Citadel Interview Questions

1. **How to use a random number generator RAND7() that generates numbers 0-6 (both inclusive) to create a random number generator RAND10() that generates numbers 0-9 (both inclusive)?** (Solution : see image below)

RAND7() will generate all numbers from 0 to 6 uniformly and randomly. (equally likely with probability $= \frac{1}{7}$)

Now, let's define a function

$$\underbrace{RAND7()}_{unif\ (0,6)} + \underbrace{7 \times RAND7()}_{[0,7,14,21,28,35,42]}$$

As both sets has numbers which are equally likely with probability $\frac{1}{7}$, the cross product has probability $\frac{1}{49}$ and if we apply the function on cross product we get numbers from 0 to 48.

Sum of two uniform random distributions is also uniform random. So, we have unif (0, 48).

Now, we use method of <u>rejection sampling technique</u>.

We divide the set into 10 divisions with 4 numbers each and discard the remaining elements from the end.

if num < 40 : (num is random number from 0 to 48)
return num % 10

else we do the whole computation again and reject this trial.

In this manner our final output will be uniformly random from 0 to 9 with probability $= \frac{1}{10}$. This is RAND10() which we wanted to generate.

2. **500 ants are randomly put on a 1-foot string (independent uniform distribution for each ant between 0 and 1). Each ant randomly moves toward one end of the string (equal probability to the left or the right) at a constant speed of 1 foot/minute until it reaches one end of the string. Also assume that the size of the ant is infinitely small. When two ants collide head-on, they both immediately change directions and keep on moving at 1 foot/min. What is the expected time for all ants to fall off the string?**

We first use the principle that for two ants collision has the same effect as two ants passing through each other, which means collisions have no effect on an ant. Also since there is uniform random distribution of ants and left and right ends are symmetrical, we just consider all ants to be facing left and placed uniformly on the stick (IID events).

The speed of ants is $1 foot/min$ and the question simplifies to calculating the expected length of the farthest ant from the left end. As that is the ant which will fall at last.

Now we can find the expected value either by integration or using Linearity of expectation. Here we show it by integration method.

$Z_n = max(X_1, X_2..X_n)$

$P(Z_n <= x) = (P(X_i <= x))^n = x^n$

From the CDF, we obtain the pdf by differentiation

$f_{Z_n}(x) = nx^{n-1}$

$E[Z_n] = \int_0^1 x f_{Z_n}(x)dx = \int_0^1 nx^n dx = \frac{n}{n+1}[x^{n+1}]_0^1 = \frac{n}{n+1}$

Hence the Expected value of longest length from the end is $\frac{500}{501}$. Dividing with $1 foot/min$ gives the expected value of time till the last ant drops off the stick = $\frac{500}{501}$