A Momentum-accelerated Hessian-vector-based Latent Factor Analysis Model Supplementary File

Weiling Li, Xin Luo, *Senior Member*, *IEEE*, Huaqiang Yuan, and MengChu Zhou, *Fellow*, *IEEE*

This is the supplementary file for the paper entitled *a momentum-accelerated hessian-vector-based latent factor analysis model*. Some supplementary proofs for convergence analysis are put in this file and cited in the manuscript.

A. Proof of Proposition 1

Proof. First of all, we reformulate the objective $S(\boldsymbol{x})$ as:

$$S(\boldsymbol{x})=\frac{1}{2}\sum_{(u,i)\in\boldsymbol{K}} z_{u,i}^2(\boldsymbol{x})=\sum_{(u,i)\in\boldsymbol{K}}\varepsilon_{u,i}(\boldsymbol{x})=\sum_{(u,i)\in\boldsymbol{K}}\varepsilon_{u,i}\left(\boldsymbol{x}_{(u)},\ \boldsymbol{x}_{(i)}\right), \tag{A1}$$

where $z_{u,i}=r_{u,i}-\boldsymbol{x}_{(u)}\boldsymbol{x}_{(i)}$, and $\varepsilon_{u,i}(\boldsymbol{x}_{(u)}, \boldsymbol{x}_{(i)})$ denotes the partial objective defined on $r_{u,i}$ with $\boldsymbol{x}_{(u)}$ and $\boldsymbol{x}_{(i)}$, respectively. Thus, according to *Definition* 2 and *Lemma* 3, $H_S(\boldsymbol{x})$ is Lipschitz continuous if each $H_\varepsilon(\boldsymbol{x})$ is Lipschitz continuous.

$$\begin{aligned}H_\varepsilon(\boldsymbol{x}_1)\text{-}H_\varepsilon(\boldsymbol{x}_2)&=J_\varepsilon(\boldsymbol{x}_1)^{\mathrm{T}}J_\varepsilon(\boldsymbol{x}_1)\text{-}J_\varepsilon(\boldsymbol{x}_2)^{\mathrm{T}}J_\varepsilon(\boldsymbol{x}_2)\\&=\left(r_{u,i}-\boldsymbol{x}_1\boldsymbol{x}_{(i)}^{\mathrm{T}}\right)^2\boldsymbol{x}_{(i)}^{\mathrm{T}}\boldsymbol{x}_{(i)}-\left(r_{u,i}-\boldsymbol{x}_2\boldsymbol{x}_{(i)}^{\mathrm{T}}\right)^2\boldsymbol{x}_{(i)}^{\mathrm{T}}\boldsymbol{x}_{(i)}\\&=\left(2r_{u,i}-\boldsymbol{x}_2\boldsymbol{x}_{(i)}^{\mathrm{T}}-\boldsymbol{x}_1\boldsymbol{x}_{(i)}^{\mathrm{T}}\right)\left(\boldsymbol{x}_2-\boldsymbol{x}_1\right)\boldsymbol{x}_{(i)}^{\mathrm{T}}\left(\boldsymbol{x}_{(i)}^{\mathrm{T}}\boldsymbol{x}_{(i)}\right)\\&\Rightarrow\left\|H_\varepsilon(\boldsymbol{x}_1)\text{-}H_\varepsilon(\boldsymbol{x}_2)\right\|_F\le\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|\left\|\left(2r_{u,i}-\boldsymbol{x}_2\boldsymbol{x}_{(i)}^{\mathrm{T}}-\boldsymbol{x}_1\boldsymbol{x}_{(i)}^{\mathrm{T}}\right)\boldsymbol{x}_{(i)}^{\mathrm{T}}\left(\boldsymbol{x}_{(i)}^{\mathrm{T}}\boldsymbol{x}_{(i)}\right)\right\|_F\\&\le\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|\left\|2z_{\max}(\boldsymbol{x})\cdot\boldsymbol{x}_{(i)}^{\mathrm{T}}\left(\boldsymbol{x}_{(i)}^{\mathrm{T}}\boldsymbol{x}_{(i)}\right)\right\|_F.\end{aligned} \tag{A2}$$

Then we replace $\boldsymbol{x}_{(u)}$ in $\varepsilon_{u,i}(\boldsymbol{x})$ with arbitrary vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ which are independent of each other, and fix $\boldsymbol{x}_{(u)}$'s coupled LF vector $\boldsymbol{x}_{(i)}$ in the non-convex term. Recall that $H_\varepsilon(\boldsymbol{x})\approx J_\varepsilon(\boldsymbol{x})^{\mathrm{T}}J_\varepsilon(\boldsymbol{x})$, we have the inferences in (A2). By applying the inferences to $\forall u\in\boldsymbol{U}$ and $i\in\boldsymbol{I}$, we conclude that $H_S(\boldsymbol{x})$ is Lipschitz continuous. □

B. Proof of Theorem 1

Proof. Note that $\nabla S(\boldsymbol{x})$ is twice continuously differentiable. Then following the Taylor's *Theorem*, we have:

$$\nabla S(\boldsymbol{x}_k)-\nabla S(\boldsymbol{x}_{k\text{-}1})=\left(J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})+\rho\mathbf{I}\right)\cdot\boldsymbol{\gamma}_k+\int_0^1\left[\nabla S(\boldsymbol{x}_{k\text{-}1}+t\boldsymbol{\gamma}_k)-\nabla S(\boldsymbol{x}_{k\text{-}1})\right]\cdot\boldsymbol{\gamma}_k dt. \tag{A3}$$

By combining (A3) with (13), we have:

$$\nabla S(\boldsymbol{x}_k)=\nabla S(\boldsymbol{x}_{k\text{-}1})+\left(J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})+\rho\mathbf{I}\right)\cdot\Delta\boldsymbol{x}_k+\left(J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})+\rho\mathbf{I}\right)\times\mu\left(\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right)+o\left(\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\right). \tag{A4}$$

Note that $\Delta\boldsymbol{x}_k$ is the solution of $\boldsymbol{l}=\nabla S(\boldsymbol{x}_{k\text{-}1})+(J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})+\rho\mathbf{I})\Delta\boldsymbol{x}_k$, which is achieved by CGD. Hence, we induce that:

$$\nabla S(\boldsymbol{x}_k)=\boldsymbol{l}+\mu\cdot\left(J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})+\rho\mathbf{I}\right)\times\left(\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right)+o\left(\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\right). \tag{A5}$$

Recall that CGD terminates with the following condition:

$$\left\|\boldsymbol{l}\right\|\le\tau_k\left\|\nabla S(\boldsymbol{x}_k)\right\|, \tag{A6}$$

where $\{\tau_k\}$ is the forcing sequence with $0<\tau_k<1$ for all k. By substituting (A6) into (A5) and taking the norms of both sides of the equation, we achieve the following inequality:

$$\left\|\nabla S(\boldsymbol{x}_k)\right\|\le\tau_{k\text{-}1}\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|+\mu\left(\left\|J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}}J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F+\left\|\rho\mathbf{I}\right\|_F\right)\times\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\|+o\left(\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\right). \tag{A7}$$

Note that $J_F(\boldsymbol{x})$ is formulated as:

$$J_F(\boldsymbol{x})=\left[-\nabla z_{u,i}(\boldsymbol{x})^{\mathrm{T}}\Big|_{(u,i)\in\boldsymbol{K}}\right]. \tag{A8}$$

Thus, we have the following inference:

$$\nabla S(\boldsymbol{x})=\sum_{(u,i)\in\boldsymbol{K}}z_{u,i}(\boldsymbol{x})\nabla z_{u,i}(\boldsymbol{x})=-J_F(\boldsymbol{x})^{\mathrm{T}}z(\boldsymbol{x}), \tag{A9}$$

where $z(\boldsymbol{x})=(z_1(\boldsymbol{x}), z_2(\boldsymbol{x}), ..., z_{|K|}(\boldsymbol{x}))^{\mathrm{T}}$. Suppose that $\|z(\boldsymbol{x})\|=1/h$ for $h>0$. Based on (A8) and (A9), we have following inference:

$$\begin{aligned}\left\|\nabla S(\boldsymbol{x}_k)\right\| &\le \tau_{k\text{-}1}\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\| + \mu\left\|J_F(\boldsymbol{x}_{k\text{-}1})^{\mathrm{T}} J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F \times \left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + \mu\left\|\rho\mathbf{I}\right\|_F\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + o\left(\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\right)\\ &\le \tau_{k\text{-}1}\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\| + \mu h\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F \times \left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + \mu\left\|\rho\mathbf{I}\right\|_F\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + o\left(\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\right).\end{aligned} \tag{A10}$$

Since ρ is an adjustable parameter, we have two cases:

Case 1.

$$\left\|\nabla S(\boldsymbol{x}_k)\right\| \le \left(\tau_{k\text{-}1} + 2\mu h\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + o(1)\right)\times\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|,\ if\ \left\|\rho\mathbf{I}\right\|_F \le h\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F. \tag{A11}$$

Case 2.

$$\left\|\nabla S(\boldsymbol{x}_k)\right\| \le \left(\tau_{k\text{-}1} + \theta\mu h\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + o(1)\right)\times\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|,\ if\ \left\|\rho\mathbf{I}\right\|_F > h\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F. \tag{A12}$$

where $\theta = \dfrac{2\left\|\rho\mathbf{I}\right\|_F}{h\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F}$. Note that θ is naturally bigger than 2 since $\dfrac{\left\|\rho\mathbf{I}\right\|_F}{h\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F} > 1$ in Case 2, hence we have:

$$\begin{aligned}\left\|\nabla S(\boldsymbol{x}_k)\right\| &\le \left(\tau_{k\text{-}1} + \theta\mu h\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| + o(1)\right)\times\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|\\ &\le \left(\frac{1+\tau_{k\text{-}1}}{2} + \theta\mu h\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\|\right)\times\left\|\nabla S(\boldsymbol{x}_{k\text{-}1})\right\|.\end{aligned} \tag{A13}$$

Note that when $\boldsymbol{x}_k$ is close enough to $\boldsymbol{x}^*$, the $o(1)$ term is bounded by $(1-\tau_k)/2$ and recall that $\tau_k \le \tau$. Hence, according to *Lemma* 2, MHLFA model converges if there exists k such that $\nabla S(\boldsymbol{x}_k)=0$, i.e.,

$$\begin{aligned}&\frac{1+\tau}{2} + \theta\mu h\left(\left\|J(\boldsymbol{x}_{k\text{-}1})\right\|_F + 1\right)\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\| \le 1\\ &\Rightarrow \mu \le \frac{1-\tau}{2\theta h\left(\left\|J(\boldsymbol{x}_{k\text{-}1})\right\|_F + 1\right)\left\|\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k-2}\right\|} = \frac{1-\tau}{2\theta h\left(\left\|J(\boldsymbol{x}_{k\text{-}1})\right\|_F + 1\right)\left\|\boldsymbol{\gamma}_{k\text{-}1}\right\|}.\end{aligned} \tag{A14}$$

Note that $\boldsymbol{\gamma}_k=-\Delta\boldsymbol{x}_k-\mu(\boldsymbol{x}_{k\text{-}1}-\boldsymbol{x}_{k\text{-}2})$ forms a sequence where

$$\left\{\boldsymbol{\gamma}_k = \sum_{i=1}^{k}\mu^{k\text{-}i}\Delta\boldsymbol{x}_i\right\}. \tag{A15}$$

By reasonably assuming that $\mu=1/2$, we have:

$$\begin{aligned}\left\|\boldsymbol{\gamma}_{k\text{-}1}\right\| &= \left\|\sum_{i=1}^{k\text{-}1}\frac{1}{2^{k\text{-}1\text{-}i}}\Delta\boldsymbol{x}_i\right\|\\ &= \left\|\Delta\boldsymbol{x}_{k\text{-}1} + \frac{1}{2}\Delta\boldsymbol{x}_{k-2} + \frac{1}{2^2}\Delta\boldsymbol{x}_{k-3} + \cdots + \frac{1}{2^{k-2}}\Delta\boldsymbol{x}_1\right\|\\ &\le \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-2}}\right)\left\|\Delta\boldsymbol{x}_m\right\|\\ &\le \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-2}}\right)2G\left\|\nabla S(\boldsymbol{x}_m)\right\| = 4G\left\|\nabla S(\boldsymbol{x}_m)\right\|,\end{aligned} \tag{A16}$$

where $\|\Delta\boldsymbol{x}_m\|$ and $\|\nabla S(\boldsymbol{x}_m)\|$ are the maximum values of $\|\Delta\boldsymbol{x}_k\|$ and $\|\nabla S(\boldsymbol{x}_k)\|$, respectively. By combining (A14) with (A16) and adopting *Lemma* 4, we have:

$$\begin{aligned}\frac{1-\tau}{2\theta h\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|\left\|\boldsymbol{\gamma}_{k\text{-}1}\right\|} &\ge \frac{1-\tau}{8\theta Gh\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|\left\|\nabla S(\boldsymbol{x}_{\mathrm{m}})\right\|}\\ &= \frac{1-\tau}{8\theta Gh\left\|J_F(\boldsymbol{x}_{k\text{-}1})\right\|_F\left\|J_F(\boldsymbol{x}_{\mathrm{m}})^{\mathrm{T}} z(\boldsymbol{x}_{\mathrm{m}})\right\|}\\ &\ge \frac{1-\tau}{8\theta G\left|\boldsymbol{K}\right|^2 q^2}.\end{aligned} \tag{A17}$$

Note that (A17) indicates that with the following condition:

$$\kappa = \min\left(\frac{1-\tau}{8\theta G\left|\boldsymbol{K}\right|^2 q^2}, \frac{1}{2}\right). \tag{A18}$$

Hence, *Theorem* 1 holds. □