2024Spring Probability & Statistics for EECS 2024/04/01

Homework 5

Professor: Ziyu Shao *Due: 2024/04/07 10:59pm*

1. Nick and Penny are independently performing independent Bernoulli trials. For concreteness, assume that Nick is flipping a nickel with probability p_1 of Heads and Penny is flipping a penny with probability p_2 of Heads. Let $X_1, X_2, \cdots$ be Nick's results and $Y_1, Y_2, \cdots$ be Penny's results, with $X_i \sim \text{Bern}(p_1)$ and $Y_j \sim \text{Bern}(p_2)$.

   (a) Find the distribution and expected value of the first time at which they are simultaneously successful, *i.e.*, the smallest n such that $X_n = Y_n = 1$.

   Hint: Define a new sequence of Bernoulli trials and use the story of the Geometric.

   (b) Find the expected time until at least one has a success (including the success).

   Hint: Define a new sequence of Bernoulli trials and use the story of the Geometric.

   (c) For $p_1 = p_2$, find the probability that their first successes are simultaneous, and use this to find the probability that Nick's first success precedes Penny's.

2. A building has n floors, labeled $1, 2, ..., n$. At the first floor, k people enter the elevator, which is going up and is empty before they enter. Independently, each decides which of floors $2, 3, ..., n$ to go to and presses that button (unless someone has already pressed it).

   (a) Assume for this part only that the probabilities for floors $2, 3, ..., n$ are equal. Find the expected number of stops the elevator makes on floors $2, 3, ..., n$.

   (b) Generalize (a) to the case that floors $2, 3, ..., n$ have probabilities $p_2, ..., p_n$ (respectively); you can leave your answer as a finite sum.

3. Given a random variable $X \sim \text{Pois}(\lambda)$ where $\lambda > 0$, show that for any non-negative integer k, we have the following identity:

$$E\left[\binom{X}{k}\right] = \frac{\lambda^k}{k!}.$$

4. (a) Use LOTUS to show that for $X \sim \text{Pois}(\lambda)$ and any function g,

$$E\left(Xg(X)\right) = \lambda E\left(g(X+1)\right).$$

   This is called the *Stein-Chen identity* for the Poisson.

(b) Find the moment $E(X^4)$ for $X \sim \text{Pois}(\lambda)$ by using the identity from (a) with the fact that X has mean λ and variance λ.

5. Suppose a fair coin is tossed repeatedly, and we obtain a sequence of H and T (H denotes Head and T denotes Tail). Let N denote the number of tosses to observe the first occurrence of the pattern "HTHT". Find $E(N)$ and $Var(N)$.

6. (**Optional Challenging Problem**) An Erdos-Renyi random graph is formed on n vertices. Each unordered pair (edge) (i, j) of vertices is connected with probability p, independently of all the other pairs.

   (a) A wedge (or path of length 2) is a tuple $(i, \{j, k\})$ where i, j, k are distinct and each of the edges (i, j) and (i, k) is connected. Let W denote the number of wedges contained in the random graph. Find appropriate condition under which W is approximately Poisson distributed.

   (b) A triangle is a set of three vertices $\{i, j, k\}$ such that each of the three edges (i, j), (j, k) and (i, k) is connected. Let T denote the number of triangles contained in the random graph. Find appropriate condition under which T is approximately Poisson distributed.