**2024Spring Probability & Statistics for EECS** 2024/04/14

# Homework 6

*Professor: Ziyu Shao* *Due: 2024/04/21 10:59pm*

1. The Cauchy distribution has PDF

$$f(x) = \frac{1}{\pi\left(1+x^2\right)}$$

for all $x$. Find the CDF of a random variable with the Cauchy PDF. Hint: Recall that the derivative of the inverse tangent function $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

2. The Pareto distribution with parameter $a > 0$ has PDF

$$f(x) = \frac{a}{x^{a+1}}$$

for $x \geq 1$ (and 0 otherwise). This distribution is often used in statistical modeling. Find the CDF of a Pareto r.v. with parameter $a$; check that it is a valid CDF.

3. The *Beta distribution* with parameters $a = 3, b = 2$ has PDF

$$f(x) = 12x^2(1-x), \text{ for } 0 < x < 1.$$

Let $X$ have this distribution.

(a) Find the CDF of $X$.

(b) Find $P(0 < X < 1/2)$.

(c) Find the mean and variance of $X$ (without quoting results about the Beta distribution).

4. The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times 0, $\Delta t$, $2\Delta t$, ..., where $\Delta t$ is a small positive number. Let the probability of success of each trial be $\lambda\Delta t$, where $\lambda$ is a positive constant. Let $G$ be the number of failures before the first success (in discrete time), and $T$ be the time of the first success (in continuous time).

(a) Find a simple equation relating $G$ to $T$. Hint: Draw a timeline and try out a simple example.

(b) Find the CDF of $T$. Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of $T$ converges to the Expo($\lambda$) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

5. Let $Z \sim \mathcal{N}(0, 1)$, and $c$ be a nonnegative constant. Find $E[\max(Z - c, 0)]$, in terms of the standard Normal CDF $\Phi$ and PDF $\varphi$.

6. (Optional Challenging Problem) Let $X \sim \mathcal{N}(0, 1)$, its corresponding CDF is denoted as $\Phi$ and the corresponding PDF is denoted as $\varphi$.

   (a) If $x > 0$, show the following inequality holds:

$$\frac{x}{x^2+1}\varphi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\varphi(x)$$

   (b) Define the function $g(x)$ as follows:

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt, \ \forall x \geq 0.$$

   Show the following inequality holds:

$$g(x) \leq e^{-x^2}, \ \forall x \geq 0.$$