Probability & Statistics for EECS: Homework #06

Due on Apr 21, 2024 at 23:59

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Problem 1

Let X be a continuous random variable with a PDF f. In order to calculate the CDF of this random variable, we use the definition of CDF.

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f(\xi)\, d\xi = \int_{-\infty}^{x} \frac{1}{\pi(1+\xi^2)}\, d\xi = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{(1+\xi^2)}\, d\xi \\ &= \frac{1}{\pi}\left[\tan^{-1}(\xi)\right]_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1}(x) - \tan^{-1}\left(-\frac{\pi}{2}\right)\right] \\ &= \frac{\tan^{-1}(x)}{\pi} + \frac{1}{2} \end{aligned}$$

Problem 2

For $x \leq 1$ we have that

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u) du = \int_{-\infty}^{x} 0 du = 0$$

Now, for $x > 1$ we have

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u) du = \int_{1}^{x} \frac{a}{u^{a+1}} du = a\int_{1}^{x} u^{-a-1} du = a\left.\frac{u^{-a}}{-a}\right|_{1}^{x}$$

$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(1-x^{-a}) = 1$ and $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+}(1-x^{-a}) = 1-F$ is certainly continuous on $\mathbb{R}$ (eventual additional check is needed for $c = 1$ where we have $\lim_{x\to 1^-} F(x) = 0$ and $\lim_{x\to 1^+} F(x) = 1 - 1 = 0$). So F is a valid CDF.

Problem 3

(a)

$$P(X \leq x) = \int_{-\infty}^{x} f(s)\, ds = \int_{0}^{x} f(s)\, ds = \int_{0}^{x} (12s^2 - 12s^3)\, ds = 4x^3 - 3x^4$$

(b) Using part (a), we have following

$$P(0 < X < 1/2) = F(1/2) - F(0) = \frac{5}{16}$$

(c)

$$E(X) = \int_{\mathbb{R}} x f(x)\, dx = \int_{0}^{1} x(12x^2 - 12x^3)\, dx = \int_{0}^{1} 12x^3 - 12x^4\, dx = \frac{3}{5}$$

and

$$E(X^2) = \int_{\mathbb{R}} x^2 f(x)\, dx = \int_{0}^{1} x^2(12x^2 - 12x^3)\, dx = \int_{0}^{1} 12x^4 - 12x^5\, dx = \frac{2}{5}$$

Finally, we have

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = \frac{2}{5} - \left(\frac{3}{5}\right)^2 = \frac{2}{5} - \frac{9}{25} = \frac{1}{25}$$

Problem 4

(a) Suppose that there was G failures before first success. Each failure has taken Δt time to happen. At the time when Gth failure happen, we have already pasted $(G-1)\Delta t$ time. Then, there has to pass Δt time and then we have successful trial. Finally, we have that

$$T = (G-1)\Delta t + \Delta t = G\Delta t$$

(b) Use part (a) to obtain following

$$P(T > t) = P(G\Delta t > t) = P\left(G > \frac{t}{\Delta t}\right) = (1-\lambda\Delta t)^{\frac{t}{\Delta t}}$$

$$P(T \leq t) = 1 - P(T > t) = 1 - (1-\lambda\Delta t)^{\frac{t}{\Delta t}}$$

(c) Let's fix $t \geq 0$. Use the definition of exponential function to obtain that

$$\lim_{\Delta t \to 0} P(T \leq t) = \lim_{\Delta t \to 0} \left[1 - (1-\lambda\Delta t)^{\frac{t}{\Delta t}}\right]$$

$$= \lim_{\Delta t \to 0} \left[1 - \left((1 + \frac{-\lambda}{\frac{1}{\Delta t}})^{\frac{1}{\Delta t}}\right)^t\right] = 1 - e^{-\lambda t}$$

Observe that on limit we have got that CDF of T is exactly PDF of exponential distribution with parameter λ.

Problem 5

Observe that it can be written as following

$$\max(Z-c, 0) = (Z-c)\mathbf{x}_{\{Z-c>0\}} = (Z-c)\mathbf{x}_{\{Z>c\}}$$

So using LOTUS, we have following

$$E(\max(Z-c,0)) = \int_{\mathbb{R}} (z-c)\mathbf{x}_{\{z>c\}} f(z)\,dz = \int_c^\infty (z-c)f(z)\,dz$$

$$= \int_c^\infty zf(z)\,dz - \int_c^\infty cf(z)\,dz = I_1 - I_2$$

Let's calculate first integral. We have following

$$I_1 = \int_c^\infty \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz$$

$$= \int_{\frac{c^2}{2}}^\infty \frac{1}{\sqrt{2\pi}} e^{-u}\,du = \frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2}} = \varphi(c)$$

For the second integral, we have

$$I_2 = cP(Z > c) = c(1 - P(Z \leq c)) = c(1 - \Phi(z))$$

Finally, we have obtained that

$$E(\max(Z-c,0)) = \varphi(c) - c(1 - \Phi(z))$$