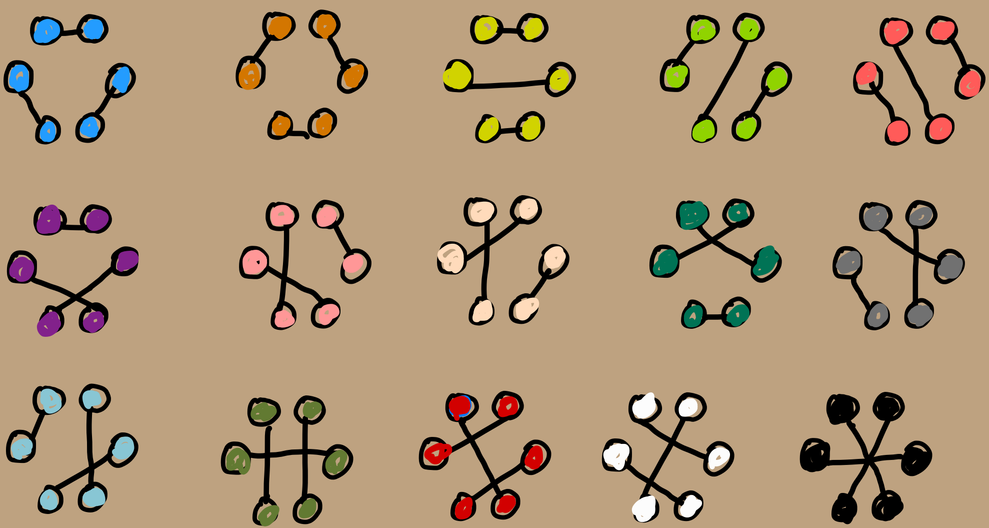
Counting / Combinatorics



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Lecture Notes

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Sets & Counting / Combinatorics.

① Intro

There are lots of things in CS we want to count. E.g. steps in an algorithm - is one algorithm for sorting better than another.

A motivating example: The security of passwords? How many passwords are their given different rules.

An ATM card might have 4-digits. How many pins? 10,000 $= 10^4$.

Each digit has 10 possibilities and there are 4 digits.

4-digits: $10{,}000 = 10^4$

4 - lower case: $26^4 = 456{,}976$

4 - upper/lower $52^4 = 7{,}311{,}616$

More passwords ⇒ more security. There is a reason why you can't have a 2-digit pin!!

"Brute force" attacks (attempting all possibilities) should be hard.

NSF Rules:

7-10 characters
uppercase, lowercase, digits
≥ 2 letters
≥ 2 digits

} less obvious how to count the size of this search space.

The restrictions decrease size of the PW space.

Why? To combat human nature: short passwords are easy to remember

Common Passwords in data breaches:

123456
123456789
qwerty
⋮
patterns on keyboard.

Is the shrinkage of the PW space substantial?

Binary #s (

n bits $\Rightarrow$ 2^n values

unsigned $0 \ldots 2^{n-1}$

e.g. $n = 4$

0000	0
⋮	⋮
1111	15 = 8+4+2+1
8 4 2 1	

___ ___ ___ ___

0/1 0/1 0/1 0/1

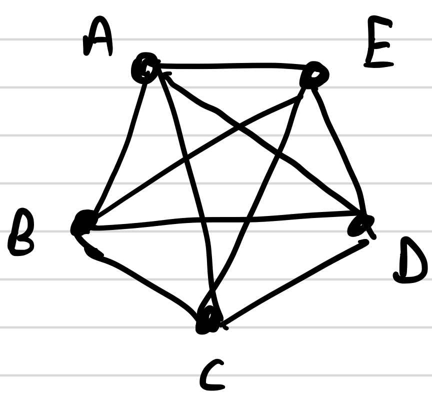
$2 \times 2 \times 2 \times 2 = 2^4$ possibilities

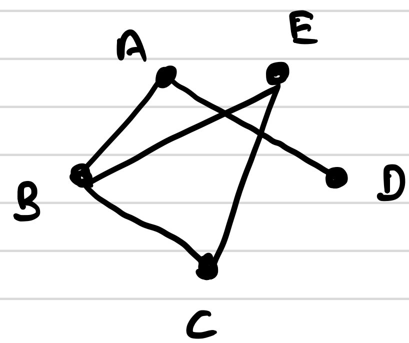
Counting Social Networks

How many graphs are there on the vertices $V = \{A, B, C, D, E\}$

$|V| = 5$

$$|E| = \frac{|V|(|V|-1)}{2} = 10$$





	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE
Complete graph:	1	1	1	1	1	1	1	1	1	1
Other Graph	1	0	1	0	1	0	1	0	1	0

For each pair (u, v) we decide whether to include an edge, so we have

$$\underset{\text{choices}}{2 \cdot 2 \cdot 2} \cdots\cdots \cdot 2 = 2^{10} = \underline{\underline{1024}}$$

(10) Sum and Product Rule

Pants or Shorts 8 pants, 6 pairs of shorts

I can wear one or another. so I have $8+6 = 14$ options.

Pants and Shirts 8 pants, 12 shirts

$8 \times 12 \Rightarrow 96$ options.

A) **Product Rule:** If A & B are finite sets then the number of ways to pick an $a \in A$ and an element $b \in B$ is:

$$|A \times B| = |A| \times |B|$$

more generally, $A_1, A_2, \ldots A_n$

$$|A_1 \times A_2 \times \cdots A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

4 char passwords: upper/lower/digit.
How many pw's do you have?

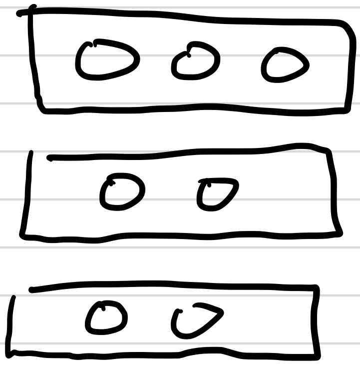
$A = \{a, b, c, \ldots z, A, B, C, \ldots Z, 0 \ldots 9\}$

$A \times A \times A \times A \qquad |A| = 62$

$62 \times 62 \times 62 \times 62 = 62^4 = 14{,}776{,}336$

"For each position I have 62 choices"
62 choices for position 1 **AND** position 2
AND ....

Pick Representative from each of three tables



$$\begin{array}{r} 3 \\ \times \\ 2 \\ \times \\ 2 \\ \hline 12 \end{array}$$ possibilities

Think about the problem in English:

"AND" $\Rightarrow$ multiplication / product

"OR" $\Rightarrow$ addition / sum.

B) <u>Sum Rule</u> "Pick this or that"

If A & B are <u>disjoint</u> finite sets, then the # ways of picking an object from A <u>OR</u> B is $|A|+|B|$

Generally $A_1, A_2, \dots A_n$ all <u>mutually disjoint</u> then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$$

Example: Roulette

2 Green

18 Red

18 Black.

(a)

5-spin sequences:

___ ___ ___ ___ ___

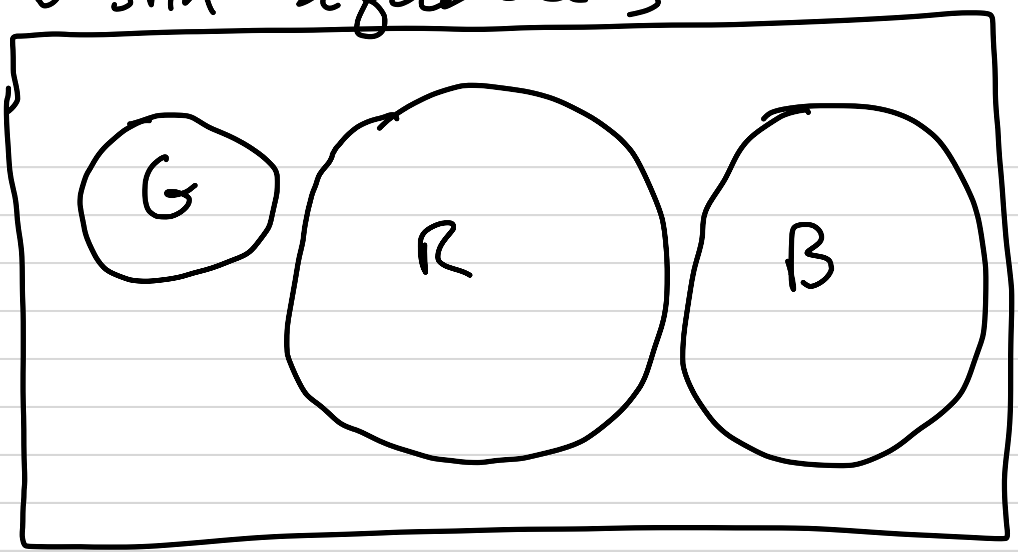
$$38 \times 38 \times 38 \times 38 \times 38 = 38^5$$

(Product Rule)

(b)

5-spin sequences all of same color?

5 Spin Sequences



all same color $= |G| + |R| + |B|$

(Sum Rule)

$|G| = \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ } = 32$

$2 \times 2 \times 2 \times 2 \times 2$

$|R| = |G| = \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ } = 18^5$

$18 \times 18 \times 18 \times 18 \times 18$

$2 \cdot 18^5 + 32$

(c) # Not All the same colors

(At least 2 G or 2R or 2B)

$|\mathcal{S}| - |G| - |R| - |B| = 38^5 - 2 \cdot 18^5 - 32$

(d) All distinct values ?

e.g 2 , 5 , 32 , 7 , 25

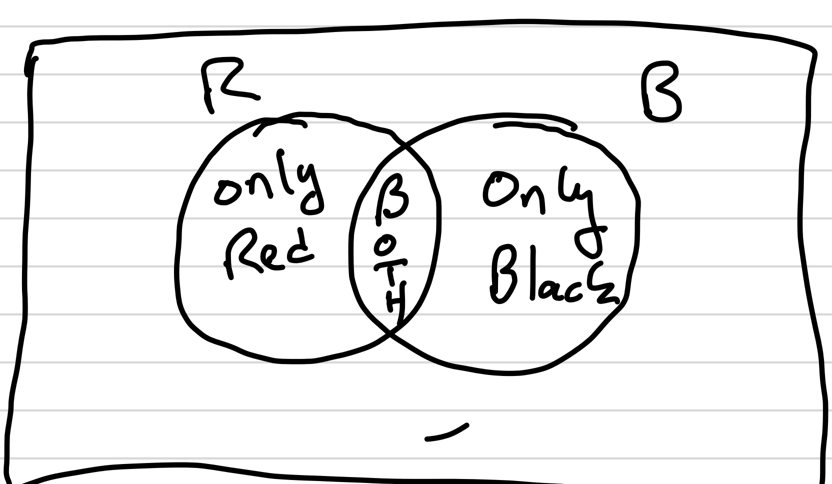
___ ___ ___ ___ ___

$38 \times 37 \times 36 \times 35 \times 34$

(This is a 5 - Permutation on 38 values - more about this soon.)

"Partial Permutations"

(e) A mix of Red & Black? (no green)



We know that there are

36^5 sequences of R & B

values. (no green)

18^5 Red Only

18^5 Black Only

So $36^5 - 2 \cdot 18^5 = \#$

having some Red & some black

(But no Green)

Example : Song Playlists

6 songs by Regina Spektor

5 songs by Ingrid Michaelson

a) # 7-song playlists?

$$11 \times 11 \times \cdots\cdots = 11^7$$

b) no repeats?

$$11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

c) Only IM or only RS?

$$|P| = |RS| + |IM|$$

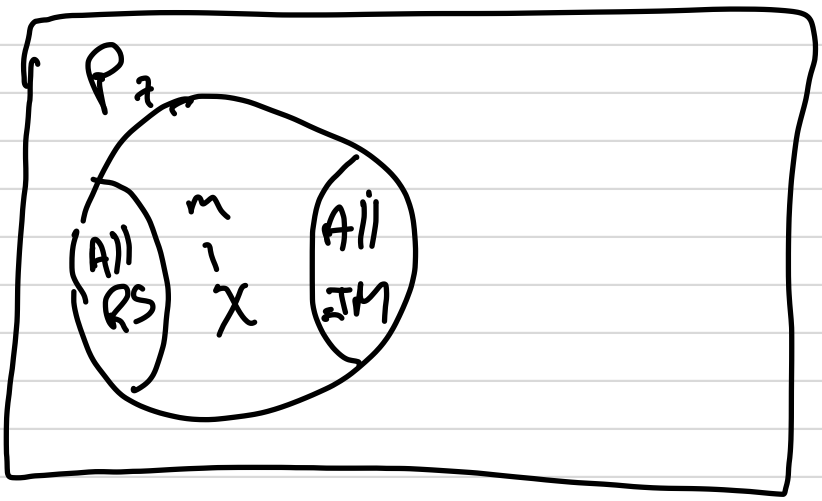
$$= 6^7 + 5^7$$

d) 7 .... 10 songs ?

$$|P| = |P_7| + |P_8| + |P_9| + |P_{10}|$$

$$= 11^7 + 11^8 + 11^9 + 11^{10}$$

e) 7 songs but At least 1 song by each artist ?



$$|Mix| = |P_7| - |\text{All RS}| - |\text{All JM}|$$

$$= 11^7 - 6^7 - 5^7$$

f) Must Alternate Artists in a 7-song Playlist? <u>and</u> all songs are distinct?

R	I	R	I	R	I	R
6 ·	5 ·	5	4	4	3	3

= 21,600

<u>OR</u>

I	R	I	R	I	R	I
5	6	4	5	3	4	2

= 14,400

By Sum Rule: 21,600 + 14,400 = 3600[illegible]

c) **Principle of Inclusion/Exclusion:**

What if the sets aren't disjoint?

R : set of red cards = 26
F : set of face cards = 12

How many ways can I pick a card that is red <u>or</u> a face card?

<u>NOT</u> $26 + 12 = 38$ ✗ (I'm double counting red face cards)

Rather: $26 + 12 - 6 = 32$ ✓

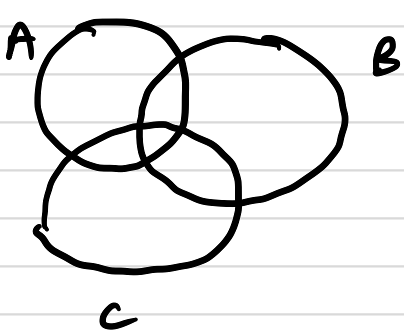
Principle of Inclusion/Exclusion says:

2 sets A, B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

It still works when the sets are disjoint because then $|A \cap B| = 0$

<u>Three Sets</u>



"Pick an element from A or B or C"

$$\boxed{\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}}$$

For larger sets, it's a continuing series of inclusions & exclusions

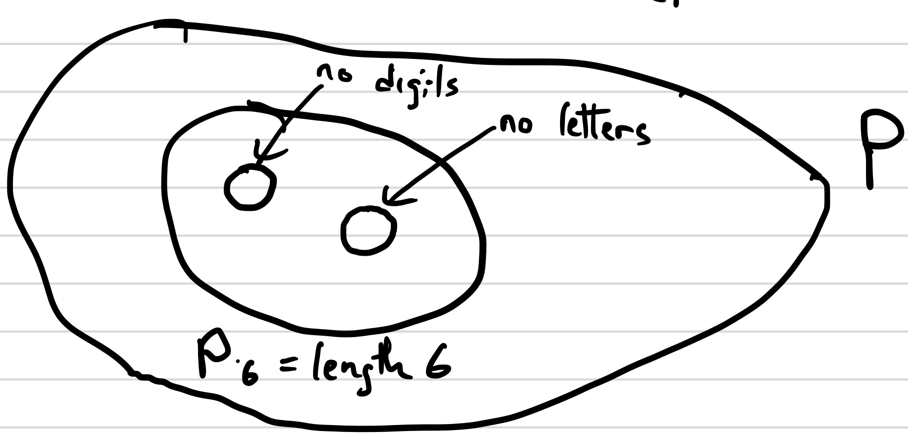
11 Examples - continued

a) PW^s between 6 & 10 chars upper, lower, digits.
w/ at least one letter
at least one digit

We have to break up into pieces.

Step 1 Pick length: 6 or 7 or 8 or 9 or 10
mutually disjoint ⇒ sum rule.

Step 2 Focus on length 6 password.
Pick 1st char and 2nd and .... and 6th
(product rule)



$$|P_6| = \underbrace{62^6}_{\text{all}} - \left(\underbrace{10^6}_{\text{no letters}} + \underbrace{52^6}_{\text{no digits}}\right)$$

$$\text{all} - \underbrace{}_{\text{illegal}} = \text{legal}$$

Step 3 : Apply sum rule to account for $A_7 \ldots A_{10}$

By sum rule:

$$|P| = \sum_{i=6}^{10} P_6 = \sum_{i=6}^{10} \left(62^i - 10^i - 52^i\right) \approx 7 \times 10^{17}$$

≈ 700 million billion

Permutations

Traveling Salesman Problem: A salesman has to visit 10 cities. How many ways can you visit the cities.

Delivery services care about this!

We might be trying to minimize time, cost, gas, # of left turns, etc.

We could try all possibilities to find the best solution. How many ways are there?

	10	options for first	10 ×
AND	9	options for second	9 ×
AND	8	options for third	8 ×
⋮	⋮		⋮
AND	1	option for 10th	1
			$10!$

"10 factorial"

$10! = 3,628,800$

$25! \approx 1.55 \times 10^{25}$ (4.12×10^{17} seconds since big bang.)

So ⇒ 37 million choices per second since the beginning of time will find us our optimal.

For many problems, best solutions require we consider all possibilities to find guaranteed best.

(So we find something close instead.)

# of ways to order n objects = $n!$
Each ordering is a "permutation"

How many ways can you visit 3 out of 10 cities?

$$\underset{\substack{\uparrow \\ \text{1st} \\ \text{choice}}}{10} \cdot \underset{\substack{\uparrow \\ \text{2nd}}}{9} \cdot \underset{\substack{\uparrow \\ \text{3rd}}}{8} = \frac{10!}{7!} = \frac{10!}{(10-3)!}$$

Def: Permutations : # permutations = $n!$

$$n! = n(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

k-Permutation:

$${}_nP_k = \boxed{P(n,k) = \frac{n!}{(n-k)!}} = \underbrace{n\cdot(n-1)\cdots(n-k+1)}_{k \text{ terms}}$$

Examples: a) 10 packages $\Rightarrow$ $10! \approx 3.6$ million

b) $6/10$ packages $\Rightarrow$ $10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5$

$$= \frac{10!}{(10-6)!} = 151{,}200$$

c) 6 on north - together
4 on south - together

$$\underset{\substack{\uparrow \\ \text{N or S}}}{2} \cdot \underset{\substack{\uparrow \\ \text{S}}}{4!} \cdot \underset{\substack{\uparrow \\ \text{N}}}{6!} = 34{,}560$$

Permutations with identical objects.

How many permutations of the letters "banana"

$n = 6$, but many of the $n! = 720$ arrangements are the same.

$b_1 \; a_1 \; n_1 \; a_2 \; n_2 \; a_3$

$b_1 \; a_3 \; n_2 \; a_1 \; n_1 \; a_2$

$\Downarrow$

$$\frac{6!}{\underset{a\text{'s}}{3!}\;\underset{n\text{'s}}{2!}} = \frac{\cancel{6}\cdot 5\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\;\;\cancel{2}\cdot\cancel{1}}$$

$= 60$ arrangements.

In general:

n objects, k distinct.

where m_k = # of occurrences of the k^{th} type of object

Then:

$$\text{\# permutations} = \frac{n!}{m_1!\, m_2! \cdots m_k!}$$

Combinations

An <u>unordered</u> subset of k out of n objects.

e.g. we have to pick $3/10$ packages to put on the truck, <u>but we don't care what order we put the packages on the truck</u>

$$P(10,3) = \binom{10}{3} \cdot 3!$$

deliver 3 of 10 packages

Pick the 3 packages

Define the possible delivery sequence for the 3 packages.

And in general:

$$P(n,k) = \binom{n}{k} \cdot k! = \frac{n!}{(n-k!)}$$

$$\therefore \quad \# \text{ combinations} = \boxed{C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}}$$

Note: Calculating Combinations Quickly and by hand:

$$\binom{8}{5}$$

a) $\binom{8}{5} = \binom{8}{3}$ Since $\binom{n}{k} = \binom{n}{n-k}$

b) Write out 3 #s: (The rest cancel)

c) Divide by 3! (Dont write the 1s!)

$$\frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3} \cdot \cancel{2}}$$

d) Cancel $\Rightarrow$ $8 \cdot 7 = 56$ ✓

$$\binom{11}{9} = \binom{11}{2} = \frac{11 \cdot 10}{2} = 55$$

$$\binom{12}{7} = \binom{12}{5} = \frac{\cancel{12} \cdot 11 \cdot \cancel{10} \cdot 9 \cdot 8}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}$$

$$= 11 \cdot 72$$

$$= 792$$

(using the ×11 trick

$$\begin{array}{ccccccc} \text{"}a & b\text{"} \cdot 11 & = & \text{"}a & (a+b) & b\text{"} \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ 7 & 2 & = & 7 & 9 & 2 \end{array}$$

Examples

e.g "01101"

① Consider bit strings of length 5.

How many? $2^5 = 32$

How many have exactly 3 1's?

a) Plug into the formula:

$$\binom{5}{3} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{2\cdot 1\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = 10$$

b) Reason it out

5-length bit strings

# of 1's	# of strings		
0	1		00000
1	5		10000, 01000 . . .
2	10	} must be equal	01100 ↔ 10011
3	10		01011 ↔ 10100
4	5		11110, 11101, ...
5	1		11111
	32 ← Total		

c) Think Recursively

$$\binom{5}{3}$$

0- - - - $\binom{4}{3} + \binom{4}{2}$ 1- - - -

$$10 = \binom{3}{3} + \binom{3}{2} + \binom{3}{2} + \binom{3}{1}$$

① ③

$$+ \binom{2}{2} + \binom{2}{1} + \binom{2}{2} + \binom{2}{1} +$$

① ② ① ②

In general:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

(2) Let $A = \{a, b, c, d, e\}$

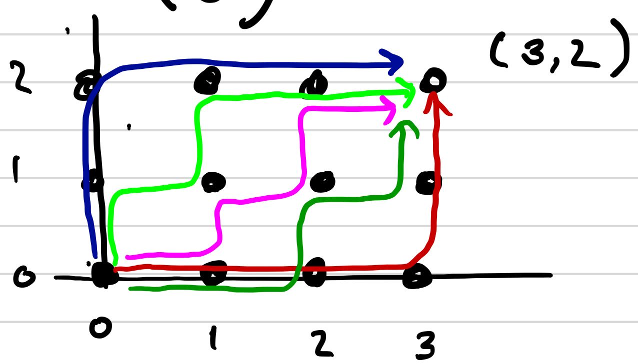
How many elements of $\mathcal{P}(A)$ are size 3?

a	b	c	d	e		
0	1	1	0	1	=	$\{b, c, e\}$
1	0	0	1	1	=	$\{a, d, e\}$
						$\vdots$

$\equiv$ # of 5-length bit strings having 3 ones.

$= \binom{5}{3} = 10$

(3)



How many lattice paths (shortest paths) from $(0,0) \to (3,2)$

All lattice paths are length 5, move up 2 times and right 3 times.

We encode up = 0, right = 1.

Eg, the bright green path is 01011.

So again this is equivalent to finding 5-bit strings with 3 ones.

$\binom{5}{3} = 10$

How many permutations of the letters A, B, C, D, E, F which do not contain the substring BAD

For example F BAD C E, F C E BAD, etc.

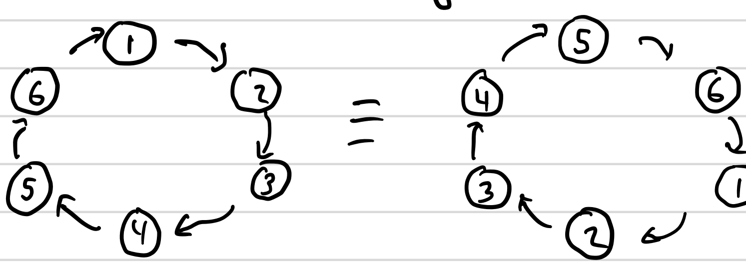
$$\#\text{ Legal Permutations} = \#\text{ Total Permutations} - \#\text{ Illegal}$$

$$= 6! - \underbrace{4!}_{\text{Think of permutations of the 4 elements C, E, F, BAD}}$$

$$= 720 - 24$$

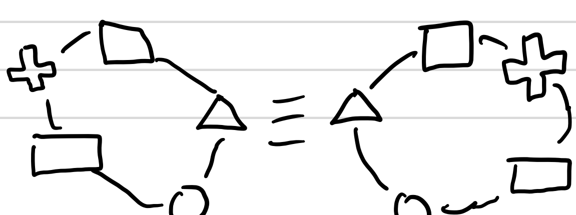
$$= 696$$

A circular permutation is a permutation arranged in a circle. By rotational symmetry.



$$\#\text{ Permutations} = \frac{n!}{n} = (n-1)!$$

5 beads chosen from 9 distinct beads for a necklace. The necklace can be flipped over too! How many necklaces?



$$\frac{P(9,5)}{2 \cdot 5} = \frac{P(n,k)}{\underbrace{2}_{\text{flip symmetry}}\ \underbrace{k}_{\text{rotational symmetry}}} = 1512 \text{ necklaces}$$

← k beads permuted from n choices