Probability & Statistics for EECS: Homework #05

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Problem 1

(a) Define event Z_1: both nickles are Heads. So we have:

$$Z_1 \sim Geom(p_1 p_2)$$

$$P(Z_1 = k) = (1 - p_1 p_2)^{k-1} p_1 p_2$$

$$E(Z_1) = \frac{1}{p_1 p_2}$$

(b) Define event Z_2: ≥ 1 nickle is Head. So we have:

$$Z_2 \sim Geom(1 - (1 - p_1)(1 - p_2))$$

$$E(Z_2) = \frac{1}{p_1 + p_2 - p_1 p_2}$$

(c)

$$\mathbb{P}[X = Y] = \sum_{k=1}^{\infty} p^2 (q^2)^{k-1} = p^2 \sum_{k=0}^{\infty} (q^2)^k = \frac{p^2}{1 - q^2} = \frac{p}{2 - p}$$

$$\mathbb{P}[X > Y] = \frac{1 - \frac{p}{2-p}}{2} = \cdots = \frac{1 - p}{2 - p}$$

Problem 2

(a) Let X be the random variable for the total amount of stops and let I_k be the indicator variable such that $I_k = 1$ if someone stopped at the k^{th} floor. Then

$$X = I_2 + I_3 + \cdots + I_n,$$

and it follows that:

$$E[X] = E[I_2 + I_3 + \cdots + I_n] = E[I_2] + E[I_3] + \cdots + E[I_n].$$

Then $E[I_j] = P(\text{at least someone hit the } j^{th} \text{ button}) = 1 - \left(\frac{n-2}{n-1}\right)^k$

$$E[X] = (n - 1)\left(1 - \left(\frac{n-2}{n-1}\right)^k\right)$$

(b) We use the same reasoning except we identify that

$$E[I_j] = 1 - (1 - p_j)^k$$

and it follows that

$$E[X] = \sum_{j=2}^{n} 1 - (1 - p_j)^k = n - 1 - \sum_{j=2}^{n} (1 - p_j)^k.$$

Problem 3

Using LOTUS, let $j = i - k$:

$$
\begin{aligned}
E\binom{n}{k} &= \sum_i f(i)P(X=i) \\
&= \sum_{i=k}^{\infty} \frac{i!}{k!(i-k)!} \cdot \frac{\lambda^i(e^{-\lambda})}{i!} \\
&= \lambda^k(e^{-\lambda}) \sum_{j=0}^{\infty} \frac{1}{k!} \cdot \frac{\lambda^j}{j!} \\
&= \lambda^k(e^{-\lambda}) \frac{1}{k!} e^{\lambda} \\
&= \frac{\lambda^k}{k!}
\end{aligned}
$$

Problem 4

(a)

$$E(Xg(X)) = \sum_{k=0}^{\infty} kg(k)\frac{e^{-\lambda}\lambda^{k+1}}{k!} = \sum_{k=1}^{\infty} g(k)\frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$\lambda E(g(X+1)) = \lambda \sum_{k=0}^{\infty} g(k+1)\frac{e^{-\lambda}\lambda^{k+1}}{k!} = \sum_{k=1}^{\infty} g(k)\frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$E(Xg(X)) = \lambda E(g(X+1))$$

(b)

$$
\begin{aligned}
E(X^3) &= E(X \cdot X^2) = \lambda E((X+1)^2) \\
\mathrm{Var}(X+1) &= E((X+1)^2) - E^2(X+1) \\
\lambda &= E((X+1)^2) - (\lambda+1)^2 \\
E(X^3) &= \lambda[\lambda + (\lambda+1)^2] \\
&= \lambda^2 + \lambda(\lambda+1)^2 \\
&= \lambda^3 + 3\lambda^2 + \lambda
\end{aligned}
$$

$$
\begin{aligned}
E(X^4) &= \lambda E[(X+1)^3] \\
&= \lambda[E(X^3) + E(3X) + E(3X^2) + E(1)] \\
&= \lambda(\lambda^3 + 3\lambda^2 + \lambda + 3(\lambda+\lambda^2) + 3\lambda + 1) \\
&= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda
\end{aligned}
$$

Problem 5

Let $p = q = \frac{1}{2}$ so that the expressions are close to those in class.
Define S_1: the result of the first toss.
Define $p_k = P(N = k)$
So we have $p_0 = p_1 = p_2 = p_3 = 0$, $p_4 = pqpq$.
For $k \geq 5$, we use the first-step method to analyze:

$$p_k = P(N = k) = P(N = k, S_1 = H) + P(N = k, S_1 = T)$$

We first deal with $P(N = K, S_1 = H)$:

$$\begin{aligned} P(N = k, S_1 = H) &= P(N = k, S_1 = H, S_2 = T) + P(N = k, S_1 = H, S_2 = H) \\ &= P(N = k, S_1 = H, S_2 = T, S_3 = H) + P(N = k, S_1 = H, S_2 = T, S_3 = T) + p \cdot P(N = k-1, S_2 = H) \\ &= p^2 q \cdot P(N = k-3, S_3 = H) + pq^2 \cdot P(N = k-3) + p \cdot P(N = k-1, S_2 = H) \end{aligned}$$

To compute the form $P(N = k, S_1 = H)$:

$$P(N = k) = qP(N = k-1) + P(N = k, S_1 = H)$$

$$P(N = k, S_1 = H) = P(N = k) - qP(N = k-1)$$

So we have:

$$p^2 q \cdot P(N = k-3, S_3 = H) = p^2 q \cdot [P(N = k-3) - qP(N = k-4)]$$

$$p \cdot P(N = k-1, S_2 = H) = p \cdot [P(N = k-1) - qP(N = k-2)]$$

Thus we obtain $P(N = K, S_1 = H)$:

$$P(N = K, S_1 = H) = p^2 q \cdot P(N = k-3) - p^2 q^2 P(N = k-4) + pq^2 \cdot P(N = k-3) + p \cdot P(N = k-1) - pq \cdot (N = k-2)$$

Also, we know $P(N = K, S_1 = T) = qP(N = k-1)$. So we obtain:

$$P(N = k) = p_{k-1} - pq \cdot p_{k-2} + pq \cdot p_{k-3} - p^2 q^2 \cdot p_{k-4}$$

According to Generating Function:

$$g(t) = p^2 q^2 t^4 + \sum_{k=5}^{\infty} p_k t^k$$

$$g(t) - p^2 q^2 t^4 = (t - pqt^2 + pqt^3 - p^2 q^2 t^4) g(t)$$

$$g(t) = \frac{p^2 q^2 t^4}{1 - t + pqt^2 - pqt^3 + p^2 q^2 t^4}$$

Thus:

$$E(N) = g'(t)\big|_{t=1} = \frac{1}{p^2 q^2} + \frac{1}{pq} = 20$$

$$Var(N) = g''(1) + g'(1) - [g'(1)]^2 = 276$$