0.1 Full FDTD for the 1D Case

We're starting with the previously [REF] derived six equations that stem from Maxwell. They're listed here again for convenience:

$$\partial_y E_z - \partial_z E_y = -\mu \partial_t H_x \tag{1}$$
$$\partial_z E_x - \partial_x E_z = -\mu \partial_t H_y \tag{2}$$
$$\partial_x E_y - \partial_y E_x = -\mu \partial_t H_z \tag{3}$$
$$\partial_y H_z - \partial_z H_y = \varepsilon \partial_t E_x + \sigma E_x \tag{4}$$
$$\partial_z H_x - \partial_x H_z = \varepsilon \partial_t E_y + \sigma E_y \tag{5}$$
$$\partial_x H_y - \partial_y H_x = \varepsilon \partial_t E_z + \sigma E_z \ . \tag{6}$$

The operator ∂_x is a shorthand for the partial derivative $\frac{\partial}{\partial x}$.

Again, we're only considering a discretization along one axis (*z-axis*), but now we won't force the conditions a plane wave. Thus spatial derivatives along other axes (*x-axis* and *y-axis*) will be disregarded, but the electric and magnetic field will not be constricted to a single axis. This reasoning leads to the following modified equations:

$$\partial_z E_y = \mu \partial_t H_x \tag{7}$$
$$\partial_z E_x = -\mu \partial_t H_y \tag{8}$$
$$0 = -\mu \partial_t H_z \tag{9}$$
$$-\partial_z H_y = \varepsilon \partial_t E_x + \sigma E_x \tag{10}$$
$$\partial_z H_x = \varepsilon \partial_t E_y + \sigma E_y \tag{11}$$
$$0 = \varepsilon \partial_t E_z + \sigma E_z \ . \tag{12}$$

Note that equations 3 and 6 [REF] can be dropped, since they only state that there is time-variance in the electric and magnetic field along the z-axis. The discretization is done in a staggered manner (*Yee-grid*) as in the case of the plane wave and leads to the equations

$$\frac{E_y^{n+1/2}(k+1) - E_y^{n+1/2}(k)}{\Delta z} = \mu \frac{H_x^{n+1}(k+1/2) - H_x^{n}(k+1/2)}{\Delta t} \ , \tag{13}$$
$$\frac{E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)}{\Delta z} = -\mu \frac{H_y^{n+1}(k+1/2) - H_y^{n}(k+1/2)}{\Delta t} \ , \tag{14}$$
$$-\frac{H_y^{n}(k+1/2) - H_y^{n}(k-1/2)}{\Delta z} = \varepsilon \frac{E_x^{n+1/2}(k) - E_x^{n-1/2}(k)}{\Delta t} + \sigma E_x^n(k) \ , \tag{15}$$
$$\frac{H_x^{n}(k+1/2) - H_x^{n}(k-1/2)}{\Delta z} = \varepsilon \frac{E_y^{n+1/2}(k) - E_y^{n-1/2}(k)}{\Delta t} + \sigma E_y^n(k) \ , \tag{16}$$

where a new unknown variable $E_x^n(k)$ appears – originally – due to the contribution of the current density term in Ampere's circuital law [REF]. This new unknown variable can be eliminated by averaging between two time steps as in

$$E_x^n(k) = \frac{E_x^{n+1/2}(k) - E_x^{n-1/2}(k)}{2} \ . \tag{17}$$

Rearranging gives the iterative algorithm:

$$H_x^{n+1}(k+1/2) = H_x^n(k+1/2) + \frac{\Delta t}{\mu \Delta z}\left(E_y^{n+1/2}(k+1) - E_y^{n+1/2}(k)\right) \tag{18}$$

$$H_y^{n+1}(k+1/2) = H_y^n(k+1/2) - \frac{\Delta t}{\mu \Delta z}\left(E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)\right) \tag{19}$$

$$E_x^{n+1/2}(k) = E_x^{n-1/2}(k) - \frac{\Delta t}{\varepsilon \Delta z + \frac{1}{2}\sigma \Delta z \Delta t}\left(H_y^n(k+1/2) - H_y^n(k-1/2)\right) \tag{20}$$

$$E_y^{n+1/2}(k) = E_y^{n-1/2}(k) + \frac{\Delta t}{\varepsilon \Delta z + \frac{1}{2}\sigma \Delta z \Delta t}\left(H_x^n(k+1/2) - H_x^n(k-1/2)\right)\ . \tag{21}$$

In each time step, first [20] and [21] are determined for the electric field, then the magnetic field is determined with [18] and [19]. For numerical stability, the CFL condition

$$\Delta t \leq \frac{\Delta z}{c} \tag{22}$$

has to also be fulfilled. For this case we arrive at the following factors

$$\frac{\Delta t}{\mu \Delta z} = \frac{1}{Z_0 \mu_r} \tag{23}$$

$$\frac{\Delta t}{\varepsilon \Delta z + \frac{1}{2}\sigma \Delta z \Delta t} = \frac{\frac{\Delta z}{c}}{\varepsilon \Delta z + \frac{1}{2}\sigma \Delta z \frac{\Delta z}{c}} = \frac{Z_0}{\varepsilon_r + \frac{1}{2}Z_0 \sigma \Delta z} \tag{24}$$

for the spatial derivative terms in 18-21.

0.2 Simulations