Probability & Statistics for EECS: Homework #01

Due on Feb 29, 2024 at 23:59

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Problem 1

(1) Consider we are forming k groups of $n+1$ people and I am among the $n+1$ people. There are two possibilites: the other n people form $k-1$ groups and I am in a group by myself, or the other n people form k groups already, so I only have to join one of the k groups.

(2) Consider we are forming $k+1$ groups of $n+1$ people and I am among the $n+1$ people. There possibilities that 1, 2,$\cdots$, k people are not in my group. So we can divide it into k circumstances and the j people(not in my group) form k groups and I lead a group myself.

Problem 2

There are P_{26}^{26} *norepeatword*s with 26 alphabets, and $P_{26}^{1}+P_{26}^{2}+\cdots$ *norepeatword*s.

$$P=\frac{P_{26}^{26}}{P_{26}^{1}+P_{26}^{2}+\cdots+P_{26}^{26}}$$

$$\frac{1}{P}=1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{25!}=\sum_{n=1}^{25}\frac{1}{n!}\approx\sum_{i=1}^{\infty}\frac{1}{n!}=e$$

$$So\ P\approx\frac{1}{e}.$$

Problem 3

(a) Apparently, we have n^n possible bootstrap samples.

(b) Assume $(a_1,a_2,\cdots a_n)$ and $(b_1,b_2,b_3,\cdots b_n)$.
Each b_n presents the number of times that each a_n is chosen. So $\sum_{i=1}^{n}b_i=n$.

Then it turns out to be a partition problem: We should put $n-1$ partitions to seperate the numbers into k different boxes. So, there are $2n-1$ places to be inserted, which is a $\binom{2n-1}{n-1}$. So there are $\frac{(2n-1)!}{n!(n-1)!}$kinds.

(c) Assume the bootstrap samples have orders so there are n^n kinds.

$b1$: only one kind:$a_1,a_1,a_1\cdots,a_1$.
$b2$: each number appears once: $a_1,a_2,a_3\cdots a_n$ is $n!$.

So $p_1=\frac{n!}{n^n}$, $p_2=\frac{1}{n^n}$. $\frac{p_1}{p_2}=n!$

Probability of getting p_1: $\frac{n!}{n^n}$.

probability of getting p_2: $n\times\frac{1}{n^n}$.

The ratio is $(n-1)!$

Problem 4

Assume the length of the stick is k and the three pieces are a, b, and $k-a-b$. To form a triangle, we have to satisfy:

$$a+b>k-a-b$$
$$a+k-a-b>b$$
$$b+k-a-b>a$$

Problem 4 continued on next page...

The fig below gives a way to calculate the possibility, which is the proportion of the area of the shadow.

$$So\ P = \frac{1}{4}.$$

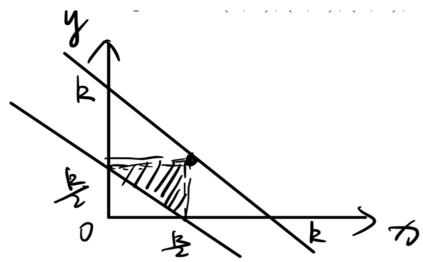


Figure 1: LP

Problem 5

(a) There are k people,then the probability of at least one holiday match is

$$1 - k!e_k(p_1, p_2, p_3, \cdots, p_{365}) = 1 - k!e_k(\vec{p}).$$

(b) Consider $k = 2, P = 1 - 2e_2(\vec{p}) = \sum_{j=1}^{365} p_j^2 \geq \frac{(p_1+p_2+\cdots+p_{365})^2}{365}$
We know that it is only equal when $p_1 = p_2 = \cdots = p_{365} = \frac{1}{365}$.

(c) 1: A's birthday is on 1 day,B's birthday is on another day, others' don't match and are not on day1 or day2 $\Rightarrow x_1x_2e_{k-2}(x_3\cdots, x_n)$

2:A's birthday and B's birthday are on one day, others' on another day $\Rightarrow (x_1 + x_2)e_{k-1}(x_3, \cdots, x_n)$

3:All people's birthday don't match.

So we have $e_k(x_1\cdots, x_n) = x_1x_2e_{k-2}(x_3\cdots, x_n) + (x_1 + x_2)e_{k-1}(x_3, \cdots, x_n) + e_k(x_3\cdots, x_n)$.

$e_k(\vec{p}) = p_1p_2e_{k-2}(p_3\cdots, p_n) + (p_1 + p_2)e_{k-1}(p_3, \cdots, p_n) + e_k(p_3\cdots, p_n)$
$e_k(\vec{r}) = \frac{(p_1+p_2)^2}{4}e_{k-2}(p_3\cdots, p_n) + (p_1 + p_2)e_{k-1}(p_3, \cdots, p_n) + e_k(p_3\cdots, p_n)$

refer to$\frac{x+y}{2} \geq \sqrt{xy}$ (only equal when x=y)

$$so\ e_k(\vec{p}) \leq e_k(\vec{r}), only\ equal\ when\ p_1 = p_2$$

So $P(at\ least\ one\ birthday\ match|p) \geq P(at\ least\ one\ birthday\ match \mid r)$.

then we define a vector $\vec{r} = (r_1, r_2, p_3, p_4, p_5, \cdots)$, $\vec{m} = (r_1, m_2, m_3, p_4, p_5, \cdots), m_2 = m_3 = \frac{r_2+p_3}{2}$

$$so\ e_k(\vec{r}) \leq e_k(\vec{m}), only\ equal\ when\ m_2 = m_3 = \frac{p_3 + r_2}{2} = \frac{p_3 + \frac{p_1+p_2}{2}}{2}$$

then $e_k(\vec{p}) \leq e_k(\vec{r}) \leq e_k(\vec{m}) \leq \cdots e_k(\vec{P})$.

So only equal when $P_{n+1} = \frac{P_n+P_{n-1}}{2}$,and $P_1 = \frac{P_{365}+P_{364}}{2}$

$\sum_{k=1}^{365} P_j = 1 \Rightarrow P_j = \frac{1}{365}$

So $e_k(\vec{P})$is max value and the value of p that minimizes the probability of at least one birthday match is given by $p_j = 365$ for all j.

Problem 6

$$P = \frac{108! \left\{ n \atop 108 \right\}}{108^n} = \frac{\sum_{k=0}^{108} (-1)^k C_{108}^k (108-k)^n}{108^n} = \sum_{k=0}^{108} (-1)^k C_{108}^k (\frac{108-k}{108})^n$$

Then we use python to plot an image:

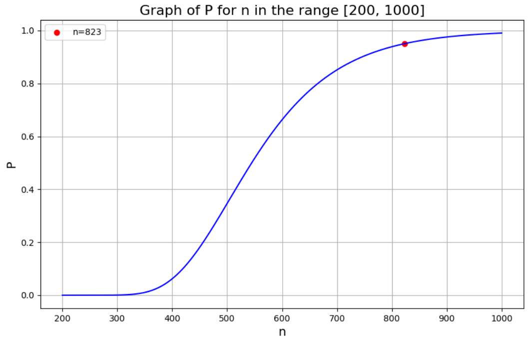
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import comb

def P_function(n):
    result = 0
    for k in range(109):
        result += (-1)**k * comb(108, k) * ((108 - k) / 108)**n
    return result

n_values = np.arange(200, 1001)
P_values = [P_function(n) for n in n_values]

first_index_above_095 = next((i for i, p in enumerate(P_values) if p > 0.95), None)
n_value_above_095 = n_values[first_index_above_095]

plt.figure(figsize=(10, 6))
plt.plot(n_values, P_values, color='blue', linestyle='-')
plt.scatter(n_value_above_095, P_values[first_index_above_095], color='red', label=f'n={
                                                            n_value_above_095}')
plt.xlabel('n', fontsize=14)
plt.ylabel('P', fontsize=14)
plt.title('Graph of P for n in the range [200, 1000]', fontsize=16)
plt.grid(True)
plt.legend()
plt.show()
```



We can tell from the fig that the minimum number of n is 823 when such probability is no less than 95%.