Lecture 10: Nonhomogeneous Linear ODE's And The Annihilator Method

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Here we consider certain constant coefficient, nonhomogeneous linear ODE's of order l:

$$L[y] = f(x), \qquad \text{i.e.,} \tag{1}$$

$$L[y] = \left(a_l D^l + a_{l-1} D^{l-1} + \cdots + a_1 D + a_0\right) y.$$

In this lecture we will solve (1) where we know how to annihilate $f(x)$ with similar operators. The trick is to apply the minimal nontrivial annihilator, say $M = (b_m D^m + \cdots + b_1 D + b_0)$ to both sides of (1) to get a homogeneous equation $M[L[y]] = 0$, or just

$$ML[y] = 0, \tag{2}$$

whose solution will not be quite the same as that of (1), but will contain the solution of (1). Fortunately we can extract the solution of (1) from that of (2). That process is where most of the work is for the method.[1]

In the sections below, we will first look at some very general linear-algebraic theory of nonhomogeneous equations, move on to a quick revisiting of annihilators with constant coefficients, and then develop the method for solving a large class of nonhomogeneous equations (1) with several examples.

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[1]Any time an operator which includes a derivative acts upon a function, it loses some information about that function. Consider the example below, where $D = d/dx$:

$$f(x) = x^2 + 25x + 9 \implies Df(x) = 2x + 25 \iff f(x) = x^2 + 25x + C, \quad \text{some } C \in \mathbb{R}.$$

By taking a derivative we lose some information about $f(x)$, namely the exact identity of the additive constant. Differential operators are not strictly invertible, i.e., one-to-one (unless appropriate initial data is specified), so we do lose information in applying them. Simlarly

$$f(x) = 2\sin 3x \implies (D^2 + 9)f(x) = 0 \iff f(x) = A\sin 3x + B\cos 3x, \qquad \text{some } A, B \in \mathbb{R}.$$

For our method here, we have

$$L[y] = f(x) \implies ML[y] = 0 \iff$$
$$y = C_1 y_1 + C_2 y_2 + \cdots + C_{m+l} y_{m+l}, \qquad \text{some } C_1, \cdots, C_{m+l} \in \mathbb{R},$$

where L is of order l and M is of order m, and $y_1, \cdots, y_{m+l}$ are linearly independent solutions to the $(m+l)$-order LHODE $ML[y] = 0$. So the solution to the original nonhomogeneous equation $L[y] = f(x)$ is contained in the new homogeneous equation $ML[y] = 0$, but some information was lost in applying M to the original.

An algebraic setting where we lose some information is in squaring both sides of an equation:

$$x = -5 \implies x^2 = 25 \iff x = \pm 5.$$

The solution is not entirely lost, but we have to test both $x = \pm 5$ in the original, since we can not follow the implication arrows both directions from first to last statements. Similarly

$$\sqrt{2x+3} = x \implies 2x + 3 = x^2 \iff 0 = x^2 - 2x - 3 \iff 0 = (x-3)(x+1) \iff x = 3, -1.$$

Notice that $x = 3$ is a solution, where $x = -1$ is not. Squaring both sides lost that information, but it was a useful thing to do. We just have to reconcile the solution with the original problem.

1 Some Linear Theory

In this section we will look at some very general linear algebraic facts concerning solutions to nonhomogeneous linear equations, be they linear differential equations or any other type of linear equation of the form $L[y] = f$.

Recall that by definition, L is a linear operator if and only if

1. $L[y_1 + y_2] = L[y_1] + L[y_2]$ for all y_1, y_2 in the domain of L;
2. $L[\alpha y] = \alpha L[y]$, for any y in the domain of L and any scalar α (so for real-variable ODE purposes, $\alpha \in \mathbb{R}$).

Recall also that these are summarized by $L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2]$. In particular, if we let $\alpha = 1$ and $\beta = -1$, then we have $L[y_1 - y_2] = L[y_1] - L[y_2]$.

Now suppose we would like to solve any nonhomogeneous linear equation

$$L[y] = f. \tag{3}$$

(You can think instead $L[y] = f(x)$, with $y = y(x)$, for our ODE setting.)

On one hand, suppose we have found one solution y_p to $L[y] = f$, i.e., suppose we found y_p such that $L[y_p] = f$. If y_h solves the homogeneous equation, i.e., $L[y_h] = 0$, then

$$L[y_p + y_h] = L[y_p] + L[y_h] = f + 0 = f, \tag{4}$$

so $y_p + y_h$ is also a solution to $L[y] = f$.

On the other hand, suppose y_{p_1} and y_{p_2} both solve the nonhomogeneous equation, i.e., $L\left[y_{p_1}\right] = f$, $L\left[y_{p_2}\right] = f$. Then

$$L\left[y_{p_2} - y_{p_1}\right] = L\left[y_{p_2}\right] - L\left[y_{p_1}\right] = f - f = 0. \tag{5}$$

In other words, the difference of any two solutions to $L[y] = f$ is a solution to the homogeneous equation $L[y] = 0$.

Putting this together, we see that the general solution to (3) can be written

$$y = y_p + y_h, \tag{6}$$

where y_p is a *particular* solution to (3) and y_h is the *homogeneous part*, i.e., the general solution to $L[y] = 0$.

Indeed, if we are dealing with a nonhomogeneous linear ODE with constant coefficients, and the order is l, then (as before) the solution will still be an l-parameter family of curves, with all the parameters contained in the expression for y_h. (y_p will contain no parameters in the final solution; y_p truly is a *particular* solution.)

2 Annihilation: A Review

Here we summarize some of the rules for finding annihilators.

1. To annihlate 1, use the operator D.

   To annihilate $x^n, x^{n-1}, \cdots, 1$, use D^{n+1}.

2. To annihilate e^{kx} use the operator $(D - k)$.

   To annihilate $x^n e^{kx}, x^{n-1} e^{kx}, \cdots, e^{kx}$, use the operator $(D - k)^{n+1}$.

3. To annihilate $\sin kx$ or $\cos kx$ (or both at once), use the operator $(D^2 + k^2)$.

   To annihlate $x^n \sin kx, x^n \cos kx, x^{n-1} \sin kx, x^{n-1} \cos kx, \cdots, \sin kx, \cos kx$ use the operator $(D^2 + k^2)^{n+1}$.

4. To annihilate $e^{kx}\sin lx$ or $e^{kx}\cos lx$ (or both at once), we need the operator whose characteristic polynomial is zero at $k \pm li$, i.e., $(m-k-li)(m-k+li) = m^2 - 2km + (k^2+l^2)$, i.e., the operator $(D^2 - 2kD + k^2 + l^2)$.

   To annihilate $x^n e^{kx}\sin lx, x^n e^{kx}\cos lx, x^{n-1}e^{kx}\sin lx, x^{n-1}e^{kx}\cos lx, \cdots, e^{kx}\sin lx$ or $e^{kx}\cos lx$, use $(D^2 - 2kD + k^2 + l^2)^{n+1}$.

Fortunately, in the last case it is enough to know we want a power of the annihilator of $Ee^{kx}\sin lx$ and $Fe^{kx}\cos lx$. Actually this is true for all four cases, but the first three are easy enough to implement in longhand, which helps avoid mistakes, so we do not take this shortcut for those cases.

Example 1 *Find a linear differential operator with constant coefficients, with minimal positive degree, which will annihilate the given function.*

- $e^{3x} - 64e^{9x}$: $\quad (D-3)(D-9)$.
- $5xe^{-4x} + 6x^4$: $\quad (D+4)^2 D^5$.
- $6\sin 2x - 5\cos 7x$: $\quad (D^2+4)(D^2+49)$.
- $22 + 2x - 6x^4 + 9x^5$: $\quad D^6$. *(Note how the annihilator requires a sixth power for the highest-order term. This operator is sufficient, since it will operate on each term individually and is more than enough to annihilate the lower-order terms.[2])*
- $1 + x\sin 4x + \cos 4x$: $\quad D(D^2+16)^2$.
- $x + x^2\cos 3x + x\sin 2x + \cos 5x - \sin 5x$: $\quad D^2(D^2+9)^3(D^2+4)^2(D^2+25)$.
- $x^3 + e^{-x} + e^x \sin 2x$: *The only difficult function is $e^x\sin 2x$, which comes from $m = 1 \pm 2i$, i.e., the factor $(m-1-2i)(m-1+2i) = m^2 - 2m + 1 + 4 = m^2 - 2m + 5$. We need $D^4(D+1)(D^2-2D+5)$.*
- $x^3 e^x \sin 2x + 3xe^x\cos 2x$: $\quad (D^2 - 2D + 5)^4$.

For the cases with complex roots of the characteristic polynomial, it can help to note a general fact. Note that if $z = a + bi$ (where $a, b \in \mathbb{R}$), and its complex conjugate is $\bar{z} = a - bi$, then $z\bar{z} = (a+bi)(a-bi) = a^2 + b^2$, so that $z\bar{z} = |z|^2$ (where $|z|$ is the length, magnitude, modulus, etc., of z). This helps to speed up the computations such as

$$(m - \underbrace{(1+2i)}_{z})(m - \underbrace{(1-2i)}_{\bar{z}}) = m^2 - (1 + 2i + 1 - 2i)m + (1^2 + 2^2) = m^2 - 2m + 5.$$

With practice, the middle term (where the complex parts cancel) can be computed quickly as well.

3 The Annihilator Method

The basic method outlined below can be varied somewhat, but each step will need to be carried out eventually. We will again suppose that we have an lth order equation of the form

$$L[y] = f(x).$$

The outline we will follow is basically as follows:[3]

[2] A common mistake is to use a different annihilator factor for each function, making for a higher-degree annihilator than necessary. It is true that one annihilator for $22 + 2x - 6x^4 + 9x^5$ is—using a different factor for each term—the product $(D)(D^2)(D^5)(D^6) = D^{14}$, but it is enough to use D^6, which will annihilate all polynomial terms of order ≤ 5, and will avoid complications from dealing with operators and equations of higher order than necessary in our methods introduced later in this lecture.

[3] Zill, and most other textbooks, use y_c instead of y_h. The "c" in y_c signifies that that part of the solution comes from the characteristic equation for the homogeneous part of the original ODE, i.e., for the related LHODE $L[y] = 0$. Zill calls y_c the "complementary function," presumably because it is what is missing if we only report y_p as the solution.

1. Identify l linearly independent functions which span y_h, which is the l-parameter solution to the related homogeneous equation $L[y_h] = 0$.

2. Derive a new, homogeneous ODE (LHODE) by applying the minimal annihilator M of the RHS $f(x)$ to both sides of the original ODE to get $ML[y] = 0$.

   (a) Solve the new LHODE, which will have an $(l + m)$-parameter family as its solution.

   (b) Identify those functions in the solution which were already contained in y_h. (They were already annihilated by L.)

   (c) The remaining functions form y_p for some choices of their parameters. Those particular choices are found in the next steps.

3. Plug y_p with its (still undetermined) coefficients into the *original* ODE, i.e.,

   (a) Set $L[y_p] = f(x)$, and compute and expand $L[y_p]$ on the LHS.

   (b) Compare coefficients of the various functions on the LHS with those on the RHS to determine the values of those coefficients, to find the exact form of y_p.

4. The solution to the original ODE, $L[y] = f(x)$, will then be

$$y = y_p + y_h.$$

   Note that this will be an l-parameter solution (from the order of the original ODE), with y_h containing all of the parameters. (The other parameters from $ML[y] = 0$, i.e., those from y_p, are no longer "parameters" but fixed constants determined in the previous step.)

Example 2 *Consider $y'' - 3y' - 40y = 6e^{2x}$.*
<u>*Solution*</u>*: This is of the form*

$$\begin{aligned} L[y] \equiv& (D^2 - 3D - 40)y = 6e^{2x}, \qquad \textit{or} \\ L[y] \equiv& \ (D-8)(D+5)y = 6e^{2x}. \end{aligned} \tag{7}$$

Finding the functions whose span is the solution to $(D-8)(D+5)y_h = 0$ is fairly easy for this example:

$$e^{8x},\ e^{-5x}.$$

Next we annihilate the RHS of (7) using $(D-2)$:

$$(D-2)[(D-8)(D+5)y] = (D-2)\left[6e^{2x}\right],$$

i.e.,

$$(D-2)(D-8)(D+5)y = 0. \tag{8}$$

The solution to this new LHODE (8) is then

$$y = \underbrace{Ae^{2x}}_{y_p} + \underbrace{Be^{8x} + Ce^{-5x}}_{y_h}.$$

The y_h part will keep its parameters, but we will have to find the particular coefficient A for y_p.[4]

$$\left.\begin{aligned} y_p &= Ae^{2x} \\ y_p' &= 2Ae^{2x} \\ y_p'' &= 4Ae^{2x} \end{aligned}\right| \qquad \begin{aligned} & y_p'' - 3y_p' - 40y_p = 6e^{2x} \\ \iff & \left(4Ae^{2x}\right) - 3\left(2Ae^{2x}\right) - 40\left(Ae^{2x}\right) = 6e^{2x}. \end{aligned}$$

[4] One nice thing about the method of annihilators is that we could put the whole form $y = Ae^{2x} + Be^{8x} + Ce^{-5x}$ into the original equation $y'' - 3y' - 40y = 6e^{2x}$, and if we are good with our linear algebra, find out that $A = -1/7$ is determined, and B and C are not and therefore can be any constants (since Be^{8x} and Ce^{-5x} disappear, i.e., are annihilated by the operator on the LHS of the original ODE), again giving us the 2-parameter form of the solution. Of course it saves some work to instead identify those terms in y_h, and then to focus on just those left-over terms which then comprise y_p to plug back into the original ODE.

From the latter equation on the right we can then write

$$\begin{aligned} && e^{2x}A(4-6-40) &= 6e^{2x} \\ \iff && -42A &= 6 \\ \iff && A &= -\frac{1}{7}. \end{aligned}$$

Putting all this together gives us, as always, $y = y_p + y_h$*, i.e.,*

$$y = -\frac{1}{7}e^{2x} + Be^{8x} + Ce^{-5x}, \qquad B, C \in \mathbb{R}. \tag{9}$$

Note that this is a two-parameter solution to a second-order ODE, which is to be expected.

Looking at the previous example, it is tempting to believe we could have *guessed* that the solution to the particular part should be of some form $y_p = Ae^{2x}$, since such a function will keep repeating itself (except for multiplicative constants) as we take derivatives in the LHS of the original equation $y'' - 3y' - 40y = 6e^{2x}$. Still, what if the RHS is something which would be annihilated by the operator on the left? It seems our "guess" would disappear and be useless. If we do not short-circuit the method—by replacing it with guesses—we still get the solution.

Example 3 *Solve the nonhomogeneous linear ODE* $y'' - 3y' - 40y = 2e^{-5x}$*.*

Solution: *We proceed as before. Since the equation can be written in the form* $(D-8)(D+5)y = 2e^{-5x}$*, we know that* y_h *will contain the two functions* e^{8x} *and* e^{-5x}*. Now we apply* $(D+5)$ *to annihilate the RHS to get*

$$\begin{aligned} (D+5)[(D-8)(D+5)y] &= (D+5)\left[2e^{-5x}\right], \quad \textit{or,} \\ (D+5)^2(D-8)y &= 0. \end{aligned} \tag{10}$$

(It is of utmost importance we group like factors.) The solution to (10) *is then*

$$y = \underbrace{Ae^{-5x}}_{\textit{part of } y_h} + \underbrace{Bxe^{-5x}}_{y_p} + \underbrace{Ce^{8x}}_{\textit{part of } y_h}.$$

We see that the only term which was not annihilated by the original operator is the term we will call $y_p = Bxe^{-5x}$*. We then plug this term into the original equation, by first calculating its first two derivatives and then substituting them in the appropriate places in the original ODE. Here we will need the product rule in a couple of places.*

$$\begin{aligned} y_p &= Bxe^{-5x} \\ y_p' &= Bx(-5e^{-5x}) + Be^{-5x} = e^{-5x}(B - 5Bx) \\ y_p'' &= e^{-5x}(-5B) - 5e^{-5x}(B - 5Bx) = e^{-5x}(-10B + 25Bx) \end{aligned}$$

Putting this into the original ODE gives us

$$\begin{aligned} && \overbrace{e^{-5x}(-10B+25Bx)}^{y''}\overbrace{-3e^{-5x}(B-5Bx)}^{-3y'}\overbrace{-40Bxe^{-5x}}^{-40y} &= 2e^{-5x} \\ \iff && e^{-5x}(-10B + 25Bx - 3B + 15Bx - 40Bx) &= 2e^{-5x} \\ \iff && e^{-5x}(-13B) &= 2e^{-5x} \\ \iff && -13B &= 2 \\ \iff && B &= -2/13. \end{aligned}$$

Thus $y_p = -\frac{2}{13}xe^{-5x}$*, and so* $y = y_p + y_h$ *becomes*

$$y = -\frac{2}{13}xe^{-5x} + Ae^{-5x} + Ce^{8x}, \qquad A, C \in \mathbb{R}.$$

Here we saw that we could not just *guess* that y_p would be of the form Ae^{-5x}, because that would be annihilated by the original operator, and have no hope to ever return the RHS, $2e^{-5x}$. We had to have a function one step more complicated: annihilated by the original operator *times* $(D+5)$, but not the original operator.

It was also important that a *constant* B gave us the correct RHS. If that had been impossible (for instance, if we had $Bx = -2/13$ instead of $B = -2/13$), then the method would have been incorrectly employed. The linear independence of the functions involved plays a role. Consider the following example, for instance.

Example 4 *Solve $y'' - 3y' - 40y = \sin 2x$.*

Solution: Again we notice that the given ODE can be written

$$(D-8)(D+5)y = \sin 2x,$$

and so the independent functions spanning y_h are e^{8x}, e^{-5x} as before.

Next we annihilate the RHS by applying (D^2+4) to both sides, giving us $(D^2+4)(D-8)(D+5)y = (D^2+4)\sin 2x$, i.e.,

$$(D^2+4)(D-8)(D+5)y = 0.$$

The solution to this new LHODE is then

$$y = \underbrace{A\sin 2x + B\cos 2x}_{y_p} + \underbrace{Ee^{8x} + Fe^{-5x}}_{y_h}.$$

We see that the latter two functions are contained in y_h, regardless of $E, F \in \mathbb{R}$, since they are annihilated by the original operator, namely $(D-8)(D+5)$, and so will only contribute zero on the RHS of that equation. The first two functions will not be annihilated by the original operator, but instead contain the y_p for some specific $A, B \in \mathbb{R}$. Our task is then to find A and B:

$$\begin{aligned} y_p &= A\sin 2x + B\cos 2x \\ y_p' &= 2A\cos 2x - 2B\sin 2x \\ y_p'' &= -4A\sin 2x - 4B\cos 2x. \end{aligned}$$

Using $y_p'' - 3y_p' - 40y_p = \sin 2x$ gives us

$$\begin{aligned} &(-4A\sin 2x - 4B\cos 2x) - 3(2A\cos 2x - 2B\sin 2x) \\ &\qquad\qquad - 40(A\sin 2x + B\cos 2x) = \sin 2x, \end{aligned}$$

which, after expanding and then combining like terms on the left, gives us

$$(-4A + 6B - 40A)\sin 2x + (-4B - 6A - 40B)\cos 2x = \sin 2x,$$

i.e.,

$$(-44A + 6B)\sin 2x + (-6A - 44B)\cos 2x = \sin 2x,$$

Because of the linear independence of $\sin 2x$ and $\cos 2x$, the only way for this last equation to hold (as an identity*, i.e., an equality of functions) is for the coefficients of $\sin 2x$ and $\cos 2x$ on the left to be the same as on the right. In other words, the following two equations must hold:*[5]

$$\begin{array}{rcrcl} -44A & + & 6B & = & 1 \\ -6A & - & 44B & = & 0. \end{array} \tag{11}$$

[5] This is the same idea we use when looking at vectors in $\mathbb{R}^3$. If, for instance, $a\vec{i} + b\vec{j} + c\vec{k} = 2\vec{i} + 3\vec{j}$, then we know for certain that $a = 2$, $b = 3$ and $c = 0$.

Now we have to solve the above system of two linear equations with two unknowns, A and B. There are many methods: elimination, substitution, Cramer's Rule come to mind immediately. Because of the second equation, we will use substitution. The second equation gives us $-6A = 44B$, and so $A = -\frac{44}{6}B = -\frac{22}{3}B$. Substituting this for A into the first equation gives

$$-44\left(-\frac{22}{3}B\right) + 6B = 1,$$

which, after multiplying by 3, then gives $968B + 18B = 3$, or $986B = 3$, or $B = 3/986$.

With that, we go back to our substitution to get

$$A = -\frac{22}{3}B = -\frac{22}{3}\cdot\frac{3}{986} = -\frac{22}{986} = -\frac{11}{493}.$$

Thus $y_p = A\sin 2x + B\cos 2x = -\frac{11}{493}\sin 2x + \frac{3}{986}\cos 2x$. Setting $y = y_p + y_h$ finally gives

$$y = -\frac{11}{493}\sin 2x + \frac{3}{986}\cos 2x + Ee^{8x} + Fe^{-5x}.$$

Actually, as is often the case of two equations and two unknowns, here Cramer's Rule makes for simpler calculations.[6]

Example 5 *Solve $y''' + 9y' = 4x^2 + \cos x + e^{2x}$.*

Solution: First we rewrite the equation into the form $(D^3 + 9D)y = 4x^2 + \cos x + e^{2x}$, which can then be written

$$D(D^2 + 9)y = 4x^2 + \cos x + e^{2x}. \tag{14}$$

Now we apply the annihilator of the RHS to the equation to give the new LHODE $D^3(D^2+1)(D-2)[D(D^2+9)]y = D^3(D^2+1)(D-2)[4x^2 + \cos x + e^{2x}]$, or

$$D^4(D^2+1)(D-2)(D^2+9)y = 0.$$

This (or the related characteristic equation in m) gives us

$$y = \underbrace{A}_{\text{in } y_h} + \underbrace{Bx + Cx^2 + Ex^3 + F\sin x + G\cos x + He^{2x}}_{y_p} + \underbrace{I\sin 3x + J\cos 3x}_{\text{in } y_h}.$$

[6] Recall Cramer's Rule for solving such equations as in Example 4 says that

$$A = \frac{\Delta_A}{\Delta}, \qquad B = \frac{\Delta_B}{\Delta}, \tag{12}$$

where Δ_A, Δ_B and Δ are sometimes called *the determinants of A, B, and the general determinant*. For our case, the system (11) gives us

$$\Delta = \begin{vmatrix} -44 & 6 \\ -6 & -44 \end{vmatrix}, \qquad \Delta_A = \begin{vmatrix} 1 & 6 \\ 0 & -44 \end{vmatrix}, \qquad \Delta_B = \begin{vmatrix} -44 & 1 \\ -6 & 0 \end{vmatrix}. \tag{13}$$

Notice how Δ derives from the coefficients of A and B in the two equations (11), Δ_A comes from replacing the first (or "A") column with the numbers on the right of the equations, and Δ_B comes similarly from using the RHS's entries to replace the second (or "B") column. For this particular example, three quick computations give $\Delta = 44^2 + 6^2 = 1972$, $\Delta_A = -44$, and $\Delta_B = 6$, and so (12) gives us $A = -44/1972 = -11/493$, $B = 6/1972 = 3/986$, as before.

Cramer's Rule, with the same general pattern, also works for higher dimensional square systems (n linear equations, n unknowns) but is often unwieldy once one gets beyond a system of three equations, since the determinant of an $n \times n$ matrix has $n!$ different products (of n terms each!) to add. Even with three equations, elimination (a.k.a. row reduction in linear algebra) is computationally competitive and quite often more efficient, particularly for hand calculations. For $n > 3$ elimination is almost always best.

Example 4 is one example where Cramer's Rule can save us from some of the messier calculations.

From the original ODE, we see that y_h *contains the linearly independent functions* $1, \sin 3x, \cos 3x$, *so what is left for* y_p *are* $y_p = Bx + Cx^2 + Ex^3 + F\sin x + G\cos x + He^{2x}$. *Now*

$$\begin{aligned}
y_p &= Bx + Cx^2 + Ex^3 + F\sin x + G\cos x + He^{2x}\\
y_p' &= B + 2Cx + 3Ex^2 + F\cos x - G\sin x + 2He^{2x}\\
y_p'' &= 2C + 6Ex - F\sin x - G\cos x + 4He^{2x}\\
y_p''' &= 6E - F\cos x + G\sin x + 8He^{2x}.
\end{aligned}$$

Plugging this into the original ODE, i.e., $y_p''' + 9y_p' = 4x^2 + \cos x + e^{2x}$ *gives*

$$\begin{aligned}
&\left(6E - F\cos x + G\sin x + 8He^{2x}\right)\\
&\qquad + 9\left(B + 2Cx + 3Ex^2 + F\cos x - G\sin x + 2He^{2x}\right) = 4x^2 + \cos x + e^{2x}.
\end{aligned}$$

Now we compare coefficients of the functions involved which appear in the above equation.

$$\begin{aligned}
x^2 :& \qquad 27E = 4\\
x :& \qquad 18C = 0\\
1 :& \qquad 6E + 9B = 0\\
\sin x :& \qquad G - 9G = 0\\
\cos x :& \qquad -F + 9F = 1\\
e^{2x} :& \qquad 8H + 18H = 1.
\end{aligned}$$

Taking these in turn, we see that $E = 4/27$*;* $C = 0$*;* $9B = -6E = -6(4/27) = -8/9 \iff B = -8/81$*;* $-8G = 0 \iff G = 0$*;* $8F = 1 \iff F = 1/8$*;* $26H = 1 \iff H = 1/26$. *This then gives*

$$y_p = -\frac{8}{81}x + \frac{4}{27}x^3 + \frac{1}{8}\sin x + \frac{1}{26}e^{2x},$$

which finally gives us $y = y_p + y_h$, *or*

$$y = -\frac{8}{81}x + \frac{4}{27}x^3 + \frac{1}{8}\sin x + \frac{1}{26}e^{2x} + A + I\sin 3x + J\cos 3x, \qquad A, I, J \in \mathbb{R}.$$

Homework 10-A

1. Find a minimal annihilator for each of the following functions.

   (a) $1 + 2x + x^3 - 10x^5$.

   (b) $e^{2x} - 5e^{6x}$.

   (c) $\sin 2x - 9\cos 5x$.

   (d) x^3e^{10x}.

   (e) $x^4 \sin 9x$.

   (f) $xe^{2x} - 2x^2e^{2x} - 9x^2 + 5x^2 \sin 2x + 4x^2 \cos 2x$.

   (g) $e^{3x} \cos 2x$.

   (h) $x^5e^{3x} \cos 2x$.

2. Without finding the coefficients of y_p, use annihilators to find the form of y_p and y_h.

   For example, for $y'' + y = xe^x$, we have $(D^2 + 1)y = xe^x$, so we annihilate RHS using $(D - 1)^2$ to get $(D - 1)^2(D^2 + 1)y = 0$, and so

   $$y = \underbrace{Ae^x + Bxe^x}_{y_p} + \underbrace{E\sin x + F\cos x}_{y_h}. \qquad \text{Done!}$$

   (a) $(D + 1)(D - 4)(D^2 + 9)y = 3\sin 2x + 5\cos 3x + e^{2x} - 7e^{9x}$.

   (b) $(D + 1)(D - 4)(D^2 + 9)y = x\cos 3x + e^{-x} + x^4$.

   (c) $(D^2 + 1)y = x^2e^x \sin 2x$. (You do not really need the exact form of the annihilator on the RHS. It is enough to know how many factors of the annihilator of $e^x \sin 2x$ you need, and what else would be annihilated in the process.)

3. Solve the given differential equation using the method of annihilators.

   (a) (From Zill) $y'' + y' = 3$.

   (b) $y'' - 2y' - 3y = 4e^{-x} - 9$.

   (c) $y'' + 4y' = \sin 3x + x^2 + 9e^{-4x}$.