2024Spring Probability & Statistics for EECS 2024/04/29

Homework 8

Professor: Ziyu Shao *Due: 2024/05/05 10:59pm*

1. Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of constant c.

   (b) Find the conditional probability $P(Y \le \frac{X}{4} \big| Y \le \frac{X}{2})$.

2. Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6 \cdot 2^{\min(x,y)}} & \text{if } x, y \ge 0, |x-y| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal distributions of X and Y.

   (b) Are X and Y independent?

   (c) Find $P(X = Y)$.

3. Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign 1 or -1, with equal probabilities) independent of (X, Y).

   (a) Determine whether or not $(X, Y, X + Y)$ is Multivariate Normal.

   (b) Determine whether or not $(X, Y, SX + SY)$ is Multivariate Normal.

   (c) Determine whether or not (SX, SY) is Multivariate Normal.

4. Let Z_1, Z_2 be two *i.i.d.* random variables satisfying standard normal distributions, *i.e.*, $Z_1, Z_2 \sim \mathcal{N}(0,1)$. Define

$$X = \sigma_X Z_1 + \mu_X;$$
$$Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y,$$

   where $\sigma_X > 0$, $\sigma_Y > 0$, $-1 < \rho < 1$.

   (a) Show that X and Y are bivariate normal.

   (b) Find the correlation coefficient between X and Y, *i.e.*, $\text{Corr}(X, Y)$.

(c) Find the joint PDF of X and Y.

5. (a) Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and $Z = \frac{X}{Y}$. Find the PDF of Z.

   (b) Let X and Y be i.i.d. Unif$(0,1)$, $W = X \cdot Y$, and $Z = \frac{X}{Y}$. Find the joint PDF of (W, Z).

   (c) A point (X, Y) is picked at random uniformly in the unit circle. Find the joint PDF of R and X, where $R = \sqrt{X^2 + Y^2}$.

   (d) A point (X, Y, Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint PDF of Z and R, where $R = \sqrt{X^2 + Y^2 + Z^2}$.

6. (Optional Challenging Problem) Let X and Y be i.i.d. Unif$(0,1)$, and $Z = \frac{X}{Y}$. Find the probability that the integer close to Z is odd.