Text, letter

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Section 2: Counting

1. Choose 4 from 9 letters, each selection is independent event,

The total amount of sequences of length 4 are 9 \* 9 \* 9 \* 9 = 9 ^4.

1. Sequences of length 4 contain exactly THREE different letters can be converted to subtracting sequences of length of 4 with exactly 1/exactly 2/exactly4 letter(s), total possible sequences are 9^4 from question above.

Exactly 1: 9 (AAAA, BBBB …….IIII)

Exactly 2: C (9, 2) \* 2^4 – 2 (9 choose 2, and each slot can be filled with any of the two, subtracting the common one to avoid overcounting)

Exactly 4 = 4 different: 9 \* 8 \* 7 \* 6 = P(9, 4), order matters here

9^4 – 9 – (C (9, 2) \* 2^4 – 2) – 9\*8\*7\*6 = 6561 – (36\*16 -2) – 3024 = 6561 – 9 – 574 – 3024 = 2954

1. A puzzle can be arranged as 9! Possible ways.

Only scenarios that “BAD” or “DAB” existed are at diagonal, horizontal and vertical.

Diagonal: 2 \* 2 (two diagonals with forward and backward spelling)

Horizontal: 2 \* 3 (three horizontals with forward and backward spelling)

Vertical: 2 \* 3 (three verticals with forward and backward spelling)

After grouping “BAD” or “DAB” to one, the rest of 6 letters (CEFGHI) can be chosen randomly, so 6!

The number of puzzles do not contain the word BAD is

9! – 14 \*6!

Section 3: Probability

NS= not spam, S = spam, F = contains FREE, NF = not contains FREE

1. P(S) = 0.9, P(F|S) = 0.5, P(F|NS) = 0.2

P(NS) = 1 – P(S) = 0.1

According to the complement of an event and conditional probability,

If F and S are both events in the same sample space (Emails), the probability of F and the probability of not F (NF) still sum to 1, even when conditioned on the event S.

So does apply to F and NS.

So that

P(NF|S) = 1 – 0.5 = 0.5

P(NF|NS) = 1 – 0.2 = 0.8

P(S|F) = P(F|S) \* P(S) / (P(F|S) \* P(S) + P(F|NS) \* P(NS)) = 0.5 \* 0.9 / ( 0.5 \* 0.9 + 0.2 \* 0.01) = 45/47

1. 9 intersections can be split into 3 categories, Vertex with 2, 3, 4 intersections.

Vertex with 2 intersections: 1, 3, 7, 9

Vertex with 3 intersections: 2, 4, 6, 8

Vertex with 4 intersections: 5

|E| = 4/9 \* 2 + 4/9 \* 3 + 1/9 \* 4 = 24 / 9 = 8/3

1. If two distinct intersections are chosen at random from the Vertex above, the probability can be calculated as the |Event| / |Space|

|Space| = |S| = C (9, 2) = 36

Event that the chosen cities are not adjacent to each other is any two cities in categories of cities with 2 and 3 intersections.

For example, 1 and 3 are not adjacent, 2 and 8 are not adjacent

So, choose 2 from Vertex with 2 intersections, we have C (4, 2);

Or choose 2 from Vertex with 3 intersections, we have C (4, 2);

Or choose 1 from Vertex with 2 intersections and choose 1 from Vertex with 4 intersections: C (4,1) \* 1

The probability of choosing two cities that are adjacent are (C (9,2) – C (4, 2) – C (4, 2) – C (4, 1))/ C (9, 2) = (36 – 6 – 6 – 4) / 36 = 5/ 9

1. Corner vertex intersections are 1, 3, 7, 9.

|E| = |E1| + |E2| +|E3| +|E4| +|E5| +|E6| +|E7|

Each selection is independent to each other, so the |E1| = | E2| =|E3| =|E4| =|E5| =|E6| =|E7|

|E| = 7|E1|

Let corner vertex intersection selection to be 1, and not selected to be 0

|E1| = 4/ 9 \* 1 + 5/9 \* 0 = 4/ 9

|E| = 7 \* 4/9 = 28/9

Section 4: Sequence and Recurrences

1. M1 ~M5 = 2, 7, 15, 26, 40

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Then we can generate a quadratic polynomial function that has a, b, c in a form of Mn = an^2 + bn + c

M1 = a + b + c = 2

M2 = 4a + 2b + c = 7

M3 = 9a + 3b + c = 15

M2 – M1 = 3a + b = 5

M3 – M2 = 5a + b = 8

M3 – M2 – (M2 - M1) = 2a = 3

After arranging the equations above, we get a = 3/2, b = 1/2, c = 0. Then Mn = (3/2)\*n^2 + (1/2)\*n

1. It’s a quadratic polynomial functions, since the highest degree of the variable in the function is 2.
2. Known the recursive definition of Mn where Mn = Mn-1 + 3n – 1 is true for all n >= 1,

let S(n) be the statement that Mn = (3/2)\*n^2 + (1/2)\*n

Base case, S(1) is true because:

M1 = 3/2 + 1/ 2 = 2 = 2 which is given

Inductive step:

We assume S(k) is true for some fixed k >= 1. We wish to prove S(k+1)

Consider:

M(k+1) = 3/2 \* (k + 1)^2 + 1/2 \* (k + 1)

= 3/2 \* (k^2 + 1 + 2k) + 1/2 \* k + 1/2

= 3/2 \*k^2 + 3/2 + 3k + 1/2 \* k + 1/ 2

= 3/2 \* k^2 + 1/2 \* k + 3k + 2 known: 3/2\*k^2+ 1/2\*k = Mk

= Mk + 3k + 3 – 3 + 2

=Mk + 3(k + 1) - 1

Which S(k+1) corresponds to the recursive definition and is true,

Thus, by axiom of induction, since S(1) and all k >=1, S(k) -> S(k+1), we have that for all n>=1, S(n).

Section 5: Graphs

1.

Diagram

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2. Adjacency-list representation for graph above.

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3. Let Mn be the total edges of a n X n array, we can find that for all n >=2.

M(2) = 4,

M(3) = 12,

M(4) = 24,

M(5) = 40…..

The second order of the difference between each M is a constant which equals to 4.

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Then we can generate a quadratic polynomial function that has a, b, c in a form of Mn = an^2 + bn + c

M2 = 4a + 2b + c = 4

M3 = 9a + 3b + c = 12

M4 = 16a + 4b + c = 24

After arranging the equations above, we get a = 2, b = -2, c = 0. Then Mn = 2n^2 - 2n

If we plug 3 back in to the closed-form formula of Mn, we get M3 = 2 \* 3^2 – 2 \* 3 = 18 – 6 = 12, which corresponds to the correct amount of the edges we counted on the graph.