~~TRG~~- Lista 2

Exercício 1

a) $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

$\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+(y/x)^2} \cdot \left(\dfrac{-y}{x^2}\right) = -\dfrac{y}{x^2+y^2}$

$\dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2}$

$J = \dfrac{1}{(x^2+y^2)^{3/2}} \begin{vmatrix} x & y \\ -y & x \end{vmatrix} = \dfrac{1}{\sqrt{x^2+y^2}}$

Singular em $x=0$, $y=0$.

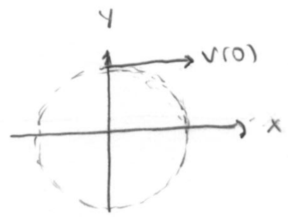
b) $\dfrac{\partial \xi}{\partial x} = \dfrac{1}{x}$, $\dfrac{\partial \xi}{\partial y} = 0$

$\dfrac{\partial \eta}{\partial x} = 0$, $\dfrac{\partial \eta}{\partial y} = 1$

$J = \begin{vmatrix} 1/x & 0 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{x}$

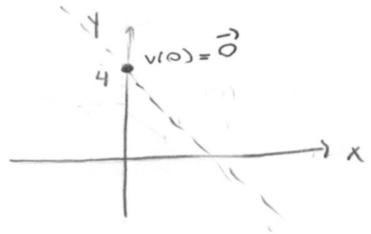
Singular na reta $x=0$.

Exercício 2



a) $V = \left(\dfrac{dx}{d\lambda}, \dfrac{dy}{d\lambda}\right) = (\cos\lambda, -\operatorname{sen}\lambda) \Rightarrow V(0) = (1,0)$

$x(0)=0$, $y(0)=1$



b) $V = (2s, -2s) \Rightarrow V(0) = (0,0)$

$x(0) = 0$, $y(0) = 4$

$y = -s^2 + 4 = -x + 4$

Exercício 3

a) $\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases} \Rightarrow \begin{cases} dx = \cos\theta\, dr - r\operatorname{sen}\theta\, d\theta \\ dy = \operatorname{sen}\theta\, dr + r\cos\theta\, d\theta \end{cases} \Rightarrow$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$\therefore g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$, $g^{ab} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$

b) A única derivada não nula da métrica é $\partial_1 g_{22} = 2r$ ($\partial_r g_{\theta\theta}$).

Símbolos de Christoffel não nulos:

$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} (\cancel{\partial_\theta g_{r\theta}} + \cancel{\partial_\theta g_{r\theta}} - \partial_r g_{\theta\theta}) = \frac{1}{2}(-2r) = -r$

$\Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta} (\partial_r g_{\theta\theta} + \cancel{\partial_\theta g_{r\theta}} - \cancel{\partial_\theta g_{r\theta}}) = \dfrac{1}{2r^2} \cdot 2r = \dfrac{1}{r} = \Gamma^\theta_{\theta r}$

Exercício 4

- $\alpha(\lambda) = (\lambda, (\lambda-1)^2, -\lambda)$
  $V = (1, 2(\lambda-1), -1)$

$$\left\{ \begin{array}{l} \alpha(1) = (1,0,-1) = P \\ V|_P = (1,0,-1) \end{array} \right.$$

Em p: $V|_p(f) = v^i \partial_i f = (1,0,-1)\cdot(2,1,0) = 2$.

$\partial_1 f = 2x,\ \partial_2 f = 2y - z,\ \partial_3 f = -y \Rightarrow \partial_i f|_p = (2,1,0)$

- $\beta(\lambda) = (\cos\lambda, \operatorname{sen}\lambda, -1)$
  $V = (-\operatorname{sen}\lambda, \cos\lambda, 0)$

$$\left\{ \begin{array}{l} \beta(0) = (1,0,-1) = P \\ V|_P = (0,1,0) \end{array} \right.$$

$\Rightarrow V|_p(f) = (0,1,0)\cdot(2,1,0) = 1$.

Exercício 5

$x = r\cos\theta,\ y = r\operatorname{sen}\theta$

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\theta\,\partial_x + \operatorname{sen}\theta\,\partial_y$$

$$\partial_\theta = \frac{\partial x}{\partial\theta}\partial_x + \frac{\partial y}{\partial\theta}\partial_y = -r\operatorname{sen}\theta\,\partial_x + r\cos\theta\,\partial_y$$

Exercício 6

$$\left| \begin{array}{l} t = a\operatorname{senh}\lambda \\ x = a\cosh\lambda \end{array} \right. \rightarrow \begin{array}{l} dt = \operatorname{senh}\lambda\, da + a\cosh\lambda\, d\lambda \\ dx = \cosh\lambda\, da + a\operatorname{senh}\lambda\, d\lambda \end{array} \rightarrow ds^2 = -a^2 d\lambda^2 + da^2$$

$$g_{ij} = \begin{bmatrix} -a^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} -a^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

Os símbolos de Christoffel não nulos são dados por: $\Gamma^{\lambda}_{a\lambda} = \Gamma^{\lambda}_{\lambda a} = \frac{1}{a}$, $\Gamma^{a}_{\lambda\lambda} = a$.

Exercício 7

$$U^\alpha \nabla_\alpha V_\beta = U^\alpha \nabla_\alpha (g_{\beta\gamma} V^\gamma) = U^\alpha g_{\beta\gamma} \nabla_\alpha V^\gamma = g_{\beta\gamma} W^\gamma = W_\beta .$$

Exercício 8

$$V^\mu V_\mu = 1 \Rightarrow 0 = \nabla_\nu (V^\mu V_\mu) = (\nabla_\nu V^\mu) V_\mu + V^\mu (\nabla_\nu V_\mu)$$

Mas: $(\nabla_\nu V^\mu) V_\mu = \nabla_\nu (g^{\mu\sigma} V_\sigma) V_\mu = g^{\mu\sigma} V_\mu \nabla_\nu V_\sigma = V^\sigma \nabla_\nu V_\sigma = V^\mu \nabla_\nu V_\mu$

Substituindo na 1.ª linha, segue o resultado.

Exercício 9

A transformação inversa é dada por:

$$t = t' \ , \quad x = \sqrt{x'^2 + y'^2} \cos(\phi' + wt') \ , \quad y = \sqrt{x'^2+y'^2} \ \text{sen}(\phi' + wt') , \quad z' = z ,$$

com $\tan\phi' = y'/x' \Rightarrow \phi' = \phi - wt$. Temos que:

$$\phi' = \arctan\frac{y'}{x'} \Rightarrow d\phi' = \frac{1}{1+(y'/x')^2}\left(-\frac{y'dx'}{x'^2} + \frac{dy'}{x'}\right) = \frac{-y'dx' + x'dy'}{x'^2+y'^2}$$

$$dx = \frac{x'dx' + y'dy'}{\sqrt{x'^2+y'^2}}\cos(\phi'+wt') - \sqrt{x'^2+y'^2}\ \text{sen}(\phi'+wt')\left[\frac{-y'dx'+x'dy'}{x'^2+y'^2} + w\,dt'\right]$$

$$dy = \frac{x'dx' + y'dy'}{\sqrt{x'^2+y'^2}}\ \text{sen}(\phi'+wt') + \sqrt{x'^2+y'^2}\cos(\phi'+wt')\left[\frac{-y'dx'+x'dy'}{x'^2+y'^2} + w\,dt'\right]$$

Substituindo na expressão para o elemento de linha do espaço de Minkowski:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= -dt'^2 + dz'^2 + \left(\frac{x'dx' + y'dy'}{\sqrt{x'^2+y'^2}}\right)^2 + \left[\sqrt{x'^2+y'^2}\left(\frac{-y'dx'+x'dy'}{x'^2+y'^2} + w\,dt'\right)\right]^2$$

$$= -\left[1 - w^2(x'^2+y'^2)\right]dt'^2 + dx'^2 + dy'^2 + dz'^2 - 2wy'\,dx'dt' + 2wx'\,dy'dt'$$

$$\therefore \ g_{\mu\nu} = \begin{bmatrix} -1+w^2r'^2 & -wy' & wx' & 0 \\ -wy' & 1 & 0 & 0 \\ wx' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad r' = \sqrt{x'^2+y'^2} = r$$

$$\Rightarrow \ g^{\mu\nu} = \begin{bmatrix} -1 & -wy' & wx' & 0 \\ -wy' & 1-w^2y'^2 & w^2x'y' & 0 \\ wx' & w^2x'y' & 1-w^2x'^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercício 10

$$\Gamma^{\mu}{}_{\rho\nu} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\rho\nu} \right)$$

$$\Rightarrow \Gamma^{\mu}{}_{\mu\nu} = \frac{1}{2} g^{\mu\sigma} \left(\cancel{\partial_\mu g_{\sigma\nu}} + \partial_\nu g_{\sigma\mu} - \cancel{\partial_\sigma g_{\mu\nu}} \right) = \frac{1}{2} g^{\rho\sigma} \partial_\nu g_{\rho\sigma} .$$

Exercício 11

- $[U,V]^\sigma = U^\mu \nabla_\mu V^\sigma - V^\mu \nabla_\mu U^\sigma = -\left(V^\mu \nabla_\mu U^\sigma - U^\mu \nabla_\mu V^\sigma \right) = -[V,U]^\sigma$

- $[U,V]^\sigma = U^\mu \left(\partial_\mu V^\sigma + \Gamma^\sigma_{\mu\nu} V^\nu \right) - V^\mu \left(\partial_\mu U^\sigma + \Gamma^\sigma_{\mu\nu} U^\nu \right)$

$$= U^\mu \partial_\mu V^\sigma - V^\mu \partial_\mu U^\sigma + \underbrace{\Gamma^\sigma_{\mu\nu} \left(U^\mu V^\nu - U^\nu V^\mu \right)}_{=0,\ \text{pela simetria de } \Gamma}$$

- $[U,fV]^\sigma = U^\mu \nabla_\mu (f V^\sigma) - f V^\mu \nabla_\mu U^\sigma$

$$= U^\mu \left(\nabla_\mu f \cdot V^\sigma + f \nabla_\mu V^\sigma \right) - f V^\mu \nabla_\mu U^\sigma$$

$$= f \underbrace{\left(U^\mu \nabla_\mu V^\sigma - V^\mu \nabla_\mu U^\sigma \right)} + V^\sigma U^\mu \nabla_\mu f$$

$$= f\, [U,V]^\sigma + V^\sigma U^\mu \nabla_\mu f$$

$\therefore [U,fV] = f\,[U,V] + V\, U^\mu \nabla_\mu f$

Exercício 12

$$\frac{dx^\mu}{dt} = (1,0,0,\omega) \Rightarrow \Delta\tau = \int \sqrt{\left(1 - \frac{2GM}{R}\right) \cdot 1 - R^2 \operatorname{sen}^2\theta\, \omega^2}\, dt$$

$$= \sqrt{1 - \frac{2GM}{R} - R^2 \operatorname{sen}^2\theta\, \omega^2}\ \Delta t$$

A quantidade dentro da raiz quadrada deve ser adimensional. Mas:

$$\left[\frac{GM}{R} \right] = \frac{(m^3 \cdot kg^{-1} \cdot s^{-2}) \cdot kg}{m} = \frac{m^2}{s^2} , \quad [R^2 \operatorname{sen}^2\theta\, \omega^2] = \frac{m^2}{s^2} ,$$

logo é preciso dividir ambos os termos por c^2:

$$\Delta\tau = \sqrt{1 - \frac{2GM}{Rc^2} - \frac{R^2 \operatorname{sen}^2\theta\, \omega^2}{c^2}}\ \Delta t \xrightarrow{R^2\omega^2 \ll GM/r} \sqrt{1 - \frac{2GM}{Rc^2}}\ \Delta t$$

Em um período: $(t = 24\,hs)$

$$\Delta\tau - 24\,hs = \left(\sqrt{1 - \frac{2GM}{Rc^2}} - 1 \right) 24 \cdot 3600\ s \simeq -6 \times 10^{-5}\ s$$

$G = 6.674 \times 10^{-11}\ m^3\, kg^{-1}\, s^{-2}$

$M = 5.972 \times 10^{24}\ kg$

$R = 6.371 \times 10^{6}\ m$