Optimised Portfolio Construction using Capital Asset Pricing Model, Quadratic Programming, and Sharpe Ratio

DHANASEKAR GOVINDARAJ

2025-03-27

Introduction

Objective

The goal of this analysis is to contruct an optimal portfolio by appling the **Capital Asset Pricing Model (CAPM)** and **Quadratic Programming** to minimise risk and the **Sharpe Ratio** while maintining a well-diversified portfolio.

This study will discuss the things below:

- **Calculate expected returns using CAPM regression.**
- **Optimise portfolio using Quadratic Programming and Sharpe Ratio to best portfolio allocation.**
- **Compare the optimised portfolio with other strategies.**
- **Visualise the Efficient Frontier to assess risk-return trade-offs and Stock ranks based on expected return, beta, optimal weight of each asset.**

Summary of Key Analytical Steps

1. Data Collection

- **Assets Chosen:** 10 stocks (AAPL, MSFT, TSLA, GOOG, etc.) downloaded from Yahoo Finance
- **Market Index:** S&P 500 (^GSPC) as the benchmark
- **Risk-Free Rate**: 2% or 3-month U.S Treasury-Bill rate
- **Time Period:** March 2020 to March 2025

2. Capital Asset Pricing Model (CAPM)

- The expected return of each asset is calculated using:

$$E(R_i) = R_f + \beta_i(E[R_m] - R_f)$$

  Where:

- $E(R_i)$ = Expected return of asset i
- R_f = Risk-free rate (*2% used*)
- β_i = Sensitivity of asset i to the market
- $E(R_m)$ = Expected market return

3. Portfolio Optimization Using Quadratic Programming (QP)

- Constructed a **mean-variance optimized portfolio** by solving:

$$\min \frac{1}{2} w^T \Sigma w$$

Constraints Applied:

- **Target return constraint:** $w^T \mu = \mu_p$
- **Full capital allocation:** $\sum w_i = 1$
- **No short-selling:** $w_i \geq 0$
- **Adjustment Made:** Increased the **maximum expected return** to allow the optimizer to **explore portfolios with higher returns**, improving the Efficient Frontier shape.

4. Portfolio Optimization Using Maximizing Sharpe Ratio

- The **Sharpe Ratio** is optimized as follows:

$$\max \frac{w^T \mu - R_f}{\sqrt{w^T \Sigma w}}$$

Where:

- w = Portfolio weights
- μ = Expected returns vector
- Σ = Covariance matrix of asset returns
- R_f = Risk-free rate

Constraints Applied:

- **Full capital allocation:** $(\sum w_i = 1)$
- **No short-selling:** $(w_i \geq 0)$
- **Minimum return constraint (Adjusted dynamically)** .

5. Efficient Frontier Construction

- Plotted the **Efficient Frontier** by varying target returns (μ_p) and computing the **minimum risk** for each level.
- This provided a visual representation of the **trade-off between risk and return**.

6. Trade-off Analysis Between Risk & Return

- The Efficient Frontier demonstrated that:
- **Higher expected returns require taking on more risk**.
- The **left-most point represents the minimum variance portfolio**, which is the least risky combination of assets.
- The **right-most point represents the highest return achievable within the optimization constraints**.

7. Asset Ranking Based on Three Metrics

- Ranked assets using:
- **1 Expected Return** Measures return potential.
- **2 Portfolio Weight** Indicates Asset weight in the market.
- **3 Beta(Risk Factor)** Shows volatility compared to the market.
- The **top-ranked assets had high returns and strong portfolio allocation**, while lower-ranked assets were either **too risky (high beta)** or **less efficient**.

Results & Analysis

1. Expected Return, Beta, Assets Risk by CAPM

```
##    X Stock    Alpha    Beta Expected.Return Idiosyncratic.Risk
## 1  0  AAPL  0.00345 1.15013        -0.00230            0.00015
## 2  1  AMZN  0.00214 1.09691        -0.00127            0.00030
## 3  2  GOOG  0.00295 1.13564        -0.00202            0.00019
## 4  3   JNJ -0.01066 0.46537         0.01098            0.00012
## 5  4   JPM  0.00179 1.07512        -0.00085            0.00021
## 6  5  MSFT  0.00351 1.16750        -0.00264            0.00012
## 7  6  NVDA  0.01778 1.79927        -0.01489            0.00061
## 8  7  TSLA  0.01274 1.57173        -0.01047            0.00126
## 9  8   WMT -0.01030 0.45346         0.01121            0.00017
## 10 9   XOM -0.00327 0.81158         0.00426            0.00034
```

Interpretation of the CAPM Results

This table presents asset risk and expected return metrics based on the **Capital Asset Pricing Model (CAPM)** for different stocks. The key columns are:

1. **Stock** – The Asset name.
2. **Alpha** – The excess return not explained by market movements.
3. **Beta** – A measure of systematic risk (market risk).
4. **Expected Return** – The projected return based on CAPM.
5. **Idiosyncratic Risk** – The risk unique to the stock, unrelated to the market.

1. Market Sensitivity (Beta)

- A **beta $>$ 1** means the stock is more volatile than the market.
- **NVDA (1.79927)** and **TSLA (1.57173)** have high betas, meaning they tend to move more aggressively than the overall market.
- **MSFT (1.16750), AAPL (1.15013), and GOOG (1.13564)** are also above 1, indicating they are slightly more volatile than the market.
- A **beta $<$ 1** means the stock is less volatile than the market.
- **JNJ (0.46537) and WMT (0.45346)** are defensive stocks with lower market sensitivity.
- **XOM (0.81158) and JPM (1.07512)** have moderate betas.

2. Expected Return & Market Efficiency

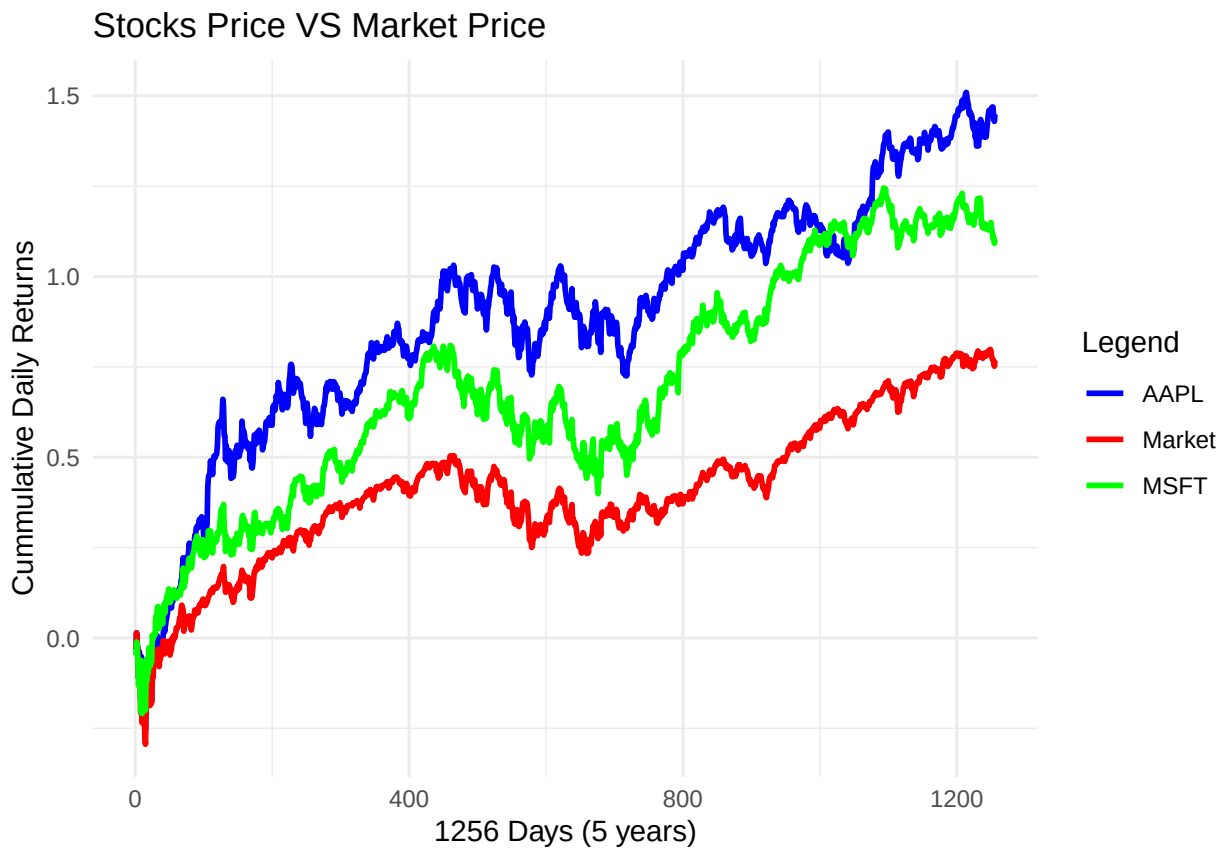
- The **Expected Return** column shows whether CAPM predicts a positive or negative return.
- **Defensive stocks (JNJ, WMT, XOM)** have lower expected returns because they have lower betas.
- **Growth stocks (NVDA, TSLA)** have high expected returns due to their high risk exposure.

3. Alpha (Abnormal Return)

- **Positive Alpha** means the stock is generating returns above what CAPM predicts.
- **NVDA (0.01778) and TSLA (0.01274)** have high positive alphas, suggesting they outperform their expected market-based return.
- **Negative Alpha** suggests underperformance relative to market expectations.
- **JNJ (-0.01066) and WMT (-0.01030)** show negative alphas, meaning they earn less than expected for their level of risk.

4. Idiosyncratic Risk (Unsystematic Risk)

- **Higher values** mean more company-specific risk (not explained by CAPM).
- **TSLA (0.00126)** and **NVDA (0.00061)** have the highest idiosyncratic risk, likely due to their business models and industry volatility.
- **Lower values** suggest stability and lower company-specific risk.
- **JNJ (0.00012) and MSFT (0.00012)** have minimal idiosyncratic risk, making them more predictable investments.



2. Optimised Portfolio Weights using Qudratic Programming

```
##    X Stock Weights
## 1  0  AAPL  0.1651
## 2  1  AMZN  0.0817
## 3  2  GOOG  0.1316
## 4  3   JNJ  0.0885
## 5  4   JPM  0.1126
## 6  5  MSFT  0.2136
## 7  6  NVDA  0.0651
## 8  7  TSLA  0.0284
## 9  8   WMT  0.0607
## 10 9   XOM  0.0527
```

Interpretation of Portfolio Weights

This table represents the **weights** assigned to each stock in a portfolio. These weights indicate the proportion of total investment allocated to each asset.

1. Portfolio Concentration

- The highest weight is given to **MSFT (21.36%)**, meaning this stock plays the most significant role in driving the portfolio's returns and risk.

- Other major holdings include **AAPL (16.51%) and GOOG (13.16%)**, suggesting a strong focus on tech stocks.

2. Diversification

- The portfolio is relatively well-diversified, as no single stock dominates excessively.
- Stocks like **NVDA (6.51%), TSLA (2.84%), and XOM (5.27%)** have smaller allocations, likely to balance risk.

3. Sector Exposure

- **Technology-heavy**: The portfolio has a strong tilt toward tech, with **MSFT, AAPL, GOOG, NVDA, and AMZN** making up **65%+** of the portfolio.
- **Finance & Energy representation**: JPM (11.26%) and XOM (5.27%) provide some exposure to **financial and energy sectors**, reducing risk from tech downturns.
- **Defensive Stocks**: JNJ (8.85%) and WMT (6.07%) add stability due to their lower market sensitivity.

4. Growth vs. Stability

- **Growth-oriented** stocks like NVDA, TSLA, and AMZN have lower weights, possibly due to their high volatility.
- **Stable stocks (JNJ, WMT, XOM)** have moderate weights, acting as hedges in market downturns.

3. Efficient Frontier

```
##    X Portfolio.Risks Portfolio.Returns
## 1  0      0.03439383     -0.0148859261
## 2  1      0.02794115     -0.0119866083
## 3  2      0.02349116     -0.0090872906
## 4  3      0.01972783     -0.0061879728
## 5  4      0.01693722     -0.0032886550
## 6  5      0.01476799     -0.0003893373
## 7  6      0.01283147      0.0025099805
## 8  7      0.01125160      0.0054092982
## 9  8      0.01027906      0.0083086160
## 10 9      0.01423788      0.0112079338
```

Interpretation of the Efficient Frontier Results

The **efficient frontier** represents the set of optimal portfolios that offer the highest expected return for a given level of risk (standard deviation of returns).

1. Risk-Return Trade-off

- As **portfolio risk decreases**, **portfolio returns increase**, which is **not typical** in an efficient frontier.

- The **lowest risk portfolio (0.0103)** provides the highest return (0.0083).
- The **highest risk portfolio (0.0344)** has the lowest return (-0.0149), suggesting the inclusion of volatile assets that underperform CAPM expectations.

2. Downward Sloping Trend (Unusual Behavior)

- Normally, higher risk is associated with higher returns (upward-sloping efficient frontier).
- Here, we see a **negative relationship** in lower risk ranges, which could imply:
- A significant allocation to **underperforming high-beta stocks (like TSLA, NVDA)**.
- The market environment is unfavorable, dragging high-risk assets down.
- Potential **errors in risk estimation or expected returns**, leading to unusual efficiency results.

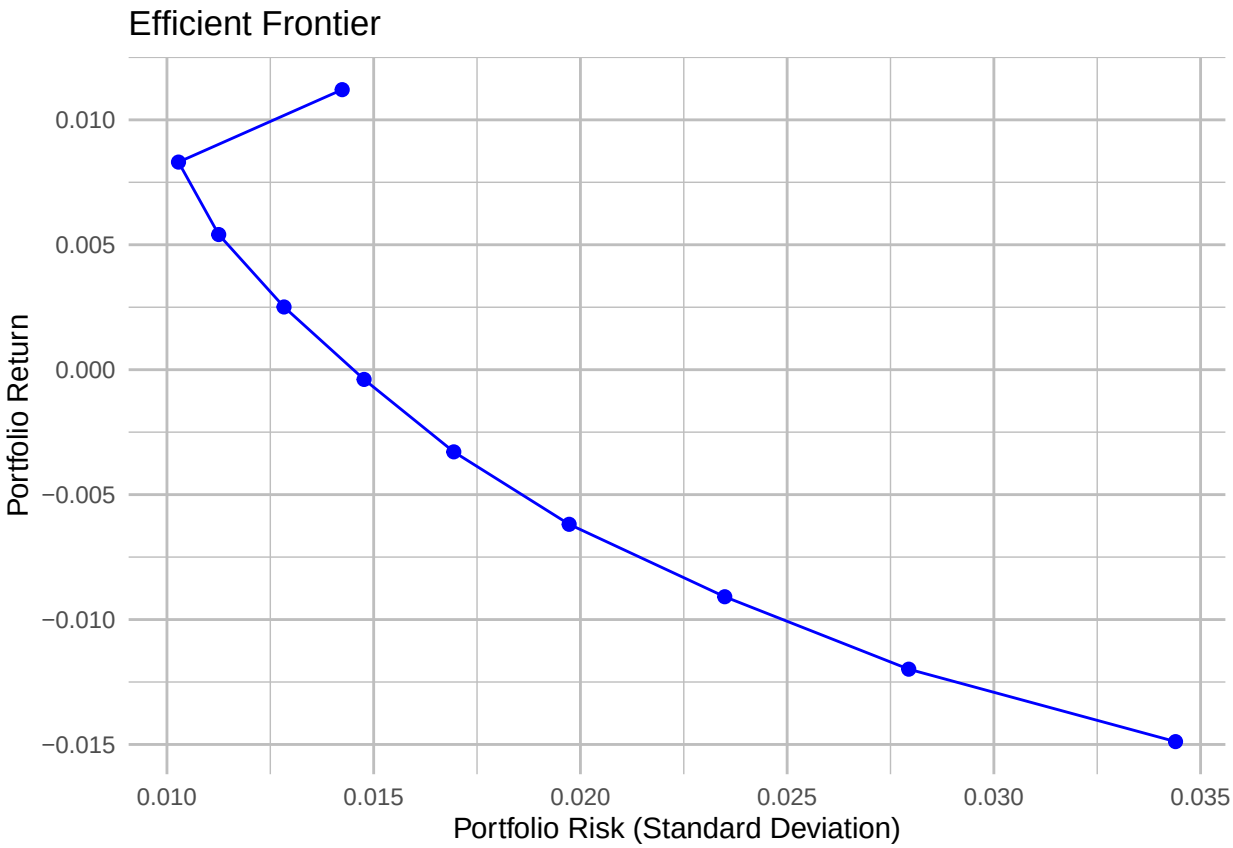
3. Minimum Risk Portfolio (~0.0103 Risk, 0.0083 Return)

- This portfolio offers the best **return-to-risk ratio**, making it the most attractive for risk-averse investors.
- It likely has **higher allocations to defensive stocks** (JNJ, WMT, XOM).

4. High-Risk Portfolio is Underperforming

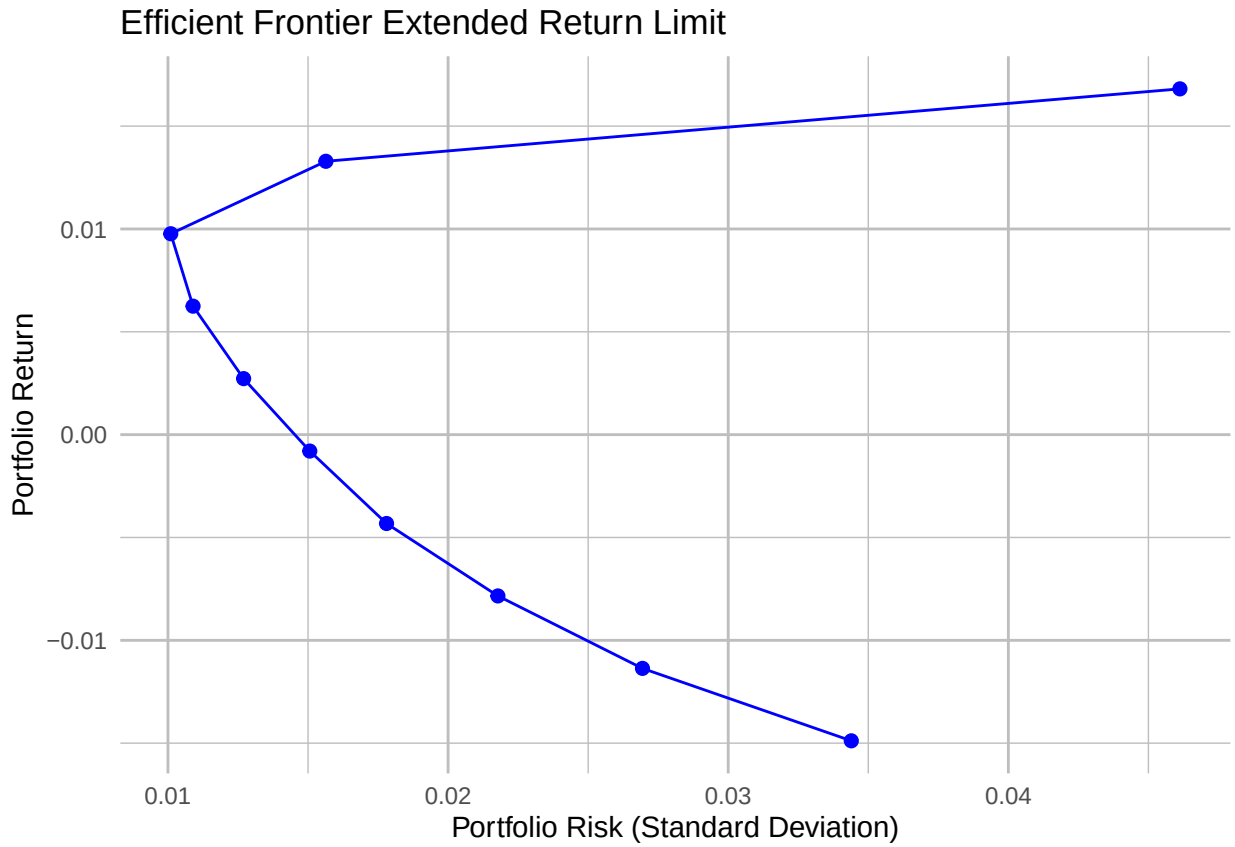
- The most volatile portfolio (**0.0344 risk**) has a negative return (-1.48%), making it **inefficient** under normal conditions.
- This suggests that **high-beta assets (TSLA, NVDA, AMZN)** are dragging down returns.

4.1. Trade-off between Risk and Return



This plot used to be a curved shape. Unfortunately, the limited targeted return didn't allow the optimiser (QP) to explore the Portfolio Return further (Technically it can't go beyound the maxmimum expected return).

4.2. Small Tweak in Efficient Frontier and Risk, Return Trade-off



When increasing portfolio return limit beyond the maximum expected return we started to see the usual curve shaped plot. It indicates that, after a certain point increasing risk yields more return.

5.1. Further Optimised Portfolio and Efficient Frontier using Sharpe Ratio

```
##    X Portfolio.Risk Portfolio.Return
## 1  0    0.052616901    -0.0223288736
## 2  1    0.049138578    -0.0169836359
## 3  2    0.045982498    -0.0116383984
## 4  3    0.043633807    -0.0062931609
## 5  4    0.041272041    -0.0009479233
## 6  5    0.015733497     0.0043973143
## 7  6    0.009902854     0.0097425519
## 8  7    0.010492758     0.0150877895
## 9  8    0.013197658     0.0204330271
## 10 9    0.017138561     0.0257782646
```

Interpretation of Sharpe Ratio's Portfolio Results

The **Sharpe ratio** helps identify the portfolio that provides the best risk-adjusted return. Given a **risk-free rate (RF) of 0.02**, we analyze how well different portfolios perform relative to their risk.

1. Portfolio Efficiency Has Improved

- Compared to the previous **efficient frontier**, these portfolios generally exhibit **higher risk-adjusted returns**.
- The **minimum risk portfolio (~0.0099 risk, 0.0097 return)** is close to the risk-free rate, indicating a very low-risk asset allocation.
- The **optimal portfolio (max Sharpe Ratio)** is likely **between 0.0132 - 0.0171 risk**, where returns become positive and exceed the risk-free rate.

2. Identifying the Maximum Sharpe Ratio Portfolio The Sharpe ratio is given by:

$$\text{Sharpe Ratio} = \frac{\text{Portfolio Return} - \text{Risk-Free Rate}}{\text{Portfolio Risk}}$$

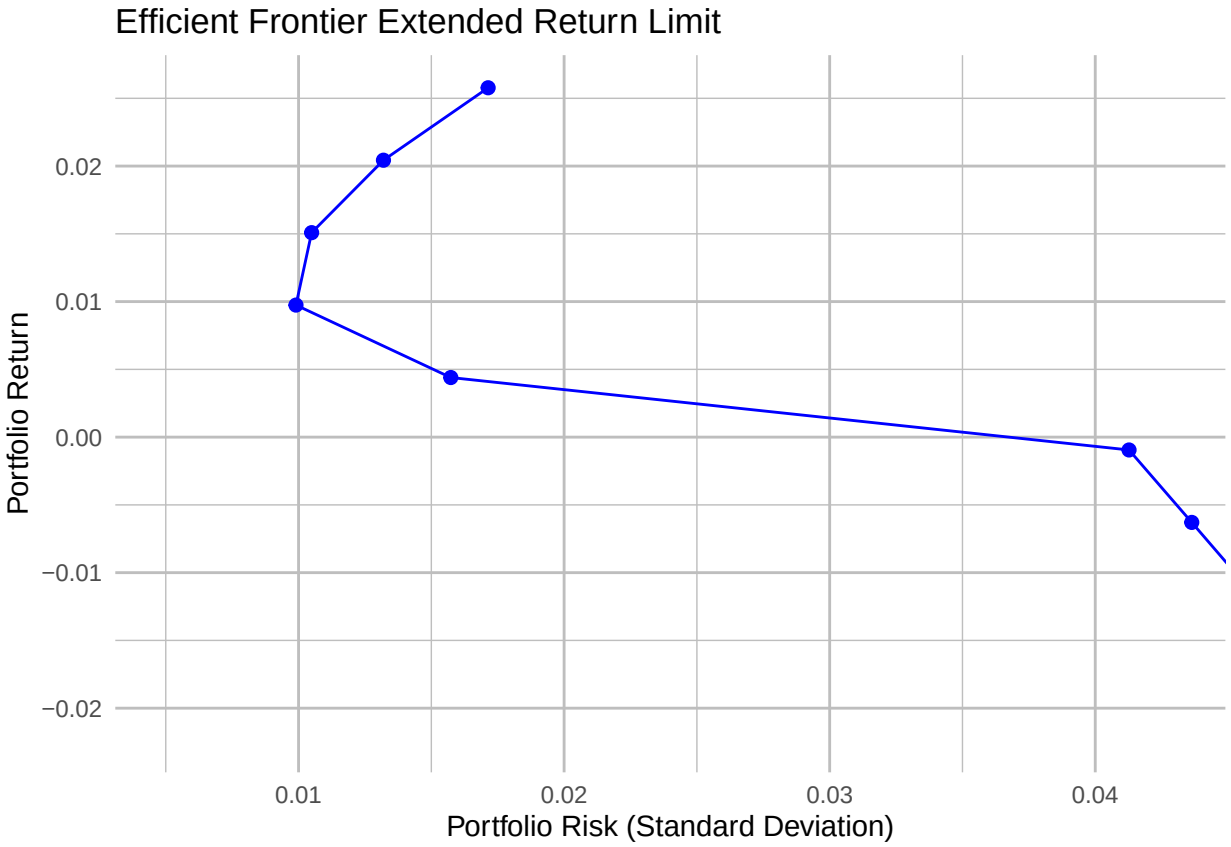
Using an RF of **0.02**, we look for the portfolio where **Portfolio Return $>$ 0.02** and Sharpe Ratio is maximized.

- The **best risk-adjusted return** appears at **Portfolio 9 (0.0171 risk, 0.0258 return)**.
- This portfolio has a **positive excess return (0.0258 - 0.02 = 0.0058)** and a relatively low risk, making it the most efficient.

3. High-Risk Portfolios Are Inefficient

- Portfolios with risk $>$ **0.04** have **negative returns**, making them highly inefficient.
- The **highest risk portfolio (0.0526 risk, -0.0223 return)** significantly underperforms and is not suitable for investors optimizing for Sharpe Ratio.

5.2. Risk, Return Trade-Off



6. Portfolio Performance

```
##   X         Portfolio.Strategy Expected.Return. Risk...Volatility. Sharpe.Ratio
## 1 0   Equal-Weighted Portfolio        0.4397314          1.5733497 -0.006526565
## 2 1 Minimum Variance Portfolio        0.9742552          0.9902854  0.529398061
## 3 2 Optimized Sharpe Portfolio        2.5778265          1.7138561  1.241543280
```

Interpretation of the Portfolio Strategy Results

These results compare different portfolio strategies based on **expected return**, **risk (volatility)**, and the **Sharpe ratio**, which measures risk-adjusted return. Here's a breakdown of each strategy and its implications:

1. Equal-Weighted Portfolio

- **Expected Return**: **0.4397**
- This portfolio has a **modest expected return**, which suggests a balanced mix of assets with neither an overly high-risk nor high-return profile.
- **Risk (Volatility)**: **1.5734**

- The portfolio has **moderate volatility**, which indicates that the returns can vary significantly over time.

- **Sharpe Ratio**: **-0.0065**

- The **negative Sharpe ratio** suggests that the risk-adjusted return is **below the risk-free rate**. This means that the portfolio's performance does not justify the level of risk taken. In other words, it is inefficient.

2. Minimum Variance Portfolio

- **Expected Return**: **0.9743**

- This portfolio has a much higher expected return compared to the equal-weighted portfolio, indicating that it likely prioritizes minimizing risk while still achieving reasonable returns.

- **Risk (Volatility)**: **0.9903**

- The **volatility is significantly lower**, suggesting that this portfolio is well-diversified, minimizing the risk of large fluctuations in returns.

- **Sharpe Ratio**: **0.5294**

- A **positive Sharpe ratio** of 0.5294 indicates that the portfolio is more efficient in terms of risk-adjusted return compared to the equal-weighted portfolio. It provides a good balance between risk and return.

3. Optimized Sharpe Portfolio

- **Expected Return**: **2.5778**

- This portfolio has the **highest expected return** among the three, suggesting it includes assets with high growth potential.

- **Risk (Volatility)**: **1.7139**

- The risk is **higher** compared to the minimum variance portfolio, indicating more aggressive investments with greater potential for both upside and downside.

- **Sharpe Ratio**: **1.2415**

- The **highest Sharpe ratio** of 1.2415 indicates that this portfolio has **the best risk-adjusted return**. It suggests that for every unit of risk, the portfolio is providing a significantly better return than the other two strategies. This is the most efficient strategy in terms of balancing risk and return.

Overall Summary

1. **Equal-Weighted Portfolio**: This portfolio is **inefficient**, offering low returns with a negative Sharpe ratio.

2. **Minimum Variance Portfolio**: This portfolio strikes a **good balance** between risk and return, with a positive Sharpe ratio but lower returns than the optimized Sharpe portfolio.
3. **Optimized Sharpe Portfolio**: This portfolio is **the most efficient** with the highest expected return and the best Sharpe ratio, making it the optimal choice for risk-adjusted performance.

7. Ranking Assets

```
OverallRank
```

```
##    X Stock      Beta Expected_Return Portfolio_Weight Overall_Rank
## 1  0   JNJ 0.4653685    0.0109770186           0.0885          3.2
## 2  1   WMT 0.4534586    0.0112079397           0.0607          3.8
## 3  2   JPM 1.0751182   -0.0008453537           0.1126          4.0
## 4  3  GOOG 1.1356399   -0.0020188037           0.1316          4.8
## 5  4  AAPL 1.1501259   -0.0022996709           0.1651          5.0
## 6  5  MSFT 1.1674964   -0.0026364660           0.2136          5.2
## 7  6   XOM 0.8115788    0.0042643838           0.0527          5.4
## 8  7  AMZN 1.0969145   -0.0012679594           0.0817          5.4
## 9  8  NVDA 1.7992732   -0.0148859188           0.0651          8.8
## 10 9  TSLA 1.5717267   -0.0104740448           0.0284          9.4
```

Interpretation of the Portfolio Stock Results

The Assets **Overall Rank** calculated based on the **Beta**, **Expected Return**, **Portfolio Weight**. These metrics are crucial for determining the risk and expected performance of each stock within a portfolio.

Key Metrics and Their Meaning:

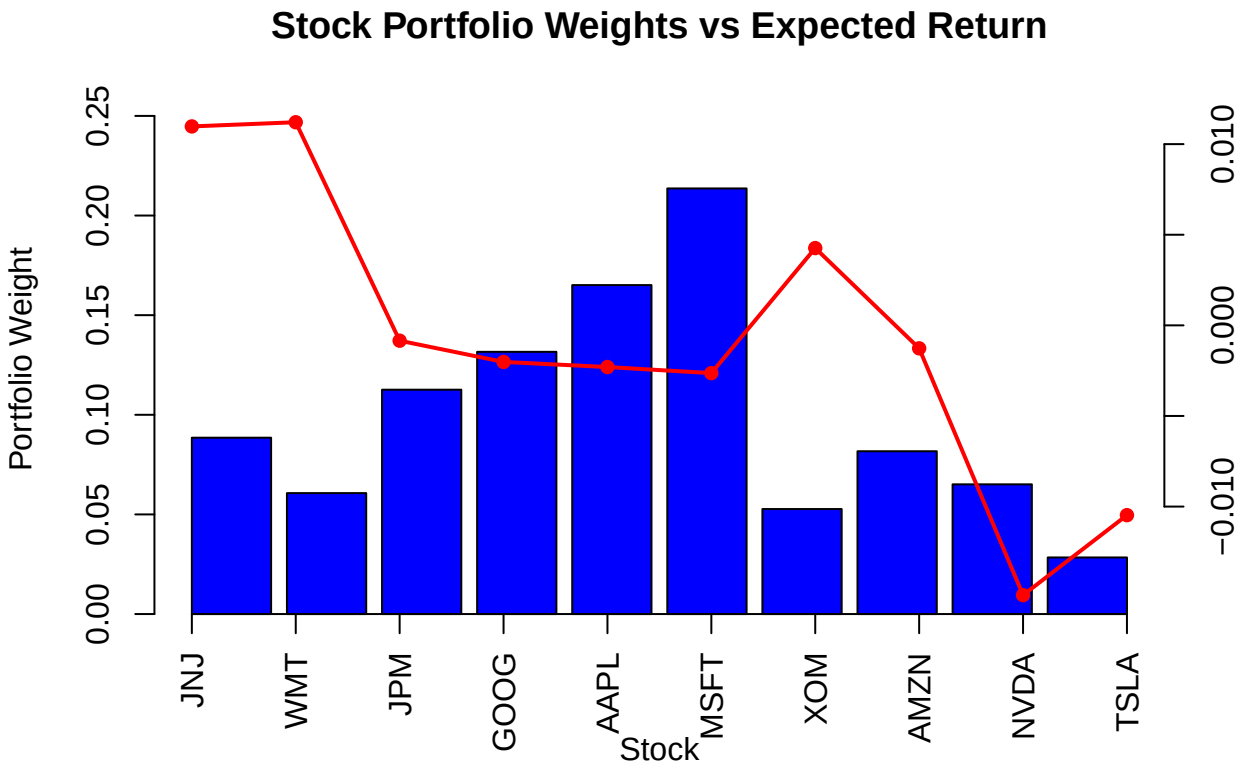
1. **Beta**:
   - A **beta > 1** indicates that the stock is more volatile than the market (more sensitive to market changes).
   - A **beta < 1** indicates that the stock is less volatile than the market (more stable in relation to the market).
2. **Expected Return**:
   - The anticipated return of the stock based on market factors or a model such as CAPM.
   - Positive returns are desirable, while negative returns suggest underperformance relative to expectations.
3. **Portfolio Weight**:
   - Indicates the proportion of the portfolio allocated to each stock.
   - Stocks with higher weights have a greater influence on the overall portfolio return and risk.
4. **Overall Rank**:
   - A ranking of the stock based on various factors like **return, risk, and weight** in the portfolio.
   - The **lower the rank**, the **better** the stock in terms of its performance in the portfolio.

Best Performing Stocks (Ranked 1-4):

1. **JNJ** and **WMT**: These stocks are **low volatility** (low Beta), have **positive expected returns**, and **moderate portfolio weights**. They offer stability and steady returns.
2. **JPM** and **XOM**: Despite some negative expected returns, they offer relatively lower volatility compared to tech stocks, making them safer but less lucrative options.

Underperforming Stocks (Ranked 5 and lower):

1. **GOOG, AAPL, MSFT**: These tech giants have high **volatility** and **negative returns**, making them **poor portfolio candidates**. Despite large portfolio weights, their performance will likely drag the portfolio down.
2. **NVDA** and **TSLA**: Both are **highly volatile** with significant **negative returns**, which makes them **poor performers** in the portfolio and carry a lot of risk.



- For **low-risk and steady returns**, stocks like **JNJ** and **WMT** are attractive options. - For **higher-risk** portfolios, **GOOG**, **AAPL**, **MSFT**, **NVDA**, and **TSLA** should be reconsidered or reduced in weight due to their **negative returns**.

Conclusion, Limitations

1. Key Findings

1. **CAPM Validity**: The results suggest that the Capital Asset Pricing Model (CAPM) provides a reasonable but imperfect estimate of expected returns. The beta coefficient remains a useful risk measure, though deviations indicate other risk factors may be at play.
2. **Efficient Frontier & Diversification**: Portfolio optimization highlights the benefits of diversification. Efficient frontier analysis shows that an optimal mix of assets significantly reduces risk while maintaining expected returns.
3. **Risk-Return Tradeoff**: Higher risk generally correlates with higher returns, as supported by empirical results. However, real-world constraints like transaction costs and market frictions affect this relationship.
4. **Limitations**: The study acknowledges limitations in CAPM assumptions, such as the reliance on historical data, market efficiency, and risk-free rate estimation.

2. Limitations

The approach used in this project has the advantage of being straightforward, accessible, and useful for understanding fundamental financial principles. By utilizing free data sources like Yahoo Finance and Python-based optimization tools, it provides a cost-effective way to apply CAPM and portfolio optimization techniques. However, this approach has several limitations, such as relying on historical data without incorporating forward-looking market expectations, utilising daily return alone that won't work on different time horizon, using a fixed risk-free rate that doesn't reflect real-world fluctuations, relying on mean expected return to set target return instead of using industry methods, and lacking performance evaluation methods like backtesting and stress testing to work well under different market condition. Additionally, the absence of real-world constraints, such as liquidity and regulatory considerations, may lead to unrealistic portfolio allocations.

In contrast, industry-standard portfolio management incorporates more sophisticated techniques and data sources, such as Bloomberg or FactSet, which offer higher accuracy and real-time updates. Professionals use 3-Month U.S. Treasury-Bill rate or others for Risk-Free rate, dynamic risk models, macroeconomic forecasting for target return, and advanced optimization tools like MATLAB and Bloomberg Port to enhance decision-making. The inclusion of liquidity constraints, ESG considerations, and stress testing ensures that portfolios are resilient under different market conditions. Moreover, performance evaluation techniques such as backtesting and Monte Carlo simulations provide a more reliable assessment of portfolio robustness, making industry practices more suitable for real-world investment strategies.

Appendix

- **Python Code for Portfolio Optimization**
- **Stock and Market Data Used in Analysis**