Probability & Statistics for EECS: Homework #08

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Problem 1

(a)

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\,dx = \int_0^1 \int_0^x cx^2y\,dy\,dx = \int_0^1 \frac{cx^4}{2}\,dx = \frac{c}{10}$$

So $c = 10$.

(b)

$$P\left(Y \le \frac{X}{4} \mid Y \le \frac{X}{2}\right) = \frac{P(Y \le \frac{X}{4}, Y \le \frac{X}{2})}{P(Y \le \frac{X}{2})} = \frac{P(Y \le \frac{X}{4})}{P(Y \le \frac{X}{2})} = \frac{\int_0^1 \int_0^{\frac{x}{4}} 10x^2y\,dy\,dx}{\int_0^1 \int_0^{\frac{x}{2}} 10x^2y\,dy\,dx} = \frac{1}{4}$$

Problem 2

(a) The marginal distributions of X is

$$P_X(X) = \sum_{y=0}^{\infty} P_{X,Y}(X,Y).$$

$X = 0$:

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{3}.$$

$X \neq 0$:

$$P(X=x) = P(X=x, Y=x-1) + P(X=x, Y=x) + P(X=x, Y=x+1) = \frac{1}{6 \cdot 2^{x-2}}$$

The marginal distribution of X is

$$P_X(X) = \begin{cases} \frac{1}{3}, & x = 0 \\ \frac{1}{6 \cdot 2^{x-2}}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

According to the symmetric property:

$$P_Y(Y) = \begin{cases} \frac{1}{3}, & y = 0 \\ \frac{1}{6 \cdot 2^{x-2}}, & y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$P_{X,Y}(0,0) = \frac{1}{6}, P(X=0)P(Y=0) = \frac{1}{9}$$

So X and Y are not independent.

(c) Since symmetric, we have $P(X = Y) = P(X = Y - 1) = P(X = Y + 1)$ and $P(X = Y) + P(X = Y - 1) + P(X = Y + 1) = 1$. So

$$P(X = Y) = \frac{1}{3}.$$

Problem 3

(a) Yes, this random vector is multivariate Normal because for any $a, b, c \in \mathbb{R}$ we have that

$$aX + bY + c(X + Y) = (a + c)X + (b + c)Y$$

and any linear combination of independent normally distributed variables is again Normal.

(b) Let's consider random variable $Z = X + Y + SX + SY = (1 + S)X + (1 + S)Y$. Observe that event $Z = 0$ is in fact event $S = -1$. Hence, we have that

$$P(Z = 0) = P(S = -1) = \frac{1}{2}$$

On the other hand, none of the normally distributed random variable has probability $\frac{1}{2}$ in zero. Hence, Z is not normally distributed. So, we have found one linear combination that is not Normal. Thus, given random vector is not multivariate Normal.

(c) Observe that random vector (X, Y) is identically distributed as $(-X, -Y)$. So, take any $C \in \mathbb{R}$. We have following

$$\begin{aligned}
P((SX, SY) \in C) &= P((X, Y) \in C, S = 1) + P((-X, -Y) \in C, S = -1) \\
&= P((X, Y) \in C \mid S = 1)P(S = 1) + P((-X, -Y) \in C \mid S = -1)P(S = -1) \\
&= \frac{1}{2}P((X, Y) \in C) + \frac{1}{2}P((-X, -Y) \in C) = P((X, Y) \in C)
\end{aligned}$$

So, we obtained that (SX, SY) is equally distributed as (X, Y). Also, we know that (X, Y) is Bivariate normal. Hence, so is (SX, SY).

Problem 4

(a) For $a, b \in \mathbb{R}$, we have

$$aX + bY = (a\Sigma_X + b\Sigma_Y\rho)Z_1 + b\sqrt{1-\rho^2}\Sigma_Y Z_2 + a\mu_X + b\mu_Y.$$

Since the linear combination of two Normal distribution follows Normal distribution, X and Y are bivariate normal.

(b) Since $Z_1, Z_2 \sim N(0, 1)$, we have $Z_1 + \sqrt{1-\rho^2}Z_2 \sim N(0, 1)$. So $X \sim N(\mu_X, \Sigma_X)$, $Y \sim N(\mu_Y, \Sigma_Y)$. Thus, we have

$$\text{Cov}(X, Y) = \text{Cov}(\Sigma_X Z_1 + \mu_X, \Sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y) = \Sigma_X\Sigma_Y\text{Cov}(Z_1, \rho Z_1 + \sqrt{1-\rho^2}Z_2) = \Sigma_X\Sigma_Y\rho.$$

Then correlation coefficient between X and Y is

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\Sigma_X\Sigma_Y\rho}{\Sigma_X\Sigma_Y} = \rho.$$

(c) Since Z_1 and Z_2 are i.i.d., we have

$$f_{Z_1,Z_2}(z_1, z_2) = f_{Z_1}(z_1)f_{Z_2}(z_2) = \frac{1}{2\pi}e^{-\frac{z_1^2+z_2^2}{2}}.$$

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Since $X = \Sigma_X Z_1 + \mu_X$, $Y = \Sigma_Y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_Y$, we have

$$Z_1 = \frac{X-\mu_X}{\Sigma_X}, \quad Z_2 = \frac{Y-\mu_Y-\rho\Sigma_Y Z_1}{\sqrt{1-\rho^2}\Sigma_Y} = \frac{Y-\mu_Y}{\sqrt{1-\rho^2}\Sigma_Y} - \frac{\rho(X-\mu_X)}{\sqrt{1-\rho^2}\Sigma_X}.$$

$$f_{X,Y}(x,y) = \left|\frac{\partial(Z_1,Z_2)}{\partial(X,Y)}\right| f_{Z_1,Z_2}(z_1,z_2)$$

$$= \frac{1}{2\pi\Sigma_X\Sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\Sigma_X}\right)^2 + \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} - \left(\frac{y-\mu_Y}{\Sigma_Y}\right)^2\right]\right).$$

Problem 5

(a) $x = zy$, so

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty} f_X(zy) f_Y(y) y\,dy \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2(z^2+1)}\,dy^2 \\
&= \frac{1}{\pi(1+z^2)}.
\end{aligned}$$

(b) Given that X and Y are independent and uniformly distributed over $[0,1]$, the joint PDF of $W = XY$ and $Z = \frac{X}{Y}$ is derived as follows:

$$f_{W,Z}(w,z) = f_{X,Y}(x,y)\left|\frac{1}{\frac{\partial(w,z)}{\partial(x,y)}}\right|$$

Where the Jacobian determinant $\left|\frac{\partial(w,z)}{\partial(x,y)}\right|$ is calculated by:

$$\left|\frac{\partial(w,z)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{2x}{y}$$

$$f_{W,Z}(w,z) = \frac{y}{2x}$$

(c)

$$f_{X,R}(x,r) = f_{X|R}(x|r) f_R(r)$$

We know from the area that the CDF of R is

$$F_R(r) = r^2$$

So

$$f_R(r) = \frac{d}{dr}F_R(r) = 2r$$

We know that $X = R\cos\Theta$, where $\Theta \sim \text{Unif}(0, 2\pi)$. Thus $f_\Theta(\theta) = \frac{1}{2\pi}$.

Problem 5 continued on next page...

Thus,

$$f_{X|R}(x|r) = f_{\Theta|R}(\theta|r)\left|\frac{dx}{d\theta}\right| = \frac{1}{2\pi|r\sin\theta|} = \frac{1}{2\pi\sqrt{r^2-x^2}}$$

Thus,

$$f_{X,R}(x,r) = \frac{1}{2\pi\sqrt{r^2-x^2}}\cdot 2r = \frac{r}{\pi\sqrt{r^2-x^2}} \quad \text{for} \quad |x|<r$$

Thus

$$f_{X,R}(x,r) = \begin{cases} \frac{r}{\pi\sqrt{r^2-x^2}} & \text{if } |x|<r \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$F_R(r) = \frac{4\pi r^3}{3}$$

Thus,

$$f_R(r) = \frac{d}{dr}F_R(r) = 3r^2$$

Let $x = r\sin\theta\cos\alpha$, $y = r\sin\theta\sin\alpha$, $z = r\cos\theta$, so

$$f_{Z|R}(z|r) = \frac{1}{2\pi\sqrt{r^2-z^2}}$$

Thus,

$$f_{Z,R}(z,r) = \frac{1}{2\pi\sqrt{r^2-z^2}}\cdot 3r^2 = \frac{3r^2}{2\pi\sqrt{r^2-z^2}}$$

So,

$$f_{Z,R}(z,r) = \begin{cases} \frac{3r^2}{2\pi\sqrt{r^2-z^2}}, & \text{if } |z|<r \\ 0, & \text{otherwise} \end{cases}$$