

DISCRETE MATHEMATICS THROUGH  
GUIDED DISCOVERY:  
Classnotes for MTH 355

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# Preface

Much of your experience in lower division mathematics courses probably had the following flavor: You attended class and listened to lectures where theory and examples were presented. Your text usually gave a parallel development. Then your turn came. For every assigned problem or proof there was a method already taught in class that was the key, and your job was to decide which method applied and then to apply it. In upper-level math major courses, your goal should be to discover some of the ideas and methods for yourself – as many as you can. These notes are intended as an introduction to discrete mathematics and also as an introduction to mathematical thinking within a classroom mode of learning called Guided Discovery.

Guided Discovery approaches mathematics very much like a mathematician does when on unfamiliar ground: Looking at special cases, trying to discover patterns, wandering up blind alleys, possibly being frustrated, but finally putting it all together into a solution of a problem or a proof of a theorem. This thinking process is as important for you as the discrete mathematics you will learn in the process. The notes consist principally of sequences of problems designed for you to discover solutions to problems yourself. Often you are guided to such solutions through simplified examples that set the stage. As you work through later problems, you will recall earlier techniques that can either be used directly or be slightly modified to get a solution. The point of learning in this way is that you are not just applying methods that someone else has developed for you but rather you are learning how to discover ideas and methods for yourself. Understanding small points and taking small steps is the usual way of doing mathematics, and is the usual path to all mathematical results – including very significant ones.

The notes are designed to be worked through linearly, with the problems in the first chapter introducing you to the habit of thinking for yourself as well as introducing you to discrete mathematics. During class you will work in groups, with some class discussion that will help give an overall context.

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**Part I**

**COURSE NOTES**

# Chapter 1

## Beginning Combinatorics

In most textbooks, an introductory explanation is followed by problems which use the information just given. Since much of the learning in guided discovery occurs as you work on problems, it is natural that definitions of concepts be given within the problems and that the text material often occurs after you have already worked problems developing the concepts. Because of this, be sure to read over all the problems including any you don't work right away.

Table 1.1: The meaning of the tags on the problem numbers.

•	essential for the course
◦	motivational material
+	summary
→	especially interesting

As you flip through the pages of these notes you will see that most of the problems are marked by a symbol to the left of the number. The problems that are marked with a bullet (•) are essential for understanding later material and are where the main ideas of the book are developed. (Your instructor may leave out some of these problems because he or she plans not to cover future problems that rely on them.) Other problems are marked with open circles (◦) which indicate that they are designed to provide a motivation for important concepts by hinting at or partially developing ideas that can be helpful in solving subsequent problems. A few problems that summarize ideas that have come before are marked with a plus sign (+).

You will also find some problems marked with an arrow (→). These point

to problems that are particularly interesting; some of them are difficult but not all are. Frequently these problems are not intrinsically difficult but only might seem hard in light of what has come before and they will be easier when you return to them after working on more problems.

## 1.1 What is Combinatorics?

Combinatorial mathematics arises from studying *combining* objects into arrangements. For example, we might be combining sports teams into a tournament, samples of tires into groups for testing on cars, students into classes to compare approaches to teaching a subject, or members of a tennis club into pairs to play tennis. There are many questions that can be asked about such arrangements of objects. Here the primary focus will be questions about *how many ways* objects can be combined into arrangements of the desired type. These are called **counting problems**. One way to count is to enumerate a complete list of all the objects with the desired properties, and another way is to count how many objects have the properties without actually making a complete list. Enumerative combinatorics is usually interested in the second approach, although obtaining a solution might involve writing a partial list.

Sometimes combinatorial mathematicians ask if a certain arrangement is *possible*. For instance, if there are ten baseball teams and each team has to play each other team once, can the whole series be scheduled if the fields are available at enough hours for forty games? Sometimes combinatorial mathematicians ask if all the arrangements able to be made have a certain desirable property. For example, do all ways of testing five brands of tires on five different cars compare each brand with each other brand on at least one common car?

Counting problems (and problems of the other sorts described above) arise throughout physics, biology, computer science, statistics, and many other subjects. In order to demonstrate all these relationships detours would have to be taken into all of these subjects. Instead, although there will be some important applications, the discussions will usually be phrased around either your everyday experience or your mathematical experience so that you won't have to learn a new context before learning the mathematics.

## 1.2 Basic Counting Principles

- 1. Five schools plan to send their baseball team to a tournament in which each team must play each other team exactly once. How many games must be played?
- 2. Now some number  $n$  of schools plan to send their baseball teams to a tournament in which each team must play each other team exactly once. Think of the teams as numbered 1 through  $n$ .
  - (a) How many games does Team 1 have to play?
  - (b) How many additional games (other than the one with Team 1) does Team 2 have to play?
  - (c) How many additional games (other than those with the first  $i - 1$  teams) does Team  $i$  have to play?
  - (d) In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?

*Hint.* If you have trouble doing this problem, work on  $n = 6$  before studying the general  $n$ .

- 3. One of the schools sending its team to the tournament has to travel some distance, and so the school is making sandwiches for team members to eat along the way. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?

An **ordered pair**  $(a, b)$  consists of two *members* (which are often called *coordinates*) that are labeled here as  $a$  and  $b$ . Then  $a$  is called the first member of the pair and  $b$  is the second member of the pair. What is an **ordered triple**?

You almost certainly used ordered pairs, at least implicitly, to solve the first three problems. At the time, did you recognize you were doing so?

- + 4. (a) If  $M$  is a set with  $m$  elements and  $N$  is a set with  $n$  elements, how many ordered pairs are there whose first element is a member of  $M$  and whose second element is a member of  $N$ ? (Note that when such a question is asked in any problem in these notes, you're required both to answer the question and to provide a justification for the answer you give.)
- (b) Explain carefully how Problem 3 can be viewed mathematically as asking you to count the number of ordered pairs from two specific sets.



While working the next problems, be sure to ask yourself whether the problem amounts to counting the number of ordered pairs or ordered triples or ordered  $n$ -tuples of elements chosen from appropriate sets. What is an **ordered  $n$ -tuple**?

- 5. Since a sandwich by itself is pretty boring, students from the school in Problem 3 are offered a choice of a drink (from among five different kinds), a sandwich (with the choices as in Problem 3), and a fruit (from among four different kinds). In how many ways may a student make a choice of a lunch, if every lunch is a choice of these three items?

Now that you have worked with your group on a few problems, you understand more about the setup of the course. For instance, you've been working on the problems in groups and your instructor has principally been listening to what's going on in the groups. Any guidance from your instructor has primarily been in the form of asking a question or occasionally giving a hint—a question or hint that might at first seem unrelated to the problem at hand. The reason for this is that in this course your instructor is a *guide*. Problems in the notes (along with perhaps cryptic hints from your instructor) are designed to lead you and your group to discover for yourselves and to prove for yourselves. There is considerable pedagogical evidence that this can lead to deep learning and understanding. In addition, it's satisfying and fun to discover things for yourself.

- 6. The coach of the team in Problem 3 knows of an ice cream shop along the way where she plans to stop to buy each team member a triple-decker cone. The store offers 12 different flavors of ice cream, and triple-decker cones are made only in homemade waffle cones. (Here repeated flavors are allowed; in fact, a triple-decker with three scoops of the same flavor is even possible. Be sure to count Strawberry, Vanilla, Chocolate as different from Chocolate, Vanilla, Strawberry, etc.)
  - (a) How many possible triple-decker cones will be available to the team members?
  - (b) How many triple-deckers have three *different* kinds of ice cream?

You probably have noticed some standard mathematical words and phrases (for instance, **set**, **ordered pair**, **function**) are creeping into the problems. One of the goals of these notes is to show how the solution of most counting problems uses standard mathematical objects. As was said earlier, since

most of the intellectual content of these notes is in the problems, it is natural that definitions of concepts will often be within problems.<sup>1</sup> For example, Problem 4 is meant to suggest that the question asked in Problem 3 was really a problem of counting all the ordered pairs consisting of a bread choice and a filling choice. The notation  $A \times B$  is usually used to represent the set of all ordered pairs whose first member is in  $A$  and whose second member is in  $B$ , and  $A \times B$  is called the **Cartesian product** of  $A$  and  $B$ . Therefore you can think of Problem 3 as asking you for the size of the Cartesian product of  $M$  and  $N$ , where  $M$  is the set of all bread types and  $N$  is the set of all possible fillings; that is, the number of different kinds of sandwiches equals the number of elements in the Cartesian product  $M \times N$ .

- 7. The idea of a function is ubiquitous in mathematics. A **function**  $f$  from a set  $\mathcal{S}$  to a set  $\mathcal{T}$  is a relationship between the two sets that associates to each element  $x$  in the set  $\mathcal{S}$  exactly one member  $f(x)$  in the set  $\mathcal{T}$ . The ideas of *function* and *relationship* will be revisited in more detail and from different points of view from time to time.

- (a) Using  $f, g, \dots$  to stand for various functions, list all the different functions from the set  $\{1, 2\}$  to the set  $\{a, b\}$ . For example, you might start with the function  $f$  given by

$$f(1) = a \quad \text{and} \quad f(2) = b.$$

- (b) Let's look at the last part in a different way. Instead of asking for a *list* of all the functions, suppose you simply asked *how many* functions are there from the set  $\{1, 2\}$  to the set  $\{a, b\}$ . Now devise a way to count the number of functions without writing an exhaustive list.
- (c) How many functions are there from the 3-element set  $\{1, 2, 3\}$  to the 2-element set  $\{a, b\}$ ?
- (d) How many functions are there from the 2-element set  $\{a, b\}$  to the 3-element set  $\{1, 2, 3\}$ ?
- (e) How many functions are there from any 3-element set to any 12-element set?
- (f) Re-do Problem 6(a) by constructing a function from the 3-element set of positions in the triple-decker to the set of 12-element set of flavors. Give an explicit verbal description of your function.

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<sup>1</sup>When you come across an unfamiliar term in a problem, most likely it was defined earlier, and you should be able to find the term listed in the index.

This idea of using functions to count is very powerful, and is one of the foundations of combinatorics. It also illustrates the added insight that can be gained by looking at a problem from more than one point of view. Take time right now to check to see if this has already happened in earlier problems.

- 8. A function  $f$  is called a **one-to-one function** (often called an **injection**) if  $f$  is a function which has the property that whenever  $x$  is different from  $y$ , then  $f(x)$  is different from  $f(y)$ .
  - (a) How many one-to-one functions are there from a 2-element set to a 3-element set?  
*Hint.* Since 2 and 3 are fairly small numbers, you might first list all the functions, and then decide which are one-to-one. But you should then work out a method for counting them without an exhaustive listing.
  - (b) How many one-to-one functions are there from a 3-element set to a 2-element set?
  - (c) How many one-to-one functions are there from a 3-element set to a 12-element set?
  - (d) Re-do Problem 6(b) by showing that a solution amounts to counting all the one-to-one functions between two sets. In order to completely make this connection, you must explicitly define these two sets as well as a function between them that describes the possible ice cream cones.
- 9. A group of three hungry members of the team in Problem 6 notices it would be cheaper to buy three pints of ice cream to share among the three of them. (And they would also then get more ice cream!)
  - (a) In how many ways may they choose three pints of two different flavors? How is this problem different from Problem 6?
  - (b) In how many ways may they choose three pints with at least two different flavors?
  - (c) In how many ways may they choose three pints of ice cream with no restrictions on repeating flavors?

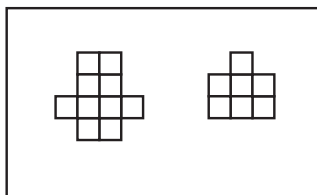
In the last part of Problem 9 you probably found it helpful to break the question into certain cases that you could solve by previous methods. After doing that you could then figure out the answer by using the answers from all the cases. Because this is a fairly common strategy, some special terminology is helpful. Two sets are said to be **disjoint** if they have no

elements in common. For example, the sets  $\{1, 3, 12\}$  and  $\{6, 4, 8, 2\}$  are disjoint, but  $\{1, 3, 12\}$  and  $\{3, 5, 7\}$  are not disjoint sets. Three or more sets are said to be **mutually disjoint** if no two of the sets have any elements in common. To solve Problem 9(c), the set of all possible choices of three pints of ice cream can be broken into three mutually disjoint sets: Three pints of different flavors; three pints with two different flavors; three pints of the same flavor. The total number of choices is the sum of the sizes of these three mutually disjoint sets.

- 10. (a) What can you say about the size of the union of a finite number of mutually disjoint finite sets?
- (b) What can you say about the size of the union of  $m$  mutually disjoint sets, each of the same size  $n$ ? This is a fundamental principle of “counting” or enumerative combinatorics.
- (c) Find at least one of the previous problems that can be solved by the counting principle in part (b). Explain how to solve your chosen problem using that principle.

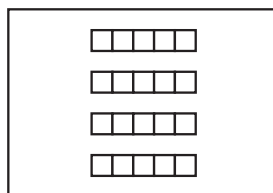
The problems you’ve just completed contain among them kernels of the fundamentals of enumerative combinatorics. For example, in your solution to Problem 10(a) you just stated the **Sum Principle** (illustrated in Figure 1.1), and in Problem 10(b), the **Product Principle** (illustrated in Figure 1.2.) These are two of the most basic principles of combinatorics, and they form a foundation on which many other counting principles are developed.

Figure 1.1: The union of these two disjoint sets has size 17.



When a set  $S$  is a union of  $m$  mutually disjoint sets  $B_1, B_2, \dots, B_m$ , then the sets  $B_1, B_2, \dots, B_m$  is said to form a **partition** of the set  $S$ . (Note that a partition of  $S$  is a set of sets.) In order that the set  $S$  is not confused with the sets  $B_i$  into which it has been divided, the sets  $B_1, B_2, \dots, B_m$  are often called the **blocks of the partition**.

Figure 1.2: The union of four disjoint sets of size 5.



- 11. Reword your solution to the last part of Problem 9 in terms of partitioning some set into blocks.

Using the language of partitions, the Sum Principle and the Product Principle translate to:

**The Sum Principle**

For any partition of a finite set  $S$ , the size of  $S$  is the sum of the sizes of the blocks of the partition.

**The Product Principle**

If a finite set  $S$  is partitioned into  $m$  blocks of the same size  $n$ , then  $S$  has size  $mn$ .

You'll notice that both of these principles refer to a partition of a *finite* set. The language could be modified a bit to cover infinite sets, but the sets considered in this book are primarily finite. In order to avoid possible complications in the future, the term “size” will only be used for finite sets.

- 12. Prove the Product Principle follows logically from the Sum Principle.
- 13. Explain how to interpret  $2 + 3 = 5$  in terms of a partitioning some set. Because of this, the Sum Principle for the case when the partition has two blocks is the definition of the binary operation of addition on the set of positive integers. Explain this in your own words.
- 14. If  $A$  is a subset of some universe  $U$ , define its **set complement** to be

$$U \setminus A := \{x \in U : x \notin A\}.$$

Use the Sum Principle for two blocks to prove  $|U \setminus A| = |U| - |A|$ .

You'll prove the Sum Principle for any finite number of blocks in the next chapter. For right now you may accept it as true and use it wherever you like. Of course you must be careful that your proofs in the next chapter do not depend on any results which you've proved using the general Sum Principle.

- 15. In a biology lab study of the effects of basic fertilizer ingredients on plants, 16 plants are treated with potash, 16 plants are treated with phosphate, and a total of eight plants among these are treated with both phosphate and potash. No other treatments are used. How many plants receive at least one treatment? If 33 plants are studied, how many receive no treatment?
- 16. Use partitions to prove a formula for the size  $|A \cup B|$  of the union  $A \cup B$  of any two (finite but not necessarily disjoint) sets  $A$  and  $B$  in terms of the sizes  $|A|$  of  $A$ ,  $|B|$  of  $B$ , and  $|A \cap B|$  of the intersection  $A \cap B$ .

The formula you proved in the last problem is a special case of the **Principle of Inclusion and Exclusion**, which is considered more thoroughly in Chapter 6.

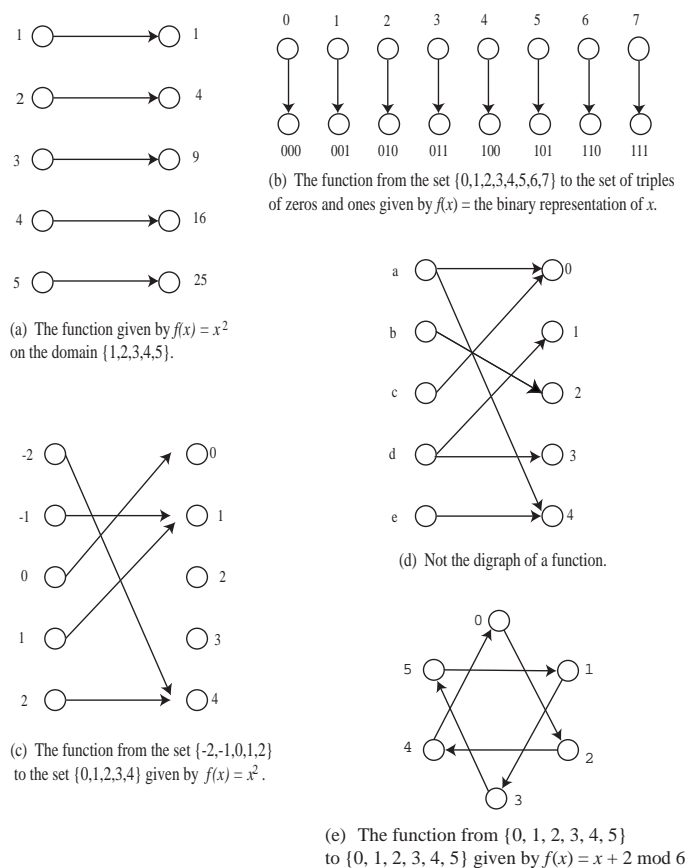
### 1.3 Functions and their Directed Graphs

A typical way to define a **function**  $f$  from a set  $\mathcal{S}$  (called the **domain** of the function) to a set  $\mathcal{T}$  (which in discrete mathematics is commonly referred to as its **co-domain**) is: A function  $f$  is a relation from  $\mathcal{S}$  to  $\mathcal{T}$  which relates each element of  $\mathcal{S}$  to one and only one member of  $\mathcal{T}$ . The notation  $f(x)$  is used to represent the element of  $\mathcal{T}$  that is related to the element  $x$  of  $\mathcal{S}$ , and the standard shorthand  $f : \mathcal{S} \rightarrow \mathcal{T}$  is used for “ $f$  is a function from  $\mathcal{S}$  to  $\mathcal{T}$ ”. Please note that the word “relation” has a precise meaning in mathematics. Do you know it? Refer to Appendix A if you need a review of this terminology.

Relations between subsets of the set of real numbers can be graphed in the Cartesian plane, and it is helpful to remember how you used a graph of a relation defined on the real numbers to determine whether it is actually a function. Namely, to do this you learned that you should check that each vertical straight line crosses the graph of the relation in at most one point. You might also recall how to determine whether such a function is one-to-one by examining its graph. If each horizontal line crosses the graph in at most one point, the function is one-to-one. If even one horizontal line crosses the graph in more than one point, the function is not one-to-one.

The domain and co-domain of the functions considered in this book will both usually be finite and will often contain objects which are not real numbers, and this means that graphs in the Cartesian plane are usually not available. But all is not lost, since there is another kind of graph called a **directed graph** (or **digraph**) that is especially useful when dealing with functions between finite sets. Figure 1.3 has several examples of digraphs of functions.

Figure 1.3: What is a digraph of a function?



In analyzing the graphs in Figure 1.3 you should notice the following: When you want to draw the digraph that represents a relation from a set  $\mathcal{S}$  to a set  $\mathcal{T}$ , you can draw a line of dots called **vertices** to represent the elements of  $\mathcal{S}$  and another (usually parallel) line of vertices to represent

the elements of  $\mathcal{T}$ . (Part (e) is slightly different.) You then can draw an arrow from the vertex for  $x \in \mathcal{S}$  to the vertex for  $y \in \mathcal{T}$  if and only if  $x$  is related to  $y$ . Such arrows are called **(directed) edges**. Because there is an inherent order in a relation, every edge is an arrow and not just a line segment. When the relation is a function,  $f : \mathcal{S} \rightarrow \mathcal{T}$ , one arrow is drawn from each  $x \in \mathcal{S}$  to its corresponding  $f(x) \in \mathcal{T}$ . Familiarize yourself with this technique by working through the digraphs in Figure 1.3. Note that in part (e) the function is from a set  $\mathcal{S}$  to itself and the picture has been simplified by drawing only one set of vertices representing the elements of  $\mathcal{S}$ . Digraphs can often be more enlightening if you experiment to find an attractive placement of the vertices rather than putting them in a row.

There is a simple test for whether a digraph of a relation from  $\mathcal{S}$  to  $\mathcal{T}$  is a digraph of a *function* from  $\mathcal{S}$  to  $\mathcal{T}$ :

- 17. Returning to the digraphs in Figure 1.3, determine which are functions. Then formulate a precise sentence stating what properties the arrows and vertices in a digraph must possess so that the digraph represents a function  $f$  from  $\mathcal{S}$  to  $\mathcal{T}$ .
- 18. (a) In how many ways can you pass out nine different candies to three children? Set up your solution as a problem about counting functions.  
 (b) In how many ways can you pass out the candy if each child must get at least one piece?  
 (c) Exactly three pieces?
- 19. Suppose you have  $n$  distinguishable balls. How many ways can you paint each of them with one color, chosen from red, black, green and blue?

The most mathematically elegant solutions to the last problems probably involve using functions. Until now, your experience with functions probably has only involved a formula in calculus. In contrast, in discrete mathematics a function can be an algorithm or it might be given by a verbal description.

From now on, for any positive integer  $n$  the notation  $[n]$  will be used for the set  $\{1, \dots, n\}$ . For example,  $[4]$  equals the set  $\{1, 2, 3, 4\}$ .

- 20. If  $B_1, B_2, \dots, B_m$  is a partition of a finite set  $\mathcal{S}$ , define a relation  $f$  from  $\mathcal{S}$  to  $[m]$  by

$$f(s) = i \iff s \in B_i .$$



- (a) Use the definition of partition to explain completely why  $f$  is a function with domain  $\mathcal{S}$ .
  - (b) This correspondence defines a relation from the set of all  $m$ -block partitions of  $\mathcal{S}$  to the set of all functions from  $\mathcal{S}$  to  $[m]$ . Is this relation a function whose domain is the set of all  $m$ -block partitions of  $\mathcal{S}$ ? Either find a counterexample or prove the relation is always a function.
- 21. A function  $f : \mathcal{S} \rightarrow \mathcal{T}$  is an **onto function** (also called a **surjection**) if each element of  $\mathcal{T}$  is  $f(x)$  for at least one  $x \in \mathcal{S}$ .
- (a) Choose finite sets  $\mathcal{S}$  and  $\mathcal{T}$  and a function from  $\mathcal{S}$  to  $\mathcal{T}$  that is one-to-one but not onto. Draw the digraph of your function.
  - (b) Choose finite sets  $\mathcal{S}$  and  $\mathcal{T}$  and a function from  $\mathcal{S}$  to  $\mathcal{T}$  that is onto but not one-to-one. Draw the digraph of your function.
- 22. Visual inspection of the digraph of a function can reveal whether or not the function is one-to-one or is onto. In each of the following parts devise a test similar to the one for testing when a digraph is the digraph of a function.
- (a) Give a verbal statement of a test for whether or not the digraph of a function is the digraph of a one-to-one function. (Remember that in order to be a one-to-one function, a relation must be a *function*.)
  - (b) Give a verbal statement of a test for whether or not the digraph of a function is the digraph of an onto function.

If you think you might need more work on functions, consider working through Appendix A outside of class. It covers functions and also digraphs in more detail.

- 23. A function from a set  $\mathcal{X}$  to a set  $\mathcal{Y}$  which is both one-to-one and onto is usually called a **bijection** (but is sometimes called a **1-1 correspondence**). What does the digraph of a bijection from a set  $\mathcal{S}$  to a set  $\mathcal{T}$  look like?
24. If  $f$  is a bijection from  $\mathcal{X}$  to  $\mathcal{Y}$  and  $g$  is a bijection from  $\mathcal{Y}$  to  $\mathcal{Z}$ , must the function composition  $g \circ f$  be a bijection from  $\mathcal{X}$  to  $\mathcal{Z}$ ? Explain.
25. If you reverse all the arrows in the digraph of a bijection  $f$  with domain  $\mathcal{S}$  and co-domain  $\mathcal{T}$ , you get the digraph of another function  $g$ . Why is  $g$  a function from  $\mathcal{T}$  to  $\mathcal{S}$ ? Why is  $g$  a bijection? What is  $f(g(x))$ ? What is  $g(f(x))$ ?

The digraphs marked (a), (b), and (e) in Figure 1.3 are digraphs of bijections. Your description in Problem 23 illustrates another fundamental principle of combinatorial mathematics:

### The Bijection Principle

Two sets have the same size if and only if there is a bijection between the sets.

It is surprising how this innocent-sounding principle frequently provides an insight into some otherwise very complicated counting arguments.

A **binary representation** of a positive integer  $m$  is an ordered list  $a_1a_2 \dots a_k$  of zeros and ones such that

$$m = a_12^{k-1} + a_22^{k-2} + \dots + a_k2^0.$$

Our definition allows “leading zeros”: for instance, both 011 and 11 represent the number 3. Often such an ordered list of  $k$  zeros and ones is called a **binary  $k$ -string**, and each of its digits is called a **bit**, which is shorthand for *binary digit*. In the above example, 011 is a binary 3-string representation of 3, while 11 is a binary 2-string representation of 3.

- 26. Let  $n$  be a fixed positive integer. For this problem, let  $\mathcal{S}$  be the set of all binary  $n$ -string representations of numbers between 0 and  $2^n - 1$ , and let  $\mathcal{T}$  be the set of all subsets of  $[n]$ . Note that the empty set  $\emptyset$  is a subset of every set.
  - (a) For  $n = 2$ , write out the sets  $\mathcal{S}$  and  $\mathcal{T}$  and then describe a natural bijection from  $\mathcal{S}$  to  $\mathcal{T}$  where in your verbal statement of the bijection, 0 and 1 play the roles of “does not belong to” and “belongs”, respectively.
  - (b) Using the strategy from part (a), describe a bijection from  $\mathcal{S}$  to  $\mathcal{T}$  for general  $n$ . Explain why your map is a bijection from  $\mathcal{S}$  to  $\mathcal{T}$ .
  - (c) Explain how part (b) enables you to find the number of subsets of  $[n]$ .

In the last problem you used the fact that the empty set is a subset of every set. This fact may seem a little strange at first, but logical reasoning will convince you that it is true. Indeed, assume by way of contradiction that there exists a set  $X$  such that  $\emptyset$  is not a subset of  $X$ . By the definition of subset this means there must be an element of  $\emptyset$  which is not an element of  $X$ . What does this contradict?

- 27. (a) Let  $n \geq 2$ . Use the Bijection Principle to prove the number of 2-element subsets of  $[n]$  equals the number of subsets of  $[n]$  that have  $n - 2$  elements. Be sure to construct a specific function which you prove is a bijection.
- (b) Let  $n \geq 3$ . Generalize the reasoning used in part (a) to prove the number of 3-element subsets of  $[n]$  equals the number of subsets of  $[n]$  which have  $n - 3$  elements.
- (c) Let  $n \geq k$ . State a natural conjecture suggested by parts (a) and (b). Is your conjecture true or false? Justify this by either proving your conjecture or providing a counterexample to the conjecture.

For each subset  $A$  of  $[n]$ , define a function (traditionally denoted by  $\chi_A$ ) as follows: <sup>2</sup>

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

The function  $\chi_A$  is called the **characteristic function** of  $A$ . Notice that the characteristic function is a function from  $A$  to  $\{0, 1\}$ . For practice, consider the function  $\chi_{\{1,3\}}$  for the subset  $S = \{1, 3\}$  of the set  $[4]$ .

- Show  $\chi_{\{1,3\}}(1) = 1$  and  $\chi_{\{1,3\}}(2) = 0$ .
  - Find  $\chi_{\{1,3\}}(3)$  and  $\chi_{\{1,3\}}(4)$ .
- 28. Let  $\mathcal{S}$  be the set of all subsets of  $[n]$ . Let  $\mathcal{T}$  be the set of all functions from  $[n]$  to  $\{0, 1\}$ . Define a map  $f : \mathcal{S} \rightarrow \mathcal{T}$  by  $f(A) = \chi_A$  for all  $A \in \mathcal{S}$ . Explain why  $f$  is a bijection. What does this say about the number of functions from  $[n]$  to  $[2]$ ?

The proofs in Problem 26 and 28 use essentially the same bijection, but they interpret sequences of zeros and ones differently and so end up being different proofs.

You'll return to the question of counting the number of functions between any two finite sets in Section 2.3.1.

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<sup>2</sup>The symbol  $\chi$  is the Greek letter chi that is pronounced Ki, where the  $i$  sounds like "eye."

## 1.4 Another Application of the Sum Principle

- 29. US coins are all marked with the year in which they were made. How many coins do you need to guarantee that on at least two of them, the date has the same last digit? (The phrase “to guarantee that on at least two of them,...” means that you can find two coins with the same last digit. You might be able to find three with that last digit, or you might be able to find one pair with the last digit 1 and one pair with the last digit 9, or any combination of equal last digits, as long as there is at least one pair with the same last digit.)

There are many ways to explain your answer to Problem 29. For example, you can separate the coins into stacks or blocks according to the last digit of their date. That is, you can put all the coins with a given last digit in a stack together (putting no other coins in that stack), and repeat this process until all coins have been placed in a stack. Using the terminology introduced earlier, this gives a partition of your set of coins into blocks of coins with the same last digit. If no two coins have the same last digit, then each block has at most one coin. Since there are only ten digits, there are at most ten non-empty blocks and by the Sum Principle there can be at most ten coins. Note that if there were only ten coins, it would be possible to have all different last digits, but with eleven coins some block must have at least two coins in order for the sum of the sizes of at most ten blocks to be 11. This is one explanation of why eleven coins are needed in Problem 29. This type of situation arises often in combinatorial situations, and so rather than always using the Sum Principle to explain your reasoning, you can use another principle which is a variant of the Sum Principle.

### The Pigeonhole Principle

If a set with more than  $n$  elements is partitioned into  $n$  blocks, then at least one block has more than one element.

The Pigeonhole Principle gets its name from the idea of a grid of little boxes that might be used to sort mail or as mailboxes for a group of people in an office. The boxes in such grids are sometimes called pigeonholes in analogy with the stacks of boxes used to house homing pigeons back when homing pigeons were used to carry messages. People will sometimes state this principle in a more colorful way as “if more than  $n$  pigeons are put into  $n$  pigeonholes, then some pigeonhole contains more than one pigeon.”

- 30. Prove the Pigeonhole Principle follows from the Sum Principle.

- 31. Prove that any function from  $[n]$  to a set of size less than  $n$  cannot be one-to-one.

*Hint.* You must prove that regardless of the function  $f$  chosen, there are always two elements, say  $x$  and  $y$ , such that  $f(x) = f(y)$ .

- 32. Prove that if  $f$  is a one-to-one function between finite sets of equal size, then  $f$  must be onto. Compare this with Problem 21(a).
  - 33. Prove that if  $f$  is an onto function between finite sets of equal size, then  $f$  must be one-to-one. Compare this with Problem 21(b).
  - 34. Let  $f$  be a function between finite sets of equal size. Then  $f$  is a bijection if and only if  $f$  is either onto or one-to-one. Find a counterexample to this result when the domain and co-domain are both infinite.
35. (Some familiarity with arithmetic modulo 10 and modulo 100 is helpful for this problem.) In this problem you prove there exists an integer  $n$  such that if you take the first  $n$  powers of any prime other than two or five, the last two digits of at least one of these powers must be “01”. (That is, the same integer  $n$  has the property for all primes  $p \neq 2, 5$ .)
- (a) All powers of 5 end in the digit 5, and all powers of 2 are even. For any prime  $p \neq 2, 5$ , prove there are at most four values of the last digit of any power  $p^i$ . What does that say about the values of  $p^i \pmod{10}$ ?
  - (b) How many values are possible for  $p^i \pmod{100}$ ? Referring to this number of values as  $N_0$ , in the remaining parts of this problem you will prove that  $n = N_0$  can be used in the statement of the result you want to prove.
  - (c) Use the Pigeonhole Principle to prove that among the first  $N_0 + 1$  powers of any prime  $p \neq 2, 5$ , there exist  $i \neq j$  such that  $p^i - p^j$  is divisible by 100. Then prove  $p^{i-j} \equiv 1 \pmod{100}$ .
  - (d) Find the smallest value of  $n$  that works in the statement of the result. Give a complete proof that this value works.

## 1.5 The Generalized Pigeonhole Principle (Optional)

Although this section is used in other optional sections of these notes, it is independent of the main body of the notes.

### The Generalized Pigeonhole Principle

If a set with more than  $kn$  elements is partitioned into  $n$  blocks, then at least one block has at least  $k + 1$  elements.

36. Prove the Generalized Pigeonhole Principle follows from the Sum Principle.
37. Draw five circles labelled Al, Sue, Don, Pam, and Jo.
- (a) Find a way to draw red and green lines between people (circles) so that every pair of people is joined by a line and there is neither a triangle consisting entirely of red lines or a triangle consisting of green lines.
  - (b) Suppose Bob joins the original group of five people. Now can you draw a combination of red and green lines that have the same property as those in part (a)? Explain.
- 38. Show that in a set of six people, there is either a subset of at least three people who all know each other or a subset of at least three people none of whom know each other. (Here it is assumed that if Person 1 knows Person 2, then Person 2 knows Person 1.) Does the conclusion hold when there are five people in the set rather than six?

Problems 37 and 38 together show that six is the smallest number  $R$  with the property that if there are  $R$  people in a room, then there is either a set of (at least) three mutual acquaintances or a set of (at least) three mutual strangers. Another way to say the same thing is to say that six is the smallest number so that no matter how you connect six points in the plane (no three on a line) with red and green lines, you can find either a red triangle or a green triangle. There is a name for this: The **Ramsey Number**  $R(m, n)$  is the smallest number  $R$  such that if you have  $R$  people in a room, then there is a set of at least  $m$  mutual acquaintances or at least  $n$  mutual strangers.

Problem 37 hints at a geometric description of Ramsey Numbers which uses the idea of a complete graph on  $R$  vertices. A **complete graph** on  $R$  vertices consists of  $R$  points in the plane, together with line segments (or curves) connecting each pair vertices. As you may guess, a complete graph is a special case of something called a graph, which will be defined more carefully in Section 4.1. The points in a graph are called **vertices** (or

**nodes**) and the line segments are called **edges**. The notation  $K_n$  is used to represent a complete graph on  $n$  vertices.

The geometric description of  $R(3, 3)$  may be translated into the language of graph theory by saying  $R(3, 3)$  is the smallest number  $R$  such that if you color the edges of a  $K_R$  with two colors, then you can find in the picture a  $K_3$  all of whose edges have the same color. The graph theory description of  $R(m, n)$  is that  $R(m, n)$  is the smallest number  $R$  such that if you color the edges of a  $K_R$  with the colors red and green, then you can find in your picture either a  $K_m$  all of whose edges are red or a  $K_n$  all of whose edges are green. Because you could have said your colors in the opposite order, you may conclude that  $R(m, n) = R(n, m)$ . In particular  $R(n, n)$  is the smallest number  $R$  such that if you color the edges of a  $K_R$  with two colors, then your picture contains a  $K_n$  all of whose edges have the same color. The results of Problems 37 and 38 combine to prove  $R(3, 3) = 6$ .

39. Since  $R(3, 3) = 6$ , an uneducated guess might be that  $R(4, 4) = 8$ . Show that this is not the case.

*Hint.* To get started, try to write down what it means to say  $R(4, 4)$  does not equal 8.

40. Show that among ten people, there are either four mutual acquaintances or three mutual strangers. What does this say about  $R(4, 3)$ ?
41. Show that among an odd number of people there is at least one person who is an acquaintance of an even number of people and therefore also a stranger to an even number of people.

*Hint.* Let  $a_i$  be the number of acquaintances of person  $i$ .

42. Find a way to color the edges of a  $K_8$  with red and green so that there is no red  $K_4$  and no green  $K_3$ .

- 43. Find  $R(4, 3)$ .

*Hint.* There is a relevant problem that you haven't used yet.

As of this writing, relatively few Ramsey Numbers are known. Some that have been found are:  $R(3, n)$  for all  $n < 10$ ;  $R(4, 4) = 18$ ;  $R(4, 5) = 25$ .

## 1.6 An Overview

Now is a good time for you to go back into this chapter with a fresh outlook, informed by your working through the problems in this chapter. Because

most of the techniques are developed in problems, you might not realize how much you have learned and most likely it will help to write a summary of important facts. Every once in a while you should do that—go through the notes with an eye for what you have learned and how you might approach older problems differently. You should review at the end of each chapter, and some of you might prefer more frequent reviews, done either on your own or in your group.

- + 44. As a group, identify four or five important principles of counting developed in this chapter. Also, identify at least four techniques used. (For this, I would call the Bijection Principle a counting principle, whereas I consider the idea of using a function to count to be a technique. There is not a clear line of separation between them.)
- + 45. When you originally solved Problems 1 to 9, you probably didn't think explicitly in terms of any basic principles, although you probably used some principles implicitly. Revisit those problems from the point of view of categorizing the counting principle used in your solution. For this, compose a  $9 \times 3$  table whose rows are labeled Problem 1, Problem 2,..., Problem 9 and columns by Sum Principle, Product Principle, Bijection Principle. In each row, indicate with an X each principle that is natural to use to solve the problem which names the row. In some problems, several principles might be used.
- + 46. As a group, discuss the function interpretation of Problems 18 and 19 and compose a similar problem of your own which can be solved using this technique. Share the problems with the other groups and evaluate their solutions. Be sure the solutions include a clear explanation of why the constructed function can be used to solve the problem. In addition, solutions must correctly identify and count the size of the domain and co-domain of the function.
- + 47. Evaluate this solution to Problem 26(b):

Define the function  $f : \mathcal{S} \longrightarrow \mathcal{T}$  by  $f(a_1 a_2 \cdots a_n)$  equals the set of all subscripts such that  $a_i = 1$ . This is a bijection since  $\mathcal{S}$  and  $\mathcal{T}$  both have  $2^n$  elements.

### 1.6.1 An overview of problem solving

You should think about how your approach to counting problems has matured over the course of this chapter. There are some fairly general tech-



niques which you have used and should continue to use. Among these are:

- As you work on a problem, think about why you are doing what you are doing. Is it helping you? If your current approach doesn't feel right, try to see why.
- Is this a problem you can decompose into simpler problems?
- Can you compose a simple example (even a silly one) of what the problem is asking you to do?
- If a problem is asking you to do something for every value of a positive integer  $n$ , then what happens with small values of  $n$  like 0, 1, and 2?
- When you're having trouble counting the number of possibilities for a specific large number  $N$ , consider temporarily replacing  $N$  with a smaller value, say  $n$ . After you solve the problem for the smaller  $n$ , see if your reasoning (or perhaps some variation of it) transfers to give a solution for the original  $N$ .

Throughout, don't worry about making mistakes, because mistakes often lead mathematicians to their best insights.

## Chapter 2

# The Principle of Mathematical Induction

### 2.1 Inductive Processes

Proof by mathematical induction is a more subtle method of reasoning than what first meets the eye. It is also much more widely applicable than you might have guessed based on your previous experience with the technique. In this chapter you'll use mathematical induction to prove the Sum Principle and the Product Principle, as well as other counting techniques you used in the first chapter.

This chapter assumes you've had some prior experience with proof by mathematical induction. If that assumption is not true for you, you should first work through Appendix B before beginning this section. Most likely the first examples of proof by induction you worked involved proving identities such as

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad (2.1)$$

or

$$\sum_{i=1}^n \frac{1}{i \cdot (i+1)} = \frac{n}{n+1}. \quad (2.2)$$

There's a common thread to these arguments: Looking at (2.1) more closely, you are asked to prove *a sequence of statements* which are indexed by positive integers  $n$ . The first four statements in this sequence of statements are

$$1 = \frac{1 \cdot 2}{2} ; 1 + 2 = \frac{2 \cdot 3}{2} ; 1 + 2 + 3 = \frac{3 \cdot 4}{2} ; 1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2} .$$

This illustrates one reason why someone might think of using induction to prove (2.1): The identity gives a sequence of statements indexed by integers  $n$  that are greater than a certain size (in this case for all integers  $n \geq 1$ ). In other words, the identity yields a family of statements  $S(n)$  parametrized by integers  $n \geq 1$ , namely, the statements

$$S(n) : \sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Any result that can be proved by induction must be able to be written as a sequence of statements which can be parameterized by integers greater than some base integer  $b$ . Because this initial requirement is satisfied by (2.1), you can see that induction might be able to be used to prove this result. Next is an explanation of why induction can be successfully used to prove (2.1).

For the moment, let's suspend belief and suppose you had no idea whether or not (2.1) is true for *any*  $n \geq 1$ , no less that it's true for *all*  $n \geq 1$ . How can you proceed? In the spirit of scientific reasoning, you might think of checking that  $S(n)$  is true for some small integers  $n$ . Maybe you would even use a computer to verify the statement  $S(n)$  for all values of  $n$  up to a quite large size. If you do this, you will find that  $S(n)$  is in fact true for every value of  $n \geq 1$  which you check, and this gives you some confidence that the statement is true in general for all  $n \geq 1$ . Of course this is not yet a proof, but rather simply a justification for continuing to try to prove the statement.

In order to get induction to work, an underlying **inductive process** must be discovered. Namely, how is the statement  $S(N)$  for any integer  $N$  related to the statements that precede it, statements that you may already have checked are true? For this particular sequence of statements, the relationship is simple: For any  $N \geq 2$ , the truth of any statement  $S(N)$  relies only on the truth of the immediately preceding statement  $S(N-1)$ . The reason for this is that the sum on the left-hand side in the statement  $S(N)$  is built up from the left-hand sum in the statement  $S(N-1)$  by adding the integer  $N$  to the sum. The uncovering of an *inductive process* for this problem is further evidence that a proof by mathematical induction might be able to be constructed.

For similar reasons, the second sequence of statements (2.2) can most likely be proved by induction. Indeed, the underlying inductive process is really the same. First identify the sequence of statements  $S(1), S(2), \dots, S(n), \dots$

Here

$$S(n) \text{ is the statement } \sum_{i=1}^n \frac{1}{i \cdot (i+1)} = \frac{n}{n+1}.$$

Then observe that the left-hand sum in statement  $S(N+1)$  is obtained by adding  $1/(N+1) \cdot (N+2)$  to the left-hand sum in the immediately preceding statement  $S(N)$ . Because of this, the *inductive process* is basically the same for (2.1) and (2.2), although the algebra involved is somewhat different.

- 48. Prove each of (2.1) and (2.2) by induction on  $n \geq 1$ .
- 49. What postage do you think can be made using only three-cent and five-cent stamps? If you have an unlimited supply of these two types of stamps, do you think that there is a number  $N$  such that for every  $n \geq N$ , you can make  $n$  cents worth of postage?

In the last problem you probably did some calculations to convince yourself that  $n$  cents worth of postage can be made for  $n = 3, 5, 6$  (but not for  $n = 1, 2, 4, 7$ ) and also that apparently postage can be made for all integers that are at least 8. A proof of this fact will now be given in full detail.

First of all, the sequence of statements to be proved is:

$S(n)$ :  $n$  cents of postage can be made using only 3- and 5-cent stamps.

A key observation (and one that highlights how this problem is different from proving the identities (2.1) and (2.2)) is that *you can't prove  $S(13)$  is true by working only with the fact that  $S(12)$  is true*. The reason for this is simple: you don't have a 1-cent stamp at your disposal. In general you can never conclude that  $n$  cents of postage can be made simply by knowing that  $n-1$  cents of postage can be made; that is, in this example the truth of the statement  $S(n)$  cannot be proved using only the truth of  $S(n-1)$ .

As is done in any proof by induction, there will first be an analysis of some small cases. You've already noted in the last problem that the base integer must be  $b \geq 8$ ; this is forced by the fact that 7 cents of postage cannot be made using only 3-cent and 5-cent stamps. Also, you've undoubtedly noted that you can make 8 cents with one 3-cent stamp and one 5-cent stamp; 9 cents can be made with three 3-cent stamps and 10 with two 5-cent stamps. Maybe you've checked larger integers as well, but at this point the truth of each of  $S(8), S(9), S(10)$  has been verified.

It's already been observed that you can't get 13 cents from the fact you can get 12 cents. One way to get 13 cents is to add a 3-cent stamp to the two fives used to get 10 cents. The surprising thing is that this simple

observation is the key to describing an inductive process. Because of this, if you can make postage for three consecutive amounts, then you can make postage of any larger amount by simply adding the correct number of 3-cent stamps to an earlier attainable amount of postage. (Don't be distracted by the fact that this might not be the *only* way to get a certain amount of postage, since for instance 15-cents can be made by three 5-cent stamps as well as the five 3-cents that would be obtained in one step from the postage for 12 cents.) Said another way, the idea here is to devise an induction statement which lumps together

$$8, 9, 10;$$

and

$$11, 12, 13;$$

and

$$14, 15, 16;$$

and so on. In general for all  $n \geq 3$ ,

$$3n - 1, 3n, 3n + 1$$

are to be considered together. This analysis leads to a revision from the original sequence of statements  $S(n)$  to the following sequence of statements  $T(n)$ : for any  $n \geq 3$ ,

$T(n)$  is the statement: "It is possible to form postage totaling exactly  $3n - 1$ ,  $3n$ , and  $3n + 1$  cents using only 3-cent and 5-cent stamps."

Notice that here the base integer is  $b = 3$ . (Why can't the base case be either  $b = 1$  or  $b = 2$ ?)

Here's an inductive proof that  $T(n)$  is true for all  $n \geq 3$ :

Since  $8 = 3 + 5$ ,  $9 = 3 + 3 + 3$  and  $10 = 5 + 5$ , it is possible to make  $k$ -cent postage for all  $k = 8, 9, 10$  and this proves the base case  $T(3)$  is a true statement. Next consider any integer  $N \geq 3$  for which  $T(N)$  is true, and then show that the truth of  $T(N + 1)$  follows from the truth of  $T(N)$ . Because  $T(N)$  is true, postage can be made for each of

$$3N - 1, 3N, \text{ and } 3N + 1.$$

If you add one 3-cent stamp to the postage for  $3N - 1$ , you obtain  $3N - 1 + 3 = 3(N + 1) - 1$  cents of postage. Likewise, by adding one 3-cent stamp to the postage for each of  $3N$  and  $3N + 1$ , you can obtain postage for each of  $3(N + 1)$  and  $3(N + 1) + 1$ . Therefore, it has been shown: if  $T(N)$  is assumed to be true, then

“It is possible to form postage totaling each of

$$3(N + 1) - 1, 3(N + 1), \text{ and } 3(N + 1) + 1 \text{ cents}$$

using only 3-cent and five-cent stamps.”

This is precisely the statement  $T(N + 1)$ . Thus by the Principle of Mathematical Induction, using only 3- and 5-cent stamps, you can make  $n$  cents in postage for every  $n \geq 8$ .

The postage problem for any finite number of stamp types is also referred to as Frobenius’ Problem and the Coin Exchange Problem.

Appendix B contains another review of the fundamentals of proofs using the Principle of Mathematical Induction.

## 2.2 The Principle of Mathematical Induction

All proofs using the Principle of Mathematical Induction have four parts: An identification of the sequence of statements to be proved, a base step, an inductive step, and the inductive conclusion.

It’s helpful to identify these four parts in the proof of the postage problem in the last section. First of all, the sequence of statements  $S(n)$  which must be proved was identified. In the postage problem this involved some trial-and-error, whereas for (2.1) and (2.2) the statement of the problem already identified the parametrized statement to be proved. For the postage problem, the **base step** is the case  $n = 3$ . Next locate the sentence “Next consider any integer  $N \geq 3$  for which  $T(N)$  is true, and then show that the truth of  $T(N + 1)$  follows from the truth of  $T(N)$ .” This is an outline of what must be accomplished in the inductive step of the proof. In particular, “Because  $T(N)$  is true, postage can be made for each of

$$3N - 1, 3N, \text{ and } 3N + 1”$$

is called the **inductive hypothesis**. In the **inductive step** the statement is derived for  $n = N + 1$  from the inductive hypothesis, proving that the truth of the statement when  $n = N$  implies the truth of the statement when  $n = N + 1$ . The last sentence in the proof, “Thus by the Principle of Mathematical Induction, using only 3- and 5-cent stamps, you can make  $n$  cents in postage for every  $n \geq 8$ ”, is the **inductive conclusion**.

One way of looking at the Principle of Mathematical Induction is that it tells you that if you know the “first” case of a theorem and you can derive every other case of the theorem from a smaller case, then the theorem is true in all cases. However, the particular way in which this reasoning process has

been used above is rather restrictive because in the inductive step you have derived the next case of the statement from the *immediately preceding* case of the statement. Instead, there is another formulation of mathematical induction which is often referred to as the *Strong* Principle of Mathematical Induction. It is equivalent to the principle of mathematical induction that you've been using and it is called "strong" because it can be more easily applied to a wider range of problems.

In the following box the basic template for a proof by mathematical induction is highlighted.

### The Principle of Mathematical Induction

In order to prove a sequence of statements indexed by integers  $k \geq b$  it is sufficient to do the following:

1. Determine the sequence of statements to be proved;
2. (Base Step) Prove the statement for  $k = b$ ;
3. (Inductive Step) Prove that (for any  $N \geq b$ ) the truth of the statements for  $k = b, k = b + 1, \dots, k = N$  implies the truth of the statement for  $k = N + 1$ ;
4. (Inductive Conclusion) By the Principle of Mathematical Induction, every statement in the sequence is true.

The only change in this formulation of the principle is in the Inductive Step. This statement of the Inductive Step is preferable because it is more easily applied to a wider range of situations than the non-strong principle. For example, this statement of the Principle of Mathematical Induction allows the postage stamp problem to be proved in a more straightforward manner by directly using the original observation that the truth of  $S(N)$  follows from the truth of  $S(N - 3)$ .

Step 1: Identify the sequence of statements to be proved.

For all  $n \geq 8$ , let

$S(n)$ :  $n$  cents of postage can be made using only 3- and 5-cent stamps.

Step 2: Base Step.

Since  $8 = 3 + 5$ ,  $9 = 3 + 3 + 3$  and  $10 = 5 + 5$ , it is possible to make  $k$ -cent postage for all  $k = 8, 9, 10$  and this proves the base cases with  $n = 8, 9, 10$ . (Why must you explicitly verify the first *three* statements?)

Step 3: The Inductive Step.

Assume that  $S(k)$  is known to be true for all  $k = 8, \dots, N$ , and then

show that the truth of  $S(N + 1)$  follows. Because you know  $S(N - 2)$  is true, postage amounting to  $N - 2$  cents can be made and adding one 3-cent stamp to this allows  $N + 1$  cents of postage to be made.

Step 4: Inductive Conclusion.

By the Principle of Mathematical Induction, using only 3- and 5-cent stamps you can make  $n$  cents in postage for every  $n \geq 8$ .

- 50. What postage do you think can be made using only 5-cent and 6-cent stamps? Do you think that there is a number  $N$  such that for any  $n \geq N$ , you can make  $n$  cents worth of postage? Either explain why such an integer  $N$  does not exist or give an inductive process inherent in the problem from which a proof by induction could be constructed. (A complete proof by induction isn't required for this problem.)
- 51. Explain why it is true that for any integer  $N$  there always exists  $n > N$  for which it is impossible to make  $n$ -cents worth of postage using only 2-cent and 4-cent stamps. Generalize this observation.
- 52. Suppose  $a$  and  $b$  are two positive integers for which there exists an integer  $N$  such that for any  $n \geq N$ ,  $n$ -cents of postage can always be made using only  $a$ -cent and  $b$ -cent stamps. Using the information you have collected in the preceding problems, conjecture a value of  $N$  which has this property. ( $N$  will depend on  $a$  and  $b$ ). Test your conjecture on some more examples.
- 53. Divide your group into pairs to play three or four games of Sylver Coinage (a.k.a Postal Kiosk),<sup>1</sup> in which players take turns naming positive integers  $n$  for which  $n$ -cents of postage cannot be made using only numbers which have already been named. The player who chooses 1 loses. For instance, 3, 4, 5, 2, 1 is a possible game, but 3, 5, 7, 6, 4, 1 is not (why?). Without referring to human mortality, explain why any game of Sylver Coinage ends after finitely many steps.
- 54. Suppose that  $f$  is a function which is defined by the inductive process  $f(1) = 1$  and  $f(n) = n + f(n - 1)$ . For practice, find  $f(2)$ ,  $f(3)$ , and  $f(4)$ , and then prove that  $f(n) = n(n + 1)/2$ . Notice that this gives another proof of (2.1), because the sum there satisfies the two conditions defining the function  $f$ .
- 55. What properties in the definition of function allow you to say that every function from  $[m + 1]$  to  $[10]$  is built up from a function from  $[m]$  to  $[10]$  in a unique way? Let  $\mathcal{S}_n$  be the set of all functions from  $[n]$

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<sup>1</sup>The authors of the book *Winning Ways for Your Mathematical Plays* attribute this the game of Sylver Coinage to a 1884 article by the mathematician Joseph J. Sylvester.



- to [10]. Give an inductive process relating the set  $\mathcal{S}_{m+1}$  to the set  $\mathcal{S}_m$ , and use that to find an equation for the size of  $\mathcal{S}_{m+1}$  in terms of the size of  $\mathcal{S}_m$ . Compare this with Problem 28.
- 56. (a) Using the simple observation that a subset of  $[n]$  either does or doesn't contain  $n$ , find an inductive relationship between the number of all subsets of  $[n]$  and the number of all subsets of  $[n-1]$ .  
 (b) Use this relationship to construct an inductive proof that the set  $[n]$  has  $2^n$  subsets. Compare this with Problem 26.
  - 57. Let  $k$  be a fixed positive integer, and let  $s_n$  be the number of functions from  $[n]$  to  $[k]$ . Find an equation relating  $s_{n+1}$  to  $s_n$ . Explain.
  - + 58. Compare the inductive relationships in the last two problems. Is there a reason for one to be a special case of the other?
  - 59. For a fixed positive integer  $n$ , a **composition of  $n$**  is an ordered list of positive integers whose sum is  $n$ . For this problem,  $c_n$  will denote the number of compositions of  $n$ .  
 (a) Find  $c_n$  for some small values of  $n$ .  
 (b) Use the calculations from part (a) to determine an inductive procedure for obtaining  $c_n$  from  $c_{n-1}$ .  
 (c) Carefully prove that  $c_n = 2^{n-1}$  holds for all  $n \geq 1$ .  
 (d) Use the result of Problem 56(b) to obtain part (c) of this problem directly.
  - 60. A roller coaster car has  $n$  rows of seats, each of which has room for two people. Suppose  $n$  men and  $n$  women get into the car with a man and a woman in each row, and let  $a_n$  be the number of configurations that can occur. Find an inductive relationship between  $a_n$  and  $a_{n+1}$ , and use that to conjecture a formula for  $a_n$ . Prove your formula by induction.
  - 61. (a) Beginning with the 3-string 000, write a list of all eight binary 3-strings which is ordered in such a way that each 3-string differs from the previous one in the list by changing just one digit. This should be considered "cyclically"; that is, the last element also differs in only one digit from the first. There are many such sequences, and are called (cyclic) Gray Code for 3-strings. That is, a **Gray Code** for binary  $n$ -strings is a listing of all  $2^n$  binary  $n$ -strings in such a way that each  $n$ -string differs from the previous one in exactly one place. Describe how to get your Gray Code for 3-strings from some Gray code for 2-strings.  
 (b) Can you describe how to get some Gray Code for 4-strings from the one you found for 3-strings?

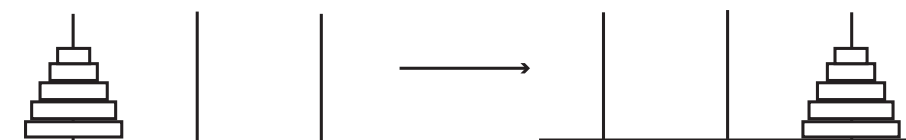
- (c) Give a written description of the inductive step of an inductive proof that of the existence of Gray codes for  $n$ -strings for all  $n \geq 1$ .

Among the original uses of Gray codes was to reduce coding errors in a pulse communication system. Frank Gray of Bell Labs received a patent for Gray codes<sup>2</sup> in 1953, but this type of sequence is now known to have been used even earlier (in the 1870s) in a telegraph device by Émile Baudot, a French engineer.

- 62. Use the definition of a Gray Code to find a bijection between the even-sized subsets of  $[n]$  and the odd-sized subsets of  $[n]$ .

*Hint.* Consider the characteristic functions of subsets.

Figure 2.1: The Towers of Hanoi Puzzle



- 63. The Towers of Hanoi Puzzle (refer to Figure 2.1) has three rods rising from a rectangular base and  $n \geq 1$  rings of different sizes. At the beginning of the puzzle all rings are stacked on one rod in order of decreasing size. A legal move consists of moving a ring from one rod to another in such a way that the ring does not land on a smaller one. Let  $m_n$  be the (minimal) number of moves required to move all the rings from the initial rod to any other rod. Find  $m_n$  for all  $n \leq 3$  and use those solutions to develop a strategy for obtaining the number of moves for  $N + 1$  rings from the number of moves for  $N$  rings. Describe this as an inductive process. Guess a formula for  $m_n$  and use induction to show your guess is correct.
- 64. Are more moves required if the Towers of Hanoi Puzzle had the additional stipulation that the stack had to wind up on a pre-determined rod? Explain.

Each of the foregoing problems had an underlying inductive process, and that process was the basis for the inductive step of the proof by mathematical induction. Next a complete proof of Problem 59 will be given.

<sup>2</sup>*Pulse Code Communication*, United States Patent Number 2632058 (March 17, 1953)

The sequence of statements to be proved is

$$S(n) : c_n = 2^{n-1} \quad \text{for all } n \geq 1 ,$$

where  $c_n$  is the number of compositions of the integer  $n$  (as was defined in Problem 59).

The base case of  $n = 1$ : Since 1 is the only composition of 1 then  $c_1 = 1 = 2^{1-1}$  and the base case is true.

Next, for fixed  $N \geq 1$  assume that

$$c_n = 2^{n-1} \quad \text{for all } n = 1, \dots, N ,$$

and use this to prove that  $c_{N+1} = 2^N$ . In other words, an inductive procedure must be identified which gives the number of compositions of the integer  $N+1$  in terms of the number of compositions for smaller  $n \leq N$ . First of all, beginning with any composition of  $N$ , if a summand of 1 is added to the end of the sum, the result is a composition of  $N+1$ . Because all of the original compositions of  $N$  are different, there are exactly  $c_N$  different compositions of  $N+1$  which end in a summand of 1. Another way to obtain compositions of  $N+1$  is to increase the last summand of a composition of  $N$  by 1. Again, these are all different from each other, and none of them can be a composition of the first type, each of which had 1 for its last summand. Since these two procedures result in mutually disjoint sets, so far,  $2c_N$  compositions of  $N+1$  have been obtained.

BUT there is still the question of whether or not *every* composition of  $N+1$  arises in one of these two ways. In other words, do these two blocks of compositions of  $N+1$  partition the set of all compositions of  $N+1$  or do they form a partition of a smaller set? Because the last summand of any composition of  $N+1$  is either 1 (and the other  $N$  summands form a composition of  $N$ ) or is greater than 1 (and becomes a composition of  $N$  when the last summand is decreased by 1), the two blocks given do partition the set of all compositions of  $N+1$ . It has thus shown that  $c_{N+1} = 2c_N = 2^N$  since  $c_N = 2^{N-1}$  by the inductive assumption. Therefore, by the Principle of Mathematical Induction,  $c_n = 2^{n-1}$  holds for all  $n \geq 1$ .

After carefully analyzing the above proof, return to your solutions to the other problems in this section to make sure you have written complete explanations in your earlier proofs. Your understanding of mathematical induction will be relied on throughout the remainder of these notes. Work through Appendix B if you'd like more practice with this proof technique.

- 65. Write down at least two ways that the Sum Principle can be expressed as a sequence of statements indexed by the positive integers. Can you

think of a third way? Settle on a way that allows you to construct an inductive process.

- 66. Prove the Sum Principle by mathematical induction. (As commented earlier in Problem 13, the base case of a partition into two blocks is the definition of addition of two positive integers.)

Since in Problem 12 you proved the Product Principle follows logically from the Sum Principle, you have now completed the proofs of both the Sum and Product Principles.

### 2.2.1 Recurrences

In the last section you considered many situations that involved counting items which are defined inductively. For instance, in Problem 56 you found the relationship

$$s_n = 2s_{n-1} . \quad (2.3)$$

Also, when  $s_n$  stands for the number of functions from  $[n]$  to  $[k]$ , for a fixed integer  $k$ , the inductive process you found in Problem 57 gave the equation

$$s_{n+1} = k s_n . \quad (2.4)$$

Equations (2.3) and (2.4) are examples of recurrence equations, which are sometimes called recurrence relations, or simply recurrences. A **recurrence** is an equation that expresses the  $n$ -th term of a sequence  $a_n$  in terms of earlier values of  $a_i$ . Other examples of recurrences are

$$a_n = a_{n-1} + 7, \quad (2.5)$$

$$a_n = 3a_{n-1} + 2^n, \quad (2.6)$$

$$a_n = a_{n-3} + 3a_{n-2}, \quad (2.7)$$

$$a_n = a_1a_{n-1} + a_2a_{n-2} + \dots + a_{n-1}a_1. \quad (2.8)$$

A **solution** to a recurrence is any sequence that satisfies the recurrence. For instance, the sequence given by  $s_n = 2^n$  is a solution to recurrence (2.3). Since  $s_n = 17 \cdot 2^n$  and  $s_n = -13 \cdot 2^n$  are two other solutions to recurrence (2.3), a recurrence can have infinitely many solutions, but in a given problem there is often only one solution of interest. For example, if you are interested in the number of subsets of a set, then the solution to recurrence (2.3) that you care about is  $s_n = 2^n$ . The reason for this is that it is the only solution that begins with  $s_0 = 1$ , the number of subsets of the empty set. (Notice that this is consistent with what you have already proved in Problem 26.) Usually  $s_0$  is called the **initial value** of the recurrence.

- 67. Use induction to show that there is one and only one solution to Recurrence (2.3) which begins with initial value  $s_0 = 1$ .
- 68. A **linear first-order recurrence** is a recurrence which expresses  $a_n$  in terms of  $a_{n-1}$  (to the first power) and other functions of  $n$ , but does not include any of  $a_0, \dots, a_{n-2}$  in the equation. Which of recurrences (2.3) through (2.8) are first-order recurrences?
- 69. Show that there is one and only one sequence  $\{a_n\}$  which has all of the following properties:
  - (a) it is defined for every nonnegative integer  $n$  (which means that it's an infinite sequence);
  - (b) it has a fixed initial value (which means  $a_0$  has a pre-determined value, say  $a$ );
  - (c) and it satisfies a linear first-order recurrence (that is,  $a_n = c a_{n-1} + d$  for some constants  $c, d$ ).
- 70. Use Recurrence 2.4 to find the number of functions from  $[n]$  to  $[k]$ .
- 71. (a) In repaying a mortgage loan with initial amount  $A$ , annual interest rate  $p\%$  (on a monthly basis), and a monthly payment of  $m$ , what recurrence describes the amount owed after  $n$  months of payments in terms of the amount owed after  $n - 1$  months? Some technical details: You make the first payment after one month. The amount of interest included in your monthly payment is  $.01p/12$  and the interest rate is applied to the amount you owed immediately after making your previous monthly payment.
- (b) Find a formula for the amount owed after  $n$  months.
- (c) Find a formula for the number of months needed to bring the amount owed to zero. Another technical point: If you were to make the standard monthly payment  $m$  in the last month, you might actually end up owing a negative amount of money. Therefore it is okay if the result of your formula for the number of months needed gives a non-integer number of months. The bank would just round up to the next integer and adjust your payment so your balance comes out to zero.
- (d) What should the monthly payment be to pay off the loan over a period of 30 years?
- 72. (a) Find a recurrence for the number of ways to divide  $2n$  people into sets of two for tennis games. Be sure to give the initial value.
- (b) Give a recurrence for the number of ways to divide  $2n$  people into sets of two for tennis games in which the first server is determined.

- (c) In each of the previous two parts, use your recurrences to write the number of ways as a product.
73. Draw  $n$  mutually intersecting circles in the plane so that each one crosses each other one exactly twice and no three intersect in the same point. Find a recurrence for the number  $r_n$  of regions into which the plane is divided by  $n$  circles. (One circle divides the plane into two regions, the inside and the outside.) Find the number of regions with  $n$  circles. For what values of  $n$  can you draw a Venn diagram showing all the possible intersections of  $n$  sets using circles to represent each of the sets?
- Hint.* Suppose  $n - 1$  circles have been drawn in such a way that they define  $r_{n-1}$  regions. When you draw a new circle, each time it crosses a new circle it finishes dividing one region into two parts and starts dividing a new region into two parts.

## 2.3 The General Product Principle

Although the Product Principle in Chapter 1 can be applied directly to solve problems such as Problems 5 and 6, the reasoning can be cumbersome. An easier way to work this type of counting problem is to think in terms of making a *sequence of choices* as in the next problem.

- 74. Suppose you make a sequence of  $m$  choices, where
- the first choice can be made in  $k_1$  ways, and
  - for each way of making the first  $i - 1$  choices, the  $i$ -th choice can be made in  $k_i$  ways.

Explain why this is an inductive process. In how many different ways may you make your sequence of  $m$  choices? At this time you need not prove your answer is correct, but simply write your answer in the next box.

### The General Product Principle

Suppose you make a sequence of  $m$  choices, where

- the first choice can be made in  $k_1$  ways, and
- for each way of making the first  $i - 1$  choices, the  $i$ -th choice can be made in  $k_i$  ways.

Then the total number of different ways to make this sequence of  $m$  choices is:

- + 75. Return to Chapter 1 and re-do Problems 5, 6, and 19 by applying the General Product Principle. In each case, write the problem as a sequence of choices, count  $k_i$  for each choice, and then apply the principle.
- 76. Use the General Product Principle to count the number of bijections on the set  $[k]$ . Explain. (It might be a good idea to work through some of the cases  $k = 2, 3$ , and 4 before doing the general case.)
  - 77. Let  $S(2)$  denote the statement of the General Product Principle for a sequence of  $m = 2$  choices. Let  $\mathcal{S}$  denote the set of all different ways to make these two choices.
    - (a) Write out  $S(2)$  explicitly.
    - (b) Remember that the first choice can be made in any of  $k_1$  ways and let  $I_1, I_2, \dots, I_{k_1}$  be the specific items that can be chosen for the first choice. Letting  $B_j$  be the subset of  $\mathcal{S}$  for which the first choice made was the item  $I_j$ , find the size of  $B_j$  for each  $j$ .
    - (c) Explain why  $B_1, \dots, B_{k_1}$  is a partition of  $\mathcal{S}$ .
    - (d) What principle allows you to conclude that the size of  $\mathcal{S}$  equals  $k_1 k_2$ ?
  - 78. Use mathematical induction to prove the General Product Principle.

#### 2.3.1 Counting the number of functions

Problem 70 is now revisited from the perspective of the General Product Principle.

- 79. Use the General Product Principle to prove the number of functions from an  $m$ -element set to a  $n$ -element set is  $m^n$ . A common notation for the set of all functions from a set  $M$  to a set  $N$  is  $N^M$ .

+ 80. Now suppose you are thinking about the set  $\mathcal{S}$  of functions  $f$  from  $[m]$  to  $[n]$ . (For example, the set of functions from the three possible places for scoops in an ice-cream cone to 12 flavors of ice cream.) Suppose  $f(1)$  can be chosen in  $k_1$  ways. (In the ice cream problem,  $k_1 = 12$  holds because there were 12 ways to choose the first scoop.) Suppose that for each choice of  $f(1)$  there are  $k_2$  choices for  $f(2)$ . (For the ice cream cones,  $k_2 = 12$  when the second flavor could be the same as the first, but  $k_2 = 11$  when the flavors had to be different.) In general, suppose that for each choice of  $f(1), f(2), \dots, f(i-1)$ , there are  $k_i$  choices for  $f(i)$ . What has been assumed so far about the functions in  $\mathcal{S}$  may be summarized as:

- There are  $k_1$  choices for  $f(1)$ .
- For each choice of  $f(1), f(2), \dots, f(i-1)$ , there are  $k_i$  choices for  $f(i)$ .

How many functions are in the set  $\mathcal{S}$ ? Is there any practical difference between the result of this problem and the General Product Principle?

The point of Problem 80 is that originally the statement the General Product Principle was somewhat informal. To be more mathematically precise it is a statement about counting sets of functions.

81. This problem revisits the question: How many subsets does a set  $S$  with  $n$  elements have? (Compare with Problems 26 and 56.)
  - (a) For the specific case of  $n = 3$ , describe a sequence of three decisions which could be made to yield subsets of  $[3]$ . Apply the General Product Principle to find the number of subsets of  $[3]$ . Re-work this from a function point of view.
  - (b) Use the functional interpretation of the General Product Principle to prove that a set with  $n$  elements has  $2^n$  subsets.
- 82. In how many ways can you pass out  $k$  distinct pieces of fruit to  $n$  children (with no restriction on how many pieces of fruit a child may get)?
- 83. Assuming  $k \leq n$ , in how many ways can you pass out  $k$  distinct pieces of fruit to  $n$  children if each child may get at most one? What is the number if  $k > n$ ? Assume for both questions that you pass out all the fruit. Note that each of these is a list of  $k$  distinct things chosen from a set  $S$  (of children).
- 84. In combinatorics, an (ordered) list of  $k$  distinct things chosen from a set  $S$  is called a **k-element permutation of S**. How many  $k$ -element permutations does an  $n$ -element set have?



- 85. Find the number of one-to-one functions from a  $k$ -element set to an  $n$ -element set. Now review your solutions to Problem 8.

Donald Knuth invented the notation  $n^{\underline{k}}$ , read “ $n$  to the  $k$  falling” or “ $n$  to the  $k$  down” for the number you just found:

$$n^{\underline{k}} = n(n-1) \cdots (n-k+1) = \prod_{i=1}^k (n-i+1). \quad (2.9)$$

## Chapter 3

# Equivalence Relations

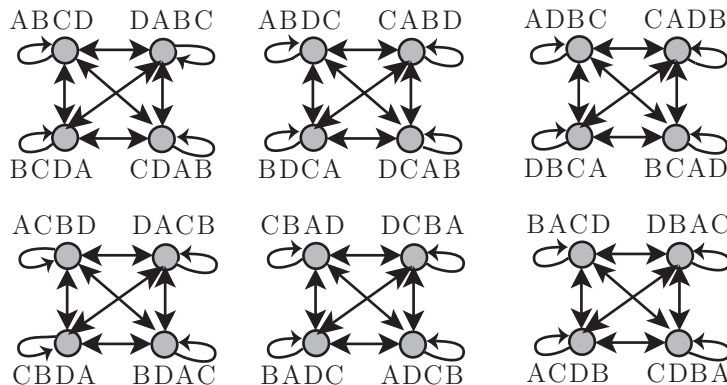
### 3.1 Equivalence Relations

Equivalence relations have been in the background of some of the problems you've already worked. For instance, in Problem 9 with three distinct flavors at first it was probably tempting to say there are 12 flavors for the first pint, 11 for the second, and 10 for the third, so there are  $12 \cdot 11 \cdot 10$  ways to choose the pints of ice cream. However, once the pints have been chosen, bought, and put into a bag, there is no way to tell which one was bought first, which second, and which third. The number  $12 \cdot 11 \cdot 10$  is the number of *lists* of three distinct flavors, in which the order in which the pints are bought makes a difference. Two of the lists become equivalent after the ice cream purchase if they have the same flavors of ice cream. In other words, two of these lists are equivalent (are related for this problem) if they list the same subset of the set of twelve ice cream flavors. To visualize this relation with a digraph, one vertex would be needed for each of the  $12 \cdot 11 \cdot 10 = 1320$  lists (which is not feasible to draw). Even with five flavors of ice cream the number of vertices would be  $5 \cdot 4 \cdot 3 = 60$ . So for now the easier-to-draw question of choosing three pints of different ice cream flavors from a choice of *four* flavors of ice cream will be considered. For this, there are  $4 \cdot 3 \cdot 2 = 24$  different lists.

- 86. Suppose you have four flavors of ice cream: V(anilla), C(hocolate), S(trawberry) and P(each). Draw the directed graph whose vertices are all lists of three distinct flavors of the ice cream, and whose edges connect two lists if they list the same three flavors. This graph makes it pretty clear in how many “really different” ways you may choose 3 flavors from four. How many?

- 87. Now again suppose you are choosing three distinct flavors of ice cream out of four possible flavors, but instead of putting scoops in a cone or choosing pints, the three scoops will be arranged symmetrically in a circular dish. The original lists are the same as in the last problem. Namely, you can describe a selection of ice cream in terms of which one goes in the dish first, which one goes in second (say to the right of the first), and which one goes in third (say to the right of the second scoop, which makes it to the left of the first scoop). But here two of these lists will be considered equivalent if once they are in the dish no one can tell which scoop went in first. Think about what makes two lists of flavors equivalent, and draw the directed graph whose vertices consist of all 24 lists of three of the flavors of ice cream and whose edges connect two lists between which you cannot distinguish as dishes of ice cream. How many dishes of ice cream can be distinguished from one another?
- 88. Consider two seating arrangements of people around a round table to be equivalent if you can get from one arrangement to the other by having everyone get up, move one chair to the right, and then sit down again. Explain how the digraph in Figure 3.1 describes the seating arrangements of four people at a round table. In your explanation, be sure to tell what the arrows signify and what the disconnected appearance of the digraph signifies. How many non-equivalent seating arrangements of four people are there?

Figure 3.1: Digraph of arranging four people at a round table.



In the last few problems, you began with a set of lists and first you had to decide when two lists were equivalent representations of the objects you were trying to count. Then you drew the directed graph for that particular relation of equivalence. Go back to your digraphs and check that each vertex has an arrow to itself. This is what is meant when it is said that a relation is **reflexive**. Also, check that whenever you have an arrow from one vertex to a second, there is an arrow from the second back to the first. This is what is meant when a relation is said to be **symmetric**. There is another property of those relations you have graphed. Namely, whenever you have an arrow from  $L_1$  to  $L_2$  and an arrow from  $L_2$  to  $L_3$ , then there is an arrow from  $L_1$  to  $L_3$ . This is what is meant when a relation is said to be **transitive**. A relation on a set is called an **equivalence relation** on the set when it satisfies all three of these properties.

89. If  $R$  is a relation on a finite set, how can the digraph be used to check whether or not the relation is reflexive? To check symmetry? To check transitivity?
90. The **adjacency matrix** of a relation  $R$  on an  $n$ -element set  $\{x_1, \dots, x_n\}$  is defined to be the  $n \times n$  matrix  $A$  whose  $(i, j)$  entry equals the number of edges from  $x_i$  to  $x_j$  in the associated digraph. For at least three relations on the set [4], find the adjacency matrix  $A$  and calculate  $A^2$ . Use this information to formulate a conjecture about what is recorded in the entries of  $A^2$ . Explain how the matrix difference  $A - A^2$  can be used to determine whether the relation is transitive.

Check that each relation of equivalence in Problems 86, 87 and 88 satisfies the three properties, and so each is an equivalence relation. Carefully visualize the same three properties in the relations of equivalence that you use in the remaining problems of this chapter. Work through Appendix C if you would like more practice with checking whether or not a relation is an equivalence relation.

You undoubtedly have noticed that for each of the equivalence relations you've considered so far, the digraph is divided into clumps of mutually connected vertices. In the next section you will show that this "clumping" property holds for all equivalence relations, and that it is exactly this property which makes equivalence relations a powerful tool for counting.

## 3.2 Equivalence Classes

Next is an investigation of the “clumping” behavior you observed in the digraphs of the equivalence relations in the last section.

- 91. Describe the equivalence relation inherent in seating  $n$  people around a round table in such a way that they’re equally spaced around the table. Check that it is an equivalence relation on the set of all  $n!$  lists of  $n$  people. This equivalence relation induces a partition of the set of lists. What size blocks occur? How many non-equivalent seating arrangements are there? Compare this problem with Problem 88.
- 92. In this problem, consider the question of making necklaces by arranging  $n$  distinguishable beads on a string. Assume that once all  $n$  beads are placed on the string its ends are carefully knotted so the knot cannot be seen and that the beads are equally spaced on the string. Regard two necklaces as equivalent if when both are placed on a table, a person can pick up one of the necklaces, move it around in space, and put it back down so that it exactly matches the other necklace. Describe this as an equivalence class on some set. What size blocks of necklaces occur? How many non-equivalent necklaces can be constructed using  $n$  distinguishable beads?
- 93. Verify explicitly that your relationship of equivalent necklaces is reflexive, symmetric, and transitive, and so is an equivalence relation.

Some standard notation is useful for working with equivalence relations:

When  $R$  is a relation on the set  $\mathcal{X}$ , the notation  $xRy$  will indicate that  $x, y \in \mathcal{X}$  are related under  $R$ . Using this notation, the relation

- $R$  is called **reflexive** if  $xRx$  for every  $x \in X$ .
- $R$  is called **symmetric** if  $xRy$  holds whenever  $yRx$  holds.
- $R$  is called **transitive** if whenever both  $xRy$  and  $yRz$ , then  $xRz$  as well.
- $R$  is an **equivalence relation** if  $R$  is reflexive, symmetric and transitive.

Suppose that  $R$  is an equivalence relation on a set  $X$  and for each  $x \in X$ , define the set  $C_x$  by  $C_x = \{y \in X : yRx\}$ . That is,  $C_x$  is the subset of  $X$  consisting of all elements of  $X$  which are equivalent to the element  $x$  under

the given equivalence relation. (Can any of the sets  $C_x$  be empty? Yes? No? Why?) The sets  $C_x$  are called the **equivalence classes** of the relation  $R$ . Find the equivalence classes for the seating relation whose digraph is given in Figure 3.1.

- 94. In Problem 91 the equivalence classes correspond to seating arrangements. For each of Problems 86, 87, and 92, describe (that is, using complete English sentences) what the equivalence classes correspond to. Three answers are expected here.
- 95. Let  $R$  be an equivalence relation on a set  $X$ .
  - (a) If the classes  $C_x$  and  $C_z$  have an element  $y$  in common, what can you conclude about the sets  $C_x$  and  $C_z$  (besides the fact that they have an element in common!)? Be explicit about what property(ies) of equivalence relations justify your answer.
  - (b) Why is every element of  $X$  in some class  $C_x$ ? Be explicit about what property(ies) of equivalence relations you are using to answer this question.
  - (c) Explain why two distinct sets  $C_x$  and  $C_z$  are disjoint. What do these sets have to do with the “clumping” you saw in the digraphs of Problems 86 and 87?

You have just proved that if  $R$  is an equivalence relation on the set  $X$ , then each element of  $X$  is in exactly one equivalence class of  $R$ . That is,

**Theorem 1.** *If  $R$  is an equivalence relation on  $X$ , then the set of equivalence classes of  $R$  forms a partition of  $X$ .*

In each of Problems 86, 87, 91, and 92 when you counted the number of equivalence classes of an equivalence relation there was a special structure to the problems that made this somewhat easy to do. For example, in Problem 86, there was a set of  $4 \cdot 3 \cdot 2 = 24$  lists of three distinct flavors chosen from V, C, S, and P. Each list was equivalent to  $3 \cdot 2 \cdot 1 = 3! = 6$  lists (including itself) since the order in which you selected the three flavors was unimportant. This says that each of the equivalence classes has size 6, and the set of all  $4 \cdot 3 \cdot 2$  lists was a union of some number  $n$  of equivalence classes, each of size 6. By the Product Principle, if you have a union of  $n$  disjoint sets, each of size 6, the union has  $6n$  elements. But you already knew that the union was the set of all 24 lists of three distinct letters chosen from the four letters. Thus,  $6n = 24$ , proving there are  $n = 4$  equivalence classes.

In Problem 87: If you choose the flavors V, C, and S, and arrange them in the dish with C to the right of V and S to the right of C, then the scoops

are in different relative positions than if you arrange them instead with S to the right of V and C to the right of S. Thus the order in which the scoops go into the dish is somewhat important, only somewhat because putting in V first, then C to its right and S to its right is the same as putting in S first, then V to its right and C to its right. In this case, each list of three flavors is equivalent to only three lists, including itself, and so if  $n$  is the number of equivalence classes, you have  $3n = 24$ . This gives  $24/3 = 8$  equivalence classes.

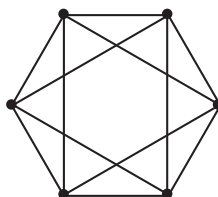
- 96. Given the partition  $\{1, 3\}, \{2, 4, 6\}, \{5\}$  of the set  $X = \{1, 2, 3, 4, 5, 6\}$ , define two elements of  $X$  to be related if they are in the same block of the partition. That is, define 1 to be related to 3 (and 1 and 3 each related to itself), define 2 and 4, 2 and 6, and 4 and 6 to be related (and each of 2, 4, and 6 to be related to itself), and define 5 to be related to itself. Show that this relation is an equivalence relation.
- 97. Suppose  $P = \{S_1, S_2, S_3, \dots, S_k\}$  is a partition of  $S$ . Define two elements of  $S$  to be related if they are in the same set  $S_i$ , and otherwise not to be related. Show that this relation is an equivalence relation on the set  $S$ . Show that the equivalence classes of the equivalence relation are the sets  $S_i$ .

In Problem 97 you just proved that every partition of a set gives rise to (or induces) an equivalence relation, and that the classes of the induced equivalence relation are the blocks of the original partition.

- 98. In how many ways can you attach two identical red beads and two identical blue beads to the corners of a square (with one bead per corner) if the square is free to move around in (three-dimensional) space? What is the underlying equivalence relation and its equivalence classes? Write out the equivalence classes as sets of lists. For this problem it might be helpful to just draw some pictures of the possible configurations since there aren't that many.
- 99. (This has already appeared as Problem 72. Equivalence relations can be used for a different approach.) A tennis club has  $2n$  members, and you want to pair the members by twos for singles matches.
  - (a) In how many different ways can you list all  $2n$  members of the club?
  - (b) Define an equivalence relation on the set of all lists which can be used to identify the lists which give rise to the same pairings.
  - (c) Find the size of an equivalence class, being careful to explain why all equivalence classes have the same size.

- (d) In how many ways can you pair up all the members of the club for singles matches?
  - (e) Suppose that in addition to specifying who plays whom for each pairing you also specify who serves first. Now in how many ways can you specify your pairs? Use your solution to part (d), if possible.
100. Suppose you plan to put six distinct computers in a network as shown in Figure 3.2 where the computers are the nodes.
- (a) What is the total number of ways to assign computers to the nodes (or the vertices) of this regular hexagon?
  - (b) The edges (line segments) show which computers can communicate directly with which others. Consider two ways of assigning computers to the nodes of the network to be different if there are at least two computers that communicate directly in one assignment and that don't communicate directly in the other. Define an equivalence relation which models the equivalence in this situation.
  - (c) Prove every equivalence class has 48 elements.
  - (d) In how many different ways can you assign computers to the network?

Figure 3.2: A computer network.



### 3.3 Counting Subsets

The symbol  $\binom{n}{k}$  is used to represent the number of ways to choose a  $k$ -element subset from an  $n$ -element set. You may have already seen this notation elsewhere, but don't be concerned if you haven't seen it because it will be developed completely here. The symbol should be read as “ $n$  choose  $k$ ”. (Another common way to read the binomial coefficient notation is “the number of combinations of  $n$  things taken  $k$  at a time” but we've found that this can be confusing and so please don't read the notation that way in this



course.) Sometimes  $\binom{n}{k}$  is written as  $C(n, k)$ , or  ${}_nC_k$ , but  $\binom{n}{k}$  is more standard in discrete mathematics and should be the only notation you use here.

- 101. Using only the given definition of  $\binom{n}{k}$  (that it equals the number of ways to choose a  $k$ -element subset from an  $n$ -element set) and the Bijection Principle, prove that

$$\binom{n}{k} = \binom{n}{n-k} \text{ for all } 0 \leq k \leq n.$$

- 102. Use the distributive law to check that

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

Explain why this means that  $\binom{4}{1} = 4$  and  $\binom{4}{2} = 6$ . (Think this through carefully!) Give a general argument for why the coefficient of  $x^k y^{n-k}$  in  $(x + y)^n$  equals  $\binom{n}{k}$ . For this reason, the symbol  $\binom{n}{k}$  is called a **binomial coefficient**.

In the next problems you will use equivalence relations to find a formula for calculating  $\binom{n}{k}$ . Although you may have seen this formula before, do not calculate the actual numbers in these problems but rather simply use binomial coefficients in your answers.

- 103. A basketball team has 12 members of which only five can play at any given time during a game.
  - (a) In how many ways may the coach choose the five players?
  - (b) To be more realistic, the five players normally consist of two guards, two forwards, and one center. If there are five guards, four forwards, and three centers on the team, in how many ways may the coach choose the team to be sent out on the court?
  - (c) If one of the centers is equally skilled at playing forward, how many different teams may the coach send out to play?
- 104. In how many ways can you pass out  $k$  (identical) ping-pong balls to  $n$  children if each child may get at most one? (Ask yourself “What is a problem like this doing in the middle of a bunch of problems about counting subsets of a set? Is it related? Or is it supposed to give a break from sets?”)
- 105. Letting  $\mathcal{S}$  denote the set of all 3-element permutations of  $\{a, b, c, d, e\}$ , Table 3.1 lists all elements of  $\mathcal{S}$  exactly once and the list is given in such a way that each row of the table lists all permutations of a certain

Table 3.1: The 3-element permutations of  $\{a, b, c, d, e\}$  organized by rows according to which 3-element set they permute.

<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>
<i>abd</i>	<i>adb</i>	<i>bad</i>	<i>bda</i>	<i>dab</i>	<i>dba</i>
<i>abe</i>	<i>aeb</i>	<i>bae</i>	<i>bea</i>	<i>eab</i>	<i>eba</i>
<i>acd</i>	<i>adc</i>	<i>cad</i>	<i>cda</i>	<i>dac</i>	<i>dca</i>
<i>ace</i>	<i>aec</i>	<i>cae</i>	<i>cea</i>	<i>eac</i>	<i>eca</i>
<i>ade</i>	<i>aed</i>	<i>dae</i>	<i>dea</i>	<i>ead</i>	<i>eda</i>
<i>bcd</i>	<i>bdc</i>	<i>cbd</i>	<i>cdb</i>	<i>dbc</i>	<i>dcb</i>
<i>bce</i>	<i>bec</i>	<i>cbe</i>	<i>ceb</i>	<i>ebc</i>	<i>ecb</i>
<i>bde</i>	<i>bed</i>	<i>dbe</i>	<i>deb</i>	<i>ebd</i>	<i>edb</i>
<i>cde</i>	<i>ced</i>	<i>dce</i>	<i>dec</i>	<i>ecd</i>	<i>edc</i>

3-element subset of  $\{a, b, c, d, e\}$ . Since each 3-element permutation appears exactly once, the rows of the table determine a partition of the set  $\mathcal{S}$ . From Problem 97 you know any partition is the set of equivalence classes of some equivalence relation. Find the equivalence relation for this partition of  $\mathcal{S}$ .

- 106. Rather than restricting to  $n = 5$  and  $k = 3$ , it is possible to partition the set of all  $k$ -element permutations of an  $n$ -element set (which can be assumed to be the set  $[n]$ ) into equivalence classes.

Let  $\mathcal{S}$  be the set of all  $k$ -element permutations of  $[n]$ , and for  $s_1, s_2 \in \mathcal{S}$ , define

$$s_1 R s_2 \iff s_1 \text{ has the same elements as } s_2 .$$

- Prove  $R$  is an equivalence relation on  $\mathcal{S}$ .
- How many elements are in any equivalence class?
- What is the size of  $\mathcal{S}$ ? (You found this earlier. In which problem?)
- Write a carefully worded sentence that describes a bijection between the set of equivalence classes of  $R$  and the set of  $k$ -element subsets of  $[n]$ .
- What formula does this give you for the number  $\binom{n}{k}$  of  $k$ -element subsets of an  $n$ -element set?

In the last problem sequence you proved the following formula.

### Calculation of Binomial Coefficients

For any  $k, n \geq 0$  with  $k \leq n$ , the number of  $k$ -element subsets of an  $n$ -element set is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

- 107. Use the above formula to find numerical answers to Problem 103.
- 108. You can write  $n$  as a sum of  $n$  ones.
  - (a) How many plus signs did you use in the sum?
  - (b) In how many ways can you write  $n$  as a sum of a list of  $k$  positive numbers? Such a list is called a **composition of the integer  $n$  into  $k$  parts**.
  - (c) Find the total number of compositions of  $n$ , that is, into any number of parts. Compare this with your solution to Problem 59.
- 109. The answer in Problem 108(b) can be expressed as a binomial coefficient. This means it should be possible to interpret a composition into  $k$  parts as a subset of some set. Find a bijection between compositions of  $n$  into  $k$  parts and certain subsets of some set. Be sure to explain explicitly what your function is; that is, tell how to get the subset from the composition.
- 110. Give a recurrence for the number of ways to divide  $4n$  people into sets of four for games of bridge. (Don't worry about how they sit around the bridge table or who is the first dealer.)
- 111. A town has  $n$  street lights running along the north side of Main Street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?  
*Hint.* First try for small  $n$ .
- 112. This is the third time you've seen this problem. Here, binomial coefficients should be used to construct a solution. A tennis club has  $2n$  members, and you want to pair up the members by twos for singles matches.
  - (a) How many ways can you list all possible pairs of these  $2n$  tennis players, where each list consists of  $n$  unordered pairs?
  - (b) Define an equivalence relation on the set of lists from part (a) which identifies lists which yield the same pairings for singles matches if you don't care who serves first.

- (c) Find the number of pairings of  $2n$  people for singles matches if you don't care who serves first.

You now have many methods for solving the following problem. Try to solve it several ways. Don't settle for doing it just one way!

113. A tennis club has  $4n$  members. To specify a doubles match, you choose two teams of two people. In how many ways can you arrange the members into doubles matches so that each player is in one doubles match? In how many ways can you do it if you also specify who serves first on each team?

### 3.3.1 Pascal's Triangle

- 114. Let  $C$  be the set of all  $k$ -element subsets of  $[n]$  that contain the number  $n$ , and let  $D$  be the set of all  $k$ -element subsets of  $[n]$  that don't contain  $n$ .
  - (a) Find the sets  $C$  and  $D$  for  $n = 5$  and  $k = 2$ .
  - (b) Let  $C'$  be the set of  $(k - 1)$ -element subsets of  $[n - 1]$ . Describe a bijection from  $C$  to  $C'$ . (A verbal description is fine.)
  - (c) Let  $D'$  be the set of  $k$ -element subsets of  $[n - 1]$ . Describe a bijection from  $D$  to  $D'$ . (A verbal description is fine.)
  - (d) Based on the two previous parts, express the sizes of  $C$  and  $D$  in terms of binomial coefficients involving  $n - 1$ .
  - (e) Apply the Sum Principle to  $C$  and  $D$  to obtain a formula that expresses  $\binom{n}{k}$  in terms of two binomial coefficients involving  $n - 1$ . This formula is a recurrence in the two variables  $n, k$ .

Write your solution to Problem 114 (e) in the following box:

**Pascal's Equation**

115. Let  $S(k, n)$  denote the number of partitions of  $[k]$ -element set into  $n$  non-empty blocks. Show that for  $k \geq n > 1$ ,  $S(k, n)$  satisfies the recurrence

$$S(k, n) = S(k - 1, n - 1) + nS(k - 1, n).$$

*Hint.* In any partition of  $[k]$ , the number  $k$  is either in a block by itself or it is not.

You will return to  $S(k, n)$  in Section 7.2.

In Problem 114 (e) you derived **Pascal's Equation** which is the basis for the famous **Pascal's Triangle**, the triangle in Figure 3.3 whose rows and columns are numbered so that the top row is the 0-th row and the initial entry of a row is called the 0-th number in the row. Then the  $n$ -th row is set up as follows: the number of  $k$ -element subsets of an  $n$ -element set is the  $k$ -th number over in the  $n$ -th row. Your formula from the last problem doesn't say anything about  $\binom{n}{k}$  when  $k = 0$  or  $k = n$ , but otherwise it says that each entry is the sum of the two that are above it and just to the left or right.

Figure 3.3: Pascal's Triangle

						1						
						1		1				
					1	2		1				
			1		3	3		1				
		1		4	6		4		1			
	1		5	10	10		5		1			
	1	6		15	20		15		6		1	
1		7		21	35		35		21		7	1

116. Just for practice, use your formula to get the 8-th row of Pascal's Triangle.
117. Without writing out any more complete rows, write enough of Pascal's Triangle to get a numerical answer for the first question in Problem 9. Try to do this as efficiently as possible. You should be able to get the answer by writing down at most 10 numbers (including the answer).
- 118. Give an inductive proof of the Binomial Theorem, using the fact that  $(x + y)^n = (x + y)(x + y)^{n-1}$ . In case you don't know the Binomial

Theorem, you can discover it as you work through the proof. Your goal is to express  $(x + y)^n$  as the sum of terms of the form

$$\text{something } x^k y^{n-k}$$

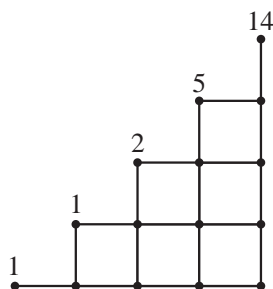
where for each  $k$  the “something” is a binomial coefficient.

### 3.3.2 Catalan numbers (Optional)

- 119. In part of a certain city, all streets run either north-south or east-west, and there are no dead ends. Suppose you are standing on a street corner. In how many ways can you walk to a corner that is four blocks north and six blocks east, using as few blocks as possible?
- 120. The last problem has a geometric interpretation in a coordinate plane. A **lattice path** in the plane is a sequence of line segments which either go from a point  $(i, j)$  to the point  $(i + 1, j)$  or from a point  $(i, j)$  to the point  $(i, j + 1)$ , where  $i$  and  $j$  are integers. (Lattice paths always move either up or to the right.) The **path length** is the number of such line segments in the path.
  - (a) What is the length of a lattice path from  $(0, 0)$  to  $(m, n)$ ?
  - (b) Show there are exactly  $\binom{m+n}{n}$  lattice paths from  $(0, 0)$  to  $(m, n)$ .
  - (c) How many lattice paths are there from  $(i, j)$  to  $(m, n)$ , assuming  $i, j, m$ , and  $n$  are all integers? Be careful: What happens to the answer if  $i > m$  or  $j > n$ ? Remember that paths go up or to the right.
- 121. Admission to a school play requires a ten-dollar donation per person. Assume that each person who comes to the play only pays for themselves and that the payment is made with either a ten-dollar bill or a twenty-dollar bill. The teacher who is collecting the money forgot to get change before the event. If there are always at least as many people who have paid with a ten as a twenty as they arrive, the teacher won't have to give anyone an IOU for change. Suppose  $2n$  people come to the play, and exactly half of them pay with ten-dollar bills.
  - (a) Describe a bijection between the set of sequences of tens and twenties given to the teacher and the set of lattice paths from  $(0, 0)$  to  $(n, n)$ . (Be sure to explain why your map is a function, one-to-one, and onto.)
  - (b) What is the geometric interpretation of a sequence that does not require the teacher to give any IOUs?

Notice that a lattice path from  $(0,0)$  to  $(n,n)$  stays inside (or on the edges of) the square whose sides are the  $x$ -axis, the  $y$ -axis, the line  $x = n$  and the line  $y = n$ . It may or may not stay within or on the triangle whose sides are the  $x$ -axis, the line  $x = n$  and the line  $y = x$ . Any lattice path that does stay within this triangle is called a **Catalan path**. Figure 3.4 shows the lattice points which form the triangle of interest for  $n = 4$ , where the sides are the  $x$ -axis, the line  $x = 4$  and the line  $y = x$ . The numbers given in the figure are the number of Catalan paths to the indicated point. Check that these numbers are correct.

Figure 3.4: The Catalan paths from  $(0,0)$  to  $(i,i)$  for  $i = 0, 1, 2, 3, 4$ . The number of paths to the point  $(i,i)$  is written just above the point.



- 122. Let  $\mathcal{P}$  be a lattice path from  $(0,0)$  to  $(n,n)$  which is not Catalan.
  - (a) Show the path  $\mathcal{P}$  must have at least one point on the line  $y = x + 1$ . Let the point  $P$  be the point whose  $x$ -coordinate is least among all these points of intersection with the line  $y = x + 1$ .
  - (b) Take the part of path  $\mathcal{P}$  which lies from  $(0,0)$  to this point  $P$ , and reflect it about the line  $y = x + 1$ . (That is, replace every upstep with a step one unit to the left and every right step with a step one unit down.) Show that this new path is a lattice path from  $(-1,1)$  to  $(n,n)$ . If you're having trouble for a general  $n$ , try it on a few non-Catalan lattice paths from  $(0,0)$  to  $(4,4)$ .

In Problem 122 you transformed a lattice path  $\mathcal{P}$  from  $(0,0)$  to  $(n,n)$  which is not Catalan into a lattice path from  $(-1,1)$  to  $(n,n)$ . This method of construction is called the **Feller Reflection Principle**.

- 123. In Problem 122 you used the Feller Reflection Principle to show that each non-Catalan path from  $(0,0)$  to  $(n,n)$  *determines* a lattice path

from  $(-1, 1)$  to  $(n, n)$ . This process therefore defines a function  $\Phi$  from the set of all non-Catalan paths from  $(0, 0)$  to  $(n, n)$  into the set of all lattice paths from  $(-1, 1)$  to  $(n, n)$ .

- (a) Check that  $\Phi$  is injective for the special case of  $n = 4$ . Write a convincing proof that  $\Phi$  is injective for all  $n$ .
- (b) Prove  $\Phi$  is a bijection.
- (c) How many non-Catalan paths are there from  $(0, 0)$  to  $(n, n)$ ?
- (d) How many Catalan paths are there from  $(0, 0)$  to  $(n, n)$ ? This is called the **Catalan number**  $C_n$ .

Refer to Richard Stanley's website in the Applied Mathematics Department at MIT for at least sixty-six combinatorial interpretations of the Catalan numbers.

### 3.4 Ordered-functions and Multisets

Suppose you wish to place  $k$  distinct books (here “distinct” will always mean that the objects are distinguishable one from the other) onto the shelves of a bookcase with  $n$  shelves. Assume that the shelves are long enough so that all of the books would fit on any of the shelves. Also, let's imagine that after you're done putting books on the shelves, you push the books on a shelf as far to the left as possible. This means that you're only thinking about how the books sit relative to each other, not about the exact places where you put any book. Since the books are all different, you can number them as the first book, the second book, and so on.

- 124. (a) In how many ways can you place the first of the  $k$  books on one of the  $n$  shelves?
- (b) When you go to place the second book, if you decide to place it on the shelf that already has a book does it matter if you place it to the left or right of the book that is already there? In how many ways can you place the second book into the bookcase?
- (c) In how many ways can you place the  $i$ th book into the bookcase?
- (d) In how many ways can you place all  $k$  books into the bookcase?
- 125. Suppose you wish to place the  $k$  distinct books so that in addition to the other requirements each shelf gets at least one book. Now in how many ways can you place the books?

*Hint.* Do something before you start the process described in Problem 124.



The suggestion intended by the hint in the last problem was to consider a two-step process. There are other ways to solve the problem. For instance, imagine first lining up the  $k$  books in a row, giving  $k - 1$  spaces between books. Choose  $n - 1$  of these spaces in which to slide a piece of paper as a divider. Now put the books before the first divider on shelf one, and the books after divider  $i$  on shelf  $i + 1$ . This gives an arrangement of the books on the shelves so that every shelf has a book.

- 126. Use the method just described to solve Problem 125.

For any given arrangement of books in the bookcase, the assignment of a book to the shelf on which it was put is a function from the set of books to the set of shelves. But this function doesn't give all information since it only records which shelf each book is on. It doesn't say which book sits to the left of which others on the shelf, information which is an important part of how the books are arranged on the shelves. In other words, the order in which the shelves receive their books matters. So, in order to record all relevant information for the arrangement, we must assign an *ordered list* of books to each shelf. In these notes such a map will be called an ordered-function, and the word is hyphenated because an ordered-function from  $\mathcal{S}$  to  $\mathcal{T}$  is in general *not a function* from  $\mathcal{S}$  to  $\mathcal{T}$ . The phrase "ordered-function" is not a standard one and in fact there is not yet a standard name for the result of an ordered distribution problem.

More precisely, an **ordered-function** from a set  $\mathcal{S}$  to a set  $\mathcal{T}$  is a map that assigns an ordered list of elements of  $\mathcal{S}$  (books) to some elements of  $\mathcal{T}$  (bookshelves) in such a way that each element of  $\mathcal{S}$  appears on one and only one of the lists. An **ordered-onto-function** is an ordered-function from  $\mathcal{S}$  to  $\mathcal{T}$  that assigns a list to *every* element of  $\mathcal{T}$ . In Problem 124 you counted the number of ordered-functions from  $[k]$  to  $[n]$  and in Problem 125 the number of ordered-onto-functions from  $[k]$  to  $[n]$ .

- 127. Let  $S$  be the set of all arrangements of  $k$  different books in an  $n$ -shelf bookcase.
  - (a) Consider the following relation on  $S$ : Two arrangements are related if and only if the two arrangements have the same number of books on each shelf. Prove this relation is an equivalence relation.
  - (b) Each equivalence class has the same number of elements. Show that this is true and find the number.
  - (c) Find the number of different equivalence classes, and write this number as a binomial coefficient.

- (d) Find the number of arrangements of  $k$  identical books on  $n$  shelves. Explain.
- 128. Use an equivalence relation to count how many ways you can put  $k$  identical books onto  $n$  distinct shelves if each shelf must get at least one book.
- A **multiset** chosen from a set may be thought of as a kind of generalized subset in which repeated elements are allowed. To determine a multiset you must say how many times (allowing the possibility of zero times) each member appears in the multiset.
- 129. (a) Find a bijection between arrangements of  $k$  identical books on  $n$  shelves and multisets chosen from  $[n]$ .  
 (b) What is the number of multisets of size  $k$  that can be chosen from  $[n]$ ?
- 130. Multisets can be used to give a more elegant solution to the earlier Problem 9: A group of three hungry members of the team in Problem 6 notices it would be cheaper to buy three pints of ice cream to share among the three of them. In how many ways may they choose three pints of ice cream with no restrictions on repeating flavors? Be sure to use multisets to solve this problem.
- 131. How many solutions are there in *nonnegative* integers to the equation  $x_1 + x_2 + \cdots + x_m = r$ , where  $m$  and  $r$  are fixed positive integers?
- 132. In how many ways can you distribute  $k$  identical objects to  $n$  distinct recipients so that each recipient gets at least  $m$  objects? (Here you assume that  $k \geq mn$ .)
- 133. Your answer to Problem 129 (b) is expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from some set  $\mathcal{S}$ .

### 3.5 The Existence of Ramsey Numbers (Optional)

In Chapter 1 the Ramsey Number  $R(m, n)$  was defined to be the smallest number  $R$  such that if there are  $R$  people in a room, then there is a subset of the set of people which has either at least  $m$  mutual acquaintances or at least  $n$  mutual strangers. However, if you return to those earlier problems (page 19) you'll see that you never proved Ramsey Numbers exist for general  $m$  and  $n$ .

Provided you can show that there is *some* integer  $R$  such that for any group of  $R$  people there are either  $m$  mutual acquaintances or  $n$  mutual strangers, the Ramsey Number  $R(m, n)$  must exist, because it is the smallest such  $R$  (and every nonempty set of positive integers has a least element). In this section you will use mathematical induction to prove that  $\binom{m+n-2}{m-1}$  is one such  $R$ . This simultaneously proves that Ramsey numbers exist and that  $R(m, n) \leq \binom{m+n-2}{m-1}$ . The question is, what should be inducted on,  $m$  or  $n$ ? In other words, do you use the fact that with  $\binom{m+n-3}{m-2}$  people in a room there are at least  $m-1$  mutual acquaintances or  $n$  mutual strangers, or do you use the fact that with at least  $\binom{m+n-3}{n-2}$  people in a room there are at least  $m$  mutual acquaintances or at least  $n-1$  mutual strangers? It turns out that both will be used. That is, it is helpful to induct on  $m$  and  $n$ . One way to do that is to use yet another variation of induction, which will be called the Principle of Double Mathematical Induction.

### The Principle of Double Mathematical Induction

In order to prove that a sequence of statements  $S(m, n)$  indexed by integers  $m \geq a$  and  $n \geq b$  are all true, you can do both of the following steps:

1. Prove the statement  $S(a, b)$  for the fixed integers  $a$  and  $b$  is true.
2. Show that the truth of the statement  $S(m, n)$  for all values of  $m$  and  $n$  with  $a + b \leq m + n \leq K$  implies the truth of the statement  $S(m, n)$  for all pairs of integers  $m, n$  with  $m + n = K + 1$ .

Then the statement  $S(m, n)$  is true for all pairs of integers  $m \geq a$  and  $n \geq b$ .

- 134. (a) Use Double Induction and Pascal's Equation to prove that for any choice of  $\binom{m+n-2}{m-1}$  people in a room there are either at least  $m$  mutual acquaintances or at least  $n$  mutual strangers.
- (b) Prove that  $R(m, n)$  exists for every pair of integers  $m, n \geq 1$  and is no more than  $\binom{m+n-2}{m-1}$ .
135. Prove that  $R(m, n) \leq R(m-1, n) + R(m, n-1)$ .
- Hint.* Begin by explicitly stating what you need to prove. You don't need to use induction.
- 136. (a) What does the equation in Problem 135 say about  $R(4, 4)$ ?
- (b) Consider 17 people arranged in a circle such that each person is acquainted with the first, second, fourth, and eighth person to the right and the first, second, fourth, and eighth person to the left.

Can you find a set of four mutual acquaintances? Can you find a set of four mutual strangers?

(c) What is  $R(4, 4)$ ?

# Chapter 4

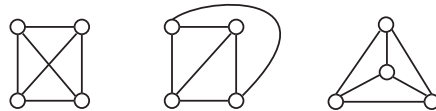
## Graph Theory

### 4.1 Undirected Graphs

The idea of a directed graph (or digraph) was introduced in Section 1.3. In this chapter you will work with undirected graphs, usually simply called graphs, which consist of vertices and edges. Vertices and edges are described in much the same way as points and lines are described in geometry, which means that vertices and edges are taken as undefined objects. Although graphs with an infinite number of vertices or an infinite number of edges can be studied, here the sets of edges and vertices are both finite.

A **graph** consists of a finite set  $\mathcal{V}$  called a vertex set and a finite set  $\mathcal{E}$  called an edge set. Each member of  $\mathcal{V}$  is called a **vertex** (plural: vertices) and each member of  $\mathcal{E}$  is called an **edge** (plural: edges). Associated with each edge are two (not necessarily different) vertices called its **endpoints**. Representations of graphs are drawn using points to represent the vertices and line segments to represent the edges. The edges can be curved if you like, but you must be sure that each endpoint is a vertex.

Figure 4.1: Three ways to draw a complete graph  $K_4$  on four vertices.



A **complete graph** on  $n$  vertices consists of  $n$  points in the plane, together with all possible edges connecting each pair vertices. The notation  $K_n$  is used to represent a complete graph on  $n$  vertices, and Figure 4.1 gives

three different ways to draw  $K_4$ .

•137. Let  $n$  be a positive integer.

- (a) How many edges are in the complete graph  $K_n$ ? Explain.
- (b) In how many ways may  $K_n$  be colored with two colors, say red and blue?

In a graph it is possible to have an edge that connects a vertex to itself (called a **loop**) and it is possible to have two or more edges between two vertices. A graph that has no loops and no multiple edges is called a **simple graph**.

138. Prove there are  $2^{\binom{n}{2}}$  simple graphs on a set of  $n$  vertices.

Figure 4.2: Three different graphs

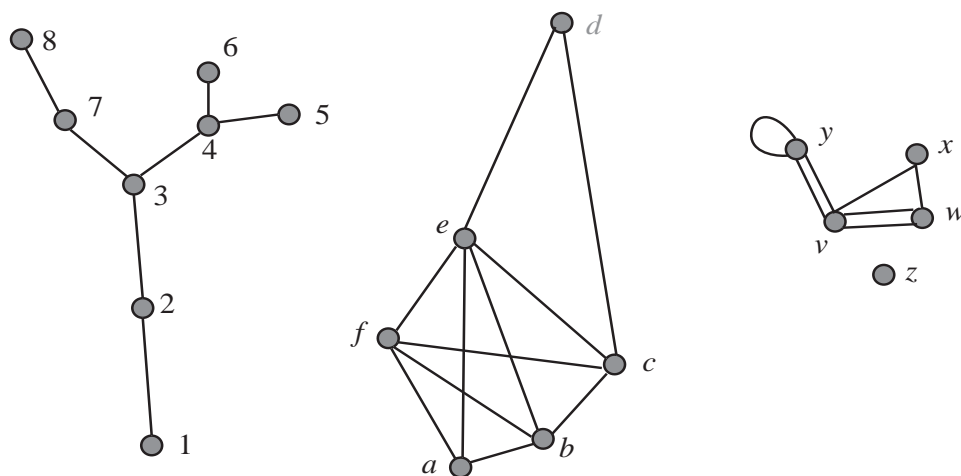


Figure 4.2 gives three pictures of graphs. Each gray circle in the figure represents a vertex and each line segment represents an edge. You will note that the vertices have been labelled, and in general you can choose to label vertices or not. The **degree** of a vertex is the number of times it appears as the endpoint of edges. For instance, the degree of  $y$  in the third graph in the figure is 4.

- 139. Find the degree of each vertex in the graph on the left in Figure 4.2. For each graph in Figure 4.2 is the number of vertices of odd degree even or odd?
- 140. The sum of the degrees of the vertices of a (not necessarily simple) graph is related in a natural way to the number of edges. In this problem, you're asked to discover this relationship, but you don't yet have to prove it.
  - (a) As a group, draw graphs with one edge, two edges, three edges, and perhaps four edges, and calculate the sum of the degrees in each graph. Conjecture a relationship between this sum and the number of edges in the graph.
  - (b) What is the contribution of a given edge to the sum of the degrees? How is the sum affected by deleting an edge (but not its endpoints) from the graph?
- 141. Explain how a graph with  $E + 1$  edges can be viewed as being constructed in a natural way from a (not necessarily unique) graph with  $E$  edges.
- 142. Use induction on the number of edges to prove your conjecture in Problem 140.

In the last two problems you capitalized on the simple observation that deleting an edge from a graph with  $E + 1$  edges results in a graph with  $E$  edges. In other words, the result in Problem 141 provided a useful inductive process for a proof in Problem 142. A word of caution: You *cannot* assume a subgraph with  $E$  edges inherits all properties of the original graph with  $E + 1$  edges. For instance, in the middle graph of Figure 4.2 every vertex has even degree, but deleting any edge from the graph results in a graph with two vertices of odd degree. In the graph on the left, the deletion of any edge results in a subgraph which is no longer connected.

So, a graph can have properties which are not necessarily inherited by its subgraphs. This affects the construction of proofs by mathematical induction: If the argument in your inductive step for graphs with  $E + 1$  edges relies on properties of a subgraph obtained by deleting an edge, then you must prove that the graph with  $E$  edges possesses every property you use. For the proof in Problem 142, the inductive assumption is: Every graph with at most  $E$  edges has the property that the sum of the degrees of its vertices equals twice the number of edges. Note this statement assumes no properties of the graph other than it has at most  $E$  edges. (For example, it might have vertices of odd degree or it might be disconnected.) Because of this,

you are able to apply the inductive assumption to any subgraph obtained by deleting an edge from the original graph with  $E + 1$  edges without proving the subgraph has any other properties. In general, you should exercise great care when using mathematical induction to prove graph-theoretical facts.

143. In Problem 142 you used induction *on the number of edges* to prove your conjecture. Now consider constructing a proof by induction *on the number of vertices*. Is there an inherent inductive process? Thinking through how you might construct such a proof, explain why inducting on the vertices is more complicated (or at least messier) than inducting on the edges.
144. Find a proof of your conjecture in Problem 140 that does not use induction.
- 145. (Refer to Problem 139.) What can you say about the number of vertices of odd degree in a graph? Explain.
- 146. Given a set of people, consider the relation of “being acquainted”.
  - (a) For any nonempty set of people, define what is meant by an “acquaintance graph” on the set by describing its vertices and which vertices are connected by edges.
  - (b) Using graph theory, prove that within any group with an odd number of people, there is at least one person who knows an even number of people.

## 4.2 Trees

A **walk** in a graph is an alternating sequence  $v_0 e_1 v_1 \dots e_k v_k$  of vertices and edges such that for all  $i$  the consecutive vertices  $v_{i-1}, v_i$  are the endpoints of the edge  $e_i$ . Note that the definition of the term “walk” does not require all of the vertices  $v_0, \dots, v_k$  to be different—any number of them might be the same. Likewise, there can be repetitions among the edges  $e_1, \dots, e_k$  in a walk. The literature in graph theory is not consistent in defining a walk. Notice that the definition used here allows a walk to have no edges, which would be the lazy walk that remains at the initial vertex.

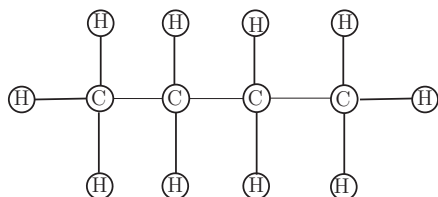
A graph is called **connected** if for any pair of vertices there is a walk which starts at one vertex and ends at the other vertex.

- 147. Which of the graphs in Figure 4.2 are connected?
- 148. A **path** in a graph is a walk with no repeated vertices. Find the longest path in the third graph of Figure 4.2.



- A **cycle** in a graph is a walk which has all of the following properties: its first and last vertices are the same but it has no other repeated vertices; it has at least one edge; it has no repeated edges.
- 149. Which graphs in Figure 4.2 have cycles? What is the largest number of edges in a cycle in the second graph in Figure 4.2? What is the smallest number of edges in a cycle in the third graph in Figure 4.2?
  - 150. A connected graph with no cycles is called a **tree**. Which graphs (if any) in Figure 4.2 are trees?
  - 151. In a tree with  $n$  vertices, given two vertices,  $v_1, v_2$ , how many paths connect  $v_1$  to  $v_2$ ? Give a complete explanation.
  - 152. Find all trees with at most 4 vertices. Give an argument which shows that every tree with at least two vertices has at least one vertex of degree 1.
  - 153. For any tree with  $n \geq 2$  vertices, remove one of its vertices of degree 1 and the edge containing that vertex (but do not remove the other end-point of the edge). Prove that the graph that remains is a tree.
  - 154. On the basis of your examples of trees with at most 4 vertices, make a conjecture about the relationship between the number of vertices and edges in a tree.
  - 155. Prove your conjecture in Problem 154 by induction. Be sure to begin by asking yourself what inductive process you can use. Is it easier to induct on the number of vertices or the number of edges? Or on something else?
  - 156. A hydrocarbon molecule is a molecule whose only atoms are either hydrogen atoms or carbon atoms. In a simple molecular model of a hydrocarbon, a carbon atom will bond to exactly four other atoms, and a hydrogen atom will bond to exactly one other atom. Such a model is shown in Figure 4.3. A hydrocarbon compound can be represented

Figure 4.3: A model of a butane molecule.



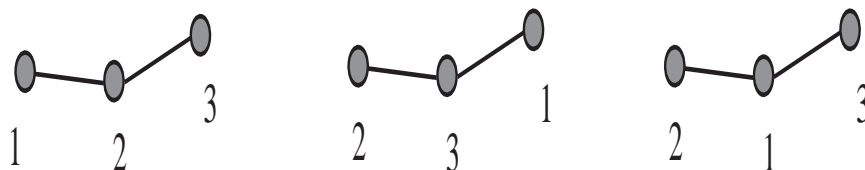
by a graph whose vertices are labelled with C's and H's where each C vertex has degree four and each H vertex has degree one. A hydrocarbon is called an “alkane” if the graph is a tree. Common examples are methane (natural gas), butane (one version of which is shown in Figure 4.3), propane, hexane (ordinary gasoline), and octane (to make gasoline burn more slowly).

- (a) How many vertices are labelled  $H$  in the graph of an alkane with exactly  $n$  vertices labelled  $C$ ?
  - (b) An alkane is called butane if it has exactly four carbon atoms. Why was it said above that one version of butane is shown in Figure 4.3?
157. What is the minimum number of vertices of degree one in a tree with  $n \geq 2$  vertices? Prove that you are correct. See if you can find (and give) more than one proof.

### 4.3 Labelled Trees and Prüfer Codes

Next you will explore the idea of *labelled* trees. Figure 4.4 gives all different labellings of a fixed tree with 3 vertices. Notice that the convention for labelling the vertices of trees is that the tree which has edges between vertices 1 and 2 and between vertices 2 and 3 is different from the tree that has edges between vertices 1 and 3 and between vertices 2 and 3.

Figure 4.4: The three labelled trees on three vertices



- 158. How many labelled trees are there on the vertex set  $[2]$ ? On the vertex set  $[3]$ ? How many labelled trees are there on four vertices? How many labelled trees are there with five vertices? You don't have a lot of data to formulate a guess, but try to guess a formula for the number of labelled trees with vertex set  $[n]$ . When you get to four and especially five vertices, draw all the unlabelled trees you can think of, and then figure out in how many different ways you can put labels on the vertices.

The next problems will develop a method for proving the formula you just guessed in the last problem. In order to do this, an auxiliary sequence is defined.

Given a tree with  $n \geq 2$  vertices which has been labelled in any way using the elements of  $[n]$ , define the auxiliary sequence  $b_1, b_2, \dots$  in the following inductive manner:

**Step 1:** If the tree has two vertices, the sequence consists of one term, the larger label, which means the sequence is  $b_1 = 2$ . Otherwise, let  $a_1$  be the lowest-numbered vertex of degree 1 in the tree. (How do you know there is such a vertex?) Let  $b_1$  be the label of the unique vertex in the tree adjacent to  $a_1$  and write down  $b_1$ . (Why is  $b_1$  unique?) For example, in the first graph in Figure 4.2,  $a_1$  is 1 and  $b_1$  is 2.

**Step 2:** Suppose  $a_1$  through  $a_{i-1}$  have already been identified, and let  $a_i$  be the lowest-numbered vertex of degree 1 in the tree you get by deleting vertices  $a_1$  through  $a_{i-1}$ . (How do you know the resulting graph is always a tree?) Let  $b_i$  be the unique vertex in this new tree adjacent to  $a_i$ . For example, in the first graph in Figure 4.2,  $a_2 = 2$  and  $b_2 = 3$ . Then  $a_3 = 5$  and  $b_3 = 4$ .

- 159. Use the letter  $B$  to stand for the sequence of  $b_i$ s inductively obtained in this way. Use your earlier work to answer the questions posed in the above two-step algorithm.
- 160. For the tree (the first graph) in Figure 4.2, the sequence  $B$  is 2344378. At this point, work with your group to draw some other labelled trees on eight vertices and construct the sequence  $B$  associated with each tree.
- 161. How long is the sequence  $B$  computed from a labelled tree with  $n$  vertices?
- 162. From your examples, decide if you can predict the last member of the sequence  $B$ . Explain.
- 163. Is it possible for  $a_1$  to be in  $B$ ? Can you tell from  $B$  what  $a_1$  is?

For a labelled tree  $T$ , the associated sequence  $P(T) := b_1, b_2, \dots, b_{n-2}$  is called a **Prüfer coding** or **Prüfer code** of  $T$ . For instance, the Prüfer code for the labelled tree  $T$  of Figure 4.2 is  $P(T) = 234437$ . Notice that the last term of  $B$  is not included in the Prüfer code because it's known to be  $n$ .

Let  $\mathcal{S}$  be the set of all labelled trees on nine vertices. For each tree  $T \in \mathcal{S}$ , define  $P(T)$  to be the Prüfer code for  $T$ .

- 164. Why is the relation  $\{(T, P(T)) : T \in \mathcal{S}\}$  a function with domain  $\mathcal{S}$ ? Find a co-domain for this function. (At this point you're not asked to find the *smallest* co-domain.)
- 165. Play the following game in your group: In turn, each of you should secretly write down a tree, determine its Prüfer code, and then share the code with the whole group. The other members of the group then should find all labelled trees that have your sequence as its Prüfer code. How many labelled trees are found? What does your answer say about the function  $P$ ?
- 166. Now, as a group write down any sequence of seven integers from  $[9]$ . Try to find a tree  $T \in \mathcal{S}$  for which your sequence is  $P(T)$ . Do this for several different sequences. Use this information to find the smallest co-domain for the function  $P$ .
- 167. Find a bijection between the set of labelled trees with  $n$  vertices and another set that you already know how to count.
- 168. Find the number of labelled trees with  $n$  vertices. Is this the formula you conjectured earlier in Problem 158?

The idea of writing the last sequence of problems as a game originated with the Fall 2003 Math 399 class at Oregon State.

In addition to providing a way to count labelled trees, there is a good bit of other interesting information encoded in the Prüfer code for a tree. You can begin to see this by working the next two problems and the problems in the optional section that follows them.

- 169. What can you say about the vertices of degree one from the Prüfer code for a tree labelled with the integers from 1 to  $n$ ; that is, what vertex or vertices in the sequence  $b_1, b_2, \dots, b_{n-1}$  can have degree 1?
- 170. What can you say about the Prüfer code for a tree in which exactly two vertices have degree 1? Does this characterize such trees?

#### 4.3.1 More information from Prüfer codes (Optional)

- 171. What can you determine about the degree of the vertex labelled  $i$  from the Prüfer code of the tree?  
*Hint.* If a vertex has degree 1, how many times does it appear in the Prüfer code of the tree? What about a vertex of degree 2?
- 172. What is the number of (labelled) trees on  $n$  vertices with three vertices of degree 1? (Assume they are labelled with the integers 1 through  $n$ .)

*Hint.* How many vertices appear exactly once in the Prüfer code of the tree and how many appear exactly twice?

- 173. How many labelled trees on  $n$  vertices have exactly four vertices of degree 1?
- 174. The **degree sequence** of a graph is a list of the degrees of the vertices in non-increasing order. For example the degree sequence of the first graph in Figure 4.2 is  $(3, 3, 2, 2, 1, 1, 1, 1)$ . For a graph with vertices labelled 1 through  $n$ , the **ordered degree sequence** of the graph is the sequence  $d_1, d_2, \dots, d_n$  in which  $d_i$  is the degree of vertex  $i$ .
- How many labelled trees have  $n$  vertices and the ordered degree sequence  $d_1, d_2, \dots, d_n$ ?
  - How many labelled trees have  $n$  vertices and a degree sequence in which the degree  $d$  appears  $i_d$  times?

## 4.4 Monochromatic Subgraphs

For a fixed positive integer  $m$ , recall  $K_m$  denotes the complete graph on  $m$  vertices. Here we consider the vertex set of  $K_m$  to be labelled using the elements of  $[m]$ . Let  $\mathcal{C}$  be the set of all colorings of the edges of  $K_m$  with two colors, say red and blue. In Problem 137 you proved  $|\mathcal{C}| = 2^{\binom{m}{2}}$ .

For  $1 \leq n \leq m$ , define the set  $\mathcal{S} := \{s : s \subseteq [m], |s| = n\}$ . (What is  $|\mathcal{S}|$ ?) For any coloring  $c \in \mathcal{C}$  and any  $s \in \mathcal{S}$ , consider the subgraph of  $K_m$  whose vertex set is  $s$  and whose edge set contains all the (colored) edges of  $K_m$  which connect pairs of vertices in  $s$ . This graph is a colored complete graph on  $n$  vertices, and it will be denoted by  $K(c, s)$ .

- 175. For  $m = 4$ ,  $n = 2$ , find  $\mathcal{C}$  and  $\mathcal{S}$ . Work together as a group to find  $K(c, s)$  for at least three choices of  $(c, s) \in \mathcal{C} \times \mathcal{S}$ .
- 176. Let  $m = 4$ .
  - For  $n = 2$ , show that for every coloring  $c \in \mathcal{C}$  there exists  $s \in \mathcal{S}$  such that every edge in  $K(c, s)$  is the same color. Such subgraphs are called **monochromatic** for the coloring  $c$ .
  - Show that  $n = 2$  is the largest possible integer which has the property given in part (a).

Define  $\text{mono}(c, \mathcal{S})$  to be 1 if  $K(c, \mathcal{S})$  is monochromatic for the coloring  $c$  and to be 0 otherwise.

◦177. Explain why

$$\frac{1}{2^{\binom{m}{2}}} \sum_{c \in \mathcal{C}} \sum_{s \in \mathcal{S}} \text{mono}(c, s) = \frac{1}{2^{\binom{m}{2}}} \sum_{s \in \mathcal{S}} \sum_{c \in \mathcal{C}} \text{mono}(c, s)$$

and why this common value is the average number of monochromatic  $n$ -subgraphs, averaged over all colorings of  $K_m$ .

- 178. For a fixed subset  $s \in \mathcal{S}$ , prove  $\sum_{c \in \mathcal{C}} \text{mono}(c, s) = 2 \cdot 2^{\binom{m}{2} - \binom{n}{2}}$ .
- 179. Show that your expression in Problem 177 equals  $2 \binom{m}{n} \cdot 2^{-\binom{n}{2}}$ .
- 180. Prove: If  $\binom{m}{n} < 2^{\binom{n}{2}-1}$  then there exists a coloring  $c^* \in \mathcal{C}$  such that no  $K(c^*, s)$  is monochromatic.  
*Hint.* Use Problem 179.

The very clever technique used in the last problem is a simple form of the “probabilistic method” of Paul Erdős.

The next problem sequence uses Problem 180 to obtain a lower bound on the Ramsey Numbers  $R(n, n)$ . These numbers were discussed earlier in the optional Section 1.5 (on page 18ff). That earlier material is only used in the next problem, which connects Ramsey Numbers with monochromatic subgraphs.

- 181. Use the definition of the Ramsey Number  $R(n, n)$  to explain why the following is true: If  $R(n, n) \leq m$ , then for all colorings  $c \in \mathcal{C}$  there exists  $s \in \mathcal{S}$  such that  $K(c, s)$  is monochromatic.
- 182. Explain why  $R(n, n) > m$  for all integers  $m$  such that  $\binom{m}{n} < 2^{\binom{n}{2}-1}$ .
- 183. Prove: If  $n \leq m$  are positive integers such that  $m < \sqrt[n]{2^{\binom{n}{2}-1} n!}$  then  $\binom{m}{n} < 2^{\binom{n}{2}-1}$ .  
*Hint.* Use the fact that  $\binom{m}{n} \leq \frac{m^n}{n!}$ .
- 184. Prove that  $R(n, n) > \sqrt[n]{2^{\binom{n}{2}-1} n!}$ .

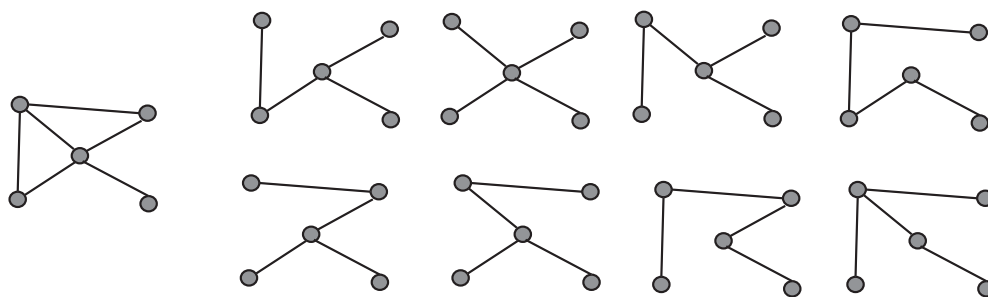
In the last section of this chapter, you will show how the inequality in Problem 184 can be used to obtain a prediction of the asymptotic size of  $R(n, n)$ .

## 4.5 Spanning Trees

Many of the applications of trees arise from trying to find an efficient way to connect all the vertices of a graph by a path. For example, in a telephone

network, at any given time there are a certain number of wires (or microwave channels, or cellular channels) available for use. These wires or channels go from one specific place to another specific place, and so the wires or channels may be thought of as edges of a graph and the places where the wires connect may be thought of as vertices of that graph. A tree whose vertices are all of the vertices of the graph  $G$  and whose edges are some of the edges of a graph  $G$  is called a **spanning tree** of  $G$ . A spanning tree for a telephone network gives a way to route calls between any two vertices in the network that uses the minimum *number* of wires. For example, Figure 4.5 contains all spanning trees of the graph on the far left of the figure.

Figure 4.5: A graph and all its spanning trees.



- 185. As a group, draw an example of a connected graph which is not a tree and has six vertices. List all of its spanning trees. (The question of counting the number of spanning trees will be considered later in Section 4.5.1.)
- 186. Explain why every connected graph has a spanning tree. It is possible to find an explanation that starts with the graph and works “down” towards the spanning tree and to find another explanation that starts with just the vertices and works “up” towards the spanning tree. Try to find both kinds.

Our motivation for talking about spanning trees was the idea of finding a minimum number of edges needed to connect all the edges of a communication network together. In many cases the edges of a communication network have costs associated with them. For example, one cell-phone operator might charge another one when a customer of one uses an antenna of the other.

Suppose a company has offices in a number of cities and wants to put together a communication network connecting its various locations with high-speed communication lines, and to do so at minimum cost. This can be modeled by a graph whose vertices are the cities in which it has offices and whose edges represent possible communications lines between the cities. Of course there will not necessarily be lines between each pair of cities—in fact the company might not want to pay for a line connecting city  $i$  and city  $j$  if it can already connect them indirectly by using other lines it has chosen.

- 187. Provide this company with a written description of a graph-theoretic model of its problem.

You will want to choose a spanning tree of minimum cost among all spanning trees of the communications graph. This special tree is often called a **minimal spanning tree** (often abbreviated as **MST**) for the graph. For this type of application, nonnegative numbers (called **weights**) are assigned to the edges of the graph and the sum of the numbers on the edges of a spanning tree is called the **cost** of the spanning tree.

- 188. (a) Put weights on the edges of your graph from Problem 185. Find a minimal spanning tree for your graph. Can you find two?  
 (b) Draw another connected graph with weighted edges, and find a minimal spanning tree for it.
- 189. Describe an inductive method (or better, two methods different in at least one aspect) for finding a minimal spanning tree in a connected graph whose edges are labelled with costs.

*Hint.* Think of forming the MST by carefully selecting one edge of the tree at a time.

The method you designed in Problem 189 is called a **greedy method**, because each time you made a choice of an edge you chose the least costly edge available to you.

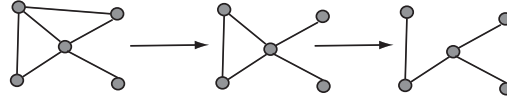
#### 4.5.1 Counting the Number of Spanning Trees (Optional)

There are two operations on graphs which can be used to get a recurrence for finding the number of spanning trees of a graph. Each operation is applied to an edge  $e$  of a graph  $G$ .

The first operation is called **deletion**: you simply *delete* the edge  $e$  from the graph by removing it from the edge set (without removing either of its endpoints). Work through Figure 4.6 for an example of how a sequence of edge deletions can be used to get a spanning tree.



Figure 4.6: Deleting two appropriate edges from this graph gives a spanning tree.



The second operation is called **contraction** of an edge. Intuitively, you contract an edge by shrinking its length until its endpoints coincide and letting the rest of the graph “go along for the ride.” To be more precise, the edge  $e$  with endpoints  $v$  and  $w$  is contracted as follows:

**Step 1** remove from the edge set all edges having either  $v$  or  $w$  (or both) for an endpoint;

**Step 2** remove  $v$  and  $w$  from the vertex set;

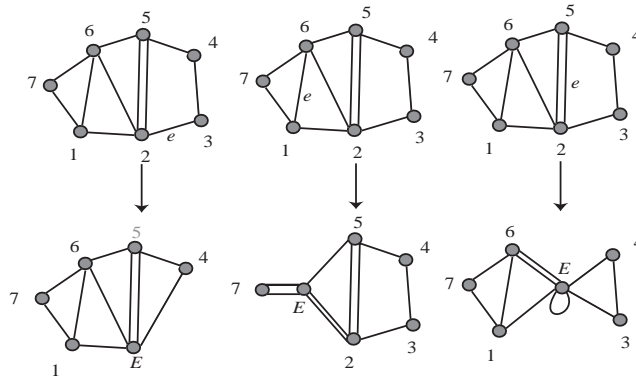
**Step 3** add a new vertex  $E$  to the vertex set;

**Step 4** for each remaining vertex that had an edge removed in Step 1, add an edge from the vertex to  $E$ ;

**Step 5** add an edge from  $E$  to  $E$  for any edge other than  $e$  whose endpoints were in the set  $\{v, w\}$ .

- 190. Work through the examples in Figure 4.7 to get a better understanding of the idea. You’ll find that the wording for this process of contraction is more complicated than the process.

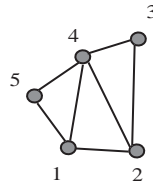
Figure 4.7: The results of contracting three different edges in a graph.



The notation  $G \setminus e$  (read as  $G$  minus  $e$ ) will be used to represent the graph that results from deleting the edge  $e$  from  $G$ , and  $G/e$  (read as  $G$  contract  $e$ ) for the result of contracting the edge  $e$  from  $G$ .

- 191. How do the number of spanning trees of  $G$  not containing the edge  $e$  relate to the number of spanning trees of  $G \setminus e$ ? How do the number of spanning trees of  $G$  containing  $e$  relate to the number of spanning trees of  $G/e$ ? Explain.
- 192. Use  $\#(G)$  to represent the number of spanning trees of a graph  $G$  (so that, for example,  $\#(G/e)$  equals the number of spanning trees of  $G/e$ ). Find an expression for  $\#(G)$  in terms of  $\#(G/e)$  and  $\#(G \setminus e)$ . The equation that results is called the **deletion-contraction recurrence**. In what sense is it a recurrence?
- 193. Use the recurrence of the last problem repeatedly to show that the graph in Figure 4.8 has twenty-one spanning trees.

Figure 4.8: A graph.



- + 194. Give a verbal description of an algorithm for counting the number of spanning trees in a connected graph.

## 4.6 Finding Shortest Paths in Graphs

Suppose that a company has a main office in one city and regional offices in other cities, and it happens that most of the communication in the company is between the main office and the regional offices. The company would like to find a spanning tree which minimizes not the total cost over all possible communication links (all edges), but rather the total cost of communication between the main office and each of the regional offices. The **weighted length** of a path in the graph is the sum of the weights of its edges, and the **distance** between two vertices is the least weighted length of any path between the two vertices. There are two optimization (actually minimization) problems inherent here:

- Given a vertex  $v$ , what is the distance between  $v$  and each other vertex?
- Given a vertex  $v$ , can you find a spanning tree in  $G$  such that the length of the path in the spanning tree from  $v$  to each vertex  $x$  is the distance from  $v$  to  $x$  in  $G$ ?

Consider the following inductive process, which is known as **Dijkstra's algorithm**. The algorithm is applied to a weighted graph whose vertices are labelled 1 to  $n$ .

**Step 1** Let  $d(1) := 0$ . Let  $d(i) := \infty$  for all other  $i$ .

Let  $v(1) := 1$ . Let  $v(j) := 0$  for all other  $j$ .

For each  $i$  and  $j$ , let  $w(i, j)$  be the minimum weight of an edge between  $i$  and  $j$ , or  $\infty$  if there are no such edges.

Let  $k := 1$ . Let  $t := 1$ .

**Step 2** For each  $i$ , if  $d(i) > d(k) + w(k, i)$  let  $d(i) = d(k) + w(k, i)$ .

**Step 3** Among those  $i$  with  $v(i) = 0$ , choose one for which  $d(i)$  is a minimum, and let  $k = i$ . Increase  $t$  by 1. Let  $v(i) = 1$ .

**Step 4** Repeat the previous two steps until  $t = n$ .

- 195. As a group, draw two connected weighted graphs which are not trees which have at least seven vertices and at least fourteen edges. Apply Dijkstra's algorithm to find spanning trees in each of the graphs.
- 196. Show that at the end of Dijkstra's algorithm each  $d(i)$  equals the distance from vertex 1 to vertex  $i$ .
- 197. In every connected graph, is there always a spanning tree such that for every vertex  $i$ , the distance from vertex 1 to vertex  $i$  given by the algorithm in Problem 195 is the distance from vertex 1 to vertex  $i$  in the tree?

## 4.7 Some Asymptotic Combinatorics (Optional)

While the formula for  $\binom{n}{k}$  given in the box on page 48 is very useful, it doesn't give a sense of how big the binomial coefficient  $\binom{n}{k}$  is. You can get a very rough idea, for example, of the size of  $\binom{2n}{n}$  by recognizing that  $(2n)!/n!$  can be written as  $\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdots \frac{n+1}{1}$ , and each quotient is at least 2, so the product is at least  $2^n$ . If this were an accurate estimate, it would mean the fraction of  $n$ -element subsets of a  $2n$ -element set would be about  $2^n/2^{2n} = 1/2^n$ , which becomes very small as  $n$  becomes large. However, it is pretty clear this approximation is not a very good one, because some of the

terms in that product are much larger than 2. In fact, if  $\binom{2n}{k}$  were the same for every  $k$ , then each would be the fraction  $\frac{1}{2n+1}$  of  $2^{2n}$ . This is much larger than the approximation  $\frac{1}{2^n}$ . But our intuition (and also Pascal's Triangle) suggests that  $\binom{2n}{n}$  is much larger than  $\binom{2n}{1}$  and is likely larger than  $\binom{2n}{n-1}$  so you can be sure the approximation is a bad one. In order to make accurate estimates of binomial coefficients, James Stirling developed a formula to approximate  $n!$  when  $n$  is large, namely  $n!$  is about  $(\sqrt{2\pi n}) n^n / e^n$ . In fact the ratio of  $n!$  to this expression approaches 1 as  $n$  becomes infinite.

### Stirling's Formula

$$n! \sim \sqrt{2\pi n} \frac{n^n}{e^n},$$

which is read as “ $n!$  is asymptotic to  $\sqrt{2\pi n} \frac{n^n}{e^n}$ .”

Proving Stirling's Formula requires more of a detour than is advisable here. However, there is an elementary proof which you can work through in the problems of the end of Section 1 of Chapter 1 of *Introductory Combinatorics* by Kenneth P. Bogart, Harcourt Academic Press (2000).

198. Use Stirling's Formula to show that the fraction of subsets of size  $n$  in an  $2n$ -element set is approximately  $1/\sqrt{\pi n}$ , which is a much bigger fraction than  $1/2^n$ .

For the final problems in this chapter, you now return to the question of calculating the asymptotic size of the Ramsey Numbers  $R(n, n)$ . Here you should restrict to large enough  $n$  so that Stirling's Formula is “reasonably” accurate, accurate enough that you may replace  $n!$  by the approximation given in Stirling's formula. This is not a tight argument—a proof requires an “ $\epsilon$ -argument”.

199. Use Stirling's Formula to convert (for large enough  $n$ ) the upper bound for  $R(n, n)$  in Problem 134(b) to an upper bound which is a multiple of a power of 2.
200. Use Stirling's Formula to convert (for large enough  $n$ ) the lower bound for  $R(n, n)$  in Problem 184 to  $\sqrt{2^n}$ .
201. Show the Ramsey Numbers  $R(n, n)$  grow exponentially with  $n$ .

## Chapter 5

# Generating Functions

### 5.1 Using Pictures to Visualize Counting

Suppose you want to choose a snack of three pieces of fruit from among apples, pears and bananas. Since choosing two or three of the same fruit has not been precluded, all your choices can be symbolically represented as

$$\mathfrak{A}\mathfrak{A}\mathfrak{A} + \mathfrak{A}\mathfrak{A}\mathfrak{P} + \mathfrak{A}\mathfrak{A}\mathfrak{B} + \mathfrak{A}\mathfrak{P}\mathfrak{A} + \mathfrak{A}\mathfrak{P}\mathfrak{P} + \mathfrak{A}\mathfrak{P}\mathfrak{B} + \mathfrak{A}\mathfrak{B}\mathfrak{A} + \mathfrak{A}\mathfrak{B}\mathfrak{P} + \mathfrak{P}\mathfrak{A}\mathfrak{A} + \mathfrak{P}\mathfrak{A}\mathfrak{P} + \mathfrak{P}\mathfrak{A}\mathfrak{B}.$$

(Why doesn't  $\mathfrak{A}\mathfrak{A}\mathfrak{A}$  appear?) Here a picture of a piece of fruit represents taking a piece of that fruit. For instance,  $\mathfrak{A}$  stands for taking an apple;  $\mathfrak{A}\mathfrak{A}$  represents taking an apple and a pear; and  $\mathfrak{A}\mathfrak{A}$  for taking two apples.

In this representation you can think of the plus sign as standing for the *exclusive or*. For example,  $\mathfrak{A} + \mathfrak{P}$  would stand for “I take an apple or a banana but not both.” This similarity with mathematical notation is extended by condensing the expression to

$$\mathfrak{A}^3 + \mathfrak{A}^2\mathfrak{P} + \mathfrak{A}^2\mathfrak{B} + \mathfrak{A}\mathfrak{P}^2 + \mathfrak{A}\mathfrak{P}\mathfrak{B} + \mathfrak{A}\mathfrak{B}^2 + \mathfrak{P}\mathfrak{A}^2 + \mathfrak{P}\mathfrak{A}\mathfrak{P} + \mathfrak{P}\mathfrak{A}\mathfrak{B}, \quad (5.1)$$

where in this notation  $\mathfrak{A}^3$  stands for choosing three apples, while  $\mathfrak{A}^2\mathfrak{P}$  represents a choice of two apples and a banana, and so on. What the notation in (5.1) is really doing is giving a convenient way to list all three-element multisets chosen from the set  $\{\mathfrak{A}, \mathfrak{P}, \mathfrak{B}\}$ . This approach was inspired by George Pólya's paper “Picture Writing,” in the December 1956 issue of *The American Mathematical Monthly*. While we are taking a somewhat more formal approach than Pólya, it is still completely in the spirit of his work.

Suppose now that you plan to choose between one and three apples, between one and two pears, and between one and two bananas, and that no

other restrictions are placed on the total number of fruit to be chosen. In a somewhat clumsy way you could describe the fruit selections as

$$\begin{aligned} & \mathfrak{A}\mathfrak{A}\mathfrak{C} + \mathfrak{A}\mathfrak{A}^2\mathfrak{C} + \mathfrak{A}\mathfrak{A}\mathfrak{C}^2 + \mathfrak{A}\mathfrak{A}^2\mathfrak{C}^2 + \mathfrak{A}^2\mathfrak{A}\mathfrak{C} + \cdots + \mathfrak{A}^2\mathfrak{A}^2\mathfrak{C}^2 + \mathfrak{A}^3\mathfrak{A}\mathfrak{C} + \cdots + \mathfrak{A}^3\mathfrak{A}^2\mathfrak{C}^2. \\ & \hspace{15em} (5.2) \end{aligned}$$

- 202. Using an  $A$  in place of the picture of an apple, a  $P$  in place of the picture of a pear, and a  $B$  in place of the picture of a banana, write out the entire expression intended in (5.2), that is, without any dots for left-out terms. (You may use pictures instead of letters if you prefer, but it gets tedious quite quickly!) Now expand the product  $(A + A^2 + A^3)(P + P^2)(B + B^2)$  and compare the result with your expression.
- 203. Substitute an  $x$  for each of  $A$ ,  $P$  and  $B$  in the expression you found in Problem 202. Expand the result in powers of  $x$  and give an interpretation of the coefficient of  $x^n$ .

You saw that expanding

$$(\mathfrak{A} + \mathfrak{A}^2 + \mathfrak{A}^3)(\mathfrak{P} + \mathfrak{P}^2)(\mathfrak{B} + \mathfrak{B}^2) \tag{5.3}$$


gives the expression in (5.2). This means that (5.2) and (5.3) each describes the number of multisets you can choose from the set  $\{\mathfrak{A}, \mathfrak{P}, \mathfrak{B}\}$  in which  $\mathfrak{A}$  appears between one and three times, and  $\mathfrak{P}$  and  $\mathfrak{B}$  each appears once or twice. Interpret (5.2) as describing each individual multiset you can choose, and interpret (5.3) as saying that you first decide how many apples you will take, and then decide how many pears to take, and then decide how many bananas. At this stage it might seem a bit magical that doing ordinary algebra with the second formula yields the first. In fact, by defining addition and multiplication with these pictures more formally we could explain in detail why things work out. This more formal exposition won't be given here.

In the descriptions of the ways to choose fruit you've seen that the pictures of the fruit can be treated as if they were variables. In the theory of generating functions (which will be developed in the next section), variables or polynomials or even power series are associated with members of a set. This is an adaptation of language introduced by George Pólya to describe how to associate variables with the members of a set. A **picture** of a member of a set  $\mathcal{S}$  means a variable, or perhaps a product of powers of variables or even a polynomial in the variables. A function  $P$  that assigns a picture  $P(s)$  to each member  $s \in \mathcal{S}$  will be called a **picture function**. The **picture enumerator** for a picture function  $P$  defined on a set  $\mathcal{S}$  will be the sum of

the pictures of the elements in  $\mathcal{S}$ , which will be written symbolically as

$$E_P(\mathcal{S}) = \sum_{s \in \mathcal{S}} P(s).$$

Using this terminology in the original problem: If  $\mathcal{S}$  is the set of the three fruit, then  $A$  is the picture of an apple, and  $A + P + B$  is the picture enumerator of the picture function on  $\mathcal{S}$ . Likewise, when  $\mathcal{S}$  is the set of all multisets of fruit which have from one to three apples, one to two pears, and one to two bananas, then in Problem 202 you found the picture enumerator for  $\mathcal{S}$ . This language has been chosen because the picture enumerator lists (that is, enumerates) all elements of  $\mathcal{S}$  according to their pictures.

204. The product  $A^2P^3$  represents taking two apples and three pears, which means choosing the picture of the ordered pair (2 apples, 3 pears) to be the juxtaposition , the product of the pictures of a multiset of two apples and a multiset of three pears.
- Show that if  $S_1$  and  $S_2$  are sets with picture functions  $P_1$  and  $P_2$  defined on them, and if the picture of an ordered pair  $(x_1, x_2) \in S_1 \times S_2$  is defined to be  $P((x_1, x_2)) = P_1(x_1)P_2(x_2)$ , then the picture enumerator of  $P$  on the set  $S_1 \times S_2$  is  $E_{P_1}(S_1)E_{P_2}(S_2)$ . This is called the **Product Principle for Picture Enumerators**.
- 205. Use the Product Principle for Picture Enumerators to explain why (5.2) and (5.3) are equal.
  - 206. What should be the picture of taking no apples? Find a polynomial in the variable  $A$  that says you may take between zero and three apples.  
*Hint.* If  $A^1$  is the picture of taking one apple and  $A^2$  is the picture of taking two apples, what would make a good picture of taking no apples?
  - 207. Write a picture enumerator that says you can take between zero and three apples, between zero and three pears, and between zero and three bananas.
  - 208. Suppose you want to choose a snack of between zero and three apples, between zero and three pears, and between zero and three bananas.
    - (a) Write a polynomial in one variable  $x$  in which the coefficient of  $x^n$  is the number of ways to choose a snack with  $n$  pieces of fruit.
    - (b) Suppose an apple costs 20 cents, a banana costs 25 cents, and a pear costs 30 cents. What should you substitute for  $A$ ,  $P$ , and  $B$  in Problem 207 in order to get a polynomial in which the coefficient

- of  $x^n$  is the number of ways to choose a selection of fruit that costs  $n$  cents?
- (c) Suppose an apple has 40 calories, a pear has 60 calories, and a banana has 80 calories. What should you substitute for  $A$ ,  $P$ , and  $B$  in Problem 207 in order to get a polynomial in which the coefficient of  $x^n$  is the number of ways to select fruit with a total of  $n$  calories?
- 209. In this problem, you want to choose a subset of the set  $[n]$ . For each  $i$  from 1 to  $n$ , use  $x_i$  to be the picture of choosing  $i$  to be in the subset.
- (a) What is the picture enumerator for either choosing  $i$  or not choosing  $i$  to be in the subset?
- (b) What is the picture enumerator for all possible choices of subsets of  $[n]$ ? What should be substituted for  $x_i$  in order to get a polynomial in  $x$  such that the coefficient of  $x^k$  is the number of ways to choose a  $k$ -element subset of  $[n]$ ? Explain.
- (c) You have just proved a special case of what theorem?

### 5.1.1 Pictures of trees (Optional)

In the following exercises, “a tree with  $n$  vertices” will always be considered to have its vertices labelled by the elements of  $[n]$ . For such a tree, define the picture of the vertex  $i$  to be  $x_i$ , and the picture of the edge with endpoints  $x_i$  and  $x_j$  to be  $x_i x_j$ . Then the **picture of the tree**  $T$  is defined to be the product

$$P(T) = \prod_{\{i,j\} \in \mathcal{T}} x_i x_j \quad (5.4)$$

where  $\mathcal{T} = \{ \{i, j\} : i \text{ and } j \text{ are connected by an edge in the tree } T \}$ .

210. Draw a tree with seven vertices, and find its picture. Show that the picture you found can be re-written as  $\prod_{i=1}^7 x_i^{\deg(i)}$ . Do this for several examples. Explain why for any tree  $T$  with  $n$  vertices, its tree picture  $P(T)$  can be re-written as  $\prod_{i=1}^n x_i^{\deg(i)}$ .
211. For each  $n \geq 1$ , let  $\mathcal{S}_n$  be the set of all trees with vertex set  $[n]$ . For each tree  $T \in \mathcal{S}_n$  use the picture  $P(T)$  given in (5.4). Find the picture enumerators  $E_P(\mathcal{S}_n)$  for each of  $n = 2, 3, 4$ . In each case, factor the polynomials as completely as possible.
212. Explain why  $x_1 x_2 \cdots x_n$  is always a factor of the picture of any tree on  $n$  vertices.



213. (a) Write down the picture of a tree on five vertices with one vertex of degree four, say vertex  $i$ .  
 (b) If a tree on five vertices has a vertex of degree three, what are the possible degrees of the other vertices? What can you say about the picture of a tree with a vertex of degree three?  
 (c) If a tree on five vertices has no vertices of degree three or four, what can you say about the picture of the tree?  
 (d) Write down the picture enumerator for all trees on five vertices.  
*Hint.* Remember the formula involving degrees and edges.
- 214. As above, for  $n \geq 1$  let  $\mathcal{S}_n$  be the set of all trees with vertex set  $[n]$ . Prove that the picture enumerator  $E_P(\mathcal{S}_n)$  equals

$$E_P(\mathcal{S}_n) = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

215. The enumerator for trees by degree sequence is the sum over all trees of  $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ , where  $d_i$  is the degree of the vertex  $i$ . Explain why  $x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$  is the enumerator by degree sequence for trees on the vertex set  $[n]$ .
216. Find the number of labelled trees on  $n$  vertices and prove your formula is correct. (You also established this formula using Prüfer codes in Section 4.3. Since you most likely used some results from that section in the proof of Problem 214, this new proof is not entirely independent of Prüfer codes.)

## 5.2 Generating Functions

### 5.2.1 Generating polynomials

In your solution to Problem 209 you saw that the process of expanding the polynomial  $(1+x)^n$  as given in the Binomial Theorem can be thought of as a way of “generating” the binomial coefficients  $\binom{n}{k}$  as the coefficients of  $x^k$  in the expansion of  $(1+x)^n$ . For this reason,  $(1+x)^n$  is called the generating polynomial for the binomial coefficients  $\binom{n}{k}$ .

More generally, the **generating polynomial** for a finite sequence  $a_0, \dots, a_n$  is the polynomial  $\sum_{i=0}^n a_i x^i$ . In Problem 208(a) you converted the picture enumerator for selecting between zero and three each of apples, pears, and bananas to the generating polynomial of the finite sequence  $a_0, \dots, a_n$  in which  $a_i$  is the number of such fruit snacks which contain  $i$  pieces of fruit. When you substituted  $x^c$  for each fruit picture (where  $c$  is the number of calories in that particular kind of fruit), the resulting polynomial was the

generating polynomial for the number of fruit selections with  $i$  calories. Also remember that the original picture enumerator was obtained by multiplying three picture enumerators:

$$1 + A + A^2 + A^3 \quad , \quad 1 + P + P^2 + P^3 \quad , \quad 1 + B + B^2 + B^3 .$$

When  $x^c$  is substituted for each fruit picture where  $c$  is now the cost of the fruit (as in Problem 208(b)), these picture enumerators become

$$1 + x^{20} + x^{40} + x^{60} \quad , \quad 1 + x^{30} + x^{60} + x^{90} \quad , \quad 1 + x^{25} + x^{50} + x^{75} ,$$

where in each case the coefficient of  $x^i$  gives the number of selections of that particular fruit which cost  $i$  cents. The Product Principle of Picture Enumerators therefore translates directly into a Product Principle for Generating Polynomials. Before stating this principle, note that in each of the above instances there was:

- a finite set  $\mathcal{S}$  of possible fruit selections (for instance, from zero to three apples);
- an associated **value function** defined from  $\mathcal{S}$  to the nonnegative integers (for instance, the cost of the fruit selection or the number of calories in the fruit selection);
- and a polynomial that is the generating polynomial for the number of elements  $s \in \mathcal{S}$  which have the value  $i$ . This polynomial will be called the generating polynomial associated with the value.

### The Product Principle for Generating Polynomials

Let  $\mathcal{S}_1, \mathcal{S}_2$  be finite sets with value functions  $v_1, v_2$ . If  $G_i(x)$  is the generating polynomial associated with the value  $v_i$  then the coefficient of  $x^k$  in the polynomial  $G_1(x)G_2(x)$  is the number of ordered pairs  $(s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2$  such that  $v_1(s_1) + v_2(s_2) = k$ .

## 5.2.2 Generating functions

Generating functions are also defined for infinite sequences, and they are defined in such a way that the generating function for an infinite sequence

$\{a_i\}_{i \geq 0}$  with only finitely many nonzero terms (say  $a_i = 0$  for all  $i > N$ ) is the same as the generating polynomial for the finite sequence  $a_0, \dots, a_N$ . The **generating function** for  $\{a_i : i \geq 0\}$  is the expression  $\sum_{i=0}^{\infty} a_i x^i$ , a **formal power series**. For *formal* power series, the series is simply a convenient way of representing the terms of sequences that interest us. You will see that they are convenient for our purposes because the sum and product of formal power series are defined in a way that captures properties of the sequences that are important in discrete mathematics. The sum of two series is defined to be coefficient-wise addition; that is,

### Addition of Formal Power Series

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) + \left( \sum_{j=0}^{\infty} b_j x^j \right) = \sum_{k=0}^{\infty} (a_k + b_k) x^k.$$

Before defining multiplication of two formal power series, remember that in calculus (and in analysis in general) you are interested in whether or not a power series is a *function*, and so in analysis it's important to know for what values of  $x$  the power series converges. On the other hand, in discrete mathematics power series can be purely formal objects, which means that even though you use the phrase generating *function*, the power series is not required to actually represent a function and so there's no need to worry about convergence. As an historical aside: Before settling on the current definition of the word “function”, the word evolved through several meanings, starting with very imprecise meanings and ending with the current definition. The terminology “generating function” may be thought of as an example of one of the earlier uses of the term function. Now on to multiplication.

◦217. (a) What is the coefficient of  $x^2$  in the polynomial

$$(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2 + b_3x^3)?$$

What is the coefficient of  $x^4$ ?

- (b) In part(a), why is there a  $b_0$  and a  $b_1$  in your expression for the coefficient of  $x^2$  but there is not a  $b_0$  or a  $b_1$  in your expression for the coefficient of  $x^4$ ?
- (c) What is the coefficient of  $x^4$  in

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4)?$$

Express this coefficient in the form

$$\sum_{i=0}^4 \text{something},$$

where the “something” is an expression you need to figure out.

- 218. The point of Problem 217 is that when the sequences  $\{a_i\}$  and  $\{b_j\}$  are finite (or, equivalently for our purposes, when  $\{a_i\}$  and  $\{b_j\}$  are infinite sequences with  $a_i = 0$  for  $i > n$  and  $b_j = 0$  for  $j > m$ ), then there is a very nice formula for the coefficient of  $x^k$  in the product

$$\left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{j=0}^m b_j x^j \right).$$

Write this formula explicitly and justify your conclusion.

- 219. Assuming that the rules of polynomial arithmetic apply to formal power series, write down a formula for the coefficient  $c_k$  of  $x^k$  in the product

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right).$$

The expression you obtained in Problem 219 *defines* the product of formal power series. That is, the product of two formal power series  $\sum_{i=0}^{\infty} a_i x^i$  and  $\sum_{j=0}^{\infty} b_j x^j$  is the unique power series  $\sum_{k=0}^{\infty} c_k x^k$ , where  $c_k$  is the expression you found in Problem 219. For convenience of referral, write the correct coefficient  $c_k$  in the following formula:

**Multiplication of Formal Power Series**

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right) = \sum_{k=0}^{\infty} \left[ \quad \quad \quad \right] x^k.$$

Since your expression for the product of two formal power series was derived using usual polynomial algebra, it should not be surprising that multiplication of formal power series satisfies the usual rules of polynomial algebra, such as the associative law and the commutative law. We could

explicitly state these rules and prove that they are all valid for addition and multiplication of formal power series. However, for the purposes of this book that's excessive, except to point out that the polynomial 1 is the multiplicative identity. (Verify that is true.)

- 220. Use the definition of multiplication to find the product  $(1-x) \sum_{k=0}^n x^k$  and the product  $(1-x) \sum_{k=0}^{\infty} x^k$ .
- 221. A formal power series  $\sum_{i=0}^{\infty} a_i x^i$  is said to be **invertible** with **inverse**  $\sum_{j=0}^{\infty} b_j x^j$  if the product equals the identity 1. Show that  $1-x$  is invertible, and that  $x+x^2$  is not invertible. Determine which powers series are invertible. Try to find a condition which is easy to verify.

The usual notation for inverse will be used here: The inverse of  $\sum_{i=0}^{\infty} a_i x^i$  is written as

$$\left( \sum_{i=0}^{\infty} a_i x^i \right)^{-1}.$$

What is  $(1-x)$ ; that is, what is the inverse of the formal power series  $1-x$ ?

Because the algebra of generating functions is the same whether the sequence is finite or infinite, the Product Principle for Generating Polynomials (as given on page 79) can be shown to hold for *all* generating functions, and mathematical induction can be used to extend this principle from two sets to any finite number of sets. (Refer to Section 5.2.3.)

### The Product Principle for Generating Functions

Suppose each of the sets  $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$  has a value function defined from the set to the nonnegative integers. For each  $i$ , let  $G_i(x)$  be the generating function associated with the value on the set  $\mathcal{S}_i$ . Then the generating function for the number of  $n$ -tuples of each possible total value is the product

$$G(x) = G_1(x)G_2(x) \dots G_n(x).$$

- 222. Suppose once again that  $i$  is an integer between 1 and  $n$ . In Problem 220 you encountered the formal power series  $\sum_{k=0}^{\infty} x^k$  in which the coefficient of every  $x^k$  is 1, an example of a **geometric series**. In this problem it will be useful to interpret this series as a generating function in which the coefficient 1 is the number of multisets of size  $k$

chosen from the singleton set  $\{i\}$ . Namely, there is only one way to choose a multiset of size  $k$  from  $\{i\}$ : choose  $i$  exactly  $k$  times. Express the generating function in which the coefficient of  $x^k$  is the number of  $k$ -element multisets chosen from  $[n]$  as a power of another power series. What does Problem 129 tell you about what this generating function equals?

- 223. Express the generating function for the number of multisets of size  $k$  chosen from  $[n]$  (where  $n$  is fixed but  $k$  can be any nonnegative integer) as the inverse of something relatively simple. Check your expression is consistent with your solution to Problem 220, where you answered this question for  $n = 1$ .

For future reference, fill in the coefficients in the following power series representation:

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} \left[ \quad \right] x^k$$

- 224. Use the above formula to write the inverse of  $(1 - x)^2$  as a formal power series. Comparing it to  $(1 - x)^{-1}$ , does this give you any insight into what might be called the formal derivative of a power series? Explain. (Again note that this differentiation process is referred to as *formal* since no underlying limit process has been established.)
- 225. (a) Write down the generating function for the number of ways to distribute identical pieces of candy to  $n = 3$  children so that no child gets more than 4 pieces.  
(b) Using the fact that

$$(1 - x)(1 + x + x^2 + \dots + x^4) = 1 - x^5,$$

write your generating function from part (a) as a quotient of polynomials.

- (c) Under the restrictions given in part (a), use the information from the last part to calculate how many ways can you pass out exactly ten pieces of candy to the three children.
- 226. Let  $m$  and  $n$  be fixed nonnegative integers. Express the generating function for the number of  $k$ -element multisets of an  $n$ -element set such that no element appears more than  $m$  times as a quotient of two

polynomials. Use this expression to get a formula for the number of  $k$ -element multisets of an  $n$ -element set such that no element appears more than  $m$  times.

- 227. Let  $j < n$  be positive integers.
  - (a) What is the generating function for the number of multisets chosen from an  $n$ -element set so that each element appears at least  $j$  times and less than  $m$  times?
  - (b) Write the generating function from part (a) as a quotient of polynomials.
  - (c) Write the quotient from part (b) as the product of a polynomial and a power series.
- 228. Let  $n$  be a fixed positive integer. Suppose there is an unlimited supply of identical pieces of candy. What is the generating function for the number of ways to pass out  $k$  pieces of candy to  $n$  children in such a way that each child gets between three and six pieces of candy (inclusive)? Use generating functions to find a formula for the number of ways to pass out  $k$  pieces of candy.
- 229. Suppose you have some chairs which you're going to paint with red, white, blue, green, yellow and purple paint. Suppose that you may paint any number of chairs red or white, at most one chair blue, at most three chairs green, only an even number of chairs yellow, and only a multiple of four chairs purple. In how many ways can you paint  $k$  chairs?

*Hint.* It is useful to write each factor in the product as a quotient of polynomials and then do some cancellation (that is, use inverses).

### 5.2.3 Product Principle for Generating Functions (Optional)

- 230. (Here is an outline of a proof of the Product Principle for Generating Functions which does not rely on the Product Principle for Picture Enumerators.) Suppose that you have two sets  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Let  $v_1$  (here  $v$  stands for value) be a function from  $\mathcal{S}_1$  to the nonnegative integers and let  $v_2$  be a function from  $\mathcal{S}_2$  to the nonnegative integers. Define a new function  $v$  on the set  $\mathcal{S}_1 \times \mathcal{S}_2$  by  $v(x_1, x_2) = v_1(x_1) + v_2(x_2)$ . Suppose further that  $\sum_{i=0}^{\infty} a_i x^i$  is the generating function for the number of elements  $x_1 \in \mathcal{S}_1$  of value  $i$ , that is, with  $v_1(x_1) = i$ . Suppose also that  $\sum_{j=0}^{\infty} b_j x^j$  is the generating function for the number of elements  $x_2 \in \mathcal{S}_2$  of value  $j$ , that is, with  $v_2(x_2) = j$ . Prove that the

coefficient of  $x^k$  in

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \left(\sum_{j=0}^{\infty} b_j x^j\right)$$

is the number of ordered pairs  $(x_1, x_2)$  in  $S_1 \times S_2$  with total value  $k$ , that is, with  $v_1(x_1) + v_2(x_2) = k$ . This is called the **Product Principle for Generating Functions**.

*Hint.* If this problem appears difficult, the most likely reason is because the definitions are all new and symbolic. Focus on what it means for  $\sum_{k=0}^{\infty} c_k x^k$  to be the generating function for ordered pairs of total value  $k$ . In particular, how do we get an ordered pair with total value  $k$ ? What do we need to know about the values of the components of the ordered pair?

231. Use mathematical induction to prove the Product Principle for Generating Functions (as given on page 82).

### 5.3 Solving Recurrences with Generating Functions

In this section you will learn how generating functions can be used to obtain a closed formula for the solution of recurrences. Such a formula is useful because it allows easier calculation of specific terms in the solution sequence. For instance, suppose the recurrence  $a_i = 3a_{i-1} + 3^i$  were used to model a certain population of bacteria, where  $a_i$  is the number of bacteria after  $i$  hours. An explicit formula for  $a_i$  (involving no previous terms) would be very handy, since it wouldn't be practical to use the recurrence itself to compute the size of the colony after many hours (why?).

Using pencil and paper, follow the following directions for algebraic manipulation using the recurrence  $a_i = 3a_{i-1} + 3^i$ . First, multiply both sides of the recurrence by  $x^i$  and then sum both the left-hand side and right-hand side from  $i = 1$  to infinity. In the left-hand side use the fact that

$$\sum_{i=1}^{\infty} a_i x^i = \left(\sum_{i=0}^{\infty} a_i x^i\right) - a_0$$

and in the right-hand side, use the fact that

$$\sum_{i=1}^{\infty} a_{i-1} x^i = x \sum_{i=1}^{\infty} a_{i-1} x^{i-1} = x \sum_{j=0}^{\infty} a_j x^j = x \sum_{i=0}^{\infty} a_i x^i$$



(where  $j$  is substituted for  $i - 1$ , a surprisingly useful trick) to rewrite the equation in terms of the power series  $\sum_{i=0}^{\infty} a_i x^i$ . Solve the resulting equation for the power series

$$\sum_{i=0}^{\infty} a_i x^i = \frac{a_0 - 1}{1 - 3x} + \frac{1}{(1 - 3x)^2}.$$

(You can save a lot of writing by using a variable such as  $y$  to represent the power series you're solving for.) Writing each summand on the right-hand side as a power series, you can equate coefficients to obtain a closed form for the recurrence in terms of the initial population,  $a_0$ .

- 232. Find a closed form for the recurrence  $a_i = 3a_{i-1} + 3^i$  with initial value  $a_0$ .

The next sequence of problems works with a mathematical model of a fictional population of rabbits. For purposes of modeling the rabbit population three assumptions are made:

- Rabbits are mature and begin to reproduce after one month.
- Each mature pair produces two new pairs at the end of each month.
- No rabbit dies during the period of observation.

The example of a rabbit population is used for historic reasons, and the goal is a classical sequence of numbers called the **Fibonacci numbers**. “Fibonacci” is the name that Leonardo de Pisa was given posthumously; it is a shortening of “son of Bonacci” in Italian. Leonardo de Pisa introduced this mathematical model of a biological population in his book, *Liber Abaci*, which was published in 1202. In time for the 500-th anniversary, Springer-Verlag published *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation* by Laurence Sigler.

- 233. Begin at the end of month 0 with 10 pairs (where a pair means one female and one male) of baby rabbits. Let  $a_n$  be the number of rabbit pairs at the end of month  $n$ . Show that  $a_0 = 10$  and  $a_n = a_{n-1} + 2a_{n-2}$ . This is an example of a **second-order recurrence** which is also linear and has constant coefficients. Using a method similar to the one used at the beginning of this section, show that

$$\sum_{i=0}^{\infty} a_i x^i = \frac{10}{1 - x - 2x^2}.$$

In the last problem you represented the generating function for  $\{a_i\}$  as a quotient of polynomials. Such a quotient is often referred to as a **rational representation** of the generating function for the recurrence. If you know how to express the reciprocal of denominator in this representation as a power series, then it is relatively easy to find the coefficients of your generating function. You did that in Problem 232 for the linear first-order recurrence given in the beginning of the section. In Problem 233 you don't have the corresponding formal power series directly available. Try to think of a way to get it.

- 234. In Fibonacci's original problem, there is one pair of baby rabbits at the end of month 0 and each pair of mature rabbits produces one new pair at the end of each month. Otherwise the situation is the same as in Problem 233.
  - (a) Find the recurrence.
  - (b) Under these assumptions, find a rational representation of the generating function for the number of pairs of rabbits at the end of  $n$  months.
- 235. (a) Use the quadratic formula to factor  $1 - x - x^2$ , and then use the factors to find the partial fraction decomposition of  $\frac{1}{1 - x - x^2}$ .  
*Hint.* It is useful to represent the roots of the equation  $1 - x - x^2 = 0$  by  $r_1, r_2$  while you're working through the partial fraction decomposition.
  - (b) Use the partial fraction decomposition you found in part (a) to write the generating function you found in Problem 234 as  $\sum_{n=0}^{\infty} a_n x^n$ , the standard form for power series.
  - (c) Solve for an explicit formula for  $a_n$ . This is called **Binet's Formula**.
- 236. Explain why there exists a real number  $b$  such that, for large values of  $n$ , the value of the  $n$ th Fibonacci number is almost exactly (but not quite) some constant times  $b^n$ . Find  $b$  and the constant.

## Chapter 6

# The Principle of Inclusion and Exclusion

### 6.1 The Size of a Union of Sets

One of the first counting principles in these notes was the Sum Principle which says that the size of a union of disjoint sets is the sum of their sizes. Computing the size of the union of *overlapping* sets quite naturally requires information about how they overlap. Taking such information into account allows the development of a powerful extension of the Sum Principle known as the Principle of Inclusion and Exclusion.

- 237. In Problem 15, just two fertilizers were used to treat all sample plants in a certain biology lab. Now suppose there are three fertilizer treatments: 15 plants are treated with nitrates, 16 with potash, 16 with phosphate, 7 with nitrate and potash, 9 with nitrate and phosphate, 8 with potash and phosphate and 4 with all three. Now how many plants have been treated? If 32 plants were studied, how many received no treatment at all?
- 238. Give a formula for the size of  $A \cup B \cup C$  in terms of the sizes of  $A$ ,  $B$ ,  $C$  and the various intersections of these sets.
- 239. Conjecture a formula for the size of a union of sets

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

in terms of the sizes of the sets  $A_i$  and their various intersections. Express your conjecture in words before attempting to write a formula.

The hardest part of generalizing your answer in Problem 238 to Problem 239 is probably finding a good notation to express your conjecture. In fact, for many people it would be easier to express the conjecture in words than to express it as mathematical formula. Some notation which will make your task easier is similar to the notation

$$E_P(S) = \sum_{s \in S} P(s)$$

that was used to stand for the sum of the pictures of the elements of a set  $S$  when picture enumerators were introduced. Here, define

$$\bigcap_{i \in I} A_i$$

to mean the intersection over all elements  $i$  in the set  $I$  of  $A_i$ ; for example,

$$\bigcap_{i \in \{1,3,4,6\}} A_i = A_1 \cap A_3 \cap A_4 \cap A_6. \quad (6.1)$$

You've already used this kind of notation (involving an operator with a descriptor below it) in summation notation for sums and in product notation for products. In this case the operator is set intersection and the descriptor identifies the values of a dummy variable that we're interested in.

- 240. Use notation similar to (6.1) to express the answer to Problem 239. Note there are many different correct ways to do this, and try to write down more than one. Choose the neatest one you can. Be sure to say why you chose it because your view of what makes a formula nice may be different from somebody else's.
- 241. A group of  $n$  students with backpacks goes to a restaurant. The manager invites everyone to check his or her backpack at the check desk and everyone does. While they are eating, a child playing in the check room randomly moves around the claim check stubs.
  - (a) What is the total number of ways to pass back the backpacks?
  - (b) Let  $A_i$  be the set of backpack distributions in which student  $i$  gets the correct backpack. In how many of the distributions of backpacks-to-students does at least one student get his or her own backpack? It might be a good idea to first consider cases with  $n = 3, 4$ , and  $5$ .

- 242. In this problem you will compute the probability that, at the end of the meal, at least one student in the previous problem receives his or her own backpack. Here the **probability** is the fraction of the total number of ways to return the backpacks in which at least one student gets his or her own backpack.
  - (a) What is the probability that at least one student gets the correct backpack?
  - (b) What is the probability that no student gets his or her own backpack?
  - (c) As the number of students becomes large, what does the probability that no student gets the correct backpack approach?

*Hint.* Think calculus and Taylor polynomials.

Problem 241 is “classically” called the Hat Check Problem—the name comes from substituting hats for backpacks. It is also sometimes called the Derangement Problem. A **derangement** of an  $n$ -element set is a permutation of that set (thought of as a bijection on  $[n]$ ) that maps no element of the set to itself; that is, a permutation with no fixed points. One can think of a way of handing back the backpacks as a permutation  $f$  of the students in which  $f(i)$  is the owner of the backpack that student  $i$  receives. Then a derangement is a way to pass back the backpacks so that no student gets his or her own.

## 6.2 The Principle of Inclusion and Exclusion

The formula you’ve discovered in Problem 240 is usually called the **Principle of Inclusion and Exclusion** for unions of sets. The reason for this name is the pattern in the formula: It first adds (includes) all the sizes of the sets, then subtracts (excludes) all the sizes of the intersections of two sets, then adds (includes) all the sizes of the intersections of three sets, and so on. There are a variety of proofs of this principle. Perhaps one of the most straightforward is an inductive proof that relies on the inductive process

$$A_1 \cup A_2 \cup \cdots \cup A_n = (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n,$$

which expresses the  $n$ -fold union as a union of two sets. What formula for  $|A \cup B|$  did you discover in Problem 16?

- 243. Use induction to give a proof of your formula for the Principle of Inclusion and Exclusion.

244. A more elegant proof can be obtained by asking for a picture enumerator for  $A_1 \cup A_2 \cup \cdots \cup A_n$ . So assume  $A$  is a set with a picture function  $P$  defined on it and that each set  $A_i$  is a subset of  $A$ .
- (a) By thinking about how the formula for the size of a union was obtained, write down instead a conjecture for the picture enumerator of a union. You could use a notation like  $E_P(\bigcap_{i \in S} A_i)$  for the picture enumerator of the intersection of the sets  $A_i$  for  $i$  in a subset  $S$  of  $[n]$ .
  - (b) If  $x \in \bigcup_{i=1}^n A_i$ , what is the coefficient of  $P(x)$  in (the inclusion-exclusion side of) your formula for  $E_P(\bigcup_{i=1}^n A_i)$ ?  
*Hint.* Let  $T$  be the set of all  $i$  such that  $x \in A_i$ . In terms of  $x$ , what is different about the  $i$  in  $T$  and those not in  $T$ ? You may come to a point where the binomial theorem would be helpful.
  - (c) If  $x \notin \bigcup_{i=1}^n A_i$ , what is the coefficient of  $P(x)$  in (the inclusion-exclusion side of) your formula for  $E_P(\bigcup_{i=1}^n A_i)$ ?
  - (d) How have you proved your conjecture for the picture enumerator of the union of the sets  $A_i$ ?
  - (e) How can you get the formula for the Principle of Inclusion and Exclusion from your formula for the picture enumerator of the union?

Frequently the Principle of Inclusion and Exclusion is applied to situations similar to that of Problem 241(b). That is, there is a set  $A$  and subsets  $A_1, A_2, \dots, A_n$  and what is required is the size of the set of elements in  $A$  which are *not* in the union. This set is known as the **complement** of the union of the  $A_i$ s in  $A$ . This will either be denoted by  $A \setminus \bigcup_{i=1}^n A_i$  or by  $\overline{\bigcup_{i=1}^n A_i}$ , where the latter is used when the universe  $A$  is clear from context. The same mathematical symbol can have different meanings in different contexts. Here an over-bar is used to denote the complement of the set, whereas in analysis or topology this sometimes means the closure of the set. There, and elsewhere, the complement of the set  $A$  might be denoted by  $A^c$ .

The Principle of Inclusion and Exclusion can refer to both the formula for the union and the one for its complement.

245. Prove the formula

$$\left| \overline{\bigcup_{i=1}^n A_i} \right| = |A| - \sum_{S \subseteq [n], S \neq \emptyset} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|.$$

A very elegant way of writing the formula in Problem 245 can be obtained by setting  $\bigcap_{i \in \emptyset} A_i$  equal to  $A$ . (Rewrite the formula using this convention.) An aside for those interested in logic and set theory: Given a family of subsets  $A_i$  of a set  $A$ , define  $\bigcap_{i \in S} A_i$  to be the set of all members  $x$  of  $A$  that are in  $A_i$  for all  $i \in S$ . (Note that this allows  $x$  to be in some other  $A_j$ s as well.) Then if  $S = \emptyset$ , the intersection consists of all members  $x$  of  $A$  that satisfy the statement “if  $i \in \emptyset$ , then  $x \in A_i$ .” But since the hypothesis of the ‘if-then’ statement is false, the statement itself is true *for all*  $x \in A$ . Therefore  $\bigcap_{i \in \emptyset} A_i = A$ .

- 246. Each person attending a party has been asked to bring a prize. The person planning the party has arranged to give out all the donated prizes (and only those prizes), but any person may win any number of prizes. Suppose there are  $n$  guests, and for each  $1 \leq i \leq n$  let  $A_i$  be the set of all distributions of prizes in which person  $i$  gets the prize he or she brought. Let  $A$  be the set of all distributions of prizes.
  - (a) Find  $|A|$  and each  $|A_i|$ .  
*Hint.* Think functions.
  - (b) In how many ways can the prizes be given out so that nobody gets the prize that he or she brought?
  - (c) What is the probability that nobody gets the prize she or he brought?
  - (d) Is there a limiting value for the probability in part (c) as the number of party guests increases? If so, find the limiting value.
- 247. There are  $m$  students attending a seminar in a room with  $n$  seats. The seminar is a long one, and in the middle the group takes a break.
  - (a) In how many ways may the students return to the room and sit down so that nobody is in the same seat as before?
  - (b) What happens in a probabilistic sense as  $m$  and  $n$  become large?
- 248. Suppose that  $n$  children join hands in a circle for a game at nursery school. The game involves everyone falling down (and letting go). In how many ways may they join hands in a circle again so that nobody has the same person immediately to the right both times the group joins hands?

### 6.3 Counting the Number of Onto Functions

- 249. Given a function  $f$  from  $[k]$  to  $[n]$ ,  $f$  will be a member of the set  $A_i$  if and only if  $f(x) \neq i$  for every  $x \in [k]$ . What do the elements of  $A_i$  look

like? For  $n = 5$  and  $k = 3$ , find  $|A_2|$  and  $|A_3|$ .

- 250. If  $f$  is an onto function from  $[k]$  to  $[n]$ , how many of the sets  $A_i$  (as defined in the previous problem) does  $f$  belong to? What is the number of onto functions from  $[k]$  to  $[n]$ ?
- 251. If a die is rolled eight times, a sequence of eight numbers from the set  $[6]$  is obtained; namely, the number of dots on top on the first roll, the number on the second roll, and so on.
  - (a) What is the number of ways of rolling the die eight times so that each of the numbers one through six appears at least once in the sequence?
  - (b) What is the probability that a sequence is rolled in which all six numbers between one and six appear?
- 252. Continuing with the last problem: How many times must a die be rolled in order to ensure the probability is at least  $1/2$  that all six numbers appear in the sequence? In order to answer this question, you will need to experiment and use a computational device like a programmable calculator or some kind of computer algebra package.

## 6.4 The Ménage Problem

A certain town has a large number of 8-year-old twins, who are all in the same third-grade class. The teacher asks  $n$  sets of twins to sit around a round table.

- 253. Let  $A_i$  be the set of all such seatings in which the children in the  $i$ -th set of twins are sitting next to each other. Find  $|A_i|$ . Find  $|A_i \cap A_j|$  for  $i \neq j$ .
- 254. For each of  $n = 4$  and  $n = 5$ , find the number of ways  $n$  sets of twins can be seated if no one may sit next to his or her twin.
- 255. For general  $n$ , in how many ways can the  $n$  sets of twins be seated if no one may sit next to his or her twin?
- 256. In this problem you are again seating  $n$  sets of twins around a round table, and now each set of twins has one boy and one girl. In how many ways can they sit so that no person is next to his or her twin and the genders alternate around the table? This problem is called the **Ménage Problem**.

*Hint.* Reason somewhat as you did in Problem 255, noting that if the set of all twins who do sit side-by-side is nonempty, then the gender



of the person at each place at the table is determined once one pair in that set is seated, or, for that matter, once one person is seated.

## 6.5 The Chromatic Polynomial of a Graph

A **coloring** of the vertices of a graph by the elements of a set  $C$  (of colors) is an assignment of an element of  $C$  to each vertex of the graph; that is, a function from the vertex set  $\mathcal{V}$  of the graph to  $C$ . A coloring is called a **proper coloring** if for each edge joining two distinct vertices<sup>1</sup>, the two vertices it joins have different colors. You may have heard of the famous Four Color Theorem of graph theory that says every drawing of a graph in the plane in which no two edges cross (though they may touch at a vertex) has a proper coloring with four colors. Here a different, though related, problem is considered: In how many ways can you properly color a graph (regardless of whether it can be drawn in the plane or not) using  $k$  or fewer colors?

- 257. Given a graph which might or might not be connected, define a relation on its set  $\mathcal{V}$  of vertices by:

$$\text{For } v, w \in \mathcal{V}, \quad vRw \iff \text{there is a path from } v \text{ to } w.$$

Prove this relation is an equivalence relation. Its equivalence classes are called the **connected components** of the graph.

Notice that the connected components depend on the edge set of the graph. That is, if a graph has vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$  and another graph has the same vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}'$ , these two graphs could have different connected components. It is traditional to use the Greek letter  $\gamma$  (called gamma) to represent the number of connected components of a graph. In particular,  $\gamma(\mathcal{V}, \mathcal{E})$  represents the number of connected components of the graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ . The Principle of Inclusion and Exclusion may be used to compute the number of ways to color a graph properly using colors from a set  $C$  of  $c$  colors.

- 258. Let  $G$  be a graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{E} = \{e_1, e_2, \dots, e_{|\mathcal{E}|}\}$ , and let  $\mathcal{F}$  be any subset of  $\mathcal{E}$ . Suppose  $C$  is a set of  $c$  colors with which to color the vertices.

---

<sup>1</sup>If a graph has a loop connecting some vertex to itself, the loop must of course connect a vertex to a vertex of the same color. Because of this, in this definition the only edges considered are those with two distinct vertices.

- (a) In terms of  $\gamma(\mathcal{V}, \mathcal{F})$ , in how many ways can you color the vertices of  $G$  so that every edge in  $\mathcal{F}$  connects two vertices of the same color?

*Hint.* Use the connected components. For each edge in  $\mathcal{F}$  to connect two vertices of the same color, we must have all the vertices in a connected component of the graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{F}$  colored the same color.

- (b) Given a coloring of  $G$ , for each edge  $e_i$  in  $\mathcal{E}$  consider the set  $A_i$  of all colorings in which both endpoints of  $e_i$  are colored the same color. In which sets  $A_i$  does a proper coloring lie?
- (c) Find a formula for the number of proper colorings of  $G$  using colors in the set  $C$ . Your formula will probably involve summing over all subsets  $\mathcal{F}$  of the edge set of the graph and using the number  $\gamma(\mathcal{V}, \mathcal{F})$  of connected components of the graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{F}$ .

The formula you found in Problem 258 involves powers of the number of colors, and so it is a polynomial function of  $c$ . People often use  $x$  as the notation for the number of colors used to color  $G$ . Frequently people will use  $\chi_G(x)$  to represent the number of ways to color  $G$  with  $x$  colors, and call  $\chi_G(x)$  the **chromatic polynomial** of  $G$ .

### 6.5.1 Deletion-Contraction (Optional)

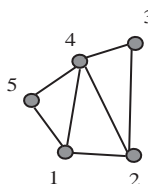
259. In Section 4.5.1 (on pages 69ff) you developed the deletion-contraction recurrence and used it to count the number of spanning trees in a graph.

- (a) Figure out how the chromatic polynomial of a graph is related to the chromatic polynomials of the graphs resulting from deletion of an edge  $e$  and from contraction of that same edge  $e$ .
- (b) Try to find a recurrence like the one for counting spanning trees that expresses the chromatic polynomial of a graph in terms of the chromatic polynomials of  $G \setminus e$  and  $G/e$  for an arbitrary edge  $e$ .
- (c) Use the recurrence from the last part to give another proof that the number of ways to color a graph with  $x$  colors is a polynomial function of  $x$ .

260. Use the deletion-contraction recurrence to reduce the computation of the chromatic polynomial of the graph in Figure 6.1 to computing chromatic polynomials that you can easily compute. (You can simplify your computations by thinking about the effect on the chromatic polynomial

of deleting an edge that is a loop, or deleting one of several edges between the same two vertices.)

Figure 6.1: A graph.



261. (a) In how many ways may you properly color the vertices of a path on  $n$  vertices with  $x$  colors? Describe any dependence of the chromatic polynomial of a path on the number of vertices.
- (b) In how many ways may you properly color the vertices of a cycle on  $n$  vertices with  $x$  colors? Describe any dependence of the chromatic polynomial of a cycle on the number of vertices.
262. In how many ways may you properly color the vertices of a tree on  $n$  vertices with  $x$  colors?
263. What do you observe about the signs of the coefficients of the chromatic polynomial of the graph in Figure 6.1? What about the signs of the coefficients of the chromatic polynomial of a path? Of a cycle? Of a tree? Make a conjecture about the signs of the coefficients of a chromatic polynomial and prove it.

## Chapter 7

# Distribution Problems

### 7.1 The Idea of Distributions

Many of the problems you solved in earlier chapters may be considered to be distribution problems—problems which involve distributing objects (such as pieces of fruit or ping-pong balls) to recipients (such as children). For example, in Problem 104 you probably worked through the fact that the number of ways to pass out  $k$  ping-pong balls to  $n$  children so that no child gets more than one ball is the number of ways that you can choose a  $k$ -element subset of an  $n$ -element set. You can think that the children are the recipients and the identical ping-pong balls are the objects you are distributing, and that the distribution is done in such a way that each recipient gets at most one ball. Those children who receive a ball form the  $k$ -element subset of the  $n$ -element set of children. (They form a *subset* because the balls are identical.)

Another popular model for distributions is to think of putting balls in boxes or books in bookcases rather than distributing objects to recipients. Passing out *identical* objects is modeled by putting identical balls into boxes. Passing out *distinct objects* is modeled by putting distinct balls into boxes. So, when you are passing out objects to recipients, you may think of the objects as being either identical or distinct. You may also think of the recipients as being either identical (grocery bags in the case of putting fruit into bags in the grocery store) or distinct (children in the case of passing fruit out to children). You may restrict the distributions to those that give at least one object to each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or you may have no such restrictions. If the objects are distinct, it may be

that the order in which the objects are received is relevant (think about ice cream in 3-decker cones) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child's trick-or-treat bag). The next three problems are a review of formulas you've already developed.

- +264. Consider the distribution of  $k$  distinct objects to  $n$  distinct recipients, with different conditions on how the objects are received. The first row in the following table is already filled in. Fill in the other rows.

Conditions	Number of Ways	Mathematical Model
No conditions	$n^k$	functions
Each gets at least one		
Each gets at most one		
Each gets exactly one		
Order matters		
Order matters Each gets at least one		

- +265. Consider the distribution of  $k$  identical objects to  $n$  distinct recipients, with different conditions on how the objects are received. Fill in all entries in the table.

Conditions	Number of Ways	Mathematical Model
No conditions		
Each gets at least one		
Each gets at most one		
Each gets exactly one		

- +266. Consider the distribution of  $k$  objects to  $n$  identical recipients, with different conditions on how the objects are received. Fill in the entries of the table (except for the entries with ?).

Objects	Conditions	Number of Ways
Distinct	Each gets at most one	
Distinct	Each gets exactly one	
Distinct	Order matters	?
Distinct	Order matters and each gets at least one	?
Identical	Each gets at most one	
Identical	Each gets exactly one	

- 267. In how many ways may you stack 5 distinct books into 3 identical boxes so that each box contains at least one book? (Assume the order of books in a stack makes a difference.) This problem different from arranging 5 distinct books in a bookcase with 3 bookshelves in such a way that each shelf gets at least one book—why?

In the last problem you might have thought to first partition the five books into three blocks and then follow by ordering the books within the blocks of the partition. This turns out not to be a useful combinatorial way of visualizing the problem because the number of ways to order the books in the various blocks depends on the sizes of the blocks and not just the number of blocks.

- 268. In this problem you want to count the number of ways to stack  $k$  distinct books into  $n$  identical boxes so that there is at least one book in every box.
  - (a) First consider the set  $\mathcal{S}$  of all arrangements of the  $k$  distinct books on  $n$  distinct shelves such that every shelf has at least one book. When are two of the arrangements in  $\mathcal{S}$  the same as far as what you're asked to count in this problem? Define this idea of “sameness” as an equivalence relation on  $\mathcal{S}$ . Explain why every equivalence class has the same size. What is this common size?
  - (b) Find the number of ways to stack  $k$  distinct books into  $n$  identical boxes so that there is a stack in every box. This number is usually denoted by  $L(k, n)$  and is called a **Lah number**.
- 269. Explain why the number of ways to stack  $k$  distinct books into  $n$  identical boxes is  $\sum_{i=1}^n L(k, i)$ .
- 270. Show the Lah numbers  $L(k, n)$  satisfy the recurrence

$$L(k, n) = L(k - 1, n - 1) + (n + k - 1)L(k, n).$$

This is similar to Pascal's Equation which you proved in Section 3.3.1. Section

*Hint.* Either  $k$  is in an ordered block by itself or it is not.

- 271. Fill in the two entries with ?s in the table in Problem 266.

## 7.2 Counting Partitions

In Problem 20 you showed any partition of  $[k]$  into  $n$  non-empty blocks corresponds to a function from  $[k]$  to  $[n]$ . For instance,  $\{1, 2, 4\}, \{3\}, \{5\}$  is

a  $k = 3$  block partition of  $[5]$  that can be described by the function

$$f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 1, f(5) = 3;$$

that is,  $f(x)$  equals the block in which  $x$  lies. Since re-ordering the partition-blocks changes the function, this correspondence is a relation from the set of  $n$ -block partitions of  $[k]$  to the set of all functions from  $[k]$  to  $[n]$  but *is not a function*. In fact, a partition of  $[k]$  into  $n$  blocks is a distribution of  $k$  distinct objects to  $n$  *indistinguishable* recipients.

We use the notation  $S(k, n)$  to stand for the number of  $n$ -block partitions of  $[k]$ , where by convention  $S(0, 0) = 1$ . For historical reasons,  $S(k, n)$  is called a **Stirling Number of the second kind**.

- 272. What is  $S(k, 1)$ ;  $S(k, k)$ ? How should you define  $S(k, n)$  for  $n > k$ ?
- 273. Find  $S(k, k - 1)$  and  $S(k, 2)$  for any  $k \geq 2$ .
- 274. Given a function  $f$  from  $[k]$  to  $[n]$ , we can define a partition of  $[k]$  by putting  $x$  and  $y$  in the same block of the partition if and only if  $f(x) = f(y)$ . (This relation is inverse to the relation in Problem 20.) How many blocks does the partition have if  $f$  is an onto function? How is the number of onto functions from  $[k]$  to  $[n]$  related to a Stirling Number? Be as precise in your answer as you can.

In Problem 115 you proved a recurrence for computing the Stirling Numbers  $S(k, n)$  that is similar to Pascal's Equation for computing binomial coefficients.

- 275. Use the Stirling recurrence to create a table of values of  $S(k, n)$  for  $1 \leq k \leq 5$  and  $1 \leq n \leq k$ . This table is sometimes called **Stirling's Triangle** because of the analogy with Pascal's Triangle.
- 276. Extend Stirling's Triangle far enough to allow you to answer the following question. A caterer is preparing three bag lunches for hikers. If the caterer has nine different sandwiches, in how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?

The total number of partitions of a  $k$ -element set is denoted by  $B(k)$  and is called the  $k$ -th **Bell Number**. Verify that  $B(1) = 1, B(2) = 2, B(3) = 5$  by explicitly exhibiting all the partitions.

- 277. For a given  $k \geq 1$ , what is the relationship between  $B(k)$  and the Stirling Numbers  $S(k, n)$ ?

- 278. (a) Prove the Bell Numbers satisfy  $B(k) = \sum_{n=0}^{k-1} \binom{k-1}{n} B(n)$  by asking yourself what happens when you delete the entire block containing  $k$ .
- (b) Use this Bell recurrence to calculate  $B(k)$  for  $k = 4, 5, 6$ .

### 7.2.1 Multinomial Coefficients (Optional)

- 279. Fix positive integers  $k_1, \dots, k_n$ . Prove there are  $n! / \prod_{i=1}^n (i!)^{k_i} k_i!$  ways to partition  $k$  distinct items into  $n$  blocks so that there are  $k_i$  blocks of size  $i$  for each  $i$ . The sequence  $k_1, k_2, \dots, k_n$  is called the **type vector** of the partition.
280. Describe how to compute  $S(n, k)$  in terms of all type vectors  $(k_1, k_2, \dots, k_n)$  such that  $k_1 + k_2 + \dots + k_n = k$ .
281. In how many ways may we label the elements of  $[k]$  with  $n$  distinct labels (numbered 1 through  $n$ ) so that label  $i$  is used  $j_i$  times? This number is called a **multinomial coefficient** and denoted by

$$\binom{k}{j_1, j_2, \dots, j_n}.$$

What if the  $j_i$ s don't add to  $k$ ?

*Hint.* Think about listing the elements of  $[k]$  and labeling the first  $j_1$  elements with label number 1.

282. Write the binomial coefficient  $\binom{k}{m}$  as a multinomial coefficient.
283. Explain how multinomial coefficients can be used to compute the number of functions from  $[k]$  to  $[n]$ .
284. Explain how to use multinomial coefficients to compute the number of onto functions from  $[k]$  to  $[n]$ .
- Hint.* How are the relevant  $j_i$ s in the multinomial coefficients you use here different from the  $j_i$ s in the previous problem?
285. How may multinomial coefficients be used to obtain an expression for  $k$ th power of a multinomial  $x_1 + x_2 + \dots + x_n$ ? This result is called the **Multinomial Theorem**.

*Hint.* Review the Binomial Theorem.

## 7.3 Additional Problems

1. Answer each of the following questions.



- (a) In how many ways can you pass out  $k$  identical pieces of candy to  $n$  children?
  - (b) In how many ways can you pass out  $k$  distinct pieces of candy to  $n$  children?
  - (c) In how many ways can you pass out  $k$  identical pieces of candy to  $n$  children so that each gets at most one? (Assume  $k \leq n$ .)
  - (d) In how many ways can you pass out  $k$  distinct pieces of candy to  $n$  children so that each gets at most one? (Assume  $k \leq n$ .)
  - (e) In how many ways can you pass out  $k$  identical pieces of candy to  $n$  children so that each gets at least one? (Assume  $k \geq n$ .)
2. The neighborhood improvement committee has been given  $r$  trees to distribute to  $s$  families living along one side of a street. Unless otherwise specified, it doesn't matter where a family plants the trees it gets.
- (a) In how many ways can the committee distribute all of them if the trees are distinct, there are more families than trees, and each family can get at most one tree?
  - (b) In how many ways can the committee distribute all of them if the trees are distinct and any family can get any number?
  - (c) In how many ways can the committee distribute all the trees if the trees are identical, there are no more trees than families, and any family receives at most one?
  - (d) In how many ways can the committee distribute all the trees if they are identical and anyone may receive any number of trees?
  - (e) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road?
  - (f) Answer the question in part (e) assuming that every family must get a tree.
  - (g) Answer the question in part (d) assuming that each family must get at least one tree.

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**Part II**

**REVIEW MATERIAL**

## Appendix A

# More on Functions and Digraphs

### A.1 Functions

**Exercise A.1.** Consider the functions from  $S = \{-2, -1, 0, 1, 2\}$  to  $T = \{1, 2, 3, 4, 5\}$  defined by  $f(x) = x + 3$ , and  $g(x) = x^5 - 5x^3 + 5x + 3$ . Write down the set of all ordered pairs  $(x, f(x))$  for  $x \in S$ , and the set of all ordered pairs  $(x, g(x))$  for  $x \in S$ . Are the two functions the same or different?

Exercise A.1 points out how two functions which appear to be different are actually the same. Most of the time when we are thinking about functions it is fine to think of a function casually as a relationship between two sets. In Exercise A.1 the set of ordered pairs you wrote down for each function is called the **relation of the function**. When we want to distinguish between the casual and the careful in talking about relationships, our casual term will be “relationship” and our careful term will be “relation.” So *relation* is a technical word in mathematics, and as such it has a technical definition:. A **relation** from a set  $S$  to a set  $T$  is a set of ordered pairs whose first elements are in  $S$  and whose second elements are in  $T$ . Another way to say this is that a **relation** from  $S$  to  $T$  is a subset of the Cartesian product  $S \times T$ .

A typical way to define a **function**  $f$  from a set  $S$  (called the **domain** of the function) to a set  $T$  (called the **co-domain**) is that  $f$  is a relation from  $S$  to  $T$  which relates each element of  $S$  to one and only one member of  $T$ . We use  $f(x)$  to stand for the element of  $T$  that is related to the element  $x$  of  $S$ , and we use the standard shorthand  $f : S \rightarrow T$  for “ $f$  is a function from  $S$  to  $T$ ”.

**Exercise A.2.** Here are some questions that will help you get used to the formal idea of a relation and the related formal idea of a function.  $S$  will stand for a finite set of size  $s$  and  $T$  will stand for a finite set of size  $t$ .

- (a) What is the size of the largest relation from  $S$  to  $T$ ?
- (b) What is the size of the smallest relation from  $S$  to  $T$ ?
- (c) What is the size of the relation of a function from  $S$  to  $T$ ? That is, how many ordered pairs are in the relation of a function from  $S$  to  $T$ ?
- (d) Before working this and the next exercise, review the definitions of one-to-one function and onto function in Chapter 1. How many different elements must appear as second elements of the ordered pairs in the relation of a one-to-one function from  $S$  to  $T$ ?
- (e) What is the minimum size that  $S$  can have if there is a onto function from  $S$  to  $T$ ?

*Sketch of solution.* (a)  $st$  because that is the size of the relation that has all the ordered pairs  $(x, y)$  with  $x \in S$  and  $y \in T$ .

- (b) 0, because the empty set of ordered pairs is a relation.
- (c)  $s$ .
- (d)  $s$ , exactly one for each element of  $S$ .
- (e) The size of  $S$  must be at least  $t$ .

**Exercise A.3.** When  $f$  is a function from  $S$  to  $T$ , the sets  $S$  and  $T$  play a big role in determining whether a function is one-to-one or onto. For the remainder of this exercise, let  $S$  and  $T$  stand for the set of nonnegative real numbers.

- (a) If  $f : S \rightarrow T$  is given by  $f(x) = x^2$ , is  $f$  one-to-one? Is  $f$  onto?
- (b) Now assume for the rest of the exercise that  $S'$  is the set of all real numbers and  $g : S' \rightarrow T$  is given by  $g(x) = x^2$ . Is  $g$  one-to-one? Is  $g$  onto?
- (c) Assume for the rest of the exercise that  $T'$  is the set of all real numbers and  $h : S \rightarrow T'$  is given by  $h(x) = x^2$ . Is  $h$  one-to-one? Is  $h$  onto?
- (d) And if the function  $j : S' \rightarrow T'$  is given by  $j(x) = x^2$ , is  $j$  one-to-one? Is  $j$  onto?
- (e) If  $f : S \rightarrow T$  is a function, we say that  $f$  **maps**  $x$  to  $y$  as another way to say that  $f(x) = y$ . Suppose  $S = T = \{1, 2, 3\}$ . Give a function from  $S$  to  $T$  that is not onto. Notice that two different members of  $S$  have mapped to the same element of  $T$ . Thus when we say that  $f$  associates one and only one element of  $T$  to each element of  $S$ , it is quite possible that the one and only one element  $f(1)$  that  $f$  maps 1 to is exactly the same as the one and only one element  $f(2)$  that  $f$  maps 2 to.



## A.2 Digraphs

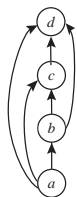


Figure A.1: The Alphabet Digraph.

Figure A.1 illustrates a digraph of the “comes before in alphabetical order” relation on the letters  $a$ ,  $b$ ,  $c$ , and  $d$ . We draw the arrow from  $a$  to  $b$ , for example, because  $a$  comes before  $b$  in alphabetical order. We try to choose the locations for the vertices so that the arrows capture what we are trying to illustrate as well as possible. Sometimes this entails redrawing our directed graph several times until we think the arrows capture the relationship well.

**Exercise A.4.** Draw the digraph of the “is a proper subset of” relation on the set of subsets of a two element set. (Remember the empty set is a subset.) How many arrows would you have had to draw if this exercise asked you to draw the digraph for the subsets of a three-element set?

**Exercise A.5.** (a) Draw the digraph of the relation from the set  $\{A, M, P, S\}$  to the set  $\{\text{Sam, Mary, Pat, Ann, Polly, Sarah}\}$  given by “is the first letter of.”  
(b) Draw the digraph of the relation from the set  $\{\text{Sam, Mary, Pat, Ann, Polly, Sarah}\}$  to the set  $\{A, M, P, S\}$  given by “has as its first letter.”

**Exercise A.6.** When we draw the digraph of a *function*  $f$ , we draw an arrow *from* the vertex representing  $x$  *to* the vertex representing  $f(x)$ . One of the relations you considered in Exercise A.5 is the relation of a function.

- (a) Which relation is the relation of a function?
- (b) How does the digraph help you visualize that one relation is a function and the other is not?

**Exercise A.7.** Digraphs of functions help you to visualize whether or not they are onto or one-to-one. For example, let both  $S$  and  $T$  be the set  $\{-2, -1, 0, 1, 2\}$  and let  $S'$  and  $T'$  both be the set  $\{0, 1, 2\}$ . Let  $f(x) = 2 - |x|$ .

- (a) Draw the digraph of the function  $f$ , assuming its domain is  $S$  and its range is  $T$ . Use the digraph to explain whether or not this function maps  $S$  onto  $T$ .
- (b) Use the digraph of the previous part to explain whether or not the function is one-to-one.
- (c) Draw the digraph of the function  $f$  assuming its domain is  $S$  and its range is  $T'$ . Use the digraph to explain whether or not the function is onto.
- (d) Use the digraph of the previous part to explain whether or not the function is one-to-one.
- (e) Draw the digraph of the function  $f$ , assuming its domain is  $S'$  and its range is  $T'$ . Use the digraph to explain whether the function is onto.
- (f) Use the digraph of the previous part to explain whether the function is one-to-one.
- (g) Suppose that the function  $f$  has domain  $S'$  and range  $T$ . Draw the digraph of  $f$  and use it to explain whether  $f$  is onto.
- (h) Use the digraph of the previous part to explain whether or not  $f$  is one-to-one.

A function from a set  $X$  to a set  $Y$  which is both one-to-one and onto is frequently called a **bijection**, especially in combinatorics. Your work in Exercise A.7 should show you that a digraph is the digraph of a bijection from  $X$  to  $Y$  when all four of the following properties hold:

- The vertices of the digraph represent the elements of  $X$  and  $Y$  (so that  $X$  is the possible domain and  $Y$  is the possible co-domain).
- Each vertex representing an element of  $X$  has one and only one arrow leaving it (so that  $f$  is indeed a function).
- Each vertex representing an element of  $Y$  has at least one arrow entering it (so that it is onto).
- Each vertex representing an element of  $Y$  has at most one arrow entering it (so it is one-to-one).

Of course, the last two properties can be combined to the requirement that each vertex representing an element of  $Y$  has exactly one arrow entering it.

## Appendix B

# More on the Principle of Mathematical Induction

You should have already seen the Principle of Mathematical Induction in other courses. If you've been able to work through Section 2.2 of Chapter 2 easily, there is no need to read this chapter. This section is provided in case you've found your background to be weak, and spending time outside class reviewing seems to be advisable.

- Exercise B.1.** (a) Write down a list of all subsets of  $\{1, 2\}$ . Don't forget the empty set! Group the sets containing 2 separately from the others.
- (b) Write down a list of the subsets of  $\{1, 2, 3\}$ . Group the sets containing 3 separately from the others.
- (c) Look for a natural way to match up the subsets containing 2 in part (a) with those not containing 2. Look for a way to match up the subsets in part (b) containing 3 with those not containing 3.
- (d) On the basis of the previous part, you should be able to find a bijection between the collection of subsets of  $\{1, 2, \dots, n\}$  containing  $n$  and those not containing  $n$ . (If you are having difficulty figuring out the bijection, try rethinking Parts (a) and (b), perhaps by doing a similar exercise with the set  $\{1, 2, 3, 4\}$ .) Describe the bijection and explain why it is a bijection. Explain why the number of subsets of  $\{1, 2, \dots, n\}$  containing  $n$  equals the number of subsets of  $\{1, 2, \dots, n-1\}$ .
- (e) Parts (a) and (b) suggest strongly that the number of subsets of a  $n$ -element set is  $2^n$ . In particular, the empty set has  $2^0$  subsets; a one-element set has  $2^1$  subsets: itself and the empty set; and in parts (a) and (b) we saw that two-element and three-element sets have  $2^2$  and  $2^3$  subsets, respectively. So there are certainly some values of  $n$  for which

an  $n$ -element set has  $2^n$  subsets. One way to prove that an  $n$ -element set has  $2^n$  subsets for all values of  $n$  is to argue by contradiction. For this purpose, suppose there is a nonnegative integer  $n$  such that an  $n$ -element set doesn't have exactly  $2^n$  subsets. In that case there may be more than one such  $n$ , and so choose  $k$  to be the smallest such  $n$ . (Notice that  $k - 1$  is still a positive integer, because  $k$  can't be 0, 1, 2, or 3.) Since  $k$  was the smallest value of  $n$  for which the statement "An  $n$ -element set has  $2^n$  subsets" is false, what do you know about the number of subsets of a  $(k - 1)$ -element set? What do you know about the number of subsets of the  $k$ -element set  $\{1, 2, \dots, k\}$  that don't contain  $k$ ? What do you know about the number of subsets of  $\{1, 2, \dots, k\}$  that do contain  $k$ ? What does the Sum Principle tell you about the number of subsets of  $\{1, 2, \dots, k\}$ ? Notice that this contradicts the way in which we chose  $k$ , and the only assumption that went into our choice of  $k$  was that "there is a nonnegative integer  $n$  such that an  $n$ -element set doesn't have exactly  $2^n$  subsets." Since this assumption has led us to a contradiction, it must be false. What can you now conclude about the statement "for every nonnegative integer  $n$ , an  $n$ -element set has exactly  $2^n$  subsets?"

**Exercise B.2.** Notice that the  $n$ th odd integer is  $2n - 1$ , and so the expression

$$1 + 3 + 5 + \cdots + 2n - 1 \tag{B.1}$$

is the sum of the first  $n$  odd integers. Experiment a bit with the sum for the first few positive integers and guess its value in terms of  $n$ . Now apply the technique of Exercise B.1 to prove that you are right.

In Exercises B.1 and B.2, our proofs had several distinct elements: We had a statement involving an integer  $n$ . We knew the statement was true for the first few nonnegative integers in Exercise B.1 and for the first few positive integers in Exercise B.2. We wanted to prove that the statement was true for all nonnegative integers in Exercise B.1 and for all positive integers in Exercise B.2. In both cases we used the method of proof by contradiction: for that purpose we assumed that there was a value of  $n$  for which our formula wasn't true. We then chose  $k$  to be the smallest value of  $n$  for which our formula wasn't true. This meant that when  $n$  was  $k - 1$ , our formula was true, (or else that  $k - 1$  wasn't a nonnegative integer in Exercise B.1 or that  $k - 1$  wasn't a positive integer in Exercise B.2). What we did next was the crux of the proof. We showed that the truth of our statement for  $n = k - 1$  implied the truth of our statement for  $n = k$ . This

gave us a contradiction to the assumption that there was an  $n$  that made the statement false. In fact, we will see that we can bypass entirely the use of proof by contradiction. We used it to help you discover the central ideas of the technique of proof by mathematical induction. The central core of mathematical induction is the proof that the truth of a statement about the integer  $n$  for  $n = k - 1$  implies the truth of the statement for  $n = k$ . For example, once we know that a set of size 0 has  $2^0$  subsets, if we have proved our implication, we can then conclude that a set of size 1 has  $2^1$  subsets, from which we can conclude that a set of size 2 has  $2^2$  subsets, from which we can conclude that a set of size 3 has  $2^3$  subsets, and so on up to a set of size  $n$  having  $2^n$  subsets for any nonnegative integer  $n$  we choose. In other words, although it was the idea of proof by contradiction that led us to think about such an implication, we can now do without the contradiction at all. What we need to prove a statement about  $n$  by this method is a place to start; that is, a value  $b$  for which we know the statement to be true, and then a proof that the truth of our statement for  $n = k - 1$  implies the truth of the statement for  $n = k$ , regardless of which  $k > b$  is considered.

### The Principle of Mathematical Induction

In order to prove a statement about an integer  $n$ , if you can

- Prove the statement when  $n = b$ , for some fixed integer  $b$ ;
- Show that the truth of the statement for  $n = k - 1$  implies the truth of the statement for  $n = k$  whenever  $k > b$ ;

then you can conclude the statement is true for all integers  $n \geq b$ .

As an example, let us return to Exercise B.1. The statement we wish to prove is the statement that “A set of size  $n$  has  $2^n$  subsets.”

Our statement is true when  $n = 0$ , because a set of size 0 is the empty set, for which the only subset is the empty set, giving  $1 = 2^0$  subsets. (This step of our proof is called a **base step**.)

Now suppose that  $k > 0$  and every set with  $k - 1$  elements has  $2^{k-1}$  subsets. Suppose  $S = \{a_1, a_2, \dots, a_k\}$  is a set with  $k$  elements. We partition the subsets of  $S$  into two blocks. Block  $B_1$  consists of the subsets that do not contain  $a_k$  and Block  $B_2$  consists of the subsets that do contain  $a_k$ . Each set in  $B_1$  is a subset of  $\{a_1, a_2, \dots, a_{k-1}\}$ , and each subset of  $\{a_1, a_2, \dots, a_{k-1}\}$  is in  $B_1$ . Thus  $B_1$  is the set of all subsets of  $\{a_1, a_2, \dots, a_{k-1}\}$ . Therefore by our assumption in the first sentence of this paragraph, the size

of  $B_1$  is  $2^{k-1}$ . Consider the function from  $B_2$  to  $B_1$  which takes a subset of  $S$  including  $a_k$  and removes  $a_k$  from it. The set  $B_2$  is the domain of this function, because every set in  $B_2$  contains  $a_k$ . This function is onto, because if  $T$  is a set in  $B_1$ , then  $T \cup \{a_k\}$  is a set in  $B_2$  which the function sends to  $T$ . This function is one-to-one because if  $V$  and  $W$  are two different sets in  $B_2$ , then removing  $a_k$  from both of them gives two different sets in  $B_1$ . Thus we have a bijection between  $B_1$  and  $B_2$ , so  $B_1$  and  $B_2$  have the same size. Therefore by the Sum Principle the size of  $B_1 \cup B_2$  is  $2^{k-1} + 2^{k-1} = 2^k$ . Therefore,  $S$  has  $2^k$  subsets. This shows that if a set of size  $k - 1$  has  $2^{k-1}$  subsets, then a set of size  $k$  has  $2^k$  subsets. Therefore by the principle of mathematical induction, a set of size  $n$  has  $2^n$  subsets for every nonnegative integer  $n$ .

The first sentence of the last paragraph is called the **inductive hypothesis**. In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last paragraph itself is called the **inductive step** of our proof. In an inductive step we derive the statement for  $n = k$  from the statement for  $n = k - 1$ , thus proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last sentence in the last paragraph is called the **inductive conclusion**.

All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion. There are a couple details worth noticing. First, in this exercise, our base step was the case  $n = 0$ , or in other words, we had  $b = 0$ . However, in other proofs,  $b$  could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for  $n = k - 1$  implies the truth of our statement for  $n = k$  required that  $k$  be at least 1, so that there would be an element  $a_k$  we could remove in order to describe our bijection. However, the second condition in the statement of the Principle of Mathematical Induction only requires that we be able to prove the implication for  $k > 0$ , so we were allowed to assume  $k > 0$ .

**Exercise B.3.** Use mathematical induction to prove your formula from Exercise B.2.

**Exercise B.4.** Experiment with various values of  $n$  in the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \sum_{i=1}^n \frac{1}{i \cdot (i+1)}.$$

Guess a formula for this sum and prove your guess is correct by induction.

**Exercise B.5.** For large values of  $n$ , which is larger,  $n^2$  or  $2^n$ ? A graph might be help to decide what “large value” means here. Use mathematical induction to prove that you are correct.

**Exercise B.6.** What is wrong with the following attempt at an inductive proof that all integers in any consecutive set of  $n$  integers are equal for every positive integer  $n$ ?

For an arbitrary integer  $i$ , all integers from  $i$  to  $i$  are equal, so our statement is true when  $n = 1$ . Now suppose  $k > 1$  and all integers in any consecutive set of  $k - 1$  integers are equal. Let  $S$  be a set of  $k$  consecutive integers. By the inductive hypothesis, the first  $k - 1$  elements of  $S$  are equal and the last  $k - 1$  elements of  $S$  are equal. Therefore all the elements in the set  $S$  are equal. Thus by the Principle of Mathematical Induction, for every positive  $n$ , every  $n$  consecutive integers are equal.

## Appendix C

# More on Equivalence Relations

**Exercise C.1.** Which of the reflexive, symmetric and transitive properties does the  $<$  relation on the integers have?

**Exercise C.2.** A relation  $R$  on the set of ordered pairs of positive integers that you learned about in grade school in another notation is the relation that says  $(m, n)$  is related to  $(h, k)$  if  $mk = hn$ . Show that this relation is an equivalence relation. In what context did you learn about this relation in grade school?

**Exercise C.3.** Another relation that you may have learned about in school, perhaps in the guise of “clock arithmetic,” is the relation of equivalence modulo  $n$ . For integers (positive, negative, or zero)  $a$  and  $b$ , we write

$$a \equiv b \pmod{n}$$

to mean that  $a - b$  is an integer multiple of  $n$ , and in this case, we say that  $a$  is **congruent to  $b$  modulo  $n$** . Show that the relation of congruence modulo  $n$  is an equivalence relation.

**Exercise C.4.** Define a relation on the set of all lists of  $n$  distinct integers chosen from  $\{1, 2, \dots, n\}$ , by saying two lists are related if they have the same elements (though perhaps in a different order) in the first  $k$  places, and the same elements (though perhaps in a different order) in the last  $n - k$  places. Show this relation is an equivalence relation.