(A) **4 Points** — Formulate the problem of identifying the mathematical model of the plant in the set-membership framework, on the basis of the following information:

- the plant can be modeled by a nonlinear Hammerstein system where the discrete-time linear time-invariant subsystem is described by the following transfer function

$$\boxed{G(z) = \frac{\beta_1 z + \beta_2}{z + \alpha_1}}$$

  and the nonlinear block $\mathcal{N}$ is such that:

$$\boxed{x_t = \gamma_1 u_t + \gamma_2 u_t^3}$$

- The following a-priori information on the system are available:
  * $\gamma_1 = 0.25$
  * $G(z)$ is a stable system;
- A set of 20 input-output data pair (available in the data file *S:\LRIC\data_exam_2A_hammer_ver2*) has been collected to describe the input-output behavior of the plant.
- Output data sequence is known to be corrupted by additive noise signals $\eta(t)$, having absolute value of amplitude bounded by $\Delta_\eta = 0.02$.

(B) **3 Points** — Provide a mathematical formulation of the optimization problems to be solved for the computation of the PUIs.

(C) **5 Points** — Provide a accurate description of the data structure to be built in order to solve the problem with the sparsePOP software.

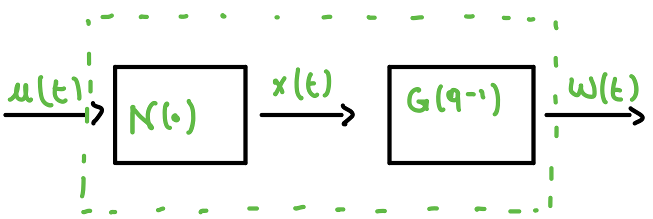
(D) **6 Points** — Write a MATLAB script for the computation of the PUIs.

Point A

- A priori information on the system

$$G(z) = \frac{\beta_1 + \beta_2 z^{-1}}{1 + \alpha_1 z^{-1}}$$

$$x(t) = \gamma_1 u(t) + \gamma_2 u^3(t)$$



- $\gamma_1 = 0{,}25$

- The system is stable

$|d_1| \le 1-\gamma \qquad \gamma = 0{,}001$

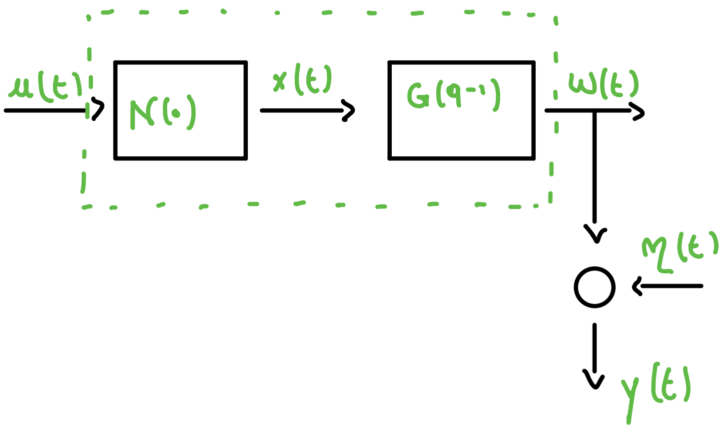
$\Downarrow$

$$\begin{cases} d_1 \le 1-\gamma \Rightarrow -d_1 + 1 - \gamma \ge 0 \\ d_1 \ge \gamma - 1 \Rightarrow d_1 + 1 - \gamma \ge 0 \end{cases}$$

- A-priori info on the noise

The noise enters the system as an output error (OE), so the output is corrupted by noise while the input is exactly known.

This structure can be represented as:



$$y(t) = \eta(t) + w(t)$$

The bound on the noise error is:

$$|\eta(t)| \le \Delta_\eta = 0{,}02$$

- I/O data collection

Now we can put all the information together to find the FPS D_θ

In this region we will find our true values for the parameters $d_1, \beta_1, \beta_2, \gamma_1, \gamma_2$

$$D_\theta = \{ \theta \in \mathbb{R}^3, \gamma \in \mathbb{R}^2 : w(t) = G(q^{-1})\, x(t), \quad \forall t = 1, \dots, N \quad *$$

$$x(t) = \gamma_1 u(t) + \gamma_2 u^2(t),$$

$$\gamma_1 = 0{,}25$$

$$-d_1 + 1 - \gamma \geq 0$$

$$d_1 + 1 - \gamma \geq 0$$

$$|\eta(t)| \leq \Delta\eta = 0{,}02 \quad \forall t = 1, \dots, N \}$$

* $w(t) = G(q^{-1})\, x(t)$

$$w(z) = \frac{\beta_1 + \beta_2 z^{-1}}{1 + d_1 z^{-1}}\, x(z)$$

$$\Downarrow$$

$$y(z) - \eta(z) = \frac{\beta_1 + \beta_2 z^{-1}}{1 + d_1 z^{-1}}\, x(z)$$

$$\Downarrow$$

$$y(t) - \eta(t) + d_1 y(t-1) - d_1 \eta(t-1) = \beta_1 x(t) + \beta_2 x(t-1)$$

$$\Downarrow$$

$$y(t) - \eta(t) + d_1 y(t-1) - d_1 \eta(t-1) - \beta_1 x(t) - \beta_2 x(t-1) = 0$$

Since we have more unknowns $(\theta_i, \eta(i), x_i)$ then we can extend D_θ and pass to the EFPS $D_{\theta,\eta,\varepsilon}$

$D_{\theta,\eta,x} = \{ \theta \in \mathbb{R}^4,\ x \in \mathbb{R}^N,\ \eta \in \mathbb{R}^N :$

$$y(t) - \eta(t) + d_1 y(t-1) - d_1 \eta(t-1) - \beta_1 x(t) - \beta_2 x(t-1) = 0, \quad \forall t = 2, \dots, N$$

$$x(t) = \gamma_1 u(t) + \gamma_2 u^2(t), \quad \forall t = 1, \dots, N$$

$$\gamma_1 = 0{,}25$$

$$-d_1 + 1 - \gamma \geq 0$$

$$d_1 + 1 - \gamma \geq 0$$

$$|\eta(t)| \leq \Delta\eta = 0{,}02 \quad \forall t = 1, \dots, N \}$$

Since the D_θ has a complex shape, we manage the solution through SM ID.

We aim to find for each parameter a maximum and a minimum value:

$$PUI_{\theta_j} = [\underline{\theta}_j, \overline{\theta}_j]$$

Point B

To find $\underline{\theta_3}$ and $\overline{\theta_3}$ we must solve two optimization problem of the form:

$\underline{\theta_3} = \min \theta_3$

s.t.

$y(t) - \eta(t) + d_1 y(t-1) - d_1 \eta(t-1) - \beta_1 x(t) - \beta_2 x(t-1) = 0, \ \forall t = 2, \dots, N$

$x(t) = \gamma_1 u(t) + \gamma_2 u^2(t), \ \forall t = 1, \dots, N$

$\gamma_1 = 0{,}25$

$-d_1 + 1 - \gamma \geq 0$

$d_1 + 1 - \gamma \geq 0$

$|\eta(t)| \leq \Delta\eta = 0{,}02 \quad \forall t = 1, \dots, N \}$

This problem is bilinear (= subjet to polynomial problems). Thus, it leads to non-convex solution for the PUI (we cannot find the minimum/maximum global optima but only local)

For this reason we can use a relaxed approach based on "moment theory" and "sum of square decomposition method"

We will find a set of convex problems that will depend on the order of relaxation δ. As this parameter is big, as we converge to the convex hull and we will be more sure that the PUI we find are the real ones

The starting value is: $\delta_{min} = \left\lceil \frac{\max \deg}{2} \right\rceil = 1$

Point C

By fixing δ, we can find through SPARSEPOP the value of the PUI

The structure (supportive matrix and coef vector) are:

$y(t) - \eta(t) + d_1 y(t-1) - d_1 \eta(t-1) - \beta_1 x(t) - \beta_2 x(t-1) = 0$

@ $t = 2$

	α_1	β_1	β_2	γ_2	$\eta(1)$	$\eta(2)$	...	$\eta(N)$	$X(1)$	$X(2)$	...	$X(N)$
$y(t)$	0	0	0	0	0	0	...	0	0	0	...	0
$-\eta(t)$	0	0	0	0	0	1	...	0	0	0	...	0
$\alpha_1 y(t-1)$	1	0	0	0	0	0	...	0	0	0	...	0
$-\alpha_1 \eta(t-1)$	1	0	0	0	1	0	...	0	0	0	...	0
$-\beta_1 x(t)$	0	1	0	0	0	0	...	0	0	1	...	0
$-\beta_2 x(t-1)$	0	0	1	0	0	0	...	0	1	0	...	0

$$\text{coef-vec} = \begin{bmatrix} y(t) & -1 & y(t-1) & -1 & -1 & -1 \end{bmatrix}^T$$

@t=1

$$x(t) - \gamma_1 u(t) - \gamma_2 u^3(t) = 0$$

	α_1	β_1	β_2	γ_2	$\eta(1)$	$\eta(2)$	...	$\eta(N)$	$X(1)$	...	$X(N)$
$x(t)$	0	0	0	0	0	0	...	0	1	...	0
$-0,25\, u(t)$	0	0	0	0	0	0	...	0	0	...	0
$-\gamma_2 u^3(t)$	0	0	0	1	0	0	...	0	0	...	0

$$\text{coef-vect} : \begin{bmatrix} 1 & -0,25\, u(t) & -(u(t))^3 \end{bmatrix}^T$$

$$-\alpha_1 + 1 - \gamma \geq 0$$

	α_1	β_1	β_2	γ_1	γ_2	$\eta(1)$	$\eta(2)$	$\dots$	$\eta(N)$	$X(1)$	$X(2)$	$\dots$	$X(N)$
$1-\gamma$	0	0	0	0	0	0	0	$\dots$	0	0	0	$\dots$	0
$-\alpha_1$	1	0	0	0	0	0	0	$\dots$	0	0	0	$\dots$	0

$$\text{coef\_vec} = \begin{bmatrix} 1-\gamma & -1 \end{bmatrix}^T$$

$\alpha_1 + 1 - \gamma \geq 0$

	α_1	β_1	β_2	γ_1	γ_2	$\eta(1)$	$\eta(2)$	$\dots$	$\eta(N)$	$X(1)$	$X(2)$	$\dots$	$X(N)$
$1-\gamma$	0	0	0	0	0	0	0	$\dots$	0	0	0	$\dots$	0
α_1	1	0	0	0	0	0	0	$\dots$	0	0	0	$\dots$	0

$$\text{coef\_vec} = \begin{bmatrix} 1-\gamma & 1 \end{bmatrix}^T$$

Point D

We can run the code and obtain the following results:

$$PUI = \begin{bmatrix} -0{,}90671 & -0{,}90349 \\ 0{,}20915 & 0{,}33101 \\ -0{,}12073 & -0{,}076917 \\ 0{,}4436 & 0{,}74106 \end{bmatrix} \qquad \theta_c = \begin{bmatrix} -0{,}9051 \\ 0{,}27008 \\ -0{,}098824 \\ 0{,}59233 \end{bmatrix}$$

The obtained results are good because θ_{min} and θ_{max} have the same sign, $\theta_{min} < \theta_{max}$ and the interval is small. The results are obtained with $\delta = 2$

$$G = \frac{0{,}27012 - 0{,}09882}{z - 0{,}9051}$$