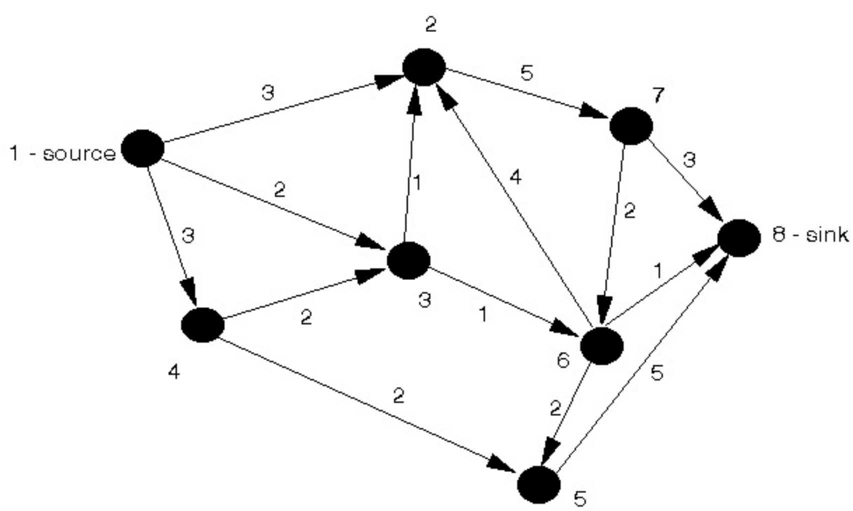
CSCI 270 Lecture 19: Network Flow

We have a weighted directed graph $G = (V, E)$, where the edge are "pipes" and their weight is their flow capacity. These values measure the rate in which fluid/data/etc can flow through the pipe. What is the maximum rate that flow can be pushed from s to t in the above graph?

Each edge e has capacity $c(e)$. An **s-t flow** is a function f which satisfies:

1. $0 \leq f(e) \leq c(e)$, for all e (capacity).
2. $\sum_{(x,v)} f((x, v)) = \sum_{(v,y)} f((v, y))$, for all nodes $v - \{s, t\}$ (conservation).

The value of a flow is $v(f) = \sum_{(s,x)} f((s, x))$



Minimum Cut

An $s - t$ **cut** is a partition of the nodes into sets $(A, V - A)$, where $s \in A$ and $t \in V - A$.

The **cutset** is the set of edges whose origin is in A and destination is in $V - A$.

The value of an $s - t$ cut is equal to the sum of the capacities of the edges in the cutset.

Problem Statement: Find the minimum cost $s - t$ cut.

1. What is the value of the mincut in the previous graph?
2. Anyone think this is a coincidence?

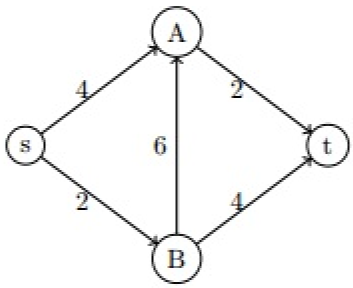
Weak Duality: Given a flow f and an $(A, V - A)$ cut, $v(f) \leq$ the value of the cut.

Corollary to Weak Duality: If $v(f) =$ the value of the cut, then f is a max-flow, and $(A, V - A)$ is a mincut.

The remaining question is, can this always be done? Is there always a flow and cut with equal values?

Let's try to solve the max-flow problem using a greedy algorithm. We'll just choose arbitrary $s - t$ paths that have a remaining capacity, and route as much flow as possible along this path.

Given the original graph G, and the flow so far f, you can construct the residual graph G_f, which contains two types of edges.



- Forward edges (u, v), which have capacity equal to the original capacity $c(u, v)$ minus the flow along the edge $f(u, v)$.
- Backward edges (v, u), which have capacity equal to the flow along the edge in the opposite direction $f(u, v)$.

Forward edges indicate you can still augment the flow along that edge. Backward edges indicate that you can "take back" your choice to push flow along that edge and find a better solution.

Why was this a valid thing to do? Are the two rules of flow maintained if we push flow "backwards"?

The **Ford-Fulkerson** algorithm looks for an **augmenting path** on the residual graph (that is, a path from s to t where you can push more flow), pushes the maximum possible amount of flow along that path, and repeats until there are no more paths.

Augmenting Path Theorem: f is a max flow iff there are no s-t paths in G_f.

Max-flow Min-cut Theorem: The value of the max flow equals the value of the mincut.

We will prove the following three statements are equivalent:

1. There is a cut $(S, V - S)$ such that $v(f) =$ the capacity of the cut.
2. f is a max flow.
3. There is no s-t path in the residual graph G_f.

$1 \to 2$: We've already proven this.

$2 \to 3$: Proof by contraposition. If there is an s-t path in G_f, then we can augment the flow, which means f is not a max flow. $\neg 3 \to \neg 2$, which means $2 \to 3$.

$3 \to 1$: Let f be a flow with no augmenting path. Let A be the set of vertices reachable in G_f from s. Note that $s \in A$ and $t \in V - A$. This forms a cut of value equal to the flow!

Thus Ford-Fulkerson optimally solves Network Flow.

Algorithm for Min-Cut
1: Run Ford-Fulkerson on the graph.
2: Create the resulting residual graph G_f
3: Let $A =$ the set of nodes reachable from s in G_f
4: Return $(A, V - A)$

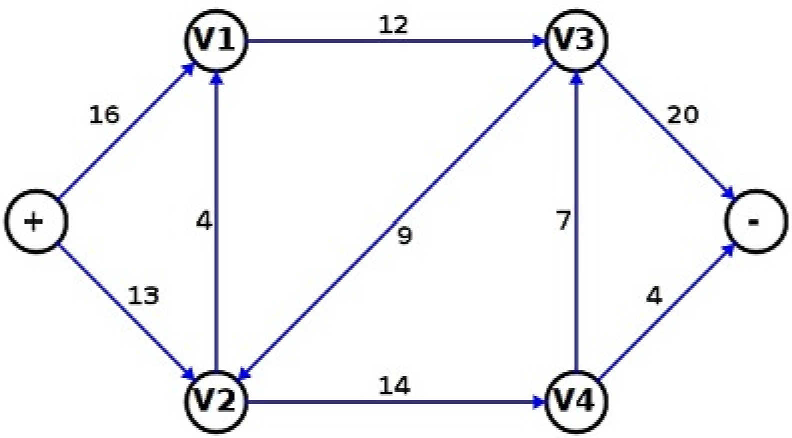
Integrality Theorem: If all capacities are integers, there is a max flow f where every value $f(e)$ is an integer.

Let's analyze the running time of Ford-Fulkerson. The capacities of edges will be integers between 1 and C.

1. Each time we do an augmentation, what is the minimum amount the total flow increases by?
2. How many total augmentations might we need?
3. How long should it take to find an augmenting path?
4. What is the runtime?
5. Is this a polynomial runtime?
6. Is it actually possible for the runtime to be this bad?
7. Any ideas on how to improve our algorithm?

Capacity-Scaling

Let $G_f(\Delta)$ be the subgraph of G_f consisting only of edges with capacity $\geq \Delta$.



- What would $G_f(\Delta)$ look like for this graph if $\Delta = 16$?
- Are there any $s - t$ paths on $G_f(16)$?
- What would $G_f(8)$ look like?
- Are there any $s - t$ paths on $G_f(8)$?
- What does G_f look like after we've augmented the flow?
- What does $G_f(8)$ look like at this point?
- What does $G_f(4)$ look like?

Ford-Fulkerson With Capacity-Scaling

1: Let $\Delta = 2^i$ for max i such that $\Delta \leq C$.
2: While $\Delta \geq 1$
3: Augment the current max flow using Ford-Fulkerson on $G_f(\Delta)$.
4: $\Delta = \frac{\Delta}{2}$