CS5002 Discrete Math
Haozhe Zhang HW 05

Problem 1 [25 (8, 8, 9)]: Ride the MBTA.

i. The Worcester line makes 10 inbound trips, 4 of which are express and 6 are local. In how many ways can we order the trips as a sequence of either express or local trips. For example, LLEELLEELL is one possibility.

ii. A train engine is to be connected with either 3, 4, or 5 passenger cars chosen from 7 available passenger cars. Assuming the passenger cars are all numbered, and that the same cars linked together in a different sequence constitutes a different train, how many possible trains could we form?

iii. Suppose an inbound train starts at station 1 and ends at station 10. The local train makes all 8 intermediate stops at stations 2 through 9. An express train is any inbound train that makes no more than 3 intermediate stops. (The starting and ending station do not count.) How many different valid express train trips are possible?

Solution:

i. In total 10 lines, in order to make up possible sequences of trips, we can count the possible subsets of the sequences.
   n= 10, r = 4

   C(10, 4) * C(6, 6) = C(10, 6) * C(4, 4) = $\binom{10}{4}\binom{6}{6} = \binom{10}{6}\binom{4}{4} = \frac{10!}{4! * 6!}$ = 210

ii. Since all the cars are numbered and we are choosing 3 distinct numbered cars from 7 available options, we are using permutation instead of choosing subsets.
   P(7, 3) + P(7, 4) +P(7, 5) = 210 + 840 + 2520 = 2570

iii. There are 4 express trains, express train is defined to be no more than 3 stops during the transportation. No more than 3 stops can be translated to less and equal than 3 or either 0, 1, 2, 3 stops.

   0 stop: A = C(8, 0) = 1
   1 stop: B = C(8, 1) = 8
   2 stops: C = C(8, 2) = 28
   3 stops: D = C(8, 3) = 56

   One express train has P = A + B + C + D = 1 + 8 + 28 + 56 = 93 possible trips
   Four express trains have 93 * 93 * 93 * 93 = 93^4 possible trips

Problem 2 [25 pts (6, 6, 6, 7)]: DNA - The code of life.

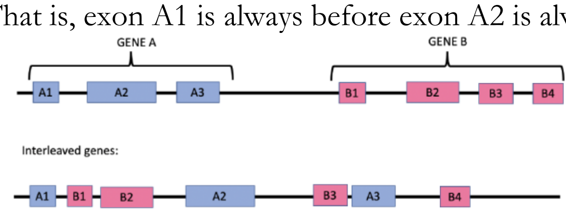
The human genome is a double helix containing some 3 billion nucleotide base pairs. For purposes of discussion we can imagine DNA as a long string of four letters (A,T,G,C) corresponding to the four nucleotides (Adenine, Thymine, Guanine, and Cytosine).

i. How many DNA sequences of length 12 are possible? Note: DNA sequences have a definite direction or orientation, so reversing the sequence produces a different DNA sequence. (Molecular biologists refer to one end as the 3' end and the other as the 5' end.)

ii. How many DNA sequences of length 12 are possible if no nucleotide ever repeats? By "repeat" we mean that the nucleotide occurs at least twice in a row (AA, CC, GG, TT).

iii. How many DNA sequences have at least one repeating nucleotide somewhere along the sequence?

iv. Human genes may be fragmented into multiple exons as shown in the figure below. In fact the exons of two genes may even be interleaved. Gene A has 3 exons and gene B has 4 exons. In how many ways can the exons of gene A and B be ordered (including interleavings of the two genes) while maintaining the relative sequence of gene A's exons and gene B's exons? That is, exon A1 is always before exon A2 is always before exon A3 and so on.



Solution:

i. Considering DNA sequence is a string consisting of 12 letters, and each letter has 4 options. The total possible sequences of DNA from one direction are 4*4*4….*4(12 times) = 4^{12} Reversing the sequence produces a different DNA sequence, which doubles the total results and make it to be $2 * 4^{12}$.

ii. Considering the scenario that no repeating nucleotides will make the new total possible DNA sequence becomes $4*3^{11}$.

iii. To calculate at least one repeating nucleotide is easier to count the complement of it, which is to calculate the total number of possible DNA sequence(i) subtracted by no repeating nucleotides (ii) which is $2 * 4^{12} - 4*3^{11}$.

iv. The order of the exons needs to be considered. We can use inclusion-exclusion to count the possible interleaved sequences. Considered all possible sequences are 2^6 starting with A1 and 2^6 starting with B1. The invalid cases are starting with A1A2A3…. and starting with B1B2B3B4…… so the final answers are $2 * 2^6 – 2 = 2^7 – 2$.

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Problem 3 [25 pts (5, 10, 10)]: A Long-expected Party

When Mr. Bilbo Baggins of Bag End announced that he would shortly be celebrating his eleventy-first birthday with a party of special magnificence, there was much talk and excitement in Hobbiton.

i. Suppose Bilbo invited 111 guests. In how many ways might the guests arrive at the party if they arrive one at a time?

ii. Suppose the guests include 11 Proudfoots (Proudfeet!) consisting of 4 women (W1:::W4), 4 men (M1:::M4), and 3 children (C1:::C3). In how many ways might the Proudfeet arrive if all the women enter together (one at a time), all the men enter together, and all the children enter together? The groups (Men, Women, Children) may still enter in any order (Men/Women/Children, Children/Women/Men, etc.) W1W4W2W3C3C2C1M2M4M1M3 is one possible arrival sequence.

iii. What if Frodo Baggins and Lobelia Sackville-Baggins are two of the eleven guests at one of the tables and they must NOT be seated next to each other. How many distinct and valid seating arrangements are there for this table? Rotating guests around the table does not count as a distinct seating arrangement.

Solution:

i. There are 111 distinct guests, so the possible ways for those guests to arrive at the party if they arrive one at a time is going to be 111! ways (hopefully not this one).

ii. W and M and C have to entered as a group of 4, 4, 3. But the orders do not matter, so we can find subsets that 11 choose 4 and 7 choose 4 and 3 choose 3.
C(11, 4) * C(7, 4) * C(3, 3)

iii. If we were bound Frodo and Lobelia to seat together, there are 2 * 10 choices for them to seat together either Frodo sits on the left or the right of Lobelia. The total seating arrangements with no limitation are 10! So, the distinct and valid seating arrangements if Frodo and Lobelia to not seat together are 10! – 2* 10.

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Problem 4 [25 pts (10, 10, 5)]: Peanut butter crackers

i. How many ways are there to pair up four distinct crackers (C1, C2, C3, and C4) to make two peanut-butter cracker sandwiches? The order in which you form the pairs makes no difference. It only matters which crackers are paired with which other crackers. One possibility is C1 paired with C4 and C2 paired with C3.

ii. How many ways are there to pair up six distinct crackers (C1……C6) to make three peanut-butter cracker sandwiches? As before, the order in which you form the pairs makes no difference.

iii. Bonus question: (Only 5 points.) Suppose we had n crackers (n is even). Derive a general expression for the number of sandwich pairings possible when making n/2 peanut butter sandwiches. Explain why your expression is correct! (Peanut butter sandwiches will never be the same - sorry!)

Solution:

i. Possible ways to pair up four distinct crackers are C(4, 2) * C(2, 2)= 6 ways.
To avoid double counting since we do not care about the orders of the pairs chosen, the possible ways to pair up four distinct crackers are 6 / 2! = 3 ways.

ii. Similarly to previous question, possible ways to pair up four distinct crackers are C(6, 2) * C(4, 2) * C(2, 2) / 3!= 90 / 6 = 15ways.

iii. $\frac{n!}{2!*(n-2)!} * \frac{(n-2)!}{2!*(n-2-2)!} * \ldots\ldots * \frac{2!}{2!} = \frac{n!}{2!*\frac{n}{2}} / \left(\frac{n}{2}\right)!$ ways to pair up the sandwiches. Every time we add two more crackers to the pool to calculate the total arrangements of pairing, we are adding C(n, 2) * n / 2 more possibilities to the pairing arrangements.

We are adding 2 more cookies to the pool which makes our total choices of picking two more cookies are having C(n, 2) choices, since we are only care about the pairs but not the orders cookies in the tray, which we can multiply by a factor of n / 2 because if n == 2, C1 C2 makes no difference to C2 C1, and n == 4, C1C2/C3C4 to C1C2/C4C3 or C1C2/C3C4 to C2C1/C3C4 makes no difference if only looking at the pair options but not the combinations of "n" distinct cookies.