HOMEWORK 3

Data Science II

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1 Question 2

1.1

We should choose $\Theta > 1$ so that all the points in the first cluster having m_1 points are detected accurately since the diameter is around 1. Also, we should have $\Theta < 1.75$ so that both the clusters are separated as the distance between them is around 1.75. Therefore, I would choose $\Theta = 1.25$ or $\Theta = 1.5$.

1.2

The first eigenvalue we get is 0. We will get $m_1 - 1$ eigenvectors which look like:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} -1 & 0 & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

$$\dots$$

$$\begin{bmatrix} -1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

and similarly $m_2 - 1$ eigenvectors for the second cluster.
We will also get m_1 as an eigenvalue with eigenvector

$$\begin{bmatrix} 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}^T$$

where we have 1 for the entries corresponding to the first cluster.
And m_2 as an eigenvalue with eigenvector

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \end{bmatrix}^T$$

where we have 1 for the entries corresponding to the second cluster.
For this problem, we have $k = 2$. So we should use the last two eigenvectors corresponding to the two non-zero eigenvalues. The rest of the $(m_1 + m_2 - k)$ eigenvalues are 0 and have the corresponding eigenvectors above.

1.3

$$L = D^{-1/2} A D^{-1/2}$$

If $\lambda, \vec{v}$ are eigenvalue and eigenvector for L, then $L\vec{v} = \lambda\vec{v}$.

$$\vec{v} = [\sqrt{d_{11}}, \sqrt{d_{22}}, \dots, \sqrt{d_{nn}}]^T$$

$$\lambda = 1$$

$$L\vec{v} = D^{-1/2} A D^{-1/2} \vec{v}$$

$$D^{-1/2}\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{d_{11}}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{d_{22}}} & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{nn}}} \end{bmatrix} \begin{bmatrix} \sqrt{d_{11}} \\ \sqrt{d_{22}} \\ \vdots \\ \sqrt{d_{nn}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d_{11} \\ d_{22} \\ \vdots \\ d_{nn} \end{bmatrix}$$

$$D^{-1/2} \begin{bmatrix} d_{11} \\ d_{22} \\ \vdots \\ d_{nn} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{d_{11}}} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{d_{22}}} & 0 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{nn}}} \end{bmatrix} \begin{bmatrix} d_{11} \\ d_{22} \\ \vdots \\ d_{nn} \end{bmatrix} = \begin{bmatrix} \sqrt{d_{11}} \\ \sqrt{d_{22}} \\ \vdots \\ \sqrt{d_{nn}} \end{bmatrix}$$

Hence $L\vec{v} = 1\vec{v}$.

1.4

To show : $P^{\infty} = D^{-1/2}\vec{v_1}\vec{v_1}^T D^{1/2}$.
Here $\vec{v}_1$ is the eigenvector corresponding to eigenvalue 1 of $L = D^{-1/2}AD^{-1/2}$.
From the previous problem, we know :

$$\vec{v}_1 = [\sqrt{d_{11}}, \sqrt{d_{22}}, \dots, \sqrt{d_{nn}}]^T$$

$$P = D^{-1}A = \begin{bmatrix} \frac{1}{d_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{d_{22}} & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & \frac{1}{d_{nn}} \end{bmatrix} \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} p_{ij} = \frac{a_{ij}}{d_{ii}} \end{bmatrix}$$

$$P^t = (D^{-1}A)^t = D^{-1/2}(D^{-1/2}AD^{-1/2})^t D^{1/2} = D^{-1/2}L^t D^{1/2}$$

Now since L is the normalized graph Laplacian, it has the property that all the eigen values except the leading one are less than 1. Using this and diagonalization, we get

$$L = PDP^{-1} = \begin{bmatrix} \vec{v_1}\vec{v_2}\dots\vec{v_n} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v_1}\vec{v_2}\dots\vec{v_n} \end{bmatrix}^T$$

$$L^{\infty} = PD^{\infty}P^{-1} = P \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & lim_{t\longrightarrow\infty}(\lambda_2)^t & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & lim_{t\longrightarrow\infty}(\lambda_n)^t \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{bmatrix} P^{-1}$$

$$P\begin{bmatrix}1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ & \vdots & & \\ 0 & 0 & \dots & 0\end{bmatrix}P^{-1} = \begin{bmatrix}\vec{v_1}\vec{v_2}\dots\vec{v_n}\end{bmatrix}\begin{bmatrix}1 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ & \vdots & & \\ 0 & 0 & \dots & 0\end{bmatrix}\begin{bmatrix}\vec{v_1}\vec{v_2}\dots\vec{v_n}\end{bmatrix}^T = \begin{bmatrix}\vec{v_1}\vec{v_2}\dots\vec{v_n}\end{bmatrix}\begin{bmatrix}\vec{v_1}\\ 0\\ \vdots\\ 0\end{bmatrix} = \vec{v_1}\vec{v_1}^T$$

Hence

$$P^{\infty} = D^{-1/2}L^{\infty}D^{1/2} = D^{-1/2}\vec{v_1}\vec{v_1}^T D^{1/2}$$

References