LECTURE NOTES ON STOCHASTIC CALCULUS FOR FINANCE FALL 2021

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Contents

Note: The page numbers and links will not be correct in the annotated version.

1. Preface.

These are the slides I used while teaching this course in Fall 2021. I projected them (spaced out) in class, and filled in the proofs by writing over them. The annotated version of these slides with handwritten proofs, blank slides (so you take notes), and the compactified un-annotated version for quick review can be found on the class website. The LaTeXsource of these slides is also available on git.

2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2021-22/944-scalc-finance1
- TA's: Shukun Long <shukunl@andrew.cmu.edu>.
- Homework Due: 10:10AM Oct 28, Nov 4, 11, 23, 30, Dec 7
- Midterm: Tue, Nov 16, in class (May be delayed to Nov 18 if we have not covered Itô's formula in time.)
- **Homework:**
  - Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - One homework assignments can be turned in 24h late without penalty.
  - Bottom homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).

  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 10% homework, 30% midterm, 60% final.

Course Outline.

- Review of Fundamentals: Replication, arbitrage free pricing.

- Quick study of the multi-period binomial model.
  - ▷ Simple example of replication / arbitrage free pricing.
  - ▷ Understand conditional expectations. (Have an explicit formula.)
  - ▷ Understand measurablity / adaptedness. (Can be stated easily in terms of coin tosses that have / have not occurred.)
  - ▷ Understand risk neutral measures. Explicit formula!
- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.

3. Replication and Arbitrage

3.1. Replication and arbitrage free pricing.

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ Mathematically: For any trading strategy such that $X_0 = 0$, and $X_n \geqslant 0$, you must also have $X_n = 0$ almost surely.
  - ▷ Equivalently: There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* If given the opportunity to trade Y at price V_0, the market remains arbitrage free, then we say V_0 is the arbitrage free price of Y.

- *We will almost always find the arbitrage free price by **replication**.*
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Try and *replicate the payoff*:
    - – Start with X_0 dollars.
    - – Use only traded assets and ensure that at maturity $X_N = V_N$.
  - ▷ Then the arbitrage free price is uniquely determined, and must be X_0.

Remark 3.1. The arbitrage free price is *unique* if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.

3.2. **Example: One period Binomial model.**

- Consider a market with a stock, and money market account.
- Interest rate for borrowing and lending is r. No transaction costs. Can buy and sell fractional quantities of the stock.
- *Model assumption:* Flip a coin that lands heads with probability $p_1 \in (0, 1)$ and tails with probability $q_1 = 1 - p_1$. Model $S_1 = uS_0$ if heads, and $S_1 = dS_0$ if tails.
  - ▷ S_0 is stock price at time 0 (known).
  - ▷ S_1 is stock price after one time period (random).
  - ▷ u, d are model parameters (pre-supposed). Called the up and down factors. (Will always assume $0 < d < u$.)

Proposition 3.2. *There's no arbitrage in this model if and only if $d < 1 + r < u$.*

Proof.

Proposition 3.3. *Say a security pays V_1 at time 1 (V_1 can depend on whether the coin flip is heads or tails). The arbitrage free price at time 0 is given by*

$$V_0 = \frac{1}{1+r}\big(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)\big) = \frac{1}{1+r}\tilde{\boldsymbol{E}} V_1\,, \quad \text{where } \tilde{p}_1 = \frac{1+r-d}{u-d}\,, \quad \tilde{q} = \frac{u-(1+r)}{u-d}\,.$$

The replicating strategy holds $\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)S_0}$ *shares of stock at time* 0.

Proof.

4. **Multi-Period Binomial Model.**

- Same setup as the one period case $0 < d < 1 + r < u$, and toss coins that land heads with probability p_1 and tails with probability q_1.
- Except now the security matures at time $N > 1$.
- Stock price: $S_{n+1} = uS_n$ if $n + 1$-th coin toss is heads, and $S_{n+1} = dS_n$ otherwise.
- To replicate it a security, we start with capital X_0.
- Buy Δ_0 shares of stock, and put the rest in cash.
- Get $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$.
- Repeat. *Self Financing Condition:* $X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$.
- *Adaptedness:* Δ_n can only depend on outcomes of coin tosses before n!

Proposition 4.1. *Consider a security that pays V_N at time N. Then for any $n \leqslant N$:*

$$V_n = \frac{1}{(1+r)^{N-n}} \tilde{\boldsymbol{E}}_n V_N \,, \qquad \Delta_n = \frac{V_{n+1}(\omega_{n+1} = H) - V_{n+1}(\omega_{n+1} = T)}{(u-d)S_n} \,.$$

- *V_n is the arbitrage free price at time $n \leqslant N$.*
- *Δ_n is the number of shares held in the replicating portfolio at time n (trading strategy).*

Question 4.2. *Why does this work?*

Question 4.3. *What is $\tilde{\boldsymbol{E}}_n$? (It's different from $\boldsymbol{E}$, and different from $\boldsymbol{E}_n$).*

4.1. **Quick review probability (finite Sample spaces).** This is just a quick reminder, to fix notation. Read one of the references, or look over the prep material / videos for a more through treatment. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 4.4. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss}\}$.

▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$. (Each ω_i could also represent the outcome of the roll of a M sided die.)

Definition 4.5. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

▷ Each sample point represents the outcome of a sequence of *all* coin tosses from 1 to N.

Definition 4.6. A *probability mass function* (PMF for short) is a function $p\colon \Omega \to [0,1]$ such that $\sum_{\omega\in\Omega} p(\omega) = 1$.

Example 4.7. Typical example: Fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$ and set $p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$. Here $H(\omega)$ is the number of heads in the sequence $\omega = (\omega_1, \dots, \omega_N)$, and $T(\omega)$ is the number of tails.

Definition 4.8. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega\in A} p(\omega)$.

▷ $\boldsymbol{P}$ is called the probability measure associated with the PMF p.

Example 4.9. $A\{\omega \in \Omega \mid \omega_1 = +1\}$. Check $\boldsymbol{P}(A) = p_1$.

4.2. **Random Variables and Independence.**

Definition 4.10. A *random variable* is a function $X : \Omega \to \mathbb{R}$.

Example 4.11. $X(\omega) = \begin{cases} 1 & \omega_2 = +1\,, \\ -1 & \omega_2 = -1\,, \end{cases}$ is a random variable corresponding to the outcome of the second coin toss.

Definition 4.12. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 4.13. Note if $\operatorname{Range}(X) = \{x_1, \ldots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 4.14. The *variance* of a random variable is $\operatorname{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 4.15. Note $\operatorname{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

Definition 4.16. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 4.17. The events $A_1, \ldots, A_n$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k})\,.$$

Remark 4.18. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Definition 4.19. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 4.20. The random variables $X_1, \ldots, X_n$ are independent if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 4.21. Independent random variables are uncorrelated, but not vice versa.

4.3. **Filtrations.**

Definition 4.22. We define a *filtration* on Ω as follows:

- ▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- ▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss. E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_1$.
- ▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses. E.g. $A = \{\omega \mid \omega_1 = 1, \omega_3 = -1, \omega_n = 1\} \in \mathcal{F}_n$.

Remark 4.23. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Remark 4.24. If $A, B \in \mathcal{F}_n$, then so do A^c, B^c, $A \cap B$, $A \cup B$, $A - B$, $B - A$.

Definition 4.25. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \dots, \omega_n$.

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Remark 4.26 (Use in Finance). For every n, the trading strategy at time n (denoted by Δ_n) must be $\mathcal{F}_n$ measurable. We can not trade today based on tomorrows price.

Example 4.27. If we represent Ω as a tree, $\mathcal{F}_n$ measurablity can be visualized by checking constancy on leaves.

4.4. **Conditional expectation.**

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \operatorname{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega))$$

$$\text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\}$$

Remark 4.29. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Proposition 4.32. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

1. *$\boldsymbol{E}_n X$ is an $\mathcal{F}_n$-measurable* random variable.
2. *For every $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.*

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X$.

Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.*

(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*

(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boldsymbol{E}_n X = X$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $A \in \mathcal{F}_n$ and $B \subseteq \mathbb{R}$, the events A and $\{X \in B\}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, ..., n^{th} coin tosses and *not* the 1^{st}, 2^{nd}, ..., n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

Proposition 4.39 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

4.5. **Martingales.**

Definition 4.40. A *stochastic process* is a collection of random variables $X_0, X_1, \dots, X_N$.

Example 4.41. Typically X_n is the wealth of an investor at time n, or S_n is the price of a stock at time n.

Definition 4.42. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Remark 4.43. Requiring processes to be adapted is fundamental to Finance. Intuitively, being adapted forbids you from trading today based on tomorrows stock price. All processes we consider (prices, wealth, trading strategies) will be adapted.

Example 4.44 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 4.45 (Stock price). Let $S_0 \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

Definition 4.46. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 4.47. Intuition: A martingale is a "fair game".

Example 4.48 (Unbiased random walk). If $\xi_1, \dots, \xi_N$ are i.i.d. and mean zero, then $X_n = \sum_{k=1}^n \xi_k$ is a martingale.

Remark 4.49. If M is a martingale, then for every $m \leqslant n$, we must have $\boldsymbol{E}_m M_n = M_m$.

Remark 4.50. If M is a martingale then $\boldsymbol{E} M_n = \boldsymbol{E} M_0 = M_0$.

4.6. **Change of measure.**

- Gambling in a Casino: If it's a martingale, then on average you won't make or lose money.
- Stock market: Bank always pays interest! Not looking for a "break even" strategy.
- Mathematical tool that helps us price securities: Find a *Risk Neutral Measure.*
  - ▷ Discounted stock price is (usually) not a martingale.
  - ▷ Invent a "risk neutral measure" which the discounted stock price is a martingale.
  - ▷ Securities can be priced by taking a conditional expectation *with respect to the risk neutral measure*. (That's the meaning of $\tilde{\boldsymbol{E}}_n$ in Proposition 4.1.)

Definition 4.51. Let $D_n = (1+r)^{-n}$ be the discount factor. (So D_n\$ in the bank at time 0 becomes 1\$ in the bank at time n.)

- Invent a new probability mass function $\tilde{p}$.
- Use a tilde to distinguish between the new, invented, probability measure and the old one.
  - ▷ $\tilde{\boldsymbol{P}}$ the probability measure obtained from the PMF $\tilde{p}$ (i.e. $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$).
  - ▷ $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ conditional expectation with respect to $\tilde{\boldsymbol{P}}$ (the new "risk neutral" coin)

Definition 4.52. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Definition 4.53. A *risk neutral measure* is an equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale. (I.e $\tilde{\boldsymbol{E}}_n(D_{n+1} S_{n+1}) = D_n S_n$.)

Remark 4.54. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 4.55. Proposition 4.1 says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n. (Called the risk neutral pricing formula.)

Proposition 4.56. *Let $\tilde{\boldsymbol{P}}$ be an equivalent measure under which the coins are i.i.d. and land heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$.*

1. *Under $\tilde{\boldsymbol{P}}$, we have $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = \frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} D_n S_n$.*
2. *$\tilde{\boldsymbol{P}}$ is the risk neutral measure if and only if $\tilde{p}_1 u + \tilde{q}_1 d = 1 + r$. (Explicitly $\tilde{p}_1 = \frac{1+r-d}{u-d}$, and $\tilde{q}_1 = \frac{u-(1+r)}{u-d}$.)*

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Proof of Proposition 4.1.

Example 4.59. Consider two stocks S^1 and S^2, $u = 2$, $d = 1/2$.

- ▷ The coin flips for S^1 are heads with probability 90%, and tails with probability 10%.
- ▷ The coin flips for S^2 are heads with probability 99%, and tails with probability 1%.
- ▷ Which stock do you like more?
- ▷ Amongst a call option for the two stocks with strike K and maturity N, which one will be priced higher?

Remark 4.60. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

Concepts that will be generalized to continuous time.

- Probability measure: Lebesgue integral, and not a finite sum. Same properties.
- Filtration: Same intuition. No easy description.
- Conditional expectation: Same properties, no formula.
- Risk neutral measure: Formula for $\tilde{\boldsymbol{P}}$ is complicated (Girsanov theorem.)
- Everything still works because of of Theorem 4.57. Understanding why is harder.

5. Stochastic Processes

5.1. Brownian motion.

- Discrete time: Simple Random Walk.
  ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\operatorname{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\operatorname{Var}(Y_t^\varepsilon) \approx t$.)
  ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).
▷ Definition is intuitive, but not as convenient to work with.

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

5.2. **Sample space, measure, and filtration.**

- Discrete time: Sample space $\Omega = (\omega_1, \dots, \omega_N)$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.

- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} \colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\Big(\bigcup_1^\infty A_n\Big) = \lim_{n \to \infty} \boldsymbol{P}(A_n)$.
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_{\Omega} X \, d\boldsymbol{P} = \int_{\Omega} X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$.
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

Proposition 5.3 (Useful properties of expectation)**.**

(1) (Linearity) $\alpha, \beta \in \mathbb{R}$, *$X, Y$ random variables,* $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E}X + \beta \boldsymbol{E}Y$.

(2) (Positivity) *If* $X \geqslant 0$ *then* $\boldsymbol{E}X \geqslant 0$. *If* $X \geqslant 0$ *and* $\boldsymbol{E}X = 0$ *then* $X = 0$ *almost surely.*

(3) (Layer Cake) *If* $X \geqslant 0$, $\boldsymbol{E}X = \displaystyle\int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt$.

(4) More generally, if φ *is increasing,* $\varphi(0) = 0$ *then* $\boldsymbol{E}\varphi(X) = \displaystyle\int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt$.

(5) (Unconscious Statistician Formula) *If PDF of X is p, then* $\boldsymbol{E}f(X) = \displaystyle\int_{-\infty}^\infty f(x)p(x)\, dx$.

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\mathcal{F}_t$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0, 1\}\}$.

5.3. **Conditional expectation.**

- Notation $\boldsymbol{E}_t(X) = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega'(s) = \omega(s)\ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

Lemma 5.8 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(Y)\,, \qquad \textit{where} \qquad g(y) = \boldsymbol{E} f(X, y)\,.$$

Remark 5.9. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X, Y) = \int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$.

5.4. **Martingales.**

Definition 5.10. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.11. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Proposition 5.12. *Brownian motion is a martingale.*

Proof.

6. Stochastic Integration

6.1. Motivation.

- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$
- Net gain/loss from changes in stock price: $\displaystyle\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  - ▷ $\|P\| = \max_k (t_{k+1} - t_k)$.
  - ▷ Riemann-Stieltjes integral: $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t$,
  - ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  - ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.

6.2. First Variation.

Definition 6.1. For any process X, define the *first variation* by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| . \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}| .$$

Remark 6.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]}X < \infty$.

Proposition 6.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

Remark 6.4. In fact, $V_{[0,T]}W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 6.5. The Riemann-Stieltjes integral $\int_0^T b_t \, dW_t$ *does not exist.*

6.3. **Quadratic Variation.**

Definition 6.6. If M is a continuous time adapted process, define

$$[M, M]_T = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 \,.$$

Proposition 6.7. *For continuous processes the following hold:*

1. *Finite first variation implies the quadratic variation is* 0
2. *Finite (non-zero) quadratic variation implies the first variation is infinite.*

Proposition 6.8. $[W, W]_T = T$ *almost surely.*

Remark 6.9. For use in the proof: $\mathrm{Var}(\mathcal{N}(0, \sigma^2)^2) = \boldsymbol{E}\mathcal{N}(0, \sigma^2)^4 - (\boldsymbol{E}\mathcal{N}(0, \sigma^2)^2)^2 = 2\sigma^2$.

Proof:.

Proposition 6.10. $W_t^2 - [W,W]_t$ *is a martingale.*

Theorem 6.11. *Let M be a continuous martingale.*

1. $\boldsymbol{E} M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M, M]_t < \infty$.
2. *In this case* $M_t^2 - [M, M]_t$ *is a continuous martingale.*
3. *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$, *then we must have* $A_t = [M, M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

Remark 6.13. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 6.14. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.

6.4. **Itô Integrals.**

- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i t_{i+1} - t_i$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$

Definition 6.15. The *Itô Integral* of D with respect to Brownian motion is defined by

$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) \,.$$

Remark 6.16. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

Theorem 6.17. *If* $\boldsymbol{E}\int_0^T D_t^2\,dt < \infty$ *a.s., then:*

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E}I(T)^2 < \infty$.

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I,I]_T = \int_0^T D_t^2\,dt$ *a.s.*

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\Big(\int_0^T D_t\, dW_t\Big)^2 = \boldsymbol{E}\int_0^T D_t^2\, dt$

Proof.

Intuition for Theorem 6.17 (2). Check $I_P(T)$ is a martingale.

Proposition 6.20. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes*

$$\int_0^T (\alpha D_s + \tilde{\alpha}\tilde{D}_s)\, dW_s = \alpha \int_0^T D_s\, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s\, dW_s$$

Proposition 6.21. $\displaystyle\int_0^{T_1} D_s\, dW_s + \int_{T_1}^{T_2} D_s\, dW_s$

Question 6.22. *If $D \geqslant 0$, then must $\int_0^T D_t\, dW_t \geqslant 0$?*

6.5. Semi-martingales and Itô Processes.

Question 6.23. *What is* $\int_0^t W_s \, dW_s$*?*

Definition 6.24. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation.
- ▷ M is a martingale.

Definition 6.25. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s \, ds$, with $\int_0^t |b_s| \, ds < \infty$
- ▷ $M_t = \int_0^t \sigma_s \, dW_s$, with $\int_0^t |\sigma_s|^2 \, ds < \infty$

Remark 6.26. Short hand notation for Itô processes: $dX_t = b_t \, dt + \sigma_t \, dW_t$.

Remark 6.27. Expressing $X = X_0+B+M$ (or $dX = b \, dt+\sigma \, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

Theorem 6.28 (Itô formula)**.** *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.29. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Proposition 6.30. *If* $X = X_0 + B + M$*, then* $[X, X] = [M, M]$.

Proposition 6.31 (Uniqueness). *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$, *with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0$.

Then $X_0 = Y_0$, $B = C$ *and* $M = N$.

Corollary 6.32. *Let* $dX_t = b_t\,dt + \sigma_t\,dW_t$ *with* $\boldsymbol{E}\int_0^t b_s\,ds < \infty$ *and* $\boldsymbol{E}\int_0^t \sigma_s^2\,ds < \infty$. *Then* X *is a martingale if and only if* $b = 0$.

Definition 6.33. If $dX = b\,dt + \sigma\,dW$, define $\displaystyle\int_0^T D_t\,dX_t = \int_0^T D_t b_t\,dt + \int_0^T D_t\sigma_t\,dW_t$.

Remark 6.34. Note $\displaystyle\int_0^T D_t b_t\,dt$ is a *Riemann integral*, and $\displaystyle\int_0^T D_t\sigma_t\,dW_t$ is a *Itô integral*.

6.6. **Itô's formula.**

Remark 6.35. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$$

Theorem (Itô's formula, Theorem 6.28)**.** *If* $f \in C^{1,2}$*, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.36*.* If $dX_t = b_t\, dt + \sigma_t\, dW_t$ then

$$df(t, X_t) = \Big(\partial_t f(t, X_t) + b_t + \frac{1}{2}\sigma_t^2\Big)\, dt + \partial_x f(t, X_t)\sigma_t\, dW_t\,.$$

Intuition behind Itô's formula.

Example 6.37. Find the quadratic variation of W_t^2.

Example 6.38. Find $\int_0^t W_s \, dW_s$.

Example 6.39. Let $M_t = W_t$, and $N_t = W_t^2 - t$.

- ▷ We know M, N are martingales.
- ▷ Is MN a martingale?

Example 6.40. Let $X_t = t\sin(W_t)$. Is $X_t^2 - [X, X]_t$ a martingale?

Example 6.41. Say $dM_t = \sigma_t \, dW_t$. Show that $M^2 - [M, M]$ is a martingale.

7. Review Problems

Problem 7.1. If $0 \leqslant r \leqslant s \leqslant t$, find $\boldsymbol{E}(W_s W_t)$ and $\boldsymbol{E}(W_r W_s W_t)$.

Problem 7.2. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2}\, ds\,, \quad Y_t = \exp\Big(\int_0^t W_s\, ds\Big)\,, \quad Z_t = tX_t^2$$

Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

Problem 7.3. Define the processes X, Y by

$$X_t \stackrel{\text{def}}{=} \int_0^t W_s \, ds \,, \quad Y_t \stackrel{\text{def}}{=} \int_0^t W_s \, dW_s \,.$$

Given $0 \leqslant s < t$, compute $\boldsymbol{E} X_t$, $\boldsymbol{E} Y_t$, $\boldsymbol{E}_s X_t$, $\boldsymbol{E}_s Y_t$.

Problem 7.4. Let $M_t = \int_0^t W_s \, dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\Big(M_t - \int_0^t f(s, W_s) \, ds\Big)$$

is a martingale.

Problem 7.5. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 \, dt$.

$$X_t = \int_0^t \sigma_u \, dW_u \, .$$

(1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leqslant s < t$ compute $\boldsymbol{E} e^{\lambda M_t}$ and $\boldsymbol{E}_s e^{\lambda M_t - M_s}$
(2) If $r \leqslant s$ compute $\boldsymbol{E} \exp(\lambda M_r + \mu(M_t - M_s))$.
(3) What is the joint distribution of $(M_r, M_t - M_s)$?
(4) *(Lévy's criterion)* If $d[M, M]_t = dt$, then show that M is a standard Brownian motion.

Problem 7.6. Define the process X, Y by

$$X = \int_0^t s\, dW_s\,, \quad Y = \int_0^t W_s\, ds\,.$$

Find a formula for $\boldsymbol{E} X_t^n$ and $\boldsymbol{E} Y_t^n$ for any $n \in \mathbb{N}$.

Problem 7.7. Let $M_t = \int_0^t W_s \, dW_s$. For $s < t$, is $M_t - M_s$ independent of $\mathcal{F}_s$? Justify.

Problem 7.8. Determine whether the following identities are true or false, and justify your answer.

(1) $e^{2t}\sin(2W_t) = 2\int_0^t e^{2s}\cos(2W_s)\,dW_s$.

(2) $|W_t| = \int_0^t \operatorname{sign}(W_s)\,dW_s$. (Recall $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 0$ if $x = 0$.)

8. Black Scholes Merton equation

8.1. Market setup and assumptions.

- Cash: simple interest rate r in a bank.
- Let Δt be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$.
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t)C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\partial_t C = rC$ and $C_t = C_0 e^{rt}$.
- r is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $r = \frac{1}{T}\ln(1+\rho)$.

- Stock price: $S_{t+\Delta t} = (1 + r\,\Delta t) S_t + \text{noise}$.
  - ▷ Variance of noise should be proportional to Δt.
  - ▷ Variance of noise should be proportional to S_t.
- $S_{t+\Delta t} - S_t = r S_t\,\Delta t + \sigma S_t (\Delta W_t)$.

Definition 8.1. A Geometric Brownian motion with parameters α, σ is defined by:

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t\,.$$

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Proposition 8.2. $S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

Market Assumptions.

- 1 stock, Price S_t, modelled by GBM(α, σ).
- Money market: Continuously compounded interest rate r.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = rC_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)

8.2. **The Black, Sholes, Merton equation.** Consider a security that pays $V_T = g(S_T)$ at maturity time T.

Theorem 8.3. *If the security can be replicated, and $f = f(t,x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{8.1}$$

$$f(t,0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{8.2}$$

$$f(T,x) = g(x) \qquad x \geqslant 0\,. \tag{8.3}$$

Theorem 8.4. *Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 8.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 8.6. $g(x) = (x - K)^+$ is a European call with strike K and maturity T.

Remark 8.7. $g(x) = (K - x)^+$ is a European put with strike K and maturity T.

Proposition 8.8. *A standard change of variables gives an explicit solution to* (8.1)–(8.3)*:*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Corollary 8.9. *For European calls,* $g(x) = (x-K)^+$, *and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy\,, \tag{8.7}$$

is the CDF of a standard normal variable.

Remark 8.10. Equation (8.1) is called a *partial differential equation.* In order to have a unique solution it needs:

(1) A terminal condition (this is equation (8.3)),
(2) A boundary condition at $x = 0$ (this is equation (8.2)),
(3) A boundary condition at infinity (not discussed yet).
   - ▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is
$$\lim_{x\to\infty} f(t,x) = 0\,.$$
   - ▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is
$$\lim_{x\to\infty} \Big[f(t,x) - (x - Ke^{-r(T-t)})\Big] = 0 \qquad \text{or} \qquad f(t,x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty\,.$$

Definition 8.11. If X_t is the wealth of a self-financing portfolio then

$$dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t) \, dt$$

for some adapted process Δ_t (called the trading strategy).

Proof of Theorem 8.3.

Proof of Theorem 8.4.

Proof of Theorem 8.4 (without discounting).

Remark 8.12. The arbitrage free price does not depend on the mean return rate!

Question 8.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (8.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

Proposition 8.14 (Put call parity)**.** *Consider a European put and European call with the same strike K and maturity T.*

- ▷ $c(t, S_t) =$ *AFP of call (given by* (8.5)*)*
- ▷ $p(t, S_t) =$ *AFP of put.*

Then $c(t, x) - p(t, x) = x - Ke^{-r(T-t)}$*, and hence* $p(t, x) = Ke^{-r(T-t)} - x - c(t, x)$*.*

8.3. **The Greeks.** Let $c(t,x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad \tau = T - t\,.$$

Definition 8.15. The *delta* is $\partial_x c$.

Remark 8.16 (Delta hedging rule). $\Delta_t = \partial_x c(t, S_t)$.

Proposition 8.17. $\partial_x c = N(d_+)$

Definition 8.18. The *Gamma* is $\partial_x^2 c$ and is given by $\partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(\frac{-d_+^2}{2}\right)$.

Definition 8.19. The *Theta* is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{2\sqrt{\tau}}N'(d_+)$

Proposition 8.20. *(1) c is increasing as a function of x.*

(2) c is convex as a function of x.

(3) c is decreasing as a function of t.

Remark 8.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Remark 8.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x) -$ tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

9. **Multi-dimensional Itô calculus**

- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),$$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t \, dY_t$.

Lemma 9.3. $[X, Y]_T = \frac{1}{4}([X+Y, X+Y]_T - [X-Y, X-Y]_T)$

Proposition 9.4 (Product rule)**.** $d(XY)_t = X_t\, dY_t + Y_t dX_t + d[X, Y]_t$

Proposition 9.5. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X, Y]_t = d[M, N]_t$.*

Remark 9.6. Recall, all processes are implicitly assumed to be *adapted* and *continuous*.

Corollary 9.7. *If X is a semi-martingale and B has bounded variation then* $[X, B] = 0$.

Notation.

- *d-dimensional vectors:* Write $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.
- *d-dimensional* random *vectors:* $X = (X_1, \dots, X_d)$, where each X_i is a random variable.
- *d-dimensional stochastic processes:* $X_t = (X_t^1, \dots, X_t^d)$, where each X_t^i is a stochastic process.
  - ▷ For scalars (or random variables): X^i denotes the i-th power of X.
  - ▷ For vectors (or random random vectors): X^i denotes the i-th *coordinate* of X.
  - ▷ There is no ambiguity (can't take powers of vectors, or coordinates of scalars)
- Alternate notation used in many books: Use $X(t)$ for the d-dimensional stochastic process, and $X_i(t)$ for the i-th coordinate.
- May sometimes write $X = (X^1, \dots, X^d)$ for random vectors, instead of $(X_1, \dots, X_d)$.

Remark 9.8 (Chain rule). If X is a differentiable function of t, then

$$d(f(t, X_t)) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i$$

Remark 9.9 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_i f = \frac{\partial f}{\partial x_i}$.

Theorem 9.10 (Multi-dimensional Itô formula)**.**

- *Let X be a d-dimensional Itô process. $X_t = (X_t^1, \ldots, X_t^d)$.*
- *Let $f = f(t, x)$ be a function that's defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.*
- *Suppose $f \in C^{1,2}$. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in each coordinate x_i*
  - ▷ *All the above partial derivatives are continuous. Then:*

$$d(f(t, X_t)) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j} \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

Remark 9.11 (Integral form of Itô's formula).

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\, dt + \sum_{i=1}^d \int_0^T \partial_i f(t, X_t)\, dX_t^i$$
$$+ \frac{1}{2} \sum_{i,j} \int_0^T \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

Remark 9.12. As with the 1D Itô, will drop the arguments (t, X_t). *Remember they are there.*

Intuition behind Theorem 9.10.

To use the d-dimensional Itô formula, we need to compute joint quadratic variations.

Proposition 9.13. *Let M, N be continuous martingales, with $\boldsymbol{E} M_t^2 < \infty$ and $\boldsymbol{E} N_t^2 < \infty$.*

(1) $MN - [M, N]$ is also a continuous martingale.

(2) Conversely if $MN - B$ is a continuous martingale for some continuous adapted, bounded variation *process B with $B_0 = 0$, then $B = [M, N]$.*

Proof.

Proposition 9.14. *(1)* (Symmetry) $[X, Y] = [Y, X]$

(2) (Bi-linearity) *If* $\alpha \in \mathbb{R}$, X, Y, Z *are semi-martingales,* $[X, Y + \alpha Z] = [X, Y] + \alpha[X, Z]$.

Proof.

Proposition 9.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \displaystyle\int_0^t \sigma_s \, dM_s$ *and* $Y_t = \displaystyle\int_0^t \tau_s \, dN_s$.
- *Then* $[X, Y]_t = \int_0^t \sigma_s \, \tau_s \, d[M, N]_s$.

Remark 9.16. Alternately, if $dX_t = \sigma_t \, dM_t$ and $dY_t = \tau_t \, dN_t$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

Proposition 9.17. *If M, N are continuous martingales, $\boldsymbol{E}M_t^2 < \infty$, $\boldsymbol{E}N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 9.18 (Warning)*.* Independence implies $\boldsymbol{E}(M_t N_t) = \boldsymbol{E}M_t \boldsymbol{E}N_t$. But it *does not* imply $\boldsymbol{E}_s(M_t N_t) = \boldsymbol{E}_s M_t \boldsymbol{E}_s N_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof.

Remark 9.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- Let $M_t = \int_0^t \mathbf{1}_{\{W_s < 0\}} \, dW_s$
- Let $N_t = \int_0^t \mathbf{1}_{\{W_s \geqslant 0\}} \, dW_s$

Definition 9.20 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1, \dots, W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 9.21. If W is a d-dimensional Brownian motion then $d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$

Theorem 9.22 (Lévy)**.** *Let M be a d-dimensional process such that:*

(1) M is a continuous martingale.

(2) The joint quadratic variation satisfies: $d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$

Then M is a d-dimensional Brownian motion.

Proof. Find $\boldsymbol{E}_s e^{\lambda M_t^i + \mu M_t^j}$ using Itô's formula, similar to Problem 7.5. $\square$

Example 9.23. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t, W_t)$. Find the Itô decomposition of X.

Question 9.24. *Let W be a 2-dimensional Brownian motion. Let $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. Is X a martingale?*

10. **Risk Neutral Pricing**

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $D_t = \exp(-\int_0^t R_s \, ds)$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

Theorem 10.4. *The (unique) risk neutral measure is given by $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big) , \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t} .$$

Theorem 10.5. *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s \, ds\Big) \, V_T)\Big) .$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \displaystyle\int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\displaystyle\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$.

Lemma 10.10. *Let* $Z_t = \boldsymbol{E}_t Z_T$. *If* X_t *is* $\mathcal{F}_t$*-measurable, then* $\tilde{\boldsymbol{E}}_s X = \frac{1}{Z_s} \tilde{\boldsymbol{E}}_s (Z_t X_t)$.

Proof. You will see this in the proof of the Girsanov theorem. ☐

Theorem 10.11 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$, and define:*

- $b_t = (b_t^1, \dots, b_t^d)$ *a d-dimensional adapted process.*
- W *a d-dimensional Brownian motion.*
- $\tilde{W}_t = W_t + \int_0^t b_s \, ds$ *(i.e. $d\tilde{W}_t = b_t \, dt + d\tilde{W}_t$).*
- $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, *where*

$$Z_t = \exp\Big(- \int_0^t b_s \cdot dW_s - \frac{1}{2} \int_0^t |b_s|^2 \, ds \Big) .$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

Remark 10.12. Note $\tilde{W}_t$ is a vector.

(1) So $\tilde{W}_t = W_t + \int_0^t b_s \, ds$ means $\tilde{W}_t^i = W_t^i + \int_0^t b_s^i \, ds$, for each $i \in \{1, \dots, d\}$.
(2) Similarly, $d\tilde{W}_t = b_t \, dt + d\tilde{W}_t$ means $d\tilde{W}_t^i = b_t^i \, dt + d\tilde{W}_t^i$ for each $i \in \{1, \dots, d\}$.

Remark 10.13. $\int_0^t b_s \cdot dW_s$ means $\int_0^t \sum_{i=1}^d b_s^i \, dW_s^i$ (dot product).

Proposition 10.14. *$dZ_t = -Z_t b_t \cdot dW_t$. Explicitly, in coordinates, $dZ_t = -Z_t \sum_{i=1}^{d} b_t^i \, dW_t^i$.*

Question 10.15. *Looks like Z is a martingale. Why did we assume it in Theorem 10.11?*

Idea behind the proof of Theorem 10.11.

Theorem (Theorem 10.4)**.** *The (unique) risk neutral measure is given by* $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$*, where*

$$Z_T = \exp\Bigl(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Bigr) \,, \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t} \,.$$

Proof of Theorem 10.4.

Theorem 10.16. *X_t represents the wealth of a self-financing portfolio if and only if $D_t X_t$ is a $\tilde{\boldsymbol{P}}$ martingale.*

Remark 10.17. The proof of the backward direction requires the *martingale representation theorem*, and is outlined on your homework.

Remark 10.18. This is the analog of Theorem 4.57

Proof of the forward direction.

Theorem (Theorem 10.5). *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t (D_T V_T) = \tilde{\boldsymbol{E}}_t \Big(\exp\Big(\int_t^T -R_s \, ds \Big) \, V_T) \Big) \,.$$

Remark 10.19. This is the analog of Proposition 4.1.

Proof of Theorem 10.5.

11. **Black Scholes Formula revisited**

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a $\mathrm{GBM}(\alpha, \sigma)$ (both α, σ are constant in time).

Theorem 11.1. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}\,dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Remark 11.2. This proves Proposition 8.8.

Theorem 11.3 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Bigl(\ln\Bigl(\frac{x}{K}\Bigr) + \Bigl(r \pm \frac{\sigma^2}{2}\Bigr)\tau\Bigr)\,, \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

is the CDF of a standard normal variable.

Remark 11.4*.* This proves Corollary 8.9.

12. **Review problems**

Problem 12.1. Let f be a deterministic function, and define

$$X_t \stackrel{\text{def}}{=} \int_0^t f(s) W_s \, ds \,.$$

Find the distribution of X.

Problem 12.2. Suppose σ, τ, ρ are three deterministic functions and M and N are two continuous martingales with respect to a common filtration $\{\mathcal{F}_t\}$ such that $M_0 = N_0 = 0$, and

$$d[M,M]_t = \sigma_t \, dt \,, \quad d[N,N]_t = \tau_t \, dt \,, \qquad \text{and} \qquad d[M,N]_t = \rho_t \, dt \,.$$

(a) Compute the joint moment generating function $\boldsymbol{E} \exp(\lambda M(t) + \mu N(t))$.

(b) *(Lévy's criterion)* If $\sigma = \tau = 1$ and $\rho = 0$, show that (M, N) is a two dimensional Brownian motion.

Problem 12.3. Consider a financial market consisting of a risky asset and a money market account. Suppose the return rate on the money market account is r, and the price of the risky asset, denoted by S, is a geometric Brownian motion with mean return rate α and volatility σ. Here r, α and σ are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$V_T = \frac{1}{T} \int_0^T S_t \, dt$$

at maturity T. Also compute the trading strategy in the replicating portfolio.

Problem 12.4. Let $X \sim N(0,1)$, and $a, \alpha, \beta \in \mathbb{R}$. Define a new measure $\tilde{\boldsymbol{P}}$ by

$$d\tilde{\boldsymbol{P}} = \exp\big(\alpha X + \beta\big)\, d\boldsymbol{P}\,.$$

Find α, β such that $X + a \sim N(0,1)$ under $\tilde{\boldsymbol{P}}$.

Problem 12.5. Let $x_0, \mu, \theta, \sigma \in \mathbb{R}$, and suppose X is an Itô process that satisfies

$$dX(t) = \theta(\mu - X_t)\,dt + \sigma\,dW_t\,,$$

with $X_0 = x_0$.

(a) Find functions $f = f(t)$ and $g = g(s,t)$ such that

$$X(t) = f(t) + \int_0^t g(s,t)\,dW_s\,.$$

The functions f, g may depend on the parameters x_0, θ, μ and σ, but should not depend on X.

(b) Compute $\boldsymbol{E} X_t$ and $\operatorname{cov}(X_s, X_t)$ explicitly.

Problem 12.6. Let $\theta \in \mathbb{R}$ and define

$$Z(t) = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big).$$

Given $0 \leqslant s < t$, and a function f, find a function such that

$$\boldsymbol{E}_s f(Z_t) = g(Z(s)).$$

Your formula for the function g can involve f, s, t and integrals, but not the process Z or expectations.

Problem 12.7. Let W be a Brownian motion, and define

$$B_t = \int_0^t \operatorname{sign}(W_s)\, dW_s\,.$$

(a) Show that B is a Brownian motion.

(b) Is there an adapted process σ such that

$$W_t = \int_0^t \sigma_s\, dB_s\,?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation $[B, W]$.

(d) Are B and W uncorrelated? Are they independent? Justify.

Problem 12.8. Let W be a Brownian motion. Does there exist an equivalent measure $\tilde{\boldsymbol{P}}$ under which the process tW_t is a Brownian motion? Prove it.