-8. (They could say they liked more than one.)

- 2 fans said they didn't like any of these droids.
- Everyone who liked C-3PO also liked R2-D2.
- Everyone who liked R2-D2 also liked BB-8.
- Three times as many fans liked BB-8 as liked R2-D2.
- 10 fans liked all three droids.

How many fans liked each droid? Explain your answer.

Solution:

Set up the problem as shown in the figure below, then solve for x and y:
$x + 10 + y = 48 = 3(x + 10)$ Therefore $x = 6$ and $y = 32$.
So $|C - 3PO| = 10$, $|R2 - D2| = 16$, and $|BB - 8| = 48$.



Problem 4: Counting

1. A grocery store has 5 apples, 4 bananas, and 3 pears on the shelf. Assume the apples are indistinguishable as are the bananas and pears. How many ways are there to purchase exactly 4 pieces of fruit? (For example, you might purchase 2 apples, 1 pear, and 1 banana, or 1 apple and 3 pears, etc.)

   Solution: The 4 pieces of fruit are balls, and the fruit types form the three bins. But we cannot choose 4 pears. So $\binom{6}{2} - 1 = 14$

2. How many ways are there to arrange the six digits 012345 so that no two adjacent digits add up to 5?

   Solution: Use the principle of inclusion / exclusion on three sets, A,B,C. Let A be the sequences containing 05 or 50, B contains 14 or 41, and C contains 23 or 32. We want $6! - |A \cup B \cup C| = 6! - (|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|) = 6! - (3 \cdot 2 \cdot 5! - 3 \cdot 4 \cdot 4! + 8 \cdot 3!) = 12 \cdot 4! - 8 \cdot 3! = 10 \cdot 4! = 240$.

   An alternative solution using a constructive argument: Construct all 6-strings of letters A, A, B, B, C, C such that first occurrence is in alphabetical order, and that no consecutive chars are the same letter, that's 5 possibilities: ABACBC,ABCABC,ABCACB,ABCBAC,ABCBCA Then replace AA with one of the pairs that sum to 5, BB with another pair, and CC with the last pair giving $3 \cdot 2 \cdot 1$ choices. Finally, permute each pair: 8 choices. So altogether we have $5 \cdot 6 \cdot 8 = 240$ possibilities.

Problem 5: Probability

1. A sock drawer contains 4 white socks and 4 black socks and 4 tan socks. What is the probability that if two socks are randomly drawn (without replacement) they will form a matching pair? Reduce your answer to a simple fraction.

   Solution: $\frac{3}{11}$
   Either intuitively: (pick any sock, then three out of eleven form a match), or formally: There are $3 \cdot \binom{4}{2}$ matching pairs out of $\binom{12}{2}$ possible pairs. $\frac{3 \cdot \binom{4}{2}}{\binom{12}{2}} = \frac{3}{11}$

2. What is the minimum number of socks that one would have to draw (without replacement) to be certain of having at least two matching pairs? Explain your answer and state the principle underlying your argument.

   Solution: 6 (by the pigeonhole principle). For example you might have picked Tan, Black, White, White, White. Any other draw produces a second matching pair.

Problem 5b: More Probability

A jar contains 3 red balls and 7 blue balls.

i. If a ball is picked at random, what is the probability that this ball is blue?

Solution: 0.7

ii. If two balls are taken out of the jar at random (without replacement), what is the probability that they are of different colors?

Solution:
Sample space S = all sets of two balls. $|S| = \binom{10}{2} = 45$.
Event E = all sets of two with one red and the other blue. $|E| = 3 \times 7 = 21$
$P[E] = 21/45 = 7/15 \approx 0.467$

iii. If three balls are taken out at random (without replacement), what is the probability that they are all of the same color?

Solution: Sample space S = all sets of 3 balls. $|S| = \binom{10}{3} = 120$.
Event E = the set of three red balls together with all sets of three blue. $|E| = 1 + \binom{7}{3} = 36$.
$P[E] = 36/120 = 3/10 = 0.3$.

Problem 5c: More Probability

i. What is the probability, if a die is rolled five times, that only two different values appear?

Solution:
Sample space: $S =$ all sequences of five numbers from $\{1, 2, 3, 4, 5, 6\}$. $|S| = 6^5$.
Event $E =$ all sequences with only two values. $|E| = \binom{6}{2} \cdot (2^5 - 2) = 15 \cdot 30 = 450$ (select the two values, then pick a sequence of 5 - excluding the two sequences that are all one color or the other).
$P[E] = |E|/|S| = 450/6^5 \approx 0.0579$

ii. Which is more likely, rolling an 8 when two dice are rolled, or rolling an 8 when three dice are rolled?

Solution: 2 dice are rolled. With 2 dice, there are 5 ways (2-6, 6-2, 3-5, 5-3, 4-4) giving a probability $5/6^2$ while with 3 dice we get (1-1-6, 1-2-5.....all the ways to get 7 with 2 dice (6) + ways to get 6 with 2 dice (5), ways to get 5 with 2 dice (4), get 4 (3), get 3 (2) and get 2 (1). This is 21 giving a probability $21/6^3$.

Problem 5d: Conditional Probability

We are given 5 cards. 3 of the cards are black and they are numbered $1, 2, 3$. The other two cards are red and they are numbered $1, 2$.

We pick 2 random cards.

i. What is the probability that both cards are red?

Solution: The sample space S consists of all sets of 2 cards selected from the given 5 cards. $|S| = \binom{5}{2} = 10$.
The event $A =$ Both cards are red consists of all sets of two cards that are red, so $|A| = 1$.
Therefore $\Pr[A] = 1/10$

ii. What is the probability that both cards are red, if we know that at least one of them is red?

Solution: Here we have a new sample space S_1 that consists of all sets of two cards with at least one red. Three sets include red1 and a black card, another three include red2 and a black card, and one includes both red cards. Thus $|S_1| = 7$.

Therefore $\Pr\left[\text{both red} \mid S_1\right] = 1/7$

iii. What is the probability that both cards are red, if we know that one of them is red card number 1?

Solution: Here the new sample space S_2 consists of all sets of 2 cards that include card red1. There are 4 such sets, thus $|S_2| = 4$ and therefore $\Pr[\text{both red} \mid S_2] = 1/4$.

Problem 5e: More Conditional Probability

Punxsutawney Phil is a weather predicting groundhog. On February 2nd, Phil makes a prediction about how soon spring comes (when weather warms up) by either observing his shadow or not each. Let's make the following assumptions:

i. If flowers bloom in April, Phil observes his shadow 30% of the time.

ii. if flowers do not bloom in April, Phil observes his shadow 80% of the time.

iii. Flowers bloom in April only 15% of the time, whether or not Phil's shadow is observed.

Given that Phil has not seen his Shadow this year, whats the probability that the flowers will bloom?

Solution: Let $S = 1$ represent the event that phil sees his shadow and $B = 1$ be the event that the flowers bloom in April. These definitions allows us to rephrase each of the statements above as a probability[1]. We list them in the same order as above:

i. $P(S = 1|B = 1) = .3$

ii. $P(S = 1|B = 0) = .8$

iii. $P(B = 1) = .15$

Before continuing on, its helpful to observe that the above three statements describe all the possible probabilities between S and B. What about $P(S = 0|B = 1)$ though, that's not on our list! Remember $P(S = 0|B = 1)$ is the probability phil has not seen his shadow given that the flowers bloom in April. Consider that, given the flowers bloom in april, Phil will either see his shadow or not:
$1 = P(S = 1|B = 1) + P(S = 0|B = 1) \implies P(S = 0|B = 1) = 1 - P(S = 1|B = 1) = 1 - .3 = .7$
By a similar logic, $P(S = 0|B = 0) = (1 - .8) = .2$ and $P(B = 0) = 1 - P(B = 1) = .85$.

How about $P(S)$ though, how do the three statements above specify this distribution? Observe that $S = 1$ (Phil sees his shadow) under exactly two conditions: either he sees his shadow and the flowers bloom in April $S = 1, B = 1$ or he sees the shadow and the flowers don't bloom, $S = 1, B = 0$[2]. We may apply the sum rule (from sets / probability) to say that:

$$P(S) = P(S = 1, B = 1) + P(S = 1, B = 0) \tag{1}$$

Finally, we apply the definition of conditional probability $P(X|Y)P(Y) = P(X, Y)$ to each of the terms on the right:

$$P(S) = P(S = 1|B = 1)P(B = 1) + P(S = 1|B = 0)P(B = 0) \tag{2}$$

[1] Translating english to probability is one of the tougher parts of a Bayes problem, remember that $P(X|Y)$ represents the probability of event X given that event Y has already happened. $P(happy|sunny)$ is the probability that one is happy given its a sunny day. $P(sunny|happy)$ is the probability that it is a sunny day given one is happy. The two are not equivilent! English is tricky because the ordering of variables might change when the meaning doesn't: "if event Y happens then the prob of X is " refers to $P(Y|X)$.

[2] Feels like there should be a conjunctive operator here, right? When random variables are listed one next to the other, you may assume it.

This equality might look like we've only made a mess of $P(S)$ at first, but observe that each of the four probabilities on the right have already been defined.

We're interested in the probability that the flowers will bloom in april given Phil has not seen his shadow: $P(B = 1|S = 0)$.

$$
\begin{aligned}
P(B = 1|S = 0) &= \frac{P(S = 0|B = 1)P(B = 1)}{P(S = 0)} \\
&= \frac{P(S = 0|B = 1)P(B = 1)}{P(S = 0|B = 0)P(B = 0) + P(S = 0|B = 1)P(B = 1)} \\
&= \frac{(1 - .3)(.15)}{(1 - .8)(1 - .15) + (1 - .3)(.15)} \\
&\approx .38
\end{aligned}
$$

The first equality is Bayes Rule, the other substitutions are explained above.

Problem 6: Expectation

A standard 52-card deck of cards contains 4 suits (Hearts, Diamonds, Clubs, and Spades). Each suit contains 13 ranks (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen,King). The Jack, Queen, and King are face cards. Two such decks are combined and shuffled together forming a 104-card deck from which you are dealt exactly five cards. How many cards would you expect to receive that are Clubs, or face cards, or both?

<u>*Solution:*</u> The fact that two 52-card decks have been combined is immaterial. There are 22 cards in a 52-card deck that are Clubs or face cards or both (13 + 12 - 3). So there are 44 such cards in a 104-card double deck, but the probability of any card being a Club or a face card or both is still $\frac{22}{52}$. Let C be a discrete random variable representing the number of club or face cards. $C = C_1 + C_2 + C_3 + C_4 + C_5$, so $E[C] = E[C_1] + E[C_2] + E[C_3] + E[C_4] + E[C_5] = 5 \cdot E[C_1] = 5 \cdot (0 \cdot \frac{30}{52} + 1 \cdot \frac{22}{52}) = \frac{110}{52} = \frac{55}{26} \approx 2.1$ cards.

Problem 7: Algorithms For the first two parts, your answer should be a single integer.

i. How many steps (compares) does it take, in the worst case, to search for a given element in an **unordered array** of length 512?

Solution:

It takes 512 compares in the worst case - element is last or not present.

ii. How many steps (compares) does Binary Search take, in the worst case, to search for a given element in an **ordered array** of length 512?

Solution:

It takes $\lceil log_2(512) \rceil = 9$ compares in the worst case - element is last or not present.

iii. If algorithms $A_1, A_2, A_3, A_4, A_5, A_6$ run on a list of length n in times $\log_2 n, n, (log_2 n)^3, n^3, 2^n$, and $n!$, miliseconds respectively, what is the **length of the largest list** that each of them could complete a run on in 1 second (1000 miliseconds)?

Your answers should be integers or integer powers of 2 or 10.

Algorithm	Run Time	Length of List
A_1	$\log_2 n$	
A_2	n	
A_3	$(\log_2 n)^3$	
A_4	n^3	
A_5	2^n	
A_6	$n!$	

Solution:

Algorithm	Run Time	Length of List
A_1	$\log_2 n$	$\mathbf{2^{1000}}$
A_2	n	**1000**
A_3	$(\log_2 n)^3$	$\log_\mathbf{2}(\mathbf{n}) = \mathbf{1000^{1/3}} = \mathbf{10}$ so $\mathbf{n} = \mathbf{2^{10}} = \mathbf{1024}$
A_4	n^3	**10**
A_5	2^n	$\lfloor \log_\mathbf{2}(\mathbf{1000}) \rfloor = \mathbf{9}$
A_6	$n!$	**6** because $6! = 720 < 1000 < 7!$

Problem 8: Mathematical Induction

Prove by mathematical induction:

$$\forall n \geq 2 : \sum_{i=2}^{n} (i-1) \cdot i = \frac{n \cdot (n-1) \cdot (n+1)}{3}$$

<u>*Solution:*</u>

Base case $n = 2:\ \sum_{i=2}^{2} (i-1) \cdot i = 1 \cdot 2 = \frac{2(2-1)(2+1)}{3} = 2$

Induction Step:
Assume $\sum_{i=2}^{k} (i-1) \cdot i = \frac{k \cdot (k-1) \cdot (k+1)}{3}$ for some $k \geq 2$. (This is our inductive hypothesis.)

We want to prove $\sum_{i=2}^{k+1} (i-1) \cdot i = \frac{(k+1)k(k+2)}{3}$.

$\sum_{i=2}^{k+1} (i-1) \cdot i = \sum_{i=2}^{k} (i-1) \cdot i + k \cdot (k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1) = \frac{k(k-1)(k+1)+3k(k+1)}{3} = \frac{k(k+1)(k-1+3)}{3} = \frac{k(k+1)(k+2)}{3} = \frac{(k+1)k(k+2)}{3}$.

We have thus shown that if the statement holds for k it holds for $k+1$. In combination with our base case, and by the principle of mathematical induction, the statement holds for all $n \geq 2$.

Problem 9: Sequences and Recurrences

i. Using the iterative, substitution method, find a closed-form formula for the recurrence: $T(n) = T(\frac{n}{2}) + n; T(1) = 1$. You may assume that n is a power of 2.

Solution:
Our scratch work:
$T(\square) = T(\frac{\square}{2}) + n$
$T(\frac{n}{2}) = T(\frac{n}{4}) + \frac{n}{2}$
$T(\frac{n}{4}) = T(\frac{n}{8}) + \frac{n}{4}$

$$
\begin{aligned}
T(n) &= T(\frac{n}{2}) + n && k = 1 \\
&= T(\frac{n}{4}) + \frac{n}{2} + n && k = 2 \\
&= T(\frac{n}{8}) + \frac{n}{4} + \frac{n}{2} + n && k = 3 \\
&\vdots \\
&= T(\frac{n}{2^k}) + \sum_{i=1}^{k} \frac{n}{2^{i-1}} \\
&= T(\frac{n}{2^k}) + \frac{n}{2^{k-1}} + \frac{n}{2^{k-2}} + \ldots + n
\end{aligned}
$$

Remember, we're looking to continue the above equalities until we get to the point where we plug in $T(1)$. Why this value? Because we know $T(1) = 1$, and this will finally break the recursion. For $T(\frac{n}{2^k})$ to be $T(1)$ then:

$$
\begin{aligned}
\frac{n}{2^k} &= 1 \\
n &= 2^k \\
\log_2 n = \log_2 2^k &= k
\end{aligned}
$$

Plugging this in for k in the last line above:

$$
\begin{aligned}
T(n) &= T(\frac{n}{2^{\log_2 n}}) + \frac{n}{2^{\log_2 n-1}} + \frac{n}{2^{\log_2 n-2}} + \ldots + n \\
&= T(1) + \frac{n}{n2^{-1}} + \frac{n}{n2^{-2}} + \ldots + n \\
&= 1 + 2 + 4 + \ldots + n \\
&= 2n - 1
\end{aligned}
$$

ii. Using the iterative / substitution method, find a closed form formula for the recurrence: $T(n) = 2T(n-1) + 1; T(0) = 0$.

Solution:
Our scratch work:

$T(\square) = 2T(\square - 1) + 1$
$T(n-1) = 2T(n-1-1) + 1$
$T(n-2) = 2T(n-2-1) + 1$

$$\begin{aligned}
T(n) &= 2T(n-1) + 1 && k = 1\\
&= 2(2T(n-1-1) + 1) + 1\\
&= 4T(n-2) + 2 + 1 && k = 2\\
&= 4(2T(n-3) + 1) + 2 + 1\\
&= 8T(n-3) + 4 + 2 + 1 && k = 3\\
&\vdots\\
&= 2^k T(n-k) + \sum_{i=0}^{k-1} 2^i
\end{aligned}$$

Unlike previous problems, our base case is $T(0) = 0$. The approach is identical, what value of k has $n - k = 0$ to break the recursion in the last line. $k = n$ will do the trick, lets apply this value for k:

$T_n = \sum_{i=1}^{n-1} 2^i$

$= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1$

$$\begin{aligned}
T(n) &= 2^n T(n-n) + \sum_{i=0}^{n-1} 2^i\\
&= 2^n * 0 + \underline{2^n - 1}\\
&= 2^n - 1
\end{aligned}$$

$2T_n = 2^n + 2^{n-1} + \cdots + 2$

$2T_n - T_n = T_n$

$= 2^n - 1$

iii. Prove your formula for the second recurrence by induction for all integers $n \geq 1$.

<u>*Solution:*</u> Let $S(n)$ be the statement that $T(n) = 2^n - 1$. Base case. $S(1)$ is true because $T(1) = 1 = 2^1 - 1$.

Inductive step. We assume $S(k)$ is true for some fixed $k \geq 1$. We wish to prove $S(k+1)$. Consider

$$\begin{aligned}
T(k+1) &= 2T(k) + 1 \text{ [by definition of T]}\\
&= 2(2^k - 1) + 1 \text{ [by inductive hypothesis]}\\
&= 2^{k+1} - 1
\end{aligned}$$

Hence $S(k+1)$ is true.

Thus by axiom of induction, since $S(1)$ and $\forall_{k\geq 1}, S(k) \implies S(k+1)$ we have that $\forall_{n\geq 1} S(n)$.

iv. Use two methods to find a closed form formula for the sequence:
$a_n = -42, -39, -34, -27, -18, ...$ The first term is a_1.

Solution:
Method 1: Solving simultaneous equations.
First-order differences are $+3, +5.+7, ...$ and second-order differences are a constant $+2, +2, +2, ...$, so the closed form formula is of the form $a_n = an^2 + bn + c$.
$n = 1$: $a + b + c = -42$.
$n = 2$: $4a + 2b + c = -39$.
$n = 3$: $9a + 3b + c = -34$.
So, $3a + b = 3$ and $5a + b = 5$. Therefore, $2a = 2$ and we get $a = 1, b = 0, c = -43$. So the final solution is $a_n = n^2 - 43$.

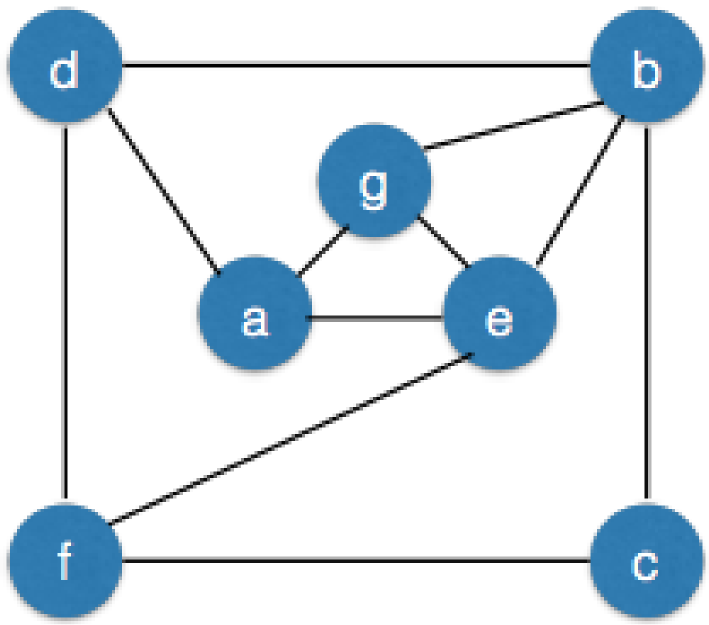
Method 2: Partial sums:
$a_1 = -42$
$a_2 = -42 + 3$
$a_3 = -42 + 3 + 5$
$a_4 = -42 + 3 + 5 + 7$
$a_n = -42 + 3 + 5 + 7 + ... + 2n - 1$
$a_n = -42 + \sum_{k=2}^{n}(2n-1) = -43 + \sum_{k=1}^{n}(2n-1)$
$a_n = -43 + 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = -43 + n(n+1) - n = n^2 - 43$ as before.

Problem 10: Graphs Consider the graph with vertex set $\{a, b, c, d, e, f, g\}$ and *adjacency lists:*

a	$\rightarrow$	g	$\rightarrow$	d	$\rightarrow$	e		
b	$\rightarrow$	g	$\rightarrow$	d	$\rightarrow$	e	$\rightarrow$	c
c	$\rightarrow$	f	$\rightarrow$	b				
d	$\rightarrow$	a	$\rightarrow$	f	$\rightarrow$	b		
e	$\rightarrow$	a	$\rightarrow$	g	$\rightarrow$	f	$\rightarrow$	b
f	$\rightarrow$	d	$\rightarrow$	e	$\rightarrow$	c		
g	$\rightarrow$	a	$\rightarrow$	e	$\rightarrow$	b		

1. Draw this graph in the plane so that no two edges intersect, except at common end vertices.

   Solution:



2. Starting at vertex a traverse the graph by *depth-first-search*, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In your drawing, color the tree edges in red and next to each vertex write its place in the order in which the vertices are first visited (with a as 1).

   Solution: The DFS ordering would be a, g, e, f, d, b, c.

3. Starting at vertex a traverse the graph by *breadth-first-search*, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In a fresh copy of your drawing of the graph, color the tree edges in blue and next to each vertex write its place in the order in which the vertices are first visited (with a as 1).

Solution: The BFS ordering is a, g, d, e, b, f, c.

4. Give a spanning tree of the graph.

   Solution: Any spanning tree will do, there are several ones.