Probability & Statistics for EECS: Homework #11

Due on May 26, 2023 at 23:59

Name: **Fei Pang**
Student ID: 2022533153

Problem 1

(a)

$$\begin{aligned} E[(Y - E[Y|X]) \cdot \phi(X)] &= E[Y\phi(X) - \phi(X)E[Y|X]] \\ &= E[Y\phi(X)] - E[\phi(X)E[Y|X]] \\ &= E[Y\phi(X)] - E[E[\phi(X) \cdot Y|X]] \\ &= E[Y\phi(X)] - E[\phi(X) \cdot Y] \\ &= 0 \end{aligned}$$

(b)

$$E[(Y - g(X))^2] = E[(Y - E(Y|X))^2] + E[(E(Y|X) - g(X))^2]$$

$$E[(Y - E(Y|X))^2] = E[(Y - g(X))^2] + E[(g(X) - E(Y|X))^2]$$

We also know:

$$E[(Y - E(Y|X))^2] \leq E[(Y - g(X))^2]$$

Therefore, we have:

$$E[(Y - g(X))^2] + E[(g(X) - E(Y|X))^2] \leq [(Y - g(X))^2]$$

$$E[(g(X) - E(Y|X))^2] \leq 0$$

$$g(X) = E(Y|X)$$

Problem 2

(a) assume that $X = 1$,$Y \sim Bern(\frac{1}{2}) + 1$

$$E(\frac{X}{X+Y}) = \frac{1}{1+1} \times \frac{1}{2} + \frac{1}{1+2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$\frac{E(X)}{E(X+Y)} = \frac{1}{1+\frac{3}{2}} = \frac{2}{5} \neq \frac{5}{12}$$

(b)

$$E(\frac{X}{X+Y}) = E(1 - \frac{Y}{X+Y}) = 1 - E(\frac{Y}{X+Y})E(\frac{X}{X+Y}) = E(\frac{Y}{X+Y}) = \frac{1}{2}$$

$$\frac{E(X)}{E(X+Y)} = \frac{E(X)}{E(X)+E(Y)} = \frac{1}{2} = E(\frac{X}{X+Y})$$

So it is necessary.

(c) $X + Y \sim Gamma(a + b, \lambda)$

$\frac{X}{X+Y} \sim Beta(a, b)$

$T = X + Y, W = \frac{X}{X+Y}$are independent

$\Rightarrow T^C = (X+Y)^C, W^C = (\frac{X}{X+Y})^C$are independent

$$E(T^C W^C) = E(T^C)E(W^C)$$

$$E(\frac{X^C}{(X+Y)^C}) = E(W^C) = \frac{E(T^C W^C)}{E(T^C)} = \frac{E(X^C)}{E[(X+Y)^C]}$$

Problem 3

(a) $I_j \sim Bern(p_1^2 p_2^2 p_3^2)$

$E(X) = \sum_{j=1}^{110} I_j = 110 \cdot p_1^2 p_2^2 p_3^2$

(b) whether generte T or G dosen't matter the result, so

$$P(A|not\ T\ or\ G) = \frac{p_1}{p_1 + p_2}, P(C|not\ T\ or\ G) = \frac{p_2}{p_1 + p_2}$$

so $P(A\ earlier\ than\ C) = \frac{p_1}{p_1+p_2}$

(c) $P_2 \sim Beta(1+1, 1+2) = Beta(2,3)$

$$E(p_2) = \frac{2}{2+3} = \frac{2}{5}$$

Problem 4

(a)

$$\begin{aligned} E(W_{HT}) &= E(W_1 + W_2) \\ W_1 &\sim FS(p) \\ W_2 &\sim FS(1-p) \\ E(W_{HT}) &= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)} \end{aligned}$$

(b)

$$\begin{aligned} E(W_{HH}) &= E(W_{HH}|O_1 = H)P(O_1 = H) + E(W_{HH}|O_1 = T)P(O_1 = T) \\ E(W_{HH}|O_1 = H) &= E(W_{HH}|O_1 = H, O_2 = H)P(O_2 = H) + E(W_{HH}|O_1 = H, O_2 = T)P(O_2 = T) \end{aligned}$$

$$\begin{aligned} E(W_{HH}|O_1 = T) &= E(W_{HH}) + 1 \\ E(W_{HH}|O_1 = H, O_2 = T) &= E(W_{HH}) + 2 \end{aligned}$$

$$\begin{aligned} E(W_{HH}) &= [2 + (1-p)E(W_{HH})]p + [E(W_{HH}) + 1](1-p) \\ &= 2p - p^2 E(W_{HH}) + E(W_{HH}) + 1 - p \\ p^2 E(W_{HH}) &= 1 + p \\ E(W_{HH}) &= \frac{1+p}{p^2} \end{aligned}$$

Problem 4 continued on next page...

(c)

$$
\begin{aligned}
E(\frac{1}{p}) &= \frac{\beta(a-1,b)}{\beta(a,b)} = \frac{a+b-1}{a-1} \\
E(\frac{1}{1-p}) &= \frac{\beta(a,b-1)}{\beta(a,b)} = \frac{a+b-1}{b-1} \\
E(\frac{1}{p^2}) &= \frac{\beta(a-2,b)}{\beta(a,b)} = \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)} \\
E(W_{HT}) &= \frac{a+b-1}{a-1} + \frac{a+b-1}{b-1} \\
E(W_{HH}) &= \frac{(a+b-1)(a+b-2)}{(a-1)(a-2)} + \frac{a+b-1}{a-1}
\end{aligned}
$$

Problem 5

(a)

$$
\begin{aligned}
\beta(a,b) &= \int_1^1 p^{a-1}(1-p)^{b-1}dp \\
E[p^2(1-p)^2] &= \frac{\int_0^1 p^2(1-p)^2 p^{a-1}(1-p)^{b-1}dp}{\int_0^1 p^{a-1}(1-p)^{b-1}dp} \\
&= \frac{\beta(a+2,b+2)}{\beta(a,b)} \\
&= \frac{\frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(a+b+4)}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \\
&= \frac{(a+1)(b+1)ab}{(a+b)(a+b+1)(a+b+2)(a+b+3)}
\end{aligned}
$$

(b) The posterior distribution of p changes:

$$P \sim Beta(a,b) \Rightarrow P \sim Beta(a+k, b+n-k)$$

It has nothing to do with the order.

(c) Let X be number of games that team A wins. We have that conditional PMF of X given p is:

$$P(X=k|p) = \binom{5}{k} p^k (1-p)^{5-k}$$

Using the Bayers theorem, we have that:

$$
\begin{aligned}
f_p(p|X=k) &= \frac{P(X=k|p)f_p(p)}{P(X=k)} \\
= \frac{1}{P(X=k)} \binom{5}{k} p^k (1-p)^{5-k} &\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1} \\
&= c \cdot p^{a+k-1}(1-p)^{b+5-k-1}
\end{aligned}
$$

where c is a constant. Thus, $p|X=k \sim Beta(a+k, b+5-k)$
Given that $p \sim Unif(0,1)$, we have that $a=b=1$, $p|X=k \sim Beta(1+k, 6-k)$.

Problem 5 continued on next page...

(d) As we have commented before, if we fix some probability p, the event that team A win the first game is independent of the event that team A wins the second game. But, this is not the case with the historical data since with the win (loss) of team A in the next game, we give more (less) probability to team A to win in the second game. Therefore,they are positively correlated with.

(e) Let Y be a random variable that marks the number of wins of team A in first four games. We know the distribution of Y given p. We have that:

$$P(Y = y|p) = \binom{4}{y} p^y (1-p)^{4-y}$$

There is no decision in the match if and only if $Y = 2$:

$$\begin{aligned} E(P(Y = 2|p)) &= E(\binom{4}{2} p^2 (1-p)^2) \\ &= 6E(p^2(1-p)^2) \\ &= \frac{1}{5} \end{aligned}$$

where $a = b = 1$ since $p \sim Unif(0, 1)$.