Probability & Statistics for EECS: Homework #03

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Problem 1

B: The villagers believe the boy.
A: The ith time that the boy said "A wolf comes".
Suppose $P(B) = p_1, P(A_i|B) = p_2, P(A_i|B^c) = p_3, p_3 > p_2$

$$\begin{aligned} P(B|A_1) &= \frac{P(A_1|B)P(B)}{P(A_1|B)P(B) + P(A_1|B^c)P(B^c)} \\ &= \frac{p_2 p_1}{p_2 p_1 + p_3(1-p_1)} \\ &= \frac{p_1}{p_1 + \frac{p_3}{p_2}(1-p_1)} \end{aligned}$$

Similar, we have:

$$\begin{aligned} P(B|A_1, A_2) &= \frac{p_1}{p_1 + (\frac{p_3}{p_2})^2(1-p_1)} \\ P(B|A_1, A_2, A_3) &= \frac{p_1}{p_1 + (\frac{p_3}{p_2})^3(1-p_1)} \end{aligned}$$

Let $\frac{p_3}{p_2} = x(x > 1)$:

$$P(B|A_1, A_2, \cdots, A_k) = \frac{p_1}{p_1 + (\frac{p_3}{p_2})^k(1-p_1)}$$

Thus, when $k \uparrow, P(B) \downarrow$.

Problem 2

(a) In previous throw we have to have a sum $n-6$ and then get number 6 or we have to have a sum $n-5$ and get number 5 and so on.
Each number is equally likely, so we can write

$$p_n = 1/6 \cdot p_{n-1} + 1/6 \cdot p_{n-2} + \cdots + 1/6 \cdot p_{n-6}$$

(b)

$$\begin{aligned} p_1 &= \frac{1}{6} \\ p_2 &= \frac{1}{6}\left(1 + \frac{1}{6}\right) \\ p_3 &= \frac{1}{6}\left(1 + \frac{1}{6}\right)^2 = \frac{1}{6}\left(1 + \frac{1}{6}\right)^2 \\ p_4 &= \frac{1}{6}\left(1 + \frac{1}{6}\right)^3 \\ &\vdots \\ p_7 &= \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 1) \\ &= \frac{1}{6}\left(\left(1 + \frac{1}{6}\right)^6 - 1\right) \end{aligned}$$

(c) In every throw, there in average falls $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$. So we can expect that for each number except that in seven throws that number will fall twice, therefore the required probability is $\frac{2}{7}$.

Problem 3

(a) S_i: The trail succeeds in the ith.

$$P(A_2) = P(S_1^c \cap S_2^c) + P(S_1 \cap S_2) = q_1 q_2 + (1-q_1)(1-q_2) = 2q_1q_2 - (q_1+q_2) + \frac{1}{2} + \frac{1}{2} = 2b_1b_2 + \frac{1}{2}$$

.

(b) It can be seen easily that:

$$\begin{aligned} P(A_{n+1}) &= P(A_n)(\frac{1}{2} + b_{n+1}) + (1 - P_{A_n})(\frac{1}{2} - b_{n+1}) \\ &= (2P_{A_n} - 1)b_{n+1} + \frac{1}{2} \end{aligned}$$

Since $P(A_n) = \frac{1}{2} + 2^{n-1} b_1 b_2 \cdots b_n$:

$$P(A_{n+1}) = \frac{1}{2} + 2^n b_1 b_2 \cdots b_n b_{n+1}$$

Thus the induction shows that the equation is right.

(c) (a) When $P_i = \frac{1}{2}$, since the probability of success and failure are at the same. P(the number of successful trials is even) $= P$(the number of successful trials is odd) $= \frac{1}{2}$
$P_i = \frac{1}{2} \implies q_i = \frac{1}{2}, b_i = 0 \Rightarrow P(A_n) = \frac{1}{2} \Rightarrow$ *correct*

(b) When $P_i = 1$, A_n directly depends on n's odd or even $\Rightarrow P(A_n) = \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases}$

$P_i = 1 \implies b_i = -\frac{1}{2} \implies P(A_n) = \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases} \Rightarrow$ *correct*

(c) When $P_i = 0$ number of success trial is always 0
$P_i = 0 \implies b_i = \frac{1}{2} \implies P(A_n) = 1 \Rightarrow$ *correct*

Problem 4

(a) solution:

$$\begin{aligned} P &= \binom{5}{2} p^2 (1-p)^3 + \binom{5}{4} p^4 (1-p) \\ &= 10 \times 0.01 \times 0.729 + 5 \times 0.0001 \times 0.9 \\ &= 0.07335 \end{aligned}$$

(b) solution:

$$P = \sum_{k \text{ is even}, k \geq 2}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

(c) solution:

$a = \sum_{k \text{ is even}, k \geq 0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$
$b = \sum_{k \text{ is odd}, k \geq 1}^{n} \binom{n}{k} p^k (1-p)^{n-k}$
Using Binomial theorem, observe that we have

$$a + b = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Problem 4 continued on next page...

$$a - b = \sum_{k=0}^{n} \binom{n}{k} (-p)^k (1-p)^{n-k} = (1-2p)^n$$

Solve this system of two equations with variables to obtain that:

$$a = \frac{1 + (1-2p)^n}{2}, \quad b = \frac{1 - (1-2p)^n}{2}$$

Finally, we have

$$\begin{aligned} \sum_{\substack{k \geq 2, \\ k \text{ even}}} \binom{n}{k} p^k (1-p)^{n-k} &= \sum_{\substack{k \geq 0, \\ k \text{ even}}} \binom{n}{k} p^k (1-p)^{n-k} - (1-p)^n \\ &= \frac{1 + (1-2p)^n}{2} - (1-p)^n \end{aligned}$$

Problem 5

(a)

$$P(X \oplus Y = 1) = p \times \frac{1}{2} + (1-p) \times \frac{1}{2} = \frac{1}{2}$$
$$P(X \oplus Y = 0) = \frac{1}{2}$$
$$X \oplus Y \sim Bern(\frac{1}{2})$$

(b)

$$P(X \oplus Y = 0|X = 0) = (1-p) \times \frac{1}{2} \times \frac{1}{1-p} = \frac{1}{2}$$
$$P(X \oplus Y = 0|X = 1) = p \times \frac{1}{2} \times \frac{1}{p} = \frac{1}{2}$$
$$P(X \oplus Y = 1|X = 0) = (1-p) \times \frac{1}{2} \times \frac{1}{1-p} = \frac{1}{2}$$
$$P(X \oplus Y = 1|X = 1) = p \times \frac{1}{2} \times \frac{1}{p} = \frac{1}{2}$$

whether p is $\frac{1}{2}$ or not
$X \oplus Y$ is independent of X
$P(X \oplus Y = 0|Y = 0) = \frac{1}{2} \times p \neq P(X \oplus Y = 0) = \frac{1}{2}$
if $p \neq \frac{1}{2}, X \oplus Y$ is dependent of Y
if $p = \frac{1}{2}, X \oplus Y$ is independent of Y

(c) we know that X_j that equals 1 when there are odd $X_j = 1$

$$P(Y_j = 1) = P(\text{odd } X_i = 1 \text{ in } X_n)$$

in Problem 3 we know that$P(oddX_j = 1inX_n) = \frac{1}{2}$ when $P(x_j = 1) = \frac{1}{2}$
So

$$P(Y_J = 1) = \frac{1}{2}, \text{ and } Y_J \sim \text{Bern}\left(\frac{1}{2}\right)$$

then proof that pairwise independent

$$P(Y_{J_a} = 1|Y_{J_b} = 1) = \frac{P(Y_{J_a} = 1 \text{ and } Y_{J_b} = 1)}{P(Y_{J_b} = 1)}$$
$$C = J_a \cap J_b, C \text{ can be } \emptyset, D = \{x_i : x_i \in J_a, x_i \notin J_b\}$$

when C isn't $\emptyset$

$$P(Y_{J_a} = 1 \text{ and } Y_{J_b} = 1) = P(Y_{J_b} = 1)P(Y_{J_c} = 1)P(Y_{J_d} = 0) + P(Y_{J_b} = 1)P(Y_{J_c} = 0)P(Y_{J_d} = 1)$$
$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Problem 5 continued on next page...

when C is $\emptyset$

$$
\begin{aligned}
P(Y_{J_a} = 1 \text{ and } Y_{J_b} = 1) &= P(Y_{J_b} = 1)P(Y_{J_d} = 1) \\
&= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
P(Y_{J_a} = 0 | Y_{J_b} = 0) &= \frac{1}{2} \\
P(Y_{J_a} = 1 | Y_{J_b} = 0) &= \frac{1}{2} \\
P(Y_{J_a} = 0 | Y_{J_b} = 1) &= \frac{1}{2} \\
P(Y_{J_a} = 1 | Y_{J_b} = 1) &= \frac{1}{2}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
P(Y_{J_a} = 0 | Y_{J_b} = 0) &= \frac{1}{2} \\
P(Y_{J_a} = 1 | Y_{J_b} = 0) &= \frac{1}{2} \\
P(Y_{J_a} = 0 | Y_{J_b} = 1) &= \frac{1}{2} \\
P(Y_{J_a} = 1 | Y_{J_b} = 1) &= \frac{1}{2}
\end{aligned}
$$

So pairwise independent. Then prove not independent.

$$
\text{If } Y_{J_1} = 1, Y_{J_2} = 1, \text{ then } Y_{J_3} = 0, \text{ where } J_1 = \{X_1\}, J_2 = \{X_2\}, J_3 = \{X_1, X_2\}
$$
$$
P(Y_{J_3} = 0 | Y_{J_1} = 1, Y_{J_2} = 1) \neq P(Y_{J_3} = 0)
$$

So not independent.