Probability & Statistics for EECS: Homework #07

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Problem 1

$$(1) P(Y=y|X=x) = \frac{P(Y=y, X=x)}{P(X=x)} = \frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$$

$$\begin{aligned}(2) f_Y(y|X=x) &= \frac{P(y \in (y-\epsilon, y+\epsilon)|X=x)}{2\epsilon} \\ &= \frac{P(X=x|y \in (y-\epsilon, y+\epsilon))P(y \in (y-\epsilon, y+\epsilon)|X=x)}{2\epsilon P(x=x)} \\ &= \frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}\end{aligned}$$

$$\begin{aligned}(3) P(Y=y|X=x) &= \frac{P(Y=y, X \in (x-\epsilon, x+\epsilon))}{P(X=x)} \\ &= \frac{P(X \in (x-\epsilon, x+\epsilon)|Y=y)P(Y=y)}{P(X=x)} \\ &= \frac{P(X \in (x-\epsilon, x+\epsilon))}{2\epsilon}\frac{P(Y=y)}{P(X \in (x-\epsilon, x+\epsilon))} \\ &= \frac{f_X(X|Y=y)P(Y=y)}{f_X(x)}\end{aligned}$$

$$(4) f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)f_Y(y)}{f_X(x)} = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

Problem 2

(a) Since $N = X + Y$, they are dependent.

Conditioning on $N = n$, we know that $X|_{N=n} \sim \text{Bin}(n, p)$, $Y|_{N=n} \sim \text{Bin}(n, 1-p)$.

For $i, j \geq 0$,

$$\begin{aligned}P(N=n, X=i, Y=j) &= P(X=i, Y=j) \\ &= \sum_{n=0}^{\infty} P(X=i, Y=j|N=n)P(N=n) \\ &= P(X=i, Y=j|N=i+j)P(N=i+j) \\ &= P(X=i|N=i+j)P(Y=j|X=i, N=i+j)P(N=i+j) \\ &= P(X=i|N=i+j)P(N=i+j) \\ &= \binom{i+j}{i}p^i(1-p)^j \cdot \frac{e^{-\lambda}\lambda^{i+j}}{(i+j)!} \\ &= \frac{e^{-\lambda p}(\lambda p)^i}{i!} \cdot \frac{e^{-\lambda(1-p)}(\lambda(1-p))^j}{j!} \\ &= e^{-\lambda}\frac{(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}\end{aligned}$$

(b) Since $N = X + Y$, they are dependent.

According to (a), $P(N=n, X=i)$ is $P(X=i, Y=n-i)$ which simplifies to the given expression.

(c) Since N is indeterminate, X, Y are independent.

According to (a), $P(X = i, Y = j)$ is given by the product of their individual probabilities, leading to X being distributed as $\text{Pois}(\lambda p)$ and similarly Y as $\text{Pois}(\lambda(1-p))$.

(d) For the covariance and correlation calculations, the following equations are used:

$$\begin{aligned}
\text{Cov}(N, X) &= \text{Cov}(X + Y, X) \\
&= \text{Cov}(X, X) + \text{Cov}(Y, X) \\
&= \text{Var}(X) + \text{Cov}(Y, X) \\
&= \lambda p \\
\text{Corr}(N, X) &= \frac{\text{Cov}(N, X)}{\sqrt{\text{Var}(N)\text{Var}(X)}} \\
&= \sqrt{p}
\end{aligned}$$

Problem 3

(a) Since $X \sim \text{Expo}(\lambda)$, $Y \sim \text{Expo}(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}, x \geq 0$; $f_Y(y) = \lambda e^{-\lambda y}, y > 0$. Suppose $T \leq t$,

When $t \leq x$, $P(T \leq t|X = x) = 0$;

When $t > x$, $P(T \leq t|X = x) = P(X + Y \leq t|X = x) = P(Y \leq t - x) = F_Y(t - x) = 1 - e^{-\lambda(t-x)}$.

Therefore, the cumulative distribution function (CDF) is given by

$$F_{T|X}(t|x) = \begin{cases} 0, & t \leq x \\ 1 - e^{-\lambda(t-x)}, & t > x \end{cases}$$

(b) The probability density function (PDF) is the derivative of the CDF with respect to t:

$$f_{T|X}(t|x) = \frac{\partial}{\partial t}(F_{T|X}(t|x)) = \begin{cases} 0, & t \leq x \\ \lambda e^{-\lambda(t-x)}, & t > x \end{cases}$$

Therefore, $f_{T|X}(t|x) \geq 0$.

The integral of the PDF over all t is 1, confirming that it is a valid PDF:

$$\int_{-\infty}^{+\infty} f_{T|X}(t|x)dt = \int_{x}^{+\infty} \lambda e^{-\lambda(t-x)}dt = -e^{-\lambda(t-x)}\Big|_{t=x}^{+\infty} = 1.$$

Therefore, it is a valid PDF.

(c) Using Bayes' rule we have that

$$\begin{aligned}
f_{X|T}(x|t) &= \frac{f(x,t)}{f_T(t)} = \frac{f_{T|X}(t|x)f_X(x)}{f_T(t)} \\
&= \frac{\alpha e^{-\lambda(t-x)}e^{-\lambda x} \cdot \lambda x}{f_T(t)} \\
&= \alpha\lambda^2 e^{-\lambda t} \cdot xt^{2x}
\end{aligned}$$

for some $\alpha > 0$. Observe that $f_{X|T}(x|t)$ is a constant function respective to x. In order to be a valid PDF, $f_{X|T}(x|t)$ has to satisfy following

$$1 = \int_{\mathbb{R}} f_{X|T}(x|t)dx = \int_0^t \alpha\lambda^2 e^{-\lambda t}dx = t\alpha\lambda^2 e^{-\lambda t}$$

So, for every $t > 0$ there has to be

$$\alpha = \frac{1}{t\lambda^2 e^{-\lambda t}}$$

and in this case it is a valid PDF.

(d) Observe that in part (c) we have that in fact $f_T(t) = \frac{1}{\alpha}$. So, we can easily obtain that

$$f_T(t) = \lambda^2 t e^{-\lambda t}$$

Problem 4

(a) Observe following for $m \in (0, 1)$

$$F_M(m) = P(M \leq m) = P(U_1 \leq m, \dots, U_3 \leq m) = P(U_1 \leq m) \cdots P(U_3 \leq m) = m^3$$

so the marginal PDF of M is

$$f_M(m) = \frac{d}{dm} F_M(m) = 3m^2$$

Now, let's find joint CDF of L and M. We have that

$$P(L \geq l, M \leq m) = P(U_i \in [l, m], \forall i) = (m - l)^3$$

for $l \leq m$. Using the LOTP, we have that

$$P(M \leq m) = P(M \leq m, L \leq l) + P(M \leq m, L > l)$$

so use that to obtain

$$F(m, l) = P(M \leq m, L \leq l) = m^3 - (m - l)^3$$

for $l \leq m$. Finally, the joint PDF is

$$f(l, m) = \frac{\partial^2}{\partial m \partial l} F(m, l) = \frac{\partial^2}{\partial m \partial l}(m^3 - (m - l)^3) = 6(m - l)$$

(b) Similarly as in (a) we can get that the CDF of L is $F_L(l) = (1 - l)^3$, so the PDF of L is $f_L(l) = 3(1 - l)^2$. So, using the definition of conditional PDF we get that

$$f_{M|L}(m|l) = \frac{f(m, l)}{f_L(l)} = \frac{6(m - l)}{3(1 - l)^2} = \frac{2(m - l)}{(1 - l)^2}$$

Problem 5

(a)

$$
\begin{aligned}
\bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \frac{1}{n}\sum_{i=1}^{n} y_i \\
r &= \frac{1}{n}\sum_{i=1}^{n}[x_i - E(X)][y_i - E(Y)] \\
&= E[(X - E(X))(Y - E(Y))] \\
&= \mathrm{Cov}(X, Y)
\end{aligned}
$$

(b)

$$
\begin{aligned}
E((X-\bar{X})(Y-\bar{Y})) &= E(XY) - E(\bar{X}Y) - E(X\bar{Y}) + E(\bar{X}\bar{Y}) \\
E(XY) &= \mathrm{Cov}(X,Y) + E(X)E(Y) \\
E(\bar{X}Y) &= \mathrm{Cov}(\bar{X},Y) + E(\bar{X})E(Y) = E(X)E(Y) \\
E(X\bar{Y}) &= \mathrm{Cov}(X\bar{Y}) + E(X)E(\bar{Y}) = E(X)E(Y) \\
E(\bar{X}\bar{Y}) &= \mathrm{Cov}(\bar{X},\bar{Y}) + E(\bar{X})E(\bar{Y}) \\
(X,Y) &\text{ is independent of } (\bar{X},\bar{Y}) \\
E(X) &= E(\bar{X}), \quad E(Y) = E(\bar{Y})
\end{aligned}
$$

$$
\begin{aligned}
E((X-\bar{X})(Y-\bar{Y})) &= 2\mathrm{Cov}(X,Y) \\
S = n^2(X_i - \bar{X})(Y_i - \bar{Y}) &= 2n^2\mathrm{Cov}(x,y) \\
\mathrm{Cov}(X,Y) &= \frac{S}{2n^2}
\end{aligned}
$$

(c)

1. After the horizontal and vertical coordinates are exchanged, the area will not change. If the area is unchanged, the covariance will not change.
2. Enlarge the base vectors of the horizontal and vertical coordinates by a_1 and a_2 times respectively, and the area of the rectangle will become a_1a_2 times of the original one, so the covariance will also become a_1a_2 times of the original one.
3. After moving the origin of the coordinate system to the left a_1 and down a_2, the area will not change. Therefore, the covariance will not change.
4. The area of a rectangle with a length of W_1 and a height of $W_2 + W_3$ is equal to the sum of the areas of two rectangles with a length of W_1 and a width of W_2 and W_3 respectively, therefore $\mathrm{Cov}(W_1, W_2 + W_3) = \mathrm{Cov}(W_1, W_2) + \mathrm{Cov}(W_1, W_3)$.