Probability & Statistics for EECS: Homework #02

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Problem 1

(a) B: Bob receives a 1.
A: Alice sends a 1.
So

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.9 \cdot 0.5}{0.9 \cdot 0.5 + 0.05 \cdot 0.5} = \frac{18}{19}$$

(b) B: Bob receives a 110.
A: Alice sends a 1.
So

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \frac{0.9 \cdot 0.9 \cdot 0.05 \cdot 0.5}{0.9 \cdot 0.9 \cdot 0.05 \cdot 0.5 + 0.05 \cdot 0.05 \cdot 0.95 \cdot 0.5} = \frac{648}{667}$$

Problem 2

(a) T: He tests positive on n of the n tests.
So

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} = \frac{a_0^n p}{a_0^n p + b_0^n q}$$

(b) T: He tests positive on n of the n tests.

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

We need to calulate $P(T|D)$:

$$\begin{aligned} P(T|D) &= P(T|D,G)P(G) + P(T|D,G^c)P(G^c) \\ &= \frac{1}{2} + \frac{1}{2}a_0^n \end{aligned}$$

Similarly, we have $P(T|D^C) = \frac{1}{2} + \frac{1}{2}b_0^n$.

So

$$P(D|T) = \frac{p(\frac{1}{2} + \frac{1}{2}a_0^n)}{p(\frac{1}{2} + \frac{1}{2}a_0^n) + q(\frac{1}{2} + \frac{1}{2}b_0^n)} = \frac{p(1 + a_0^n)}{1 + a_0^n p + b_0^n q}$$

Problem 3

$$\begin{aligned} &P(spam|W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, \cdots, W_{66}^C, \cdots W_{100}^C) \\ &= \frac{P(W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, \cdots, W_{66}^C, \cdots W_{100}^C|spam)P(spam)}{P(W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, W_{66}^C, \cdots W_{100}^C)} \\ &= \frac{P(W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, W_{66}^C, \cdots W_{100}^C|spam)P(spam)}{P(W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, W_{66}^C, \cdots W_{100}^C|spam)P(spam) + P(W_1^C, \cdots, W_{22}^C, W_{23}, W_{24}^C \cdots, W_{63}^C, W_{64}, W_{65}, \cdots W_{100}^C|not\ spam)P(not\ spam)} \\ &= \frac{(1-p_1)\cdots(1-p_{22})p_{23}(1-p_{24})\cdots(1-p_{63})p_{64}p_{65}(1-p_{66})\cdots(1-p_{100})p}{(1-p_1)\cdots(1-p_{22})p_{23}(1-p_{24})\cdots(1-p_{63})p_{64}p_{65}(1-p_{66})\cdots(1-p_{100})p + (1-r_1)\cdots(1-r_{22})r_{23}(1-r_{24})\cdots(1-r_{63})r_{64}r_{65}(1-r_{66})\cdots(1-r_{100})(1-p)} \end{aligned}$$

Problem 4

Suppose we choose door 1. If we don't switch, the probability we win the car is $p1$.
If we switch a door, as long as the car is not behind door 1, we'll win the car. So the probability is $1 - p1$.
If we choose door 2 or 3, that's the same.
So we should choose door 3 and switch, so that the maximum probability that we win turn outs to be $p1+p2$.

Problem 5

(a) Let C_j be the event that the car is hidden behind door j and let W be the event that we win using the switching strategy. Using the law of total probability, we can find the unconditional probability of winning:

$$P(W) = P(W|C_1)P(C_1) + P(W|C_2)P(C_2) + P(W|C_3)P(C_3) = \frac{2}{3}$$

(b) Let $D_i$be the event that Monty opens Door i.

$$\begin{aligned} P(C_3|D_2) &= \frac{P(D_2|C_3)P(C_3)}{P(D_2)} \\ &= \frac{P(D_2|C_3)P(C_3)}{P(D_2|C_1)P(C_1) + P(D_2|C_2)P(C_2) + P(D_2|C_3)P(C_3)} \\ &= \frac{1/3}{1/3 \cdot p + 1/3} \\ &= \frac{1}{1+p} \end{aligned}$$

(c) Replace p by $1 - p$.
So

$$P(C_2|D_3) = \frac{1}{2-p}$$

Problem 6

(a) An: The nth trail is assigned treatment A.
Bn: The nth trail is assigned treatment B.
S: The nth trail is successful.

$p_n = p(S|A_n)p(A_n) + p(S|B_n)p(B_n) = a_n \cdot a + (1 - a_n) \cdot b = (a - b)a_n + b$
$a_{n+1} = P(TreatmentA\ succeeds\ on\ the\ nth\ trial) + P(TreatmentB\ fails\ on\ the\ nth\ trial)$
$= a_n \cdot a + (1 - a_n)(1 - b)$
$= (a + b - 1)a_n + 1 - b$

(b) According to (a), we have:

$$\begin{aligned} p_{n+1} &= (a - b)a_{n+1} + b \\ &= (a - b)[(a + b - 1)a_n + 1 - b] + b \\ &= (a - b)(a + b - 1)a_n + ab + b^2 - b + a + b - 2ab \\ &= (a + b - 1)p_n + a + b - 2ab \end{aligned}$$

Problem 6 continued on next page...

(c) when $n \to \infty$, $p_{n+1} \approx p_n$
Assume $p_{n+1} \approx p_n = x$.
So

$$\begin{aligned} x &= (a+b-1)x + a + b - 2ab \\ (2-a-b)x &= a+b-2ab \\ x &= \frac{a+b-2ab}{2-a-b} \end{aligned}$$

So

$$\lim_{n\to\infty} p_n = \frac{a+b-2ab}{2-a-b}$$