CT216 : Introduction to Communication Systems

Analytic Proof

15/04/2024

Group :- 6

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Hard decision decoding

- The r_{jm} as the mth output of j th branch of the demodulator , and c_{jm} mth transmitted bit of jth branch , n_{jm} represent the additive noise , then the relation is given by

$$r_{jm} = \sqrt{\varepsilon_c}\left(2c_{jm} - 1\right) + n_{jm}$$

- The **Branch metric** is defined as logarithm of joint probability of the sequence r_{jm} , conditioned on the transmitted sequence c_{jm}

$$\mu_j^{(i)} = \log(P(\, y_j \,/\, c_j^{(i)}) \,)$$

- The **Path metric** is given by summation of all branch metric of given path

$$PM^i = \sum_{j=1}^{B} \mu_j^{(i)}$$

- The criterion for deciding between two paths through the trellis is to select the one having the larger metric. This rule maximizes the probability of a correct decision or, equivalently, it minimizes the probability of error for the sequence of information bits

Soft decision decoding

- For the soft decision decoding , the output of demodulator is normally distributed with Mean = $\sqrt{\varepsilon_c}\left(2c_{jm} - 1\right)$ and Variance = σ^2

$$p(r_{jm} \mid c_{jm}) = \frac{1}{\sqrt{2\Pi}\sigma}\, exp(-\frac{\left(r_{jm} - \sqrt{\varepsilon_c}\left(2c_{jm} - 1\right)\right)^2}{2\sigma^2})$$

- Disregarding the terms that are common across all branch metrics, the **branch metric** for the jth branch of the ith path can be expressed as:

$$\mu_j^{(i)} = \sum_{m=1}^{n} r_{jm}\left(2c_{jm} - 1\right)$$

- Also the **Path metric** is given by :

$$CM^{(i)} = \sum_{j=1}^{B} \mu_j^{(i)}$$

$$B = No.\ of\ branches\ in\ given\ path$$

- For calculating the error of probability of soft decoding , we assume that our initial codeword is all 0's , and then calculate the probability of error. Let's define the first-event error probability as the probability that another path that merges with the all-zero path at node B has a metric that exceeds the metric of the all-zero path for the first time. Suppose the incorrect path, called i=1, that merges with the all-zero path differs from the all-zero path in bits, i.e., there are 1 d's in the path i=1 and the rest are 0s. It's probability is given by

$$P_2(d) = P(CM^1 \geq CM^0)$$

- Given that the coded bits in the two paths are identical except in the d positions, above expression can be rewritten in a more concise form:

$$P_2(d) = P(\sum_{l=1}^{d} r_l^{'} \geq 0\,)$$

- In above equation $r_l^{'}$ is identically distributed gaussian random variables with

  Mean = $-\sqrt{\varepsilon_c}$, and **Variance** = $\frac{N_0}{2}$

- **<u>Let's assume Variable Z as follows :</u>**

$$Z = \sum_{l=1}^{d} r_l^{'}$$

- Since , all $r_l^{'}$ are identically independently normally distributed , then for random variable Z , Mean(β) = $-\sqrt{\varepsilon_c}d$ and Variance(α) = $\frac{N_o}{2}d$, then it's function is given by

$$P(Z = K) = \frac{1}{\sqrt{2\pi\alpha}} exp(-\frac{[K-\beta]^2}{2\alpha})$$

- To find the value of $P_2(d)$,

$$P_2(d) = P(Z \geq 0)$$

$$P_2(d) = \int_0^{\infty} \frac{1}{\sqrt{2\pi\alpha}} exp(-\frac{[K-\beta]^2}{2\alpha})\, dK$$

- **<u>The above expression can be expressed as Q function</u>**

$$P_2(d) = Q(\sqrt{\frac{2\varepsilon_c d}{N_o}})$$

$$P_2(d) = Q(\sqrt{2\gamma_b R_c d})$$

<u>Q function is defined as below :-</u>

$$Q(a) = \int_a^{\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{K^2}{2})\, dK$$

- <u>**To calculate total probability of error , we have to sum up all possible d W.R.T $P_2(d)$, calculated as follow :**</u>

$$P_e \le \sum_{d=d_{free}}^{\infty} a_d P_2(d)$$

$$P_e \le \sum_{d=d_{free}}^{\infty} a_d Q(\sqrt{2\gamma_b R_c d})$$

a_d = $number\ of\ paths\ of\ distance\ 'd'\ from\ the\ all\ zero\ path\ that\ merge\ with\ the\ all\ zero\ path$

- In the above equation , we have used an upper bound , as convolution coding is not having fixed input size. But if suppose the node is truncated periodically after B nodes, then it’s is given by

$$P_e \le \sum_{d=d_{free}}^{B} a_d Q(\sqrt{2\gamma_b R_c d})$$

- **To find the Upper-Limit to Q function , we can use the result of Chernoff bound :**

$$P(X \geq a) \;\leq\; e^{-at} M_x(t) \qquad\qquad (t > 0)$$

$$P(X \geq a) \;\leq\; e^{-at} e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

- **For above Q function , we can defined it's upper limit as :-**

$$Q(a) \;=\; P(X \geq a) \;\leq\; e^{\frac{-a^2}{2}}$$

$$Q(a) \;=\; P(X \geq a) \;\leq\; e^{\frac{-a^2}{2}}$$

$$Q(\sqrt{2\gamma_b R_c d}) \;\leq\; e^{-\gamma_b R_c d}$$

- **Bit Error Probability**

When we multiply the pairwise error probability $P_2(d)$by the number of incorrectly decoded information bits for the incorrect path at the node where they merge, we can determine the bit error rate for that path. It's important to note that the exponent in the factor N contained in the transfer function $T(D,N)$ indicates the number of information bit errors in the selection of an incorrect path that merges with an all-zero path at some node B.

$$T(D, N) \;=\; \sum_{d = d_{free}}^{\infty} a_d D^d N^{f(d)}$$

$$\frac{d}{dN} T(D, N)\ (N \;=\; 1) \;=\; \sum_{d = d_{free}}^{\infty} a_d D^d f(d)$$

$$p_b \;<\; \sum_{d = d_{free}}^{\infty} \beta_d P_2(d)$$

- **Hard decoding :-**

  Let's consider the scenario where we have two paths: Path 0, which is the all-zero path, and Parh 1, which merges with Path 0 after some transitions. Path 1 contains d ones.
  If d is odd, Path 1 will be selected over Path 0 when the number of ones in Path 1 is more than half of d. If suppose d is odd,

$$P_2(d) = \sum_{k=\frac{d+1}{2}}^{d} n_k^d p^k (1-p)^{d-k}$$

  and suppose d is even

$$P_2(d) = \sum_{k=\frac{d}{2}+1}^{d} n_k^d p^k (1-p)^{d-k} + \frac{1}{2} n_{\frac{d}{2}}^d p^{\frac{d}{2}} (1-p)^{\frac{d}{2}}$$

- **The probability of error is given by :-**

$$P_e < \sum_{d=d_{free}}^{\infty} a_d P_2(d)$$

- **Let's calculate the upper-limit to** $P_2(d)$:-

$$P_2(d) = \sum_{k=\frac{d+1}{2}}^{d} n_k^d p^k (1-p)^{d-k}$$

$$\textit{Instead of } k\,,\ \textit{let's make inequality by using } d/2 \textit{ instead of power}$$

$$P_2(d) < \sum_{k=\frac{d+1}{2}}^{d} n_k^d p^{\frac{d}{2}} (1-p)^{\frac{d}{2}}$$

$$P_2(d) < p^{\frac{d}{2}} (1-p)^{\frac{d}{2}} \sum_{k=\frac{d+1}{2}}^{d} n_k^d$$

$$P_2(d) < p^{\frac{d}{2}} (1-p)^{\frac{d}{2}} \sum_{k=0}^{d} n_k^d$$

$$P_2(d) < p^{\frac{d}{2}} (1-p)^{\frac{d}{2}} 2^d$$

$$P_2(d) < p^{\frac{d}{2}} (1-p)^{\frac{d}{2}} 2^d$$

$$P_2(d) < [4p(1-p)]^{d/2}$$

- **<u>Now , using this upper limit to probability of error :-</u>**

$$P_e < \sum_{d=d_{free}}^{\infty} a_d [4p(1-p)]^{d/2}$$

$$P_e < \sum_{d=d_{free}}^{\infty} T(D)(\, D = \sqrt{4p(1-p)})$$

- **<u>Now calculating bit error :-</u>**

$$P_b < \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

$$P_b < \frac{d}{dN} T(D, N)(D = \sqrt{4p(1-p)}\, ,\ N = 1)$$

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