CSCI 270 Homework #1

Due Date: Wednesday, January 22nd, 11:59pm

All homework will be submitted electronically through Gradescope.

1. Order the following functions from smallest asymptotic running time to greatest. Additionally, identify all pairs of functions f_i and f_j where $f_i(n) = \Theta(f_j(n))$, or explicitly state that none exist. Explain your answers.

   (a) $f_a(n) = n!$
   (b) $f_b(n) = \sqrt{\log n^{20}}$
   (c) $f_c(n) = 2^{n^3}$
   (d) $f_d(n) = n \cdot \frac{\ln n}{\ln 2}$
   (e) $f_e(n) = n(\log n)^{20}$
   (f) $f_f(n) = n^{\log n}$
   (g) $f_g(n) = \log \sqrt{n^{20}}$
   (h) $f_h(n) = \lfloor \pi^e \rfloor !$
   (i) $f_i(n) = \prod_{i=1}^{n} \frac{i+1}{i}$
   (j) $f_j(n) = \frac{n}{f_h(n)}$

2. Suppose $f(n) = O(s(n))$, and $g(n) = O(r(n))$. Prove or disprove (by giving a counterexample) the following claims:

   (a) $f(n) - g(n) = O(s(n) - r(n))$
   (b) $\frac{f(n)}{g(n)} = O(\frac{s(n)}{r(n)})$
   (c) if $g(n) = O(f(n))$, then $f(n) + g(n) = O(s(n))$
   (d) if $r(n) = O(s(n))$, then $g(n) = O(f(n))$

3. Suppose $x_1, x_2, x_3, x_4 \in R$, $x_1 + x_2 = x_3 + x_4 = 1$, and $x_1x_3 + x_2x_4 > 1$. Use proof by contradiction to show that at least one of x_1, x_2, x_3, x_4 is negative. **Hint:** What do you know about the product $(x_1 + x_2) \cdot (x_3 + x_4)$?

4. Suppose there are n locations in a city which only has one-way roads. For each pair of locations, there is exactly one road between them (so for any pair of locations u,v, there is either a road from u to v, or a road from v to u, but not both). A **nexus** is a location which can be reached from **any** location in the city, using no more than 2 roads. Use induction to prove that there must be a **nexus**. **Hint:** Think about the set S_b of cities which can reach nexus b using exactly 1 road, and explicitly refer to this set in your proof.