**2024Spring Probability & Statistics for EECS** 2024/3/18

# Homework 3

*Professor: Ziyu Shao* *Due: 2024/3/24 10:59pm*

1. Please reinterpret the following story from the Bayesian perspective.

狼来了：从前有个放羊娃，每天都把羊群带到山上去吃草，山里有狼出没. 第一天，放羊娃觉得无聊，想要作弄山下耕作的村民. 他朝着山下大喊“狼来了!狼来了”，村民们信以为真，冲上山来准备帮助他，发现被欺骗了，大家很生气. 第二天，放羊娃故技重施，村民们虽然有点迟疑，但还是冲上山来准备打狼，结果又一次 发现被欺骗了，大家非常生气. 第三天，狼真的来了，此时放羊娃慌了，哭着向山下大喊“狼来了!狼来了!”，请 求村民的帮助. 但这一次村民们认为他又在撒谎，无人相信他. 最后他所有的羊都被狼吃掉了.

2. A fair die is rolled repeatedly, and a running total is kept (which is, at each time, the total of all the rolls up until that time). Let $p_n$ be the probability that the running total is ever exactly $n$ (assume the die will always be rolled enough times so that the running total will eventually exceed $n$, but it may or may not ever equal $n$).

   (a) Write down a recursive equation for $p_n$ (relating $p_n$ to earlier terms $p_k$ in a simple way). Your equation should be true for all positive integers $n$, so give a definition of $p_0$ and $p_k$ for $k < 0$ so that the recursive equation is true for small values of $n$.

   (b) Find $p_7$.

   (c) Give an intuitive explanation for the fact that $p_n \to 1/3.5 = 2/7$ as $n \to \infty$.

3. A sequence of $n \geq 1$ independent trials is performed, where each trial ends in “success” or “failure” (but not both). Let $p_i$ be the probability of success in the $i^{th}$ trial, $q_i = 1 - p_i$, and $b_i = q_i - 1/2$, for $i = 1, 2, \ldots, n$. Let $A_n$ be the event that the number of successful trials is even.

   (a) Show that for $n = 2$, $P(A_2) = 1/2 + 2b_1 b_2$.

(b) Show by induction that

$$P(A_n) = 1/2 + 2^{n-1} b_1 b_2 \dots b_n$$

(This result is very useful in cryptography. Also, note that it implies that if $n$ coins are flipped, then the probability of an even number of Heads is $1/2$ if and only if at least one of the coins is fair.) *Hint*: Group some trials into a super-trial.

(c) Check directly that the result of (b) is true in the following simple cases: $p_i = 1/2$ for some $i$; $p_i = 0$ for all $i$; $p_i = 1$ for all $i$.

4. A message is sent over a noisy channel. The message is a sequence $x_1, x_2, \dots, x_n$ of $n$ bits ($x_i \in \{0, 1\}$). Since the channel is noisy, there is a chance that any bit might be corrupted, resulting in an error (a 0 becomes a 1 or vice versa). Assume that the error events are independent. Let $p$ be the probability that an individual bit has an error ($0 < p < 1/2$). Let $y_1, y_2, \dots, y_n$ be the received message (so $y_i = x_i$ if there is no error in that bit, but $y_i = 1 - x_i$ if there is an error there).

   To help detect errors, the $n$th bit is reserved for a parity check: $x_n$ is defined to be 0 if $x_1 + x_2 + \cdots + x_{n-1}$ is even, and 1 if $x_1 + x_2 + \cdots + x_{n-1}$ is odd. When the message is received, the recipient checks whether $y_n$ has the same parity as $y_1 + y_2 + \cdots + y_{n-1}$. If the parity is wrong, the recipient knows that at least one error occurred; otherwise, the recipient assumes that there were no errors.

   (a) For $n = 5, p = 0.1$, what is the probability that the received message has errors which go undetected?

   (b) For general $n$ and $p$, write down an expression (as a sum) for the probability that the received message has errors which go undetected.

   (c) Give a simplified expression, not involving a sum of a large number of terms, for the probability that the received message has errors which go undetected.

5. For $x$ and $y$ binary digits (0 or 1), let $x \bigoplus y$ be 0 if $x = y$ and 1 if $x \neq y$ (this operation is called exclusive or (often abbreviated to XOR), or addition mod 2).

   (a) Let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(1/2)$, independently. What is the distribution of $X \bigoplus Y$?

   (b) With notation as in sub-problem ($a$), is $X \bigoplus Y$ independent of $X$? Is $X \bigoplus Y$ independent of $Y$? Be sure to consider both the case $p = 1/2$ and the case $p \neq 1/2$.

   (c) Let $X_1, \dots, X_n$ be i.i.d. (*i.e.*, independent and identically distributed) Bern(1/2) R.V.s. For each nonempty subset $J$ of $\{1, 2, \dots, n\}$, let

   $$Y_J = \bigoplus_{j \in J} X_j.$$

   Show that $Y_J \sim \text{Bern}(1/2)$ and that these $2^n - 1$ R.V.s are pairwise independent, but not independent.

6. (**Optional Challenging Problem**) By LOTP for problems with recursive structure, we generate many difference equations.

   (a) Solve the following difference equation:

   $$p \cdot f_{i+1} - f_i + q \cdot f_{i-1} = -1, 1 \leq i \leq N - 1$$

   where $0 < p < 1$, $q = 1 - p$, $N$ is a constant, $f_0 = 0$, $f_N = 0$.

   (b) Solve the following difference equation:

   $$f_{i+1} = b \cdot f_i + a \cdot f_{i-1} + h, i \geq 1.$$

   where $h$ is a constant.

   (c) Solve the following difference equation:

   $$f_{i+1} = b \cdot f_i + a \cdot f_{i-1} + g(i), i \geq 1.$$

   where $g(i)$ is a function of $i$.