CS5002 Discrete Structures Prof. Rachlin.
Spring 2022 March 9, 2022

CS52002 Midterm Exam

Instructions (for midterm exam):

1. The exam is open book and open notes. You may use a calculator but it should not be necessary.

2. The exam window starts at 12:00pm ET/Boston (Noon) and all exams must be received by 11:59pm ET/Boston. Once started you will have 3 hours and 15 minutes to complete the exam AND submit your exam to gradescope.

3. Discussing the exam with others is strictly prohibited. Giving or receiving help from fellow students earns all parties involved an automatic F for the class. Do not test us on this.

4. Unless otherwise specified, you may leave your answer in the form of a fraction or mathematical expression that clearly shows your thought process. For credit you must show your scratch work and clearly explain how you got to your final answer. If your expression isn't clear or obvious you won't earn full credit.

5. If you have a question, post a PRIVATE question on Piazza. An instructor will respond as soon as possible.

6. When submitting your exam you must specify, for each problem, the page or pages where the solution to that problem is to be found. If we can't see your solution you will not receive credit for the problem.

7. You have been granted an extra 15 minutes solely for the purpose of uploading your exam. Exams emailed to instructors will receive a 5 percent penalty, so be sure to give yourself enough time to submit your exam! Exams received via email more than a few minutes beyond the deadline will not be graded.

8. May the force be with you!

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I have read the instructions above and understand that I may not discuss the exam with other students at any time. SIGN HERE: Haozhe Zhang

Section 1 [25 pts (5 each)]: Number Representations

1. Convert BB816 to octal. (Show your work.)

   In hexadecimal, B = 11, number inside the parenthesis represents which numeral system the number is using.
   BB8(16)= 11 * 16^2 + 11 * 16 + 8 = 3,000(10)

2. Convert D216 to binary. Now, treating the number as an 8-bit two's-complement number, convert to decimal (base-10)
   D = 13 in hexadecimal
   D2(16) = 11010010(2) <- 8=bit 2's complement number
   Flipping bits, 11010010 -> 00101101
   Adding one, 00101101 -> 00101110
   00101110(2) = 46
   Adding the negation sign = -46

3. The droid C3PO from Star Wars knows 6 million forms of communication. We assign each form of communication a unique identifier in the form of a fixed-length unsigned binary number. (By fixed-length we mean that the number of bits allocated to each identifier is the same.) How many bits must be allocated for each identifier? Give your answer in the form of an integer, and explain.

   Let n be the fixed-length of bits.
   Then the maximum unsigned n-bits integer has 2^n - 1 bits
   Let 2^n - 1 = 6,000,000
   2^22 = 4,194,304
   2^23 = 8,388,608
   In this case, we need to have a fixed-length of n = 23 bits unsigned binary to include 6,000,000
4. 2^{15} = 32; 768. How many numbers between 0 and 2^{15} are the sum of exactly three powers of 8? Each power must be a natural number. Natural numbers are the non-negative integers (0, 1, 2, 3, ...). The powers need not be district. For example, 10 is one such number because 10 = 81 + 80 + 80 = 8 + 1 + 1 and 521 is another such number because 521 = 83 + 81 +80 =512 + 8 + 1. Reduce your answer to an integer and explain your answer.

   Total number options are 2^15 + 1 == 8^5 = 32,768,
   We can treat the problem as seeking for the maximum number which each digit is not distinct at base 5 and plus 1.

   The last possible option is 8^5 + 8^0 + 8^0 = 32,768
   The largest one is 444, the smallest is 000, so the total numbers are 5* 5 * 5 + 1 = 126

5. Droids on the planet Tatooine perform calculations in base-6. What is the largest base-6 number that can be represented with three distinct digits? (Each digit in your base-6 number must be different.) Convert your answer to decimal (base-10)

   The largest base- 6 number that is represented with three distinct digits is 543
   5 * 6^2 + 4 * 6^1 + 3 = 207

Section 2 [25 pts]: Logic

Given the domains m 2 MOV IES and c 2 CHARACTERS and the following predicates:

- SW(m): m is a Star Wars movie.
- A(c; m): c is a character who appears in the movie m
- D(c): c is a Droid.
- L(m): I liked m.
- Q(c; q): c says q (c is quoted as saying q.)
- jarjar is the character Jar Jar Binks
- bad is the quote: "I've got a bad feeling about this!" (∀, ∃, ∧, ∨, →, ¬, ≡)

1. Express the following English sentence with predicate logic using Existential and/or Universal quantifiers: I didn't like any Star Wars movie if Jar Jar Binks was in it

   ∀x: ¬L(x) ∧ SW(x) → A(jarjar, x)

2. Express the following English sentence with predicate logic using Existential and/or Universal quantifiers: At least two droids appear in every Star Wars movie. You may use the notation x = y and x != y as needed

   ∀x ∃y ∃z: SW(x) → D(y) ∧ A(y, x) ∧D(z) ∧ A(z, x) ^ (y != z)

3. Express the following English sentence with predicate logic using Existential and/or Universal quantifiers: In every Star Wars movie, some character in the movie says "I've got a bad feeling about this!"

   ∀x ∃y: SW(x) → A(y, x) ∧ Q (y, bad)

4. Negate your expression from sub-problem (3). Now simplify the logical expression using the laws of logical equivalence. To simplify, eliminate implications (if any) and negations in front of quantifiers. Negations should appear only in front of single predicates. Show each step. Cite the law of logical equivalence you are applying with each step.

   ¬(∀x ∃y: SW(x) → A(y, x) ∧ Q (y, bad)) start
   ∃x ¬∃y: SW(x) → A(y, x) ∧ Q (y, bad) Negation of universal quantifier
   ∃x ∀y: ¬(SW(x) → A(y, x) ∧ Q (y, bad)) Negation of existential quantifier
   ∃x ∀y: ¬(¬SW(x) ∨ (A(y, x) ∧ Q (y, bad))) Defn. of Implication
   ∃x ∀y: ¬¬SW(x) ∧ ¬ (A(y, x) ∧ Q (y, bad))) DeMorgan's
   ∃x ∀y: SW(x) ∧ ¬ (A(y, x) ∧ Q (y, bad)) Double negation
   ∃x ∀y: SW(x) ∧ (¬ A(y, x) ∨ ¬Q (y, bad)) DeMorgan's
   ∃x ∀y: SW(x) ∧ ¬ A(y, x) ∨ SW(x) ∧ ¬Q (y, bad)) Distributive

5. Convert your final simplified expression from the previous problem back into English

   For some Star War movie, there is no such character y in the movie or y has not said such quote as "I've got a bad feeling about this!".

Section 3 [25 pts (15,10)]: Sets

1. In this problem A = {R2D2, C3PO, BB8} and B = {BB8, R5D4}. |A| denotes the cardinality (number of members) of the set A. Answer each question with an integer or an expression involving exponents. You do not need to show each step. ∪ ∩

   $|A| = 3$, $|B| = 2$;

   i. $|A \times B| =?$

      $|A \times B| = 3 * 2 = 6$

   ii. $|A - B| =?$

      $|A - B|$ = |A ∩ negation B| = 2

   iii. $|(A - B) \cap (B - A)| =?$

      $|(A - B) \cap (B - A)|$ = |A ∩ negation B ∩ B ∩ negation A| = 0

   iv. $|P(A \triangle B)| = ?$

      $|P(A \triangle B)| = |P(A- A\cap B + B - A\cap B)| = 2^3 = 8$

   v. $|\{x : x \in P(A \cup B) \land |x| = 2\}| =?$

      $|P(A \cup B)| = 16$
      Size 0 = 1 (elements)
      Size 1 = 4 (elements)
      Size 2 = 6
      Size 3 = 4
      Size 4 = 1

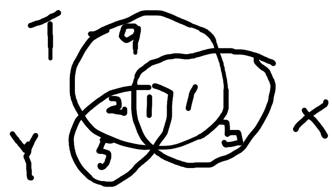
      $|x| = 2$, x = size 2, where size 2 = 6
      $|\{x : x \in P(A \cup B) \land |x| = 2\}| = 6$

2. Some rebel pilots were surveyed to determine what they thought of different star fighters. The three kinds of star fighters are the X-Wing, the Y-Wing, and Tie Fighters.

   • Every pilot liked at least one kind of star fighter
   • 1 liked all three. (The force is strong with this pilot!)
   • 1 liked X-Wings and Y-Wings
   • 2 liked X-Wings and Tie Fighters
   • 3 liked Y-Wings and Tie Fighters
   • 5 liked X-Wings
   • 8 liked Y-Wings
   • 13 liked Tie Fighters.

Use the Principle of Inclusion and Exclusion to determine how many pilots answered the survey. Draw a Venn Diagram depicting the above data. Note: when we say, for example, that n pilots liked a particular kind of Star Fighter, it means they might have liked other types as well. For example, some of the 13 pilots that like Tie Fighters liked X-Wings, Y-Wings or maybe both ∪ ∩

A ∪ B ∪ C = A + B + C – A ∩ B - A ∩ C - B ∩ C + A ∩ B ∩ C = 13 + 8 + 5 – 2 – 3 – 1 + 1 = 21



Section 4 [25 pts (5 each)]: Counting

1. Suppose droids in the Star Wars universe follow the following naming conventions:
   • Names must be 3-5 characters including uppercase letters and digits. (No dashes.)
   • All names must start with a letter and contain at least 1 digit.

   How many droid names are possible? You can leave your answer as an expression as long as you clearly explain each term in the expression. Some valid names include BB8, R2D2, C3PO, and H2SO4.

   Uppercase letter options: 26
   Digit options: 10
   Considering starting with Uppercase and no digit scenarios:
   Name with 3 characters: P = 26^3
   Name with 4 characters: P = 26^4
   Name with 5 characters: P = 26^ 5

   No digit scenario is a completement to scenarios that starting with uppercase and containing at least 1 digit.
   So the possible names are 26*36^2 + 26*36^3 + 26*36^4 – 26^3 - 26^4 – 26^5.

2. Seventeen Jedi (17) are fighting for their lives in an arena. Each Jedi uses a light saber with one of 4 colors.
   (a) Explain why five or more Jedi must be using the same color light saber. What mathematical principle are you invoking in your explanation?

   Pigeonhole principles (17/4), let 4 Jedi use 4 different colors, and the one option the fifth Jedi can choose from is one of the 4. So the five or more Jedi must be using the same color light saber.

   (b) How many Jedi would there need to be in the arena in order to guarantee that seven or more Jedi are using the same color light saber?
   Let each color of light sabers be held by 6 Jedi, there is 4 * 6 = 24 Jedis, the 25th Jedi must choose 1 of the 4 color light sabers and be the seventh person holding that color of light

saber. So 25 Jedi would there need to be in the arena in order to guarantee that seven or more Jedi are using the same color light sabes.

3. Nine Star Wars movies make up the three Star Wars trilogies:
(a) Episodes 1-3 (The prequels with Jar Jar Binks)
(b) Episodes 4-6 (The originals starring Mark Hamill as Luke Skywalker)
(c) Episodes 7-9 (The ones starring Daisy Ridley as Rey)

My weekend movie marathon consists of an ordered sequence of five distinct movies selected from these nine. How many movie marathons are possible? (The same five movies watched in a difference sequence constitute a different marathon.)

A sequence of five distinct movies selected from 9.
The first selection has 9 options, and the second selection has 8 options ….. the fifth selection has 5 options left.

Hence, we have P(9, 5) possibilities

4. What if each marathon consists of four to six movies and each marathon is distinguished by the particular movies I select and not the order in which I watch them. Furthermore, I'm allowed to pick the same movie more than once. Now how many movie marathons are possible?

Orders don't matter and movie selections are not distinct.
We have:
Marathon size 4: 9^4
Marathon size 5: 9^5
Marathon size 6: 9^6
The total movie marathon possibilities become 9^4 + 9^5 + 9^6

5. What if my movie marathon consists of exactly five(5) distinct movies: two movies from one of the trilogies, 2 movies from a 2nd trilogy, and 1 movie from a 3rd. The movies can be watched in whatever order I choose and the same five movies watched in a different order constitute a different marathon. Now how many movie marathons are possible?

Order matters.
5 distinct selections.
First pick = P(3, 2), P(3, 1), P(3, 0)
Second pick = P(3, 2) or C(2, 2)
Third pick = P(3, 1) or C(2, 1) or C(1, 1)

Total possibilities = P(3, 2) * P(3, 2) *P(3, 1) + P(3, 1) * P(2, 2) * P(3, 1) +
P(3, 1) * P(3, 2) * P(2, 1) + P(3, 0) * P(2, 2) * P(2, 1) +
P(3, 0) * P(3, 2) * P(1, 1)